Quantum Hermite-Hadamard and quantum Ostrowski type inequalities for s-convex functions with applications

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March 07, 2024

Abstract

In this study, we use quantum calculus to prove Hermite-Hadamard and Ostrowski type inequalities for s-convex functions in the second sense. The newly proven results are also shown to be an extension of comparable results in the literature, like the results of [1, 12, 16]. Furthermore, it is provided that how the newly discovered inequalities can be applied to special means of real numbers.

QUANTUM HERMITE-HADAMARD AND QUANTUM OSTROWSKI TYPE INEQUALITIES FOR s-CONVEX FUNCTIONS WITH APPLICATIONS

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ABSTRACT. In this study, we use quantum calculus to prove Hermite-Hadamard and Ostrowski type inequalities for s-convex functions in the second sense. The newly proven results are also shown to be an extension of comparable results in the literature, like the results of [1,12,16]. Furthermore, it is provided that how the newly discovered inequalities can be applied to special means of real numbers.

1. Introduction

In convex functions theory, Hermite-Hadamard (HH) inequality is very important which was discovered by C. Hermite and J. Hadamard independently (see, also [19], and [35, p.137]),

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x)dx \leq \frac{f(a)+f(b)}{2} \tag{1.1}$$

where f is a convex function. In the case of concave mappings, the above inequality is satisfied in reverse order.

Hudzik and Maligranda defined s-convex functions in the second sense in [24], which may be expressed as: a mapping $f:\mathbb{R}^{+}\rightarrow\mathbb{R}$, where $\mathbb{R}^{+}=[0,\infty)$ is called s-convex in the second sense if

$$f(tx+(1-t)y) \leq t^{s}f(x)+(1-t)^{s}f(y)$$

for all $x,y\in\mathbb{R}^{+}$ and $t\in[0,1]$. After that, Dragomir and Fitzpatrick [18] used this newly class of functions and proved the following HH inequality:

$$2^{s-1}f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_{a}^{b} f(x)\,dx \leq \frac{f(a)+f(b)}{s+1}. \tag{1.2}$$

For more recent integral inequalities for the class of s-convex functions and its generalizations via different integral opertaors, one can consult [11,17,20,21,28,30,34].

On the other hand, several studies have been carried out in the domain of q-analysis, beginning with Euler, in order to achieve proficiency in mathematics that constructs quantum computing q-calculus, which is considered a relationship between physics and mathematics. It has a wide range of applications in mathematics, including combinatorics, simple hypergeometric functions, number theory, orthogonal polynomials, and other sciences, as well as mechanics, relativity theory, and quantum theory [23,27]. Euler is thought to be the inventor of this significant branch of mathematics. He used the q-parameter in Newton's work on infinite series. Later, Jackson presented the q-calculus, which knew no limits calculus, in a methodical manner [22,25]. In 1966, Al-Salam [10] introduced a q-analogue of the q-fractional integral and q-Riemann-Liouville fractional. Since then, the related research has gradually increased. In particular, in 2013, Tariboon and Ntouyas introduced ${}_aD_q$-difference operator and q_a-integral in [36]. In 2020, Bermudo et al. introduced the notion of ${}^bD_q$ derivative and q^b-integral in [12].

Many integral inequalities have been studied using quantum integrals for various types of functions. For example, in [3,6,8,9,12–14,26,31], the authors used ${}_aD_q$, ${}^bD_q$-derivatives and q_a,q^b-integrals to prove HH integral inequalities and their left-right estimates for convex and coordinated convex functions. In [32], Noor et al. presented a generalized version of quantum HH integral inequalities.

Key words and phrases. Hermite-Hadamard inequality, Ostrowski inequality, q-integral, quantum calculus, s-convex functions.

2010 Mathematics Subject Classification 26D10, 26A51, 26D15.

For generalized quasi-convex functions, Nwaeze et al. proved certain parameterized quantum integral inequalities in [33]. Khan et al. proved quantum HH inequality using the green function in [29]. Budak et al. [15], Ali et al. [2,4] and Vivas-Cortez et al. [37] developed new quantum Simpson's and quantum Newton's type inequalities for convex and coordinated convex functions. For quantum Ostrowski's inequalities for convex and co-ordinated convex functions one can consult [5,7,16].

