Timelike ruled surface in one-parameter hyperbolic dual spherical motions.

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Abstract

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Abstract. In this paper, we analyze a certain class of timelike ruled surface in one-parameter hyperbolic dual spherical motions by means of the E. Study map. Then, some new formulae of surfaces theory into Minkowski line space and their geometrical explanations are derived. In addition to that, timelike Plücker conoid associated with the motion has been obtained and investigated in detail. Finally, we have obtained the necessary and sufficient condition for a timelike ruled surface to be a constant timelike Disteli-axis.

Key Words. Blaschke frame, E. Study map, Lie algebra, Disteli-axis.

MSC (2000). 53A04, 53A05, 53A17.

1 Introduction

A ruled surface is defined by the property that through each point in the surface, there is at least one straight line which also lies in the surface. A ruled surface may be thought of as a one-parameter family of oriented lines. This surface is significant in the research of mechanism design problems because it is generally the trajectory of the oriented line embedded in a moving rigid body in spatial motion [1-3]. One of the ultimate suitable methods to research the motion of line space seems to found a relationship between this space and dual numbers. Via the E. Study map in screw and dual number algebra, the set of all oriented lines in Euclidean 3-space $\mathbb{E}^3$ is directly linked to the set of points on the dual unit sphere in the dual 3-space $\mathbb{D}^3$. It leads to that the differential geometry of ruled surfaces can be researched by considering one-parameter dual curves lying completely on dual unit sphere. More specifics about the needful basic concepts of the dual elements, and the connection between ruled surfaces and one-parameter dual spherical motions can be found in [4-9].

In $\mathbb{E}_1^3$ (the 3-dimensional Minkowski space) the study of ruled surfaces is more interesting than the Euclidean case, since Lorentzian metric is not positive definite metric, the distance function can be negative, positive or zero, whereas the distance function in the Euclidean 3-space $\mathbb{E}^3$ can only be positive. Therefore, the kinematics and geometric interpretations can be more different. Thus, if we replace the Minkowski 3-space $\mathbb{E}_1^3$ instead of the Euclidean 3-space $\mathbb{E}^3$ the E. Study map can be defined as follows: the timelike and spacelike oriented lines are represented with the timelike and spacelike dual points on hyperbolic and Lorentzian dual unit spheres $\mathbb{H}_+^2$ and $\mathbb{S}_1^2$ in the Lorentzian Dual 3-Space $\mathbb{D}_1^3$, respectively [10-12]. It means that a differentiable curve on $\mathbb{H}_+^2$ corresponds to a timelike ruled surface at $\mathbb{E}_1^3$. Similarly the spacelike (resp. timelike) curve on $\mathbb{S}_1^2$ corresponds to any timelike (resp. spacelike) ruled surface at $\mathbb{E}_1^3$. Due to its relationship with physical sciences in Minkowski space, many geometers and engineers have studied and gained many ownerships of the ruled surfaces (see [10-16]).

In this inquiry we examine timelike ruled surfaces with constant Disteli-axis based upon the curvature theory of a dual hyperbolic (resp. Lorentzian) spherical curve which matches in a timelike ruled surface in Minkowski 3-space $\mathbb{E}_1^3$. It is shown that if timelike ruled surfaces are considered in the context of line geometry, then a definition analogous to the concept of helix can be developed for such surfaces. Especially, if all the generators of a timelike ruled surface have a constant spatial distance with a definite timelike line then the timelike ruled surface is a constant Disteli-axis ruled surface. Furthermore, the locus of a timelike line, fixed in a body undergoing a screw motion of constant pitch, is a timelike general helicoid if the striction curve is a timelike or spacelike helix, a Lorentzian sphere if the striction curve is Euclidean circle, and a timelike cone if the striction curve becomes a fixed point.

2 Basic concepts

We start with basic concepts on the Minkowski 3–space $\mathbb{E}_1^3$, the theory of dual numbers, dual Lorentzian vectors and E. Study map, for example [1-3, 17-21].

