backward stochastic differential equations driven by both standard and fractional Brownian motions with time deplayed generators

Sadibou Aidara[1], Yaya sagna[2], and Lamine Sylla[2]

[1]Université Gaston Berger UFR de Sciences Appliquées et de Technologie
[2]Universite Gaston Berger UFR de Sciences Appliquees et de Technologie

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Abstract

This paper deals with a class of backward stochastic differential equations driven by both standard and fractional Brownian motions with time deplayed generators. In this type of equation, a generator at time t can depend on the values of a solution in the past, weighted with a time delay function, for instance, of the moving average type. We establish an existence and uniqueness result of solutions for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator.

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BSDEs driven by both standard and fractional Brownian motions with time deplayed generators

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Abstract. This paper deals with a class of backward stochastic differential equations driven by both standard and fractional Brownian motions with time deplayed generators. In this type of equation, a generator at time t can depend on the values of a solution in the past, weighted with a time delay function, for instance, of the moving average type. We establish an existence and uniqueness result of solutions for a sufficiently small time horizon or for a sufficiently small Lipschitz constant of a generator.

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1. Intoduction

Nonlinear Backward stochastic differential equations (BSDEs in short) have been first introduced by Pardoux and Peng [12] in order to give a probabilistic interpretation for the solutions of semi-linear Partial differential equations (PDEs). Since this first result, it has been widely recognized that BSDEs provide a useful framework for formulating a lot of mathematical problems such as used in financial mathematics, optimal control, stochastic games and partial differential equations. We also mention that, following Pardoux and Peng [12], many papers were devoted to improving the results of Pardoux and Peng [12] by weakening the Lipschitz conditions on coefficients (for example, see Aidara [2], Wang and Huang [13], Aidara and Sow [1], Mao [10], Aidara and Sagna [3]). Based on the above important applications, specially in the field of Finance, and optimal control, recently in [6], Delong and Imkeller introduced BSDEs with time delayed generators defined by

$$Y(t)=\xi+\int_t^T f\left(s,Y_s,Z_s\right)ds-\int_t^T Z(t)dW_s \quad 0\leq t\leq T, \tag{1.1}$$

where for all $t\in[0,T]$,

$$(Y_t,Z_t)=(Y(t+u),Z(t+u))_{-T\leq u\leq 0} \tag{1.2}$$

represents all the past of the process solution until t. An existence and uniqueness result under a small global Lipschitz K or a small horizon time T has been derived. This restrictive is inevitable, since authors with two examples, proved that this result can not be extended in the general case. However, for some special class of generators, existence and uniqueness result may still hold for arbitrary both time horizon and Lipschitz constant.

On the other hand, several authors investigates BSDEs with respect to fractional Brownian motion (fractional BSDEs in short). In [4], Bender gaves one of the earliest result on fractional BSDEs. The author established an explicit solution of a class of linear fractional BSDEs with arbitrary Hurst parameter H. This is done essentially by means of solution of a specific linear parabolic PDE. There are two major obstacles depending on the properties of fractional Brownian motion: Firstly, the fractional Brownian motion is not a semimartingale except for the case of Brownian motion ($H=1/2$), hence the classical Itô calculus based on semimartingales cannot be transposed directly to the fractional case. Secondly, there is no martingale representation theorem with respect to the fractional Brownian motion. Studing nonlinear fractional BSDEs, Hu and Peng [9] overcame successfully the second obstacle in the case $H>1/2$ by means of the quasi-conditional expectation. The authors prove existence and uniqueness of the solution but with some restrictive assumptions on the generator. In this same spirit, Maticiuc and Nie [11] interesting in backward stochastic variational inequalities, improved this first result by weakening the required condition on the drift of the stochastic equation.

In this paper, our aim is to generalize the result established in [6] to the following driven by both standard and fractional Brownian motions (SFBSDEs in short):

$$Y(t)=\xi+\int_t^T f\left(s,\eta_s,Y_s,Z_{1,s},Z_{2,s}\right)ds-\int_t^T Z_1(s)dB_s-\int_t^T Z_2(s)dB_s^H,\ \ \forall t\in[0,T],$$

where $\Big(Y_t,Z_{1,t},Z_{2,t}\Big)=\Big(Y(t+u),Z_1(t+u),Z_2(t+u)\Big)_{-T\leq u\leq 0}$, B is a Brownian motion, B^H is a fractional Brownian motion with Hurst parameter greater than $1/2$ and η is a stochastic process given by

$$\eta_t=\eta_0+\int_0^t b(s)ds+\int_0^t\sigma_1(s)dB_s+\int_0^t\sigma_2(s)dB_s^H.$$

We establish an existence and uniqueness result of solutions for this kind of BSDEs by a Picard-type iteration.