Inspired by this ongoing studies, we offer some new quantum HH type inequalities and Ostrowski type inequalities for s-convex functions in the second sense.

The following is the structure of this paper: A brief overview of the concepts of q-calculus, as well as some related works, is given in Section 2. In Section 3, we show the relationship between the results presented here and comparable results in the literature by proving quantum HH inequalities for s-convex functions in the second sense. Quantum Ostrowski type inequalities for s-convex functions in the second are presented in Section 4. Section 5 concludes with some recommendations for future studies.

2. Preliminaries of q-Calculus and Some Inequalities

In this section, we recollect some formerly regarded concepts. Also, here and further we use $q \in (0,1)$ and the following notation(see [27]):

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + q^2 + \ldots + q^{n-1}, \quad q \in (0,1).$$

In [25], Jackson gave the q-Jackson integral from 0 to b as follows:

$$\int_0^b f(x) \ d_qx \ = (1-q)\, b \sum_{n=0}^{\infty} q^n f(bq^n) \tag{2.1}$$

provided the sum converge absolutely.

Definition 1. *[36] The q_a-derivative of a mapping $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is defined as:*

$$_aD_q f(x) \ = \frac{f(x) - f(qx + (1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{2.2}$$

If $x = a$, we define $_aD_q f(a) = \lim_{x \to a} \ _aD_q f(x)$ if it exists and it is finite.

Definition 2. *[12] The q^b-derivative of a mapping $f : [a,b] \to \mathbb{R}$ at $x \in [a,b]$ is defined as:*

$$^bD_q f(x) \ = \frac{f(qx + (1-q)b) - f(x)}{(1-q)(b-x)}, \ x \neq b.$$

If $x = b$, we define $^bD_q f(b) = \lim_{x \to b} \ ^bD_q f(x)$ if it exists and it is finite.

Definition 3. *[36] The q_a-integral of a mapping $f : [a,b] \to \mathbb{R}$ is defined as:*

$$\int_a^x f(t) \ _ad_qt \ = (1-q)(x-a) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)a),$$

where $x \in [a,b]$.

Definition 4. *[12] The q^b-integral of a mapping $f : [a,b] \to \mathbb{R}$ is defined as:*

$$\int_x^b f(t) \ ^bd_qt \ = (1-q)(b-x) \sum_{n=0}^{\infty} q^n f(q^n x + (1-q^n)b),$$

where $x \in [a,b]$.

In [12], Bermudo et al. established the following quantum HH type inequality:

Theorem 1. *For the convex mapping $f : [a,b] \to \mathbb{R}$, the following inequality holds*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)} \left[\int_a^b f(x) \ _ad_qx \ + \int_a^b f(x) \ ^bd_qx \right] \leq \frac{f(a)+f(b)}{2}. \tag{2.3}$$

In [16], Budak et al. proved the following Ostrowski inequality by using the concepts of quantum derivatives and integrals:

Theorem 2. *Let $f : [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a function and ${}^{b}D_q f$ and ${}_{a}D_q f$ be two continuous and integrable functions on $[a,b]$. If $|{}^{b}D_q f(t)|, |{}_{a}D_q f(t)| \leq M$ for all $t \in [a,b]$, then we have the following quantum Ostrowski inequality*

$$\left| f(x) - \frac{1}{b-a}\left[\int_a^x f(t) \ {}_{a}d_q t \ + \int_x^b f(t) \ {}^{b}d_q t \ \right] \right| \leq \frac{qM}{(b-a)}\left[\frac{(x-a)^2 + (b-x)^2}{[2]_q} \right] \tag{2.4}$$

3. Hermite-Hadamard Inequalities

In this section, we prove HH inequalities for s-convex functions in the second sense using the quantum integrals.