A dual number A is a number $a + \varepsilon a^*$, where a, a^* in $\mathbb{R}$ and ε is a dual unit with the property that $\varepsilon^2 = 0$. Then the set

$$\mathbb{D}^3 = \{A := \mathbf{a} + \varepsilon \mathbf{a}^* = (A_1, A_2, A_3)\},$$

together with the Lorentzian scalar product

$$< \mathbf{A}, \mathbf{A} >= -A_1^2 + A_2^2 + A_3^2,$$

forms the so called dual Lorentzian 3-space $\mathbb{D}_1^3$. Thus, a point $A = (A_1, A_2, A_3)^t$ has dual coordinates $A_i = (a_i + \varepsilon a_i^*) \in \mathbb{D}$. If A is spacelike or timelike dual vector the norm $\|\mathbf{A}\|$ of A is defined by

$$\begin{aligned} \|\mathbf{A}\| &= \sqrt{|< \mathbf{A}, \mathbf{A} >|} = \sqrt{|< \mathbf{a}, \mathbf{a} >|} + \varepsilon \frac{1}{2\sqrt{|< \mathbf{a}, \mathbf{a} >|}} \frac{< \mathbf{a}, \mathbf{a} >}{|< \mathbf{a}, \mathbf{a} >|} .2 < \mathbf{a}, \mathbf{a}^* > \\ &= \|\mathbf{a}\| + \varepsilon \frac{1}{\|\mathbf{a}\|} \frac{< \mathbf{a}, \mathbf{a} >}{|< \mathbf{a}, \mathbf{a} >|} < \mathbf{a}, \mathbf{a}^* > . \end{aligned}$$

If $\mathbf{a}$ is spacelike, we have

$$\|\mathbf{A}\| = \|\mathbf{a}\| + \varepsilon \frac{1}{\|\mathbf{a}\|} < \mathbf{a}, \mathbf{a}^* >= \|\mathbf{a}\| \left(1 + \varepsilon \frac{1}{\|\mathbf{a}\|^2} < \mathbf{a}, \mathbf{a}^* > \right).$$

If $\mathbf{a}$ is timelike, we have

$$\|\mathbf{A}\| = \|\mathbf{a}\| - \varepsilon \frac{1}{\|\mathbf{a}\|} < \mathbf{a}, \mathbf{a}^* >= \|\mathbf{a}\| \left(1 - \varepsilon \frac{1}{\|\mathbf{a}\|^2} < \mathbf{a}, \mathbf{a}^* > \right).$$

Therefore, A is called a spacelike dual unit vector if $<A,A>=1$ and a timelike dual unit vector if $<A,A>= -1$. The hyperbolic and Lorentzian dual unit spheres, respectively, are

$$\mathbb{H}_+^2 = \left\{ \mathbf{A} \in \mathbb{D}_1^3 \mid -A_1^2 + A_2^2 + A_3^2 = -1 \right\},$$

and

$$\mathbb{S}_1^2 = \left\{ \mathbf{A} \in \mathbb{D}_1^3 \mid -A_1^2 + A_2^2 + A_3^2 = 1 \right\}.$$

Theorem 1. There is a one-to-one correspondence between spacelike (resp. timelike) oriented lines in $\mathbb{E}_1^3$ and ordered pairs of vectors $(\mathbf{a}, \mathbf{a}^*)$ such that

$$\|\mathbf{A}\|^2 = \pm 1 \Longleftrightarrow \|\mathbf{a}\|^2 = \pm 1, \; <\mathbf{a}, \mathbf{a}^*>=0, \tag{2.1}$$

where a_i, $a_i^*(i = 1, 2, 3)$ of a, and a^* are called the normed Plücker coordinates of the line.

Via Theorem 1 we have the following map (E. Study's map): The dual unit spheres are shaped as a pair of conjugate hyperboloids. The ring shaped hyperboloid represents the set of spacelike lines, the common asymptotic cone represents the set of null (lightlike) lines, and the oval shaped hyperboloid forms the set of timelike lines, opposite points of each hyperboloid perform the pair of obverse vectors on a line (see Fig. 1).

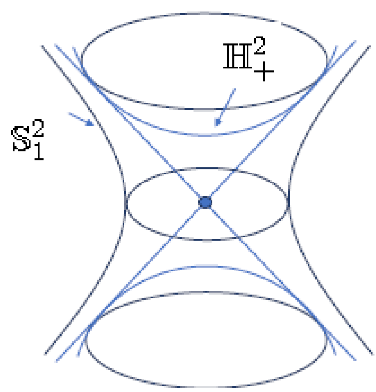

Figure 1: The dual hyperbolic and dual Lorentzian unit spheres.

Applying to E. Study map, a differentiable curve on $\mathbb{H}_+^2$ corresponds to a timelike ruled surface in $\mathbb{E}_1^3$. Similarly the dual curve on $\mathbb{S}_1^2$ corresponds to a spacelike or timelike ruled surface in $\mathbb{E}_1^3$. In view of Eq. (1), four independent parameters locating a line complex, so it is reasonable to intersect any two of line complexes and gain a finite number of lines (line congruence) with common properties. The intersection of two independent linear congruences carry outs a differentiable family of straight lines (a ruled surface). Ruled surfaces (such as cylinders and cones) include rulings where the tangent plane relates the surface over the entire line (torsal lines) [3].