This paper is organized as follows. In section 2, we introduce some preliminaries. In section 3, we prove existence and uniqueness of solutions of SFBSDEs with time deplayed generators.

2. Fractional Stochastic calculus

Let Ω be a non-empty set, F a $\sigma-$algebra of sets Ω, $\mathbf{P}$ a probability measure defined on $\mathcal{F}$ and $\{\mathcal{F}_t,\ t \in [0,T]\}$ a $\sigma-$algebra generated by both standard and fractional Brownian motions. The triplet $(\Omega, \mathcal{F}, \mathbf{P})$ defines a probability space and $\mathbf{E}$ the mathematical expectation with respect to the probability measure $\mathbf{P}$.

The fractional Brownian motion $\left(B_t^H\right)_{t\geq 0}$ with Hurst parameter $H \in (0,1)$ is a zero mean Gaussian process with the covariance function

$$\mathbf{E}[B_t^H B_s^H] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t,s \geq 0.$$

Suppose that the process $\left(B_t^H\right)_{t\geq 0}$ is independent of the standard Brownian motion $(B_t)_{t\geq 0}$. Throughout this paper it is assumed that $H \in (1/2, 1)$ is arbitrary but fixed.

Denote $\phi(t,s) = H(2H-1)|t-s|^{2H-2},\ (t,s) \in \mathbf{R}^2$.
Let ξ and η be measurable functions on $[0,T]$. Define

$$\langle \xi, \eta \rangle_t = \int_0^t \int_0^t \phi(u,v)\xi(u)\eta(v)dudv \ \text{ and } \ \|\xi\|_t^2 = \langle \xi, \xi \rangle_t.$$

Note that, for any $t \in [0,T]$, $\langle \xi, \eta \rangle_t$ is a Hilbert scalar product. Let $\mathcal{H}$ be the completion of the set of continuous functions under this Hilbert norm $\|\cdot\|_t$ and $(\xi_n)_n$ be a sequence in $\mathcal{H}$ such that $\langle \xi_i, \xi_j \rangle_T = \delta_{ij}$. Let $\mathscr{P}_T^H$ be the set of all polynomials of fractional Brownian motion. Namely, $\mathscr{P}_T^H$ contains all elements of the form

$$F(\omega) = f\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_n(t)dB_t^H\right)$$

where f is a polynomial function of n variables. The Malliavin derivative D_t^H of F is given by

$$D_s^H F = \sum_{i=1}^n \frac{\partial f}{\partial x_i}\left(\int_0^T \xi_1(t)dB_t^H, \int_0^T \xi_2(t)dB_t^H, \dots, \int_0^T \xi_n(t)dB_t^H\right)\xi_i(s) \quad 0 \leq s \leq T.$$

Similarly, we can define the Malliavin derivative $D_t G$ of the Brownian functional

$$G(\omega) = f\left(\int_0^T \xi_1(t)dB_t, \int_0^T \xi_2(t)dB_t, \dots, \int_0^T \xi_n(t)dB_t\right).$$

The divergence operator D^H is closable from $L^2(\Omega, F, \mathbf{P})$ to $L^2(\Omega, F, \mathbf{P}, H)$. Hence we can consider the space $\mathbb{D}_{1,2}$ is the completion of $\mathscr{P}_T^H$ with the norm

$$||F||_{1,2}^2 = \mathbf{E}|F|^2 + \mathbf{E}||D_s^H F||_T^2.$$

Now we introduce the Malliavin ϕ-derivative $\mathbb{D}_t^H$ of F by

$$\mathbb{D}_t^H F = \int_0^T \phi(t,s) D_s^H F ds.$$

We have the following (see[[8], Proposition 6.25]):