Theorem 3. *Assume that the mapping $f : \mathbb{R}^+ \to \mathbb{R}$ is s-convex in the second sense and $a, b \in \mathbb{R}^+$ with $a < b$. Then the following inequality holds for $s \in (0,1]$:*

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_a^b f(x) \ {}_{a}d_q x + \int_a^b f(x) \ {}^{b}d_q x \right] \leq \frac{f(a)+f(b)}{[s+1]_q}. \tag{3.1}$$

Proof. As f is s-convex in the second sense on $\mathbb{R}^+$ we have

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y),$$

for all $x, y \in \mathbb{R}^+$ and $t \in [0,1]$.

Obverse that

$$2^s f\left(\frac{x+y}{2}\right) \leq f(x) + f(y). \tag{3.2}$$

We get the following, by putting $x = tb + (1-t)a$ and $y = ta + (1-t)b$ in (3.2)

$$2^s f\left(\frac{a+b}{2}\right) \leq f(tb + (1-t)a) + f(ta + (1-t)b).$$

From Definitions 3 and 4, we have

$$2^{s-1} f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_a^b f(x) \ {}_{a}d_q x + \int_a^b f(x) \ {}^{b}d_q x \right]$$

and the first inequality in (3.1) is proved.

To proved the second inequality, we use the s-convexity and we have

$$f(tb + (1-t)a) \leq t^s f(b) + (1-t)^s f(a) \tag{3.3}$$

and

$$f(ta + (1-t)b) \leq t^s f(a) + (1-t)^s f(b). \tag{3.4}$$

By adding (3.3) and (3.4), from Definition 3 and 4, we have

$$\frac{1}{2(b-a)}\left[\int_a^b f(x) \ {}_{a}d_q x + \int_a^b f(x) \ {}^{b}d_q x \right] \leq \frac{f(a)+f(b)}{[s+1]_q}$$

and the proof is completed. □

Remark 1. *If we set $s = 1$ in Theorem 3, then we recapture the inequality (2.3).*

Remark 2. *In Theorem 3, if we take the limit as $q \to 1^-$, then inequality (3.1) becomes the inequality (1.2).*

4. Ostrowski's Inequalities

In this section, we prove Ostrowski's type inequalities for s-convex functions in the second sense. We use the following lemma to prove the new results.

Lemma 1. *[16] Let $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ be a function. If ${}^{b}D_{q}f$ and ${}_{a}D_{q}f$ are two continuous and integrable functions on $[a,b]$, then for all $x\in[a,b]$ we have*

$$\begin{aligned}
&f(x)-\frac{1}{b-a}\left[\int_{a}^{x}f(t)\ {}_{a}d_{q}t\ +\int_{x}^{b}f(t)\ {}^{b}d_{q}t\ \right] \quad (4.1)\\
&= \frac{q(x-a)^{2}}{b-a}\int_{0}^{1}t\ {}_{a}D_{q}\ f(tx+(1-t)a)\,d_{q}t\\
&-\frac{q(b-x)^{2}}{b-a}\int_{0}^{1}t\ {}^{b}D_{q}\ f(tx+(1-t)b)\,d_{q}t.
\end{aligned}$$

Theorem 4. *Assume that the mapping $f:I\subset\mathbb{R}^{+}\to\mathbb{R}$ is differentiable and $a,b\in I$ with $a<b$. If $|{}_{a}D_{q}f|$ and $|{}^{b}D_{q}f|$ are s-convex mappings in the second sense, then the following inequality holds:*

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\left[\int_{a}^{x}f(t)\ {}_{a}d_{q}t\ +\int_{x}^{b}f(t)\ {}^{b}d_{q}t\ \right]\right| \quad (4.2)\\
&\leq \frac{q(x-a)^{2}}{b-a}\left[\frac{1}{[s+2]_{q}}\left|\ {}_{a}D_{q}\ f(x)\right|+\Theta_{1}\left|\ {}_{a}D_{q}\ f(a)\right|\right]\\
&+\frac{q(b-x)^{2}}{b-a}\left[\frac{1}{[s+2]_{q}}\left|\ {}^{b}D_{q}\ f(x)\right|+\Theta_{1}\left|\ {}^{b}D_{q}\ f(b)\right|\right]
\end{aligned}$$