Definition 1. Let $\mathbf{T}$ and Z are two non-null dual vectors in $\mathbb{D}_1^3$:
i) Let us consider that T and Z are spacelike dual vectors, then

• If they span a spacelike dual plane; there is a unique dual number $\Psi = \psi + \varepsilon\psi^*$; $0 \leq \psi \leq \pi$, and $\psi^* \in \mathbb{R}$ such that $<T,Z>= \|\mathbf{T}\| \|\mathbf{Z}\| \cosh \Psi$. This number is called the spacelike dual angle between T and Z.

• If they span a timelike dual plane; there is a unique dual number $\Psi = \psi + \varepsilon\psi^* \geq 0$ such that $<T,Z>= \epsilon \|\mathbf{T}\| \|\mathbf{Z}\| \cosh \Psi$, where $\epsilon = +1$ or $\epsilon = -1$ according to $sign(T_2) = sign(Z_2)$ or $sign(T_2) \neq sign(Z_2)$, respectively. This number is called the central dual angle between T and Z.

ii) Let us consider that T and Z are timelike dual vectors, then there is a unique dual number $\Psi = \psi + \varepsilon\psi^* \geq 0$ such that $<T,Z>= \epsilon \|\mathbf{T}\| \|\mathbf{Z}\| \cosh \Psi$, where $\epsilon = +1$ or $\epsilon = -1$ according to T and Z have different time-orientation or the same time-orientation, respectively.

iii) Let us consider that T is spacelike dual, and Z is timelike dual, then there is a unique Lorentzian timelike dual angle $\Psi = \psi + \varepsilon\psi^* \geq 0$ such that $<T,Z>= \epsilon \|\mathbf{T}\| \|\mathbf{Z}\| \sinh \Psi$, where $\epsilon = +1$ or $\epsilon = -1$ according to $sign(T_2) = sign(Z_1)$ or $sign(T_2) \neq sign(Z_1)$**.**

2.1 One-parameter hyperbolic dual spherical motions

Let $\mathbb{H}^2_{+m}$ and $\mathbb{H}^2_{+f}$ be two hyperbolic dual unit spheres with dual coordinate frames $\{\mathbf{O};\ R_1(\text{timelike}),\ R_2,\ R_3\}$, and $\{\mathbf{O};\ \mathbf{F}_1(\text{timelike}),\ \mathbf{F}_2,\ \mathbf{F}_3\}$, respectively. We suppose that the elements of the set $\{\mathbf{O};\ R_1,\ R_2,\ R_3\}$ are functions of a real parameter $t \in \mathbb{R}$ (say the time) whilst the set $\{\mathbf{O}; \mathbf{F}_1,\ \mathbf{F}_2,\ \mathbf{F}_3\}$ is fixed. Then $\mathbb{H}^2_{+m}$ moves with respect to $\mathbb{H}^2_{+f}$. Such motion is called a one-parameter hyperbolic dual spherical motions, and will denoted by $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$. If the hyperbolic dual unit spheres $\mathbb{H}^2_{+m}$ and $\mathbb{H}^2_{+f}$ corresponds to the Lorentzian line spaces $\mathbb{L}_m$ and $\mathbb{L}_f$, respectively, then $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$ is being congruous with the one-parameter Lorentzian spatial motion $\mathbb{L}_m/\mathbb{L}_f$. Therefore $\mathbb{L}_m$ is the moving space with respect to the fixed space $\mathbb{L}_f$. In Lorentzian sense, by putting $< \mathbf{F}_i,R_j >= L_{ij}$ and introducing the dual matrix $L = (L_{ij})$, we can write the E. Study map in the matrix form as follows:

$$\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f} : \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{pmatrix} = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix}. \tag{2.2}$$

The dual matrix $L = (L_{ij}) + \varepsilon(L^*_{ij})$ has the possession that $L^T = \epsilon L^{-1}\epsilon$, where the matrix ϵ is a signature matrix and will be denoted by [14]

$$\epsilon = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.3}$$

Since $L^T = \epsilon L^{-1}\epsilon$, and $L^{-1} = \epsilon L^T \epsilon L$, then

$$LL^{-1} = L\epsilon L^T \epsilon = \epsilon L^T \epsilon L = I, \tag{2.4}$$

where I is the 3×3 unit matrix. Hence, the set of dual orthogonal 3×3 matrices, denoted by $\mathbb{O}(\mathbb{D}_1^{3\times3})$, form a group with matrix multiplication as the group operation (real orthogonal matrices are subgroup of dual orthogonal matrices). The identity element of $\mathbb{O}(\mathbb{D}_1^{3\times3})$ is the 3×3 unit matrix.