Theorem 2.1. *Let $F : (\Omega, \mathcal{F}, \mathbf{P}) \longrightarrow \mathcal{H}$ be a stochastic processes such that*

$$\mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T |\mathbb{D}_s^H F_t|^2 dsdt \right) < +\infty.$$

Then, the Itô-Skorohod type stochastic integral denoted by $\int_0^T F_s dB_s^H$ exists in $L^2(\Omega, \mathcal{F}, \mathbf{P})$ and satisfies

$$\mathbf{E}\left(\int_0^T F_s dB_s^H \right) = 0 \quad \textit{and}$$

$$\mathbf{E}\left(\int_0^T F_s dB_s^H \right)^2 = \mathbf{E}\left(\|F\|_T^2 + \int_0^T \int_0^T \mathbb{D}_s^H F_t \mathbb{D}_t^H F_s dsdt \right).$$

Let us recall the fractional Itô formula (see[[7], Theorem 3.1]).

Theorem 2.2. *Let $\sigma_1 \in L^2([0,T])$ and $\sigma_2 \in \mathcal{H}$ be deterministic continuous functions.*
Assume that $\|\sigma_2\|_t$ is continuously differentiable as a function of $t \in [0,T]$. Denote

$$X_t = X_0 + \int_0^t \alpha(s) ds + \int_0^t \sigma_1(s) dB_s + \int_0^t \sigma_2(s) dB_s^H,$$

where X_0 is a constant, $\alpha(t)$ is a deterministic function with $\int_0^t |\alpha(s)| ds < +\infty$. Let $F(t,x)$ be continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then

$$\begin{aligned} F(t, X_t) = F(0, X_0) &+ \int_0^t \frac{\partial F}{\partial s}(s, X_s) ds + \int_0^t \frac{\partial F}{\partial x}(s, X_s) dX_s \\ &+ \frac{1}{2} \int_0^t \frac{\partial^2 F}{\partial x^2}(s, X_s) \left[\sigma_1^2(s) + \frac{d}{ds} \|\sigma_2\|_s^2 \right] ds, \quad 0 \le t \le T. \end{aligned}$$

Let us finish this section by giving a fractional Itô chain rule (see[[7], Theorem 3.2]).

Theorem 2.3. *Assume that for $i = 1, 2$, the processes μ_i, α_i and ϑ_i, satisfy*

$$\mathbf{E}\left[\int_0^T \mu_i^2(s) ds + \int_0^T \alpha_i^2(s) ds + \int_0^T \vartheta_i^2(s) ds \right] < +\infty.$$

Suppose that $D_t\alpha_i(s)$ and $\mathbb{D}_t^H\vartheta_i(s)$ are continuously differentiable with respect to $(s,t)\in[0,T]^2$ for almost all $\omega\in\Omega$. Let X_t and Y_t be two processes satisfying

$$X_t = X_0+\int_0^t \mu_1(s)ds+\int_0^t \alpha_1(s)dB_s+\int_0^t \vartheta_1(s)dB_s^H, \qquad 0\le t\le T,$$

$$Y_t = Y_0+\int_0^t \mu_2(s)ds+\int_0^t \alpha_2(s)dB_s+\int_0^t \vartheta_2(s)dB_s^H, \qquad 0\le t\le T.$$

If for $i=1,2$, the following conditions hold:

$$\mathbf{E}\left[\int_0^T |D_t\alpha_i(s)|^2 dsdt\right]<+\infty, \qquad \mathbf{E}\left[\int_0^T |\mathbb{D}_t^H\vartheta_i(s)|^2 dsdt\right]<+\infty,$$

then

$$X_tY_t = X_0Y_0+\int_0^t X_s dY_s+\int_0^t Y_s dX_s + \int_0^t \left[\alpha_1(s)D_sY_s+\alpha_2(s)D_sX_s+\vartheta_1(s)\mathbb{D}_s^H Y_s+\vartheta_2(s)\mathbb{D}_s^H X_s\right]ds,$$

which may be written formally as

$$d(X_tY_t) = X_tdY_t+Y_tdX_t+\left[\alpha_1(t)D_tY_t+\alpha_2(t)D_tX_t+\vartheta_1(t)\mathbb{D}_t^H Y_t+\vartheta_2(t)\mathbb{D}_t^H X_t\right]dt.$$

We are now in position to move on to study our main subject.