where

$$\Theta_{1}=\int_{0}^{1}t(1-t)^{s}\,d_{q}t.$$

Proof. From Lemma 1 and properties of the modulus, we have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\left[\int_{a}^{x}f(t)\ {}_{a}d_{q}t\ +\int_{x}^{b}f(t)\ {}^{b}d_{q}t\ \right]\right| \quad (4.3)\\
&\leq \frac{q(x-a)^{2}}{b-a}\int_{0}^{1}t\left|\ {}_{a}D_{q}\ f(tx+(1-t)a)\right|d_{q}t+\frac{q(b-x)^{2}}{b-a}\int_{0}^{1}t\left|\ {}^{b}D_{q}\ f(tx+(1-t)b)\right|d_{q}t.
\end{aligned}$$

Since the mappings $|{}_{a}D_{q}f|$ and $|{}^{b}D_{q}f|$ are s-convex in the second sense, therefore

$$\begin{aligned}
(4.4)\int_{0}^{1}t\left|\ {}_{a}D_{q}\ f(tx+(1-t)a)\right|d_{q}t &\leq \int_{0}^{1}t^{s+1}\left|\ {}_{a}D_{q}\ f(x)\right|d_{q}t+\int_{0}^{1}t(1-t)^{s}\left|\ {}_{a}D_{q}\ f(a)\right|d_{q}t\\
&= \frac{1}{[s+2]_{q}}\left|\ {}_{a}D_{q}\ f(x)\right|+\Theta_{1}\left|\ {}_{a}D_{q}\ f(a)\right|
\end{aligned}$$

and

$$\begin{aligned}
(4.5)\int_{0}^{1}t\left|\ {}^{b}D_{q}\ f(tx+(1-t)b)\right|d_{q}t &\leq \int_{0}^{1}t^{s+1}\left|\ {}^{b}D_{q}\ f(x)\right|d_{q}t+\int_{0}^{1}t(1-t)^{s}\left|\ {}^{b}D_{q}\ f(b)\right|d_{q}t\\
&= \frac{1}{[s+2]_{q}}\left|\ {}^{b}D_{q}\ f(x)\right|+\Theta_{1}\left|\ {}^{b}D_{q}\ f(b)\right|.
\end{aligned}$$

We obtain the resultant inequality (4.2) by putting (4.4) and (4.5) in (4.3). □

Remark 3. *If we set $s=1$ in Theorem 4, then we obtain the following inequality*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\left[\int_a^x f(t) \ {}_a d_q t \ + \int_x^b f(t) \ {}^b d_q t \ \right]\right| \\
\leq \ & \frac{q}{(b-a)(1+q)(1+q+q^2)}\left[(x-a)^2\left((1+q)\left| \ {}_aD_q \ f(x)\right| + q^2\left| \ {}_aD_q \ f(a)\right|\right)\right. \\
&\left. +(b-x)^2\left((1+q)\left| \ {}^bD_q \ f(x)\right| + q^2\left| \ {}^bD_q \ f(b)\right|\right)\right]
\end{aligned}$$

which is given by Budak et al. in [16].

Corollary 1. *If we assume $|{}_aD_q f(x)|$, $|{}_aD_q f(a)| \leq M$ in Theorem 4, then we have following quantum Ostrowski's type inequality for s-convex functions in the second sense:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\left[\int_a^x f(t) \ {}_a d_q t \ + \int_x^b f(t) \ {}^b d_q t \ \right]\right| \\
\leq \ & \frac{Mq}{b-a}\left(\frac{1}{[s+2]_q} + \Theta_1\right)\left[(x-a)^2 + (b-x)^2\right].
\end{aligned} \tag{4.6}$$

Remark 4. *If we set $s=1$ in Corollary 1, then we recapture inequality (2.4).*

Remark 5. *In Corollary 1, if we take the limit as $q \to 1^-$, then Corollary 1 reduces to [1, Theorem 2].*