The Lie algebra $\mathbb{L}(\mathbb{O}_{\mathbb{D}_1^{3\times3}})$ of the group $\mathbb{GL}$ of 3×3 positive orthogonal dual matrices L is the algebra of skew-adjoint 3×3 dual matrices

$$\Omega(t) := L^{'}\epsilon L^T \epsilon = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ \Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} = \begin{pmatrix} -\Omega_1 \\ \Omega_2 \\ \Omega_3 \end{pmatrix}, \tag{2.5}$$

where dash indicates the differential of L with respect to the real parameter $t \in \mathbb{R}$. Then the derivative equation of $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$ is:

$$\begin{pmatrix} \mathbf{R}_1^{'} \\ \mathbf{R}_2^{'} \\ \mathbf{R}_3^{'} \end{pmatrix} = \begin{pmatrix} 0 & \Omega_3 & -\Omega_2 \\ \Omega_3 & 0 & \Omega_1 \\ \Omega_2 & -\Omega_1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix} = \mathbf{\Omega}\times \begin{pmatrix} \mathbf{R}_1 \\ \mathbf{R}_2 \\ \mathbf{R}_3 \end{pmatrix}. \tag{2.6}$$

where $\mathbf{\Omega}(t)= \omega+\varepsilon\omega^* = (\Omega_1, \Omega_2, \Omega_3)$ is called the instantaneous dual rotation vector of $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$. ω and ω^*, respectively, corresponding to the instantaneous rotational differential velocity vector and the instantaneous translational differential velocity vector of $\mathbb{L}_m/\mathbb{L}_f$.

3 Main results

During the motion $\mathbb{L}_m/\mathbb{L}_f$, any fixed timelike line $\mathbf{X} \in \mathbb{L}_m$, generally, traces a timelike ruled surface in $\mathbb{L}_f$ will be indicated by (X). In kinematics, this timelike ruled surface is indicate to as timelike line trajectory. In order to analyze its geometrical properties, we set up a moving frame coincident with the point on $\mathbb{H}^2_{+m}$. Then the Blaschke frame can be construct as:

$$\left.\begin{array}{c} \mathbf{X}=\mathbf{X}(t),\ \mathbf{T}(t)=\frac{\mathbf{X}'}{\|\mathbf{X}'\|},\ \mathbf{G}(t)=\mathbf{X}\times\mathbf{T}, \\ \mathbf{G}\times\mathbf{X}=\mathbf{T},\ \mathbf{T}\times\mathbf{G}=-\mathbf{X}, \\ <\mathbf{X},\mathbf{X}>=-1,\ <\mathbf{T},\mathbf{T}>=<\mathbf{G},\mathbf{G}>=1, \\ <\mathbf{X},\mathbf{T}>=<\mathbf{X},\mathbf{G}>=<\mathbf{G},\mathbf{T}>=0. \end{array}\right\}$$

The dual unit vectors $\mathbf{X}$, $\mathbf{T}=\mathbf{t}+\varepsilon\mathbf{t}^*$, and $\mathbf{G}=\mathbf{g}+\varepsilon\mathbf{g}^*$ match to three synchronous alternately orthogonal lines in Minkowski 3-space $\mathbb{E}^3_1$. Their point of intersection is the central point $\mathbf{C}$ on the ruling $\mathbf{X}$. $\mathbf{G}$ is the limit position of the common perpendicular to $\mathbf{X}(t)$ and $\mathbf{X}(t+dt)$, and is called the central tangent of the ruled surface at the central point. The line $\mathbf{T}$ is called the central normal of $\mathbf{X}$ at the central point. Thus, the motion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$ is given by [16]

$$\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}:\begin{pmatrix}\mathbf{X}'\\ \mathbf{T}'\\ \mathbf{G}'\end{pmatrix}=\begin{pmatrix}0 & P & 0\\ P & 0 & Q\\ 0 & -Q & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{T}\\ \mathbf{G}\end{pmatrix}=\mathbf{\Omega}\times\begin{pmatrix}\mathbf{X}\\ \mathbf{T}\\ \mathbf{G}\end{pmatrix}, \tag{3.1}$$