3. SFBSDEs with time deplayed generators

3.1. Definitions and notations

Let us consider

$$\eta_t = \eta_0+b(t)+\int_0^t \sigma_1(s)dB_s+\int_0^t \sigma_2(s)dB_s^H, \quad 0\le t\le T$$

where the coefficients η_0, b, σ_1 and σ_2 satisfy:

- η_0 is a given constant,
- $b,\sigma_1,\sigma_2:[0,T]\to\mathbf{R}$ are deterministic continuous functions, σ_1 and σ_2 are differentiable and $\sigma_1(t)\neq 0$, $\sigma_2(t)\neq 0$ such that

$$|\sigma|_t^2 = \int_0^t \sigma_1^2(s)ds+\|\sigma_2\|_t^2, \quad 0\le t\le T, \tag{3.1}$$

where $$\|\sigma_2\|_t^2 = H(2H-1)\int_0^t\int_0^t |u-v|^{2H-2}\sigma_2(u)\sigma_2(v)dudv.$$

Let $$\hat\sigma_2(t)=\int_0^t \phi(t,v)\sigma_2(v)dv, \quad 0\le t\le T.$$

The next Remark will be useful in the sequel.

Remark 3.1. The function $|\sigma|_t^2$ defined by eq.(3.1) is continuously differentiable with respect to t on $[0,T]$, and

a) $\frac{d}{dt}|\sigma|_t^2 = \sigma_1^2(t)+\frac{d}{dt}\|\sigma_2\|_t^2 = \sigma_1^2(t)+\sigma_2(t)\hat\sigma_2(t)>0, \quad 0\le t\le T.$

b) for a suitable constant $C_0 > 0$, $\inf_{0\le t\le T} \frac{\hat{\sigma}_2(t)}{\sigma_2(t)} \ge C_0$.

The goal of this paper is to study backward stochastic differential equations driven by both standard and fractional Brownian motions with time delayed generators which dynamics is described by

$$Y(t) = \xi + \int_t^T f(s,\eta_s,Y_s,Z_{1,s},Z_{2,s})ds - \int_t^T Z_1(s)dB_s - \int_t^T Z_2(s)dB_s^H, \quad 0 \le t \le T, \tag{3.2}$$

where the generator f at time $s \in [0,T]$ depends on the past values of the solution denoted by $Y_s := Y(s+u)_{-T\le u\le 0}$, $Z_{1,s} := Z_1(s+u)_{-T\le u\le 0}$ and $Z_{2,s} := Z_2(s+u)_{-T\le u\le 0}$. We always set $Z_1(t) = 0$, $Z_2(t) = 0$ and $Y(t) = Y(0)$ for $t < 0$.

Before giving the definition of the solution for the above equation, we introduce the following sets (where $\mathbf{E}$ denotes the mathematical expectation with respect to the probability measure $\mathbf{P}$):
• $\mathscr{C}^{1,2}_{\mathrm{pol}}([0,T]\times\mathbf{R})$ is the space of all $\mathscr{C}^{1,2}$-functions over $[0,T]\times\mathbf{R}$, which together with their derivatives are of polynomial growth;
• $V_{[0,T]} = \left\{ Y = \psi(\cdot,\eta) : \ \psi \in \mathscr{C}^{1,2}_{\mathrm{pol}}([0,T]\times\mathbf{R}), \frac{\partial\psi}{\partial t} \text{ is bounded, } t\in[0,T] \right\}$;
• $\widetilde{V}^{\beta}_{[0,T]}$ the completion of $V_{[0,T]}$ under the following norm ($\beta > 0$)

$$\left\|Y\right\|_\beta = \left(\int_0^T e^{\beta t}\mathbf{E}|Y_t|^2 dt\right)^{1/2} = \left(\int_0^T e^{\beta t}\mathbf{E}|\psi(t,\eta_t)|^2 dt\right)^{1/2};$$

• $\mathscr{B}^2([0,T],\mathbf{R}) = \widetilde{V}^{\beta}_{[0,T]} \times \widetilde{V}^{\beta}_{[0,T]} \times \widetilde{V}^{\beta}_{[0,T]}$ is a Banach space with the norm

$$\left\|(Y,Z_1,Z_2)\right\|^2_{\mathscr{B}^2} = \left\|Y\right\|^2_\beta + \left\|Z_1\right\|^2_\beta + \left\|Z_2\right\|^2_\beta;$$