Theorem 5. *Assume that the mapping $f : I \subset \mathbb{R}^+ \to \mathbb{R}$ is differentiable and $a, b \in I$ with $a < b$. If $|{}_aD_q f|^{p_1}$ and $|{}^bD_q f|^{p_1}$, $p_1 \geq 1$ are s-convex mappings in the second sense, then the following inequality holds:*

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\left[\int_a^x f(t) \ {}_a d_q t \ + \int_x^b f(t) \ {}^b d_q t \ \right]\right| \\
\leq \ & \frac{q}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{p_1}}\left[(x-a)^2\left(\frac{1}{[s+2]_q}\left| \ {}_aD_q \ f(x)\right|^{p_1} + \Theta_1\left| \ {}_aD_q \ f(a)\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\left. +(b-x)^2\left(\frac{1}{[s+2]_q}\left| \ {}^bD_q \ f(x)\right|^{p_1} + \Theta_1\left| \ {}^bD_q \ f(b)\right|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned} \tag{4.7}$$

Proof. From Lemma 1, using properties of the modulus and power mean inequality, we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\left[\int_a^x f(t) \ {}_a d_q t \ + \int_x^b f(t) \ {}^b d_q t \ \right]\right| \\
\leq \ & \frac{q(x-a)^2}{b-a}\int_0^1 t\left| \ {}_aD_q \ f(tx+(1-t)a)\right| d_qt + \frac{q(b-x)^2}{b-a}\int_0^1 t\left| \ {}^bD_q \ f(tx+(1-t)b)\right| d_qt \\
\leq \ & \frac{q(x-a)^2}{b-a}\left(\int_0^1 td_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\left| \ {}_aD_q \ f(tx+(1-t)a)\right|^{p_1} d_qt\right)^{\frac{1}{p_1}} \\
&+\frac{q(b-x)^2}{b-a}\left(\int_0^1 td_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\left| \ {}^bD_q \ f(tx+(1-t)b)\right|^{p_1} d_qt\right)^{\frac{1}{p_1}}.
\end{aligned} \tag{4.8}$$

Since the mappings $|_aD_qf|^{p_1}$ and $|^bD_qf|^{p_1}$ are s-convex in the second sense, therefore

$$\left(\int_0^1 td_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\left|\ {}_aD_q\ f\left(tx+(1-t)\,a\right)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}} \tag{4.9}$$
$$\leq \left(\frac{1}{[2]_q}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[s+2]_q}\left|\ {}_aD_q\ f\left(x\right)\right|^{p_1}+\Theta_1\left|\ {}_aD_q\ f\left(a\right)\right|^{p_1}\right)^{\frac{1}{p_1}}$$

and

$$\left(\int_0^1 td_qt\right)^{1-\frac{1}{p_1}}\left(\int_0^1 t\left|\ {}^bD_q\ f\left(tx+(1-t)\,b\right)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}} \tag{4.10}$$
$$\leq \left(\frac{1}{[2]_q}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[s+2]_q}\left|\ {}^bD_q\ f\left(x\right)\right|^{p_1}+\Theta_1\left|\ {}^bD_q\ f\left(b\right)\right|^{p_1}\right)^{\frac{1}{p_1}}.$$

We obtain the resultant inequality (4.7) by putting (4.9) and (4.10) in (4.8). □

Remark 6. *If we set $s=1$ in Theorem 5, then we obtain the following inequality*

$$\left|f\left(x\right)-\frac{1}{b-a}\left[\int_a^x f\left(t\right)\ {}_ad_qt\ +\int_x^b f\left(t\right)\ {}^bd_qt\right]\right|$$
$$\leq \frac{q}{\left(b-a\right)[2]_q}\left[\left(x-a\right)^2\left(\frac{[2]_q\left|\ {}_aD_q\ f\left(x\right)\right|^{p_1}+q^2\left|\ {}_aD_q\ f\left(a\right)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right.$$
$$\left.+\left(b-x\right)^2\left(\frac{[2]_q\left|\ {}^bD_q\ f\left(x\right)\right|^{p_1}+q^2\left|\ {}^bD_q\ f\left(b\right)\right|^{p_1}}{[3]_q}\right)^{\frac{1}{p_1}}\right]$$

which is proved by Budak et al. in [16].