where

$$P=p+\varepsilon p^*=\left\|\mathbf{X}'\right\|,\ Q:=q+\varepsilon q^*=\det(\mathbf{X},\mathbf{X}',\mathbf{X}'')\left\|\mathbf{X}'\right\|^{-2},$$

are Blaschke invariants of (X), and $\mathbf{\Omega}(t)=Q\mathbf{X}+P\mathbf{G}$. The tangent of the striction curve can be written as

$$\mathbf{C}'=-q^*\mathbf{x}+p^*\mathbf{g}.$$

The invariants p, p^*, q, and q^* are said to be the structure functions of the ruled surface. The distribution parameters of the ruled surfaces (X), (T), and (G), respectively, are:

$$\mu(t)=\frac{p^*}{p},\ \delta(t)=\frac{pp^*+qq^*}{q^2+p^2},\ \text{and}\ \Gamma(t)=\frac{q^*}{q}.$$

Definition 2. A non-developable ruled surface is defined as a constant parameter ruled surface if the structure functions $\mu(t)$, $\delta(t)$, and $\Gamma(t)$ are all constant.

Under the hypothesis that $|Q| > |P|$, we specify the evolute of $\mathbf{X} \in \mathbb{H}^2_+$ as:

$$\mathbf{B}(t) = \mathbf{b}(t) + \varepsilon \mathbf{b}^*(t) = \frac{\mathbf{\Omega}}{\|\mathbf{\Omega}\|} = \frac{Q\mathbf{X}+P\mathbf{G}}{\sqrt{Q^2 - P^2}}. \tag{3.2}$$

It is apparent that $\mathbf{B}$ is the Disteli-axis (striction-axis or curvature-axis) of (X). Hence, we may write $\mathbf{B}$ of the form

$$\mathbf{B}(t) = \cosh \Psi \mathbf{X} + \sinh \Psi \mathbf{G}, \tag{3.3}$$

where

$$\coth \Psi = \frac{Q}{P}.$$

Notice that $\Psi = \psi + \varepsilon \psi^*$ is the Lorentzian dual spherical radius of curvature. The trigonometric hyperbolic function Ψ can be written as:

$$\coth \Psi = \coth \psi - \varepsilon \psi^* \frac{1}{\sinh^2 \psi} = \frac{q + \varepsilon q^*}{p + \varepsilon p^*}. \tag{3.4}$$

The function

$$\Sigma(t) := \gamma + \varepsilon (\Gamma - \mu\gamma) = \frac{Q}{P}, \tag{3.5}$$

is called the dual geodesic curvature. Here $\gamma(t) = \frac{q}{p}$ is the geodesic curvature of the hyperbolic spherical image curve $t \in I \mapsto \mathbf{x}(t)$ of (X). Thus, by means of the real and dual parts of Eqs. (10), and (11), respectively, we find

$$\gamma(t) = \coth \psi = \frac{q}{p}, \tag{3.6}$$

and

$$\psi^*(t) = \frac{1}{2} (\mu - \Gamma) \sinh 2\psi, \tag{3.7}$$

ψ^* is the normal distance along $\mathbf{T}$ measured from $\mathbf{B}$ to $\mathbf{X}$.

3.1 Kinematic-geometry and timelike Plücker conoid

Now, we study the kinematic-geometry of the timelike line trajectory (X). To do this, we are do a detailed study of the Blaschke invariants $P(t)$, and $Q(t)$. From Eq. (7), we can write the following equations:

$$\mathbf{X}^{'}(t) = (\|\mathbf{\Omega}\| \, \mathbf{B}) \times \mathbf{X}, \ \mathbf{T}^{'}(t) = (\|\mathbf{\Omega}\| \, \mathbf{B}) \times \mathbf{T}, \ \mathbf{G}^{'}(t) = (\|\mathbf{\Omega}\| \, \mathbf{B}) \times \mathbf{G}. \tag{3.8}$$

Hence, at any instant, it is seen that:$\|\mathbf{\Omega}\| = \Omega = \omega + \varepsilon\omega^*$ is the dual angular speed of the motion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$ about $\mathbf{B}$. Thus,

$$\omega(t) = \sqrt{q^2 - p^2}, \text{ and } \omega^*(t) = \frac{qq^* - pp^*}{\sqrt{q^2 - p^2}}, \tag{3.9}$$

corresponding to the rotational angular speed and translational angular speed of the motion $\mathbb{L}_m/\mathbb{L}_f$ along $\mathbf{B}$, respectively.