• Finally let $\mathbf{L}^2_{-T}(\mathbf{R})$ denote the space of measurable functions $\varphi : [-T,0] \to \mathbf{R}$ satisfying

$$\int_{-T}^0 |\varphi(t)|^2\, dt < +\infty.$$

Definition 3.2. A triplet of processes (Y,Z_1,Z_2) is called a solution to SFBSDE (3.2), if $(Y,Z_1,Z_2) \in \mathscr{B}^2([0,T],\mathbf{R})$ and satisfies eq.(3.2).

More precisely we establish an existence and uniqueness result for (3.2) under the following assumptions:
• **(H1)**: $\xi = h(\eta_T)$ for some function h with bounded derivative, $\mathbf{E}\left[e^{\beta T}|\xi|^2\right] < +\infty$;
• **(H2)**: the generator $f : \Omega\times[0,T]\times\mathbf{R}\times\mathbf{L}^2_{-T}(\mathbf{R})\times\mathbf{L}^2_{-T}(\mathbf{R})\times\mathbf{L}^2_{-T}(\mathbf{R}) \to \mathbf{R}$ is product measurable, $\mathbb{F}$-adapted and Lipschitz continuous in the sense that

for some probability measure α on $([-T,0]\times\mathcal{B}([-T,0]))$

$$\left|f(t,x,y_t,z_{1,t},z_{2,t})-f(t,x,y'_t,z'_{1,t},z'_{2,t})\right|^2\leq K\int_{-T}^{0}|y(t+u)-y'(t+u)|^2\,\alpha(du)$$
$$+K\int_{-T}^{0}\left(|z_1(t+u)-z'_1(t+u)|^2+|z_2(t+u)-z'_2(t+u)|^2\right)\alpha(du)$$

for $\mathbb{P}\otimes\lambda$-a.e. $(\omega,t)\in\Omega\times[0,T]$ and for any $x\in\mathbf{R}$; $y,z_1,z_2,y',z'_1,z'_2\in\mathbf{L}^2_{-T}(\mathbf{R})$.

- **(H3)**: $f(s,\cdot,\cdot,\cdot)=0$ for $t<0$.

We have the following (see [[7], Theorem 5.3])

Theorem 3.3. *Assume that σ_1 and σ_2 are continuous and $|\sigma|_t^2$ defined by eq.*(3.1) *is a strictly increasing function of t. Let the SFBSDE* (3.2) *has a solution of the form*
$(Y(t)=\psi(t,\eta_t),\ Z_1(t)=-\varphi_1(t,\eta_t),\ Z_2(t)=-\varphi_2(t,\eta_t))$, *where* $\psi\in\mathscr{C}^{1,2}([0,T]\times\mathbf{R})$. *Then*

$$\varphi_1(t,x)=\sigma_1(t)\psi'_x(t,x),\quad \varphi_2(t,x)=\sigma_2(t)\psi'_x(t,x).$$

The next proposition will be useful in the sequel.

Proposition 3.4. *Let (Y,Z_1,Z_2) be a solution of the SFBSDE* (3.2)*. Then for almost $t\in[0,T]$,*

$$D_tY(t)=Z_1(t),\quad \textit{and}\quad \mathbb{D}_t^HY(t)=\frac{\hat{\sigma}_2(t)}{\sigma_2(t)}Z_2(t).$$

Proof. Since (Y,Z_1,Z_2) satisfies the SFBSDE (3.2) then we have $Y(\cdot)=\psi(\cdot,\eta_\cdot)$ where
$\psi\in\mathscr{C}^{1,2}([0,T]\times\mathbf{R})$. From Theorem 3.3, we have

$$Z_1(t)=\sigma_1(t)\psi'_x(t,x),\quad Z_2(t)=\sigma_2(t)\psi'_x(t,x).$$

Then we can write $D_tY(t)=\sigma_1(t)\psi'_x(t,x)=Z_1(t)$ and

$$\mathbb{D}_t^HY(t)=\int_0^T\phi(t,s)D_s^H\psi(t,\eta_t)ds=\psi'_x(t,\eta_t)\int_0^T\phi(t,s)\sigma_2(s)ds$$
$$=\widehat{\sigma}_2(t)\psi'_x(t,\eta_t)=\frac{\widehat{\sigma}_2(t)}{\sigma_2(t)}Z_2(t).$$

□

The main result of this section is the following theorem:

Theorem 3.5. *Assume that* **(H1), (H2)** *and* **(H3)** *hold. Then the SFBSDE* (3.2) *has a unique solution* $(Y,Z_1,Z_2)\in\mathscr{B}^2([0,T],\mathbf{R})$.