Corollary 2. *If we assume $|_aD_qf\left(x\right),\ |_aD_qf\left(a\right)|\leq M$ in Theorem 5, then we have following quantum Ostrowski's type inequality for s-convex functions in the second sense:*

$$\left|f\left(x\right)-\frac{1}{b-a}\left[\int_a^x f\left(t\right)\ {}_ad_qt\ +\int_x^b f\left(t\right)\ {}^bd_qt\right]\right|$$
$$\leq \frac{Mq}{b-a}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{p_1}}\left(\frac{1}{[s+2]_q}+\Theta_1\right)^{\frac{1}{p_1}}\left[\left(x-a\right)^2+\left(b-x\right)^2\right].$$

Remark 7. *In Corollary 2, if we take the limit as $q\to 1^-$, then Corollary 2 reduces to [1, Theorem 4].*

Theorem 6. *Assume that the mapping $f:I\subset\mathbb{R}^+\to\mathbb{R}$ is differentiable and $a,b\in I$ with $a<b$. If $|_aD_qf|^{p_1}$ and $|^bD_qf|^{p_1}$, $p_1>1$ are s-convex mappings in the second sense, then the following inequality* *holds:*

$$\left|f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_ad_qt\ +\int_x^b f(t)\ {}^bd_qt\ \right]\right| \tag{4.11}$$
$$\leq \frac{q}{b-a}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[(x-a)^2\left(\frac{1}{[s+1]_q}\left(\left|\ {}_aD_q\ f(x)\right|^{p_1}+\left|\ {}_aD_q\ f(a)\right|^{p_1}\right)\right)^{\frac{1}{p_1}}\right.$$
$$\left.+(b-x)^2\left(\frac{1}{[s+1]_q}\left(\left|\ {}^bD_q\ f(x)\right|^{p_1}+\left|\ {}^bD_q\ f(b)\right|^{p_1}\right)\right)^{\frac{1}{p_1}}\right],$$

where $\frac{1}{r_1}+\frac{1}{p_1}=1$.

Proof. From Lemma 1, using properties of the modulus and Hölder's inequality, we have

$$\left|f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_ad_qt\ +\int_x^b f(t)\ {}^bd_qt\right]\right| \tag{4.12}$$
$$\leq \frac{q(x-a)^2}{b-a}\int_0^1 t\left|\ {}_aD_q\ f(tx+(1-t)a)\right|d_qt+\frac{q(b-x)^2}{b-a}\int_0^1 t\left|\ {}^bD_q\ f(tx+(1-t)b)\right|d_qt$$
$$\leq \frac{q(x-a)^2}{b-a}\left(\int_0^1 t^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1\left|\ {}_aD_q\ f(tx+(1-t)a)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}$$
$$+\frac{q(b-x)^2}{b-a}\left(\int_0^1 t^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1\left|\ {}^bD_q\ f(tx+(1-t)b)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}}.$$

Since the mappings $|{}_aD_qf|^{p_1}$ and $|{}^bD_qf|^{p_1}$ are s-convex in the second sense, therefore

$$\left(\int_0^1 t^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1\left|\ {}_aD_q\ f(tx+(1-t)a)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}} \tag{4.13}$$
$$\leq \left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\frac{1}{[s+1]_q}\left(\left|\ {}_aD_q\ f(x)\right|^{p_1}+\left|\ {}_aD_q\ f(a)\right|^{p_1}\right)\right)$$

and

$$\left(\int_0^1 t^{r_1}d_qt\right)^{\frac{1}{r_1}}\left(\int_0^1\left|\ {}^bD_q\ f(tx+(1-t)b)\right|^{p_1}d_qt\right)^{\frac{1}{p_1}} \tag{4.14}$$
$$\leq \left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\frac{1}{[s+1]_q}\left(\left|\ {}^bD_q\ f(x)\right|^{p_1}+\left|\ {}^bD_q\ f(b)\right|^{p_1}\right)\right).$$