Hence, the following corollary can be given:

Corollary 1. During the motion $\mathbb{L}_m/\mathbb{L}_f$, at any instant t, the pitch of the motion can be given as

$$h(t) := \frac{< \omega, \omega^* >}{\|\omega\|^2} = \Gamma \cosh^2\psi - \mu \sinh^2\psi. \tag{3.10}$$

It is very apparent that if the dual vector $\mathbf{\Omega} = \omega + \varepsilon\omega^*$ is given, then the following can be specified:
(i) The timelike Disteli-axis $\mathbf{B}$ is specified by Eq. (9).
(ii) The dual angular speed of its dual angular velocity is $\|\mathbf{\Omega}\| = \omega(1 + \varepsilon h)$.
(iii) If $\mathbf{y}$ indicate to a point on the timelike Disteli-axis $\mathbf{B}$, then

$$\mathbf{y}(t, v) = \mathbf{b} \times \mathbf{b}^* + v\mathbf{b}, \ v \in \mathbb{R}, \tag{3.11}$$

is a non-developable timelike ruled surface (B). Note that if the motion $\mathbb{L}_m/\mathbb{L}_f$ is pure rotation, that is, $h(u) = 0$, then

$$\mathbf{B}(t) = \mathbf{b}(t) + \varepsilon\mathbf{b}^*(t) = \frac{1}{\|\omega\|}(\omega + \varepsilon\omega^*). \tag{3.12}$$

Note also that if $h(t) = 0$, and $\|\omega\|^2 = -1$, then $\mathbf{\Omega}$ is an timelike oriented line. However, in the case of the motion is pure translational, i.e. $\mathbf{\Omega} = 0 + \varepsilon\omega^*$, we set $\omega^* = \|\mathbf{\Omega}^*\|$, $\omega^*\mathbf{b} = \boldsymbol{\omega}^*$ and select an arbitrary $\mathbf{b}^*$under $\omega^* \neq 0$, otherwise the timelike unit vector $\mathbf{b}$ can be chosen arbitrarily, too.

The Eqs. in (13), and (16), respectively, are Minkowski versions of the Mannhiem and Hamilton formulae of surfaces theory in Euclidean 3-space. Now, let us to give geometrical significances of these formulae. ψ^* in Eq. (13) is timelike Plücker conoid has the parametric exemplification as follows: $\mathbf{T}$ is coincident with the spacelike $y-$axis of a fixed Minkowski frame $(oxyz)$; while the position of the timelike dual unit vector $\mathbf{B}$ is given by angle ψ

and distance ψ^* along the spacelike positive $y-$axis. The timelike dual unit vector $\mathbf{X}$ and the spacelike dual unit vector $\mathbf{G}$ can be selected in sense of x and z-axes, respectively. This shows that the dual unit vectors $\mathbf{X}$ and $\mathbf{G}$ together with $\mathbf{T}$ create the essential coordinate system of the Plücker conoid, as shown in Fig. 2. If $\mathbf{y}$ denote a point on this timelike surface, then we have

$$M : \mathbf{y}(t,v)=(0,\psi^*,0)+v(\cosh\psi,0,\sinh\psi), \quad v\in\mathbb{R}, \tag{3.13}$$

Employing this parametrization, the timelike dual unit vectors $\mathbf{B}$ are obviously visible crossing through the $y-$axis. Thus,

$$\psi^* := y = \frac{1}{2}(\mu-\Gamma)\sinh 2\psi,\ x = v\cosh\psi, \text{ and } z = v\sinh\psi, \tag{3.14}$$

where ψ^* gives us the intersection point of the principal axes $\mathbf{X}$ and $\mathbf{G}$ lies at a half of the conoid height. It can easily be confirmed by direct calculations that

$$\left(x^2-z^2\right)y-(\mu-\Gamma)\,xz=0, \tag{3.15}$$

which is the algebraic equation for timelike Plücker conoid occasionally also called the cylindroid. The timelike Plücker conoid given by Eq. (21) has two structure functions and it depends only on their difference; $\mu-\Gamma=1,\ -.9\leq\psi\leq .9,\ -1.5\leq v\leq 1.5$ (Fig. 3). Furthermore, solving for $\frac{x}{z}$, the roots of the second-order algebraic equation are given by:

$$\frac{x}{z}=\frac{1}{2y}\left[\mu-\Gamma\pm\sqrt{(\mu-\Gamma)^2-4y^2}\right]. \tag{3.16}$$

The Plücker conoid has also two torsal planes π_1, π_2, and each one of them contains one torsal line L as as follows:

1) If $h(u)\neq 0$ then there are two real torsal lines L_1, and L_2 passing through the point $(0,y,0)$ only if $y<(\mu-\Gamma)/2$; for the two limit points $y=\pm(\mu-\Gamma)/2$ they coincide with the principal axes $\mathbf{X}$ and $\mathbf{G}$,

2) If $h(u)=0$ then the two torsal lines L_1, and L_2 are represented by

$$\frac{x}{z} := \coth\psi = \pm\sqrt{\mu/\Gamma},\ y=\pm\sqrt{\Gamma\mu}. \tag{3.17}$$

Eq. (23) shows that the two torsal lines L_1, and L_2 are perpendicular each other in Lorentzian sense.