Proof. To prove the existence and uniqueness of a solution, we follow the classical idea by constructing a Picard scheme and show its convergence. To this end, we consider now the sequence $(Y^n, Z_1^n, Z_2^n)_{n\geq 0}$ given by

$$\begin{cases} Y^0(t) = Z_1^0(t) = Z_2^0(t) = 0, \\ Y^{n+1}(t) = \xi + \int_t^T f(s, \eta_s, Y_s^n, Z_{1,s}^n, Z_{2,s}^n)ds - \int_t^T Z_1^{n+1}(s)dB_s - \int_t^T Z_2^{n+1}(s)dB_s^H \end{cases} \tag{3.3}$$

Let us define for a process $\delta \in \{Y, Z_1, Z_2\}$, $n \geq 1$, $\overline{\delta}^{n+1} = \delta^{n+1} - \delta^n$ and the function

$$\Delta f^{(n)}(s) = f(s, \eta_s, Y_s^n, Z_{1,s}^n, Z_{2,s}^n) - f(s, \eta_s, Y_s^{n-1}, Z_{1,s}^{n-1}, Z_{2,s}^{n-1}).$$

Then, it is obvious that $(\overline{Y}^{n+1}, \overline{Z}_1^{n+1}, \overline{Z}_2^{n+1})$ solves the SFBSDE

$$\overline{Y}^{n+1}(t) = \int_t^T \Delta f^{(n)}(s)ds - \int_t^T \overline{Z}_1^{n+1}(s)dB_s - \int_t^T \overline{Z}_2^{n+1}(s)dB_s^H, \quad 0 \leq t \leq T.$$

By the fractional Itô chain rule, we have

$$\begin{aligned} |\overline{Y}^{n+1}(t)|^2 = 2\int_t^T \overline{Y}^{n+1}(s)\Delta f^{(n)}(s)ds - 2\int_t^T \overline{Z}_1^{n+1}(s)D_s\overline{Y}^{n+1}(s)ds \\ - 2\int_t^T \overline{Z}_2^{n+1}(s)\mathbb{D}_s^H\overline{Y}^{n+1}(s)ds - 2\int_t^T \overline{Y}^{n+1}(s)\overline{Z}_1^{n+1}(s)dB_s \\ - 2\int_t^T \overline{Y}^{n+1}(s)\overline{Z}_2^{n+1}(s)dB_s^H. \end{aligned}$$

Applying Itô formula to $e^{\beta t}|\overline{Y}^{n+1}(t)|^2$, we obtain that

$$\begin{aligned} e^{\beta t}|\overline{Y}^{n+1}(t)|^2 =& 2\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\Delta f^{(n)}(s)ds - 2\int_t^T e^{\beta s}\overline{Z}_1^{n+1}(s)D_s\overline{Y}^{n+1}(s)ds \\ &- 2\int_t^T e^{\beta s}\overline{Z}_2^{n+1}(s)\mathbb{D}_s^H\overline{Y}^{n+1}(s)ds - 2\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\overline{Z}_1^{n+1}(s)dB_s \\ &- 2\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\overline{Z}_2^{n+1}(s)dB_s^H - \beta\int_t^T e^{\beta s}|\overline{Y}^{n+1}(s)|^2ds. \end{aligned}$$

Take mathematical expectation on both sides, then we have

$$\begin{aligned} \mathbf{E}\left[e^{\beta t}|\overline{Y}^{n+1}(t)|^2\right] =& 2\mathbf{E}\left[\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\Delta f^{(n)}(s)ds - \int_t^T e^{\beta s}\overline{Z}_1^{n+1}(s)D_s\overline{Y}^{n+1}(s)ds\right] \\ &- 2\mathbf{E}\left[\int_t^T e^{\beta s}\overline{Z}_2^{n+1}(s)\mathbb{D}_s^H\overline{Y}^{n+1}(s)ds - \beta\int_t^T e^{\beta s}|\overline{Y}^{n+1}(s)|^2ds\right]. \end{aligned}$$