We obtain the resultant inequality (4.11) by putting (4.13) and (4.14) in (4.12). □

Remark 8. *If we set $s=1$ in Theorem 6, then we obtain the following inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_ad_qt\ +\int_x^b f(t)\ {}^bd_qt\ \right]\right|\\ \leq\ & \frac{q}{b-a}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[(x-a)^2\left(\frac{\left|\ {}_aD_q\ f(x)\right|^{p_1}+q\left|\ {}_aD_q\ f(a)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right.\\ &\left.+(b-x)^2\left(\frac{\left|\ {}^bD_q\ f(x)\right|^{p_1}+q\left|\ {}^bD_q\ f(b)\right|^{p_1}}{[2]_q}\right)^{\frac{1}{p_1}}\right]\end{aligned}$$

which is proved by Budak et al. in [16].

Corollary 3. *If we assume $|{}_aD_qf(x)|,|{}_aD_qf(a)|\leq M$ in Theorem 6, then we have following quantum Ostrowski's type inequality for s-convex functions in the second sense:*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_ad_qt\ +\int_x^b f(t)\ {}^bd_qt\ \right]\right|\\ \leq\ & \frac{Mq}{b-a}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\frac{2}{[s+1]_q}\right)^{\frac{1}{p_1}}\left[(x-a)^2+(b-x)^2\right].\end{aligned}\tag{4.15}$$

Remark 9. *In Corollary 3, if we set $s=1$, then we recapture the following inequality*

$$\begin{aligned}&\left|f(x)-\frac{1}{b-a}\left[\int_a^x f(t)\ {}_ad_qt\ +\int_x^b f(t)\ {}^bd_qt\ \right]\right|\\ \leq\ & \frac{qM}{b-a}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left[(x-a)^2+(b-x)^2\right]\end{aligned}$$

which is obtained by Budak et al. [16].

Remark 10. *In Corollary 3, if we take the limit as $q\to 1^-$, then Corollary 3 reduces to [1, Theorem 3].*

5. Applications to Special Means

For arbitrary positive numbers κ_1,κ_2 ($\kappa_1\neq\kappa_2$), we consider the means as follows:

(1) The arithmetic mean

$$\mathcal{A}=\mathcal{A}(\kappa_1,\kappa_2)=\frac{\kappa_1+\kappa_2}{2}.$$

(2) The logarithmic mean

$$\mathcal{L}_p^p=\mathcal{L}_p^p(\kappa_1,\kappa_2)=\frac{\kappa_2^{p+1}-\kappa_1^{p+1}}{(p+1)(\kappa_2-\kappa_1)}.$$

Proposition 1. *For $0<a<b$ and $0<q<1$, the following inequality is true:*

$$\begin{aligned}&\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b)-\mathcal{A}(\Bbbk_1,\Bbbk_2)\right]\right|\\ \leq\ & \frac{q(b-a)}{2}\left[\frac{1}{[s+2]_q}\left\{\mathcal{L}_s^s\left(q\frac{a+b}{2}+(1-q)a,\frac{a+b}{2}\right)\right.\right.\\ &\left.\left.+\mathcal{L}_s^s\left(q\frac{a+b}{2}+(1-q)b,\frac{a+b}{2}\right)\right\}+2\Theta_1\mathcal{A}(a^s,b^s)\right],\end{aligned}$$

where

$$\mathbb{k}_1 = (1-q)\sum_{n=0}^{\infty} q^n \left(q^n \frac{a+b}{2} + (1-q^n)\, a\right)^{s+1},$$
$$\mathbb{k}_2 = (1-q)\sum_{n=0}^{\infty} q^n \left(q^n \frac{a+b}{2} + (1-q^n)\, b\right)^{s+1}.$$

Proof. The inequality (4.2) in Theorem 4 with $x = \frac{a+b}{2}$ for $f(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ and $s \in (0,1)$ leads to this conclusion. □