Furthermore, transition from polar coordinates to Cartesian coordinates could be completed by substituting

$$x = \frac{\cosh \psi}{\sqrt{h}},\ z = \frac{\sinh \psi}{\sqrt{h}},$$

into Hamilton's formula, one obtain the following conic section

$$D : |\Gamma|\, x^2 - |\mu|\, z^2 = 1.$$

This conic section is Minkowski version of the Dupin indicatrix of surfaces theory in Euclidean 3-space. If (X) is a timelike developable ruled surface, that is $\mu = 0$, in this case the Dupin's indicatrix is a set of parallels lines represented by

$$y^2 = \left| \frac{1}{\Gamma} \right| \text{ with } \mu = 0.$$

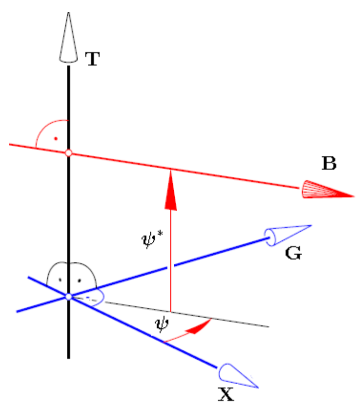

Figure 2: $\mathbf{B} = \cosh \Psi \mathbf{X} + \sinh \Psi \mathbf{G}$,

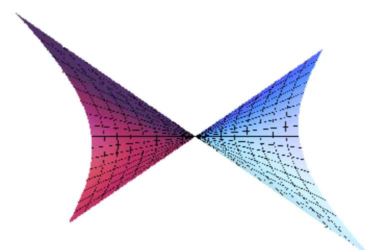

Figure 3: Timelike Plucker conoid

3.2 The constant Disteli-axis timelike ruled surfaces

A timelike ruled surface (X) is defined as a constant Disteli-axis ruled surface if the dual angle between the ruling of (X) and the Disteli-axis is always constant. Thus, when we say (X) is a timelike constant Disteli-axis, we mean that all the rulings of (X) have a constant Lorentzian dual angle from its Disteli-axis.

The dual arc length $d\widehat{s} = ds + \varepsilon ds^*$ of $\mathbf{X}(t) \in \mathbb{H}_+^2$ is

$$\widehat{s}(t) = \int_0^t P dt = \int_0^t p(1 + \varepsilon\mu) dt. \tag{3.18}$$

After that, we will use the dual arc length parameter $\widehat{s}$ instead of t. If the prime means to differentiation as $\widehat{s}$, then from Eq. (7), we get

$$\begin{pmatrix} \mathbf{X}' \\ \mathbf{T}' \\ \mathbf{G}' \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \Sigma \\ 0 & -\Sigma & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \mathbf{\Omega} \times \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}, \tag{3.19}$$

where $\mathbf{\Omega} = \mathbf{\Sigma X} + \mathbf{G}$. Thus, we may write the following relationships:

$$\widehat{\kappa}(\widehat{s}) := \kappa + \varepsilon\kappa^* = \sqrt{\Sigma^2 - 1} = \frac{1}{\sinh\psi} = \frac{1}{\widetilde{\rho}},\ \widehat{\tau}(\widehat{s}) := \tau + \varepsilon\tau^* = \pm\frac{\Sigma'}{\sqrt{\Sigma^2 - 1}} = \pm\Psi', \tag{3.20}$$

where $\widehat{\kappa}$ and $\widehat{\tau}$ are the dual curvature function and the dual torsion function of $\mathbf{X}(\widehat{s}) \in \mathbb{H}_+^2$, respectively. The terms found in Eqs. (26) are such as to their counterparts in 3-dimensional hyperbolic spherical geometry.

Definition 3. For a one-parameter hyperbolic dual motion, at an instant $\widehat{s} \in \mathbb{D}$, an oriented timelike line Z in fixed space will be said to be timelike $\mathbf{B}_k$-Disteli-axis of (X) if for all i such that $1 \leq i \leq k$, $<Z, \mathbf{X}^i(\widehat{s}) >= 0$, but $<Z, \mathbf{X}^{k+1}(\widehat{s}) > \neq 0$. Here $\mathbf{X}^i$ denotes the i-th derivatives of $\mathbf{X}$.