By Proposition 3.4, we have that

$$\begin{aligned} &\mathbf{E}\left[e^{\beta t}|\overline{Y}^{n+1}(t)|^2 + 2\int_t^T e^{\beta s}|\overline{Z}_1^{n+1}(s)|^2ds + 2\int_t^T e^{\beta s}\frac{\hat{\sigma}_2(s)}{\sigma_2(s)}|\overline{Z}_2^{n+1}(s)|^2ds\right] \\ &\qquad = 2\mathbf{E}\left[\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\Delta f^{(n)}(s)ds - \beta\int_t^T e^{\beta s}|\overline{Y}^{n+1}(s)|^2ds\right]. \end{aligned}$$

By Remark 3.1, we obtain that

$$\mathbf{E}\Big[e^{\beta t}|\overline{Y}^{n+1}(t)|^2+2\int_t^T e^{\beta s}|\overline{Z}_1^{n+1}(s)|^2ds+2C_0\int_t^T e^{\beta s}|\overline{Z}_2^{n+1}(s)|^2ds\Big]$$
$$\leq \mathbf{E}\left[2\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\Delta f^{(n)}(s)ds-\beta\int_t^T e^{\beta s}|\overline{Y}^{n+1}(s)|^2ds\right].$$

Using standard estimates $2ab\leq \lambda a^2+b^2/\lambda$ (where $\lambda>0$), we obtain that

$$2\mathbf{E}\int_t^T e^{\beta s}\overline{Y}^{n+1}(s)\Delta f^{(n)}(s)ds\leq \lambda\mathbf{E}\int_t^T e^{\beta s}\left|\overline{Y}^{n+1}(s)\right|^2ds+\frac{1}{\lambda}\mathbf{E}\int_t^T e^{\beta s}\left|\Delta f^{(n)}(s)\right|^2ds,$$

which implies

$$(\beta-\lambda)\left\|\overline{Y}^{n+1}\right\|_\beta^2+2\left\|\overline{Z}_1^{n+1}\right\|_\beta^2+2C_0\left\|\overline{Z}_2^{n+1}\right\|_\beta^2\leq\frac{1}{\lambda}\mathbf{E}\int_0^T e^{\beta s}\left|\Delta f^{(n)}(s)\right|^2ds. \tag{3.4}$$

Using the assumption **(H2)** and Fubini's Theorem, we have

$$\begin{aligned}
\frac{1}{\lambda}\mathbf{E}\int_0^T e^{\beta s}\left|\Delta f^{(n)}(s)\right|^2ds &\leq \frac{K}{\lambda}\mathbf{E}\int_0^T e^{\beta s}\int_{-T}^0\left|\overline{Y}^n(s+u)\right|^2\alpha(du)ds\\
&+\frac{K}{\lambda}\mathbf{E}\int_0^T e^{\beta s}\left[\int_{-T}^0\left(\left|\overline{Z}_1^n(s+u)\right|^2+\left|\overline{Z}_2^n(s+u)\right|^2\right)\alpha(du)\right]ds\\
&\leq\frac{K}{\lambda}\mathbf{E}\int_{-T}^0 e^{-\beta u}\int_0^T e^{\beta(s+u)}\left|\overline{Y}^n(s+u)\right|^2ds\alpha(du)\\
&+\frac{K}{\lambda}\mathbf{E}\int_{-T}^0 e^{-\beta u}\left[\int_0^T e^{\beta(s+u)}\left(\left|\overline{Z}_1^n(s+u)\right|^2+\left|\overline{Z}_2^n(s+u)\right|^2\right)ds\right]\alpha(du)\\
&\leq\frac{K}{\lambda}\mathbf{E}\int_{-T}^0 e^{-\beta u}\left[\int_v^{T+v}e^{\beta(v)}\left(\left|\overline{Y}^n(v)\right|^2+\left|\overline{Z}_1^n(v)\right|^2+\left|\overline{Z}_2^n(v)\right|^2\right)dv\right]\alpha(du)\\
&\leq\frac{K}{\lambda}\mathbf{E}\int_{-T}^0 e^{-\beta u}\alpha(du)\left(\left\|\overline{Y}^n\right\|_\beta^2+\left\|\overline{Z}_1^n\right\|_\beta^2+\left\|\overline{Z}_2^n\right\|_\beta^2\right)\\
&\leq\frac{Ke^{\beta T}}{\lambda\beta}\left\|\left(\overline{Y}^n,\overline{Z}_1^n,\overline{Z}_2^n\right)\right\|_{\mathscr{B}^2}^2
\end{aligned} \tag{3.5}$$