Proposition 2. *For $0 < a < b$ and $0 < q < 1$, the following inequality is true:*

$$\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b) - \mathcal{A}(\mathbb{k}_1, \mathbb{k}_2)\right]\right| \leq \frac{Mq(b-a)}{2}\left[\frac{1}{[s+2]_q} + \Theta_1\right].$$

Proof. The inequality (4.6) in Corollary 1 with $x = \frac{a+b}{2}$ for $f(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ and $s \in (0,1)$ leads to this conclusion. □

Proposition 3. *For $0 < a < b$ and $0 < q < 1$, the following inequality is true:*

$$\begin{aligned}
&\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b) - \mathcal{A}(\mathbb{k}_1, \mathbb{k}_2)\right]\right| \\
\leq\ & \frac{q(b-a)}{2}\left(\frac{1}{[2]_q}\right)^{1-\frac{1}{p_1}} \left[\left(\frac{1}{[s+2]_q}\left|\mathcal{L}_s^s\left(q\frac{a+b}{2} + (1-q)\, a, \frac{a+b}{2}\right)\right|^{p_1} + \Theta_1 \left|a^s\right|^{p_1}\right)^{\frac{1}{p_1}}\right. \\
&\left. + \left(\frac{1}{[s+2]_q}\left|\mathcal{L}_s^s\left(q\frac{a+b}{2} + (1-q)\, b, \frac{a+b}{2}\right)\right|^{p_1} + \Theta_1 \left|b^s\right|^{p_1}\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Proof. The inequality (4.7) in Theorem 5 with $x = \frac{a+b}{2}$ for $f(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ and $s \in (0,1)$ leads to this conclusion. □

Proposition 4. *For $0 < a < b$ and $0 < q < 1$, the following inequality is true:*

$$\begin{aligned}
&\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b) - \mathcal{A}(\mathbb{k}_1, \mathbb{k}_2)\right]\right| \\
\leq\ & \frac{q(b-a)}{2}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}} \left[\left(\frac{1}{[s+1]_q}\left(\left|\mathcal{L}_s^s\left(q\frac{a+b}{2} + (1-q)\, a, \frac{a+b}{2}\right)\right|^{p_1} + \left|a^s\right|^{p_1}\right)\right)^{\frac{1}{p_1}}\right. \\
&\left. + \left(\frac{1}{[s+1]_q}\left(\left|\mathcal{L}_s^s\left(q\frac{a+b}{2} + (1-q)\, b, \frac{a+b}{2}\right)\right|^{p_1} + \left|b^s\right|^{p_1}\right)\right)^{\frac{1}{p_1}}\right].
\end{aligned}$$

Proof. The inequality (4.11) in Theorem 6 with $x = \frac{a+b}{2}$ for $f(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ and $s \in (0,1)$ leads to this conclusion. □

Proposition 5. *For $0 < a < b$ and $0 < q < 1$, the following inequality is true:*

$$\left|\frac{1}{s+1}\left[\mathcal{A}^{s+1}(a,b) - \mathcal{A}(\mathbb{k}_1, \mathbb{k}_2)\right]\right| \leq \frac{Mq(b-a)}{2}\left(\frac{1}{[r_1+1]_q}\right)^{\frac{1}{r_1}}\left(\frac{2}{[s+1]_q}\right)^{\frac{1}{p_1}}.$$

Proof. The inequality (4.15) in Corollary 3 with $x = \frac{a+b}{2}$ for $f(x) = \frac{x^{s+1}}{s+1}$, where $x > 0$ and $s \in (0,1)$ leads to this conclusion. □

6. Conclusion

In this investigation, Hermite-Hadamard and Ostrowski type inequalities for s-convex mappings in the second sense are derived, by applying quantum integrals. It is also showed that the results established in this paper are potential generalization of the existing comparable results in the literature. As future directions, one can find similar inequalities for co-ordinated s-convex functions in the second sense.

7. Acknowledgements

The authors would like to express their sincere thanks to the editor and the anonymous reviewers for their helpful comments and suggestions.

Funding

The work was supported by the Natural Science Foundation of China (Grant No. 11971241).

Availability of data and materials

Not applicable.

Competing interests

The authors declare that they have no competing interests.

Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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