Via this definition, consider the Lorentzian dual angle

$$\widetilde{\rho} = \cosh^{-1}\left(< \mathbf{Z}, \mathbf{X} >\right),$$

such that $\mathbf{X}$ and Z have the same time-orientation, Z, and $\widetilde{\rho}$ stay fixedt up to the second order at $\widehat{s} = \widehat{s}_0$, i.e.

$$\widetilde{\rho}' \mid \widehat{s} = \widehat{s}_0 = 0,\ \ \mathbf{X}' \mid \widehat{s} = \widehat{s}_0 = \mathbf{0},$$

and

$$\widetilde{\rho}'' \mid \widehat{s} = \widehat{s}_0 = 0,\ \mathbf{X}'' \mid \widehat{s} = \widehat{s}_0 = \mathbf{0}.$$

We have for the first order

$$< \mathbf{X}', \mathbf{Z} > = 0,$$

and for the second order properties

$$< \mathbf{X}^{''}, \mathbf{Z} > |= 0.$$

Then, $\widetilde{\rho}$ will be invariant in the second approximation if and only if Z is the timelike Disteli-axis $\mathbf{B}$ of (X), that is,

$$\widetilde{\rho}^{'} = \widetilde{\rho}^{''} = 0 \Leftrightarrow \mathbf{Z} = \frac{\mathbf{X}' \times \mathbf{X}''}{\|\mathbf{X}' \times \mathbf{X}''\|} = \pm \mathbf{B}. \tag{3.21}$$

By the definition of the timelike Disteli-axis, we have the dual frame;

$$\mathbf{U}_1 = \mathbf{B}(\widehat{s}),\ \mathbf{U}_2(\widehat{s}) = \frac{\mathbf{B}^{'}}{\|\mathbf{B}'\|},\ \mathbf{U}_3(\widehat{s}) = \mathbf{B} \times \mathbf{U}_2, \tag{3.22}$$

as the Blaschke frame along $\mathbf{B}$ Thus, the calculations give that:

$$\begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix} = \begin{pmatrix} \cosh\Psi & 0 & \sinh\Psi \\ \sinh\Psi & 0 & \cosh\Psi \\ 0 & -1 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix}. \tag{3.23}$$

The variations of this frame are analogous to Eqs. (7) and is given by:

$$\begin{pmatrix} \mathbf{U}_1^{'} \\ \mathbf{U}_2^{'} \\ \mathbf{U}_3^{'} \end{pmatrix} = \begin{pmatrix} 0 & \Psi^{'} & 0 \\ \Psi^{'} & 0 & \widehat{\kappa} \\ 0 & -\widehat{\kappa} & 0 \end{pmatrix} \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix} = \widetilde{\mathbf{\Omega}} \times \begin{pmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \mathbf{U}_3 \end{pmatrix}, \tag{3.24}$$

where $\widetilde{\mathbf{\Omega}}(\widehat{s}) = \widehat{\kappa}\mathbf{U}_1 + \Psi^{'}\mathbf{U}_3$. Comparing Eq. (25) with Eq. (30) we have that the relative dual velocity is

$$\mathbf{\Omega} - \widetilde{\mathbf{\Omega}} = \Psi^{'}\mathbf{T}. \tag{3.25}$$

This shows that, the Blaschke frame involves a further rotation around the central tangent $\mathbf{T}$, whose speed equals the dual torsion $\widehat{\tau}(s)$. Hence, we obtain that: If $\widehat{\tau}(\widehat{s}) = \tau + \varepsilon\tau^{*} = 0(\Sigma^{'} = 0)$, i.e. ψ and ψ^{*} are constants, then the timelike Disteli-axis is fixed up to the second order and the timelike line $\mathbf{X}$ moves on it with constant pitch h. Thus kinematically the timelike ruled surface (X) is generated during a hyperbolic one-parameter screw motion of pitch h about the constant timelike Disteli-axis $\mathbf{B}$, by the timelike line $\mathbf{X}$ situated at a constant Lorentzian distance ψ^{*} and constant Lorentzian angle

ψ relative to $\mathbf{B}$. Hence, we have:

Theorem 2. A non-developable timelike ruled surface (X) is a constant timelike Disteli-axis if and only if (a) γ =constant, and (b) $\Gamma-\mu\gamma$ =constant.

Now, we construct the timelike ruled surfaces for which the Disteli