Hence gathering (3.4) and(3.5), we obtain

$$(\beta-\lambda)\left\|\overline{Y}^{n+1}\right\|_\beta^2+2\left\|\overline{Z}_1^{n+1}\right\|_\beta^2+2C_0\left\|\overline{Z}_2^{n+1}\right\|_\beta^2\leq\frac{Ke^{\beta T}}{\lambda\beta}\left\|\left(\overline{Y}^n,\overline{Z}_1^n,\overline{Z}_2^n\right)\right\|_{\mathscr{B}^2}^2 \tag{3.6}$$

Hence, if we choose $\lambda=\lambda_0$ satisfying $\beta\lambda_0=\beta-2\min\{1;C_0\}$, then we have

$$\left\|\left(\overline{Y}^{n+1},\overline{Z}_1^{n+1},\overline{Z}_2^{n+1}\right)\right\|_{\mathscr{B}^2}^2\leq\frac{Ke^{\beta T}}{\lambda_0\beta\left(\beta-\lambda_0\right)}\left\|\left(\overline{Y}^n,\overline{Z}_1^n,\overline{Z}_2^n\right)\right\|_{\mathscr{B}^2}^2, \tag{3.7}$$

where putting $T = \frac{1}{\beta}$,

$$\left\| \left(\overline{Y}^{n+1}, \overline{Z}_1^{n+1}, \overline{Z}_2^{n+1} \right) \right\|_{\mathscr{B}^2}^2 \leq \frac{Ke}{\lambda_0 \beta \left(\beta - \lambda_0 \right)} \left\| \left(\overline{Y}^{n}, \overline{Z}_1^{n}, \overline{Z}_2^{n} \right) \right\|_{\mathscr{B}^2}^2 . \tag{3.8}$$

Choosing K such that $K = \frac{3\lambda_0\beta(\beta-\lambda_0)}{4e}$, we obtain

$$\left\| \left(\overline{Y}^{n+1}, \overline{Z}_1^{n+1}, \overline{Z}_2^{n+1} \right) \right\|_{\mathscr{B}^2}^2 \leq \frac{3}{4} \left\| \left(\overline{Y}^{n}, \overline{Z}_1^{n}, \overline{Z}_2^{n} \right) \right\|_{\mathscr{B}^2}^2 . \tag{3.9}$$

Hence, the inequality (3.9) is a contraction, and there exists a unique limit $(Y, Z_1, Z_2) \in \widetilde{V}_{[0,T]}^{\beta} \times \widetilde{V}_{[0,T]}^{\beta} \times \widetilde{V}_{[0,T]}^{\beta}$ of a converging sequence $(Y^n, Z_1^n, Z_2^n)_{n\in\mathbf{N}}$ which satisfies the fixed point equation

$$Y(t) = \xi + \int_t^T f\left(r, \eta_s, Y_s, Z_{1,s}, Z_{2,s}\right) ds - \int_t^T Z_1(s) dB_s - \int_t^T Z_2(s) dB_s^H, \quad 0 \leq t \leq T.$$

With the same arguments, repeating the above technique we obtain a uniqueness of the solution of SFBSDE with time deplayed generators on $[0, T]$. $\square$

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Sadibou Aidara
Université Gaston Berger, Saint-Louis, Sénégal
BP 234
e-mail: sadibou.aidara@ugb.edu.sn

Yaya Sagna
Université Alioune Diop de Bambey, Sénégal
e-mail: sagnayaya88@gmail.com

Lamine Sylla
Université Gaston Berger, Saint-Louis, Sénégal
BP 234
e-mail: sylla.lamine@ugb.edu.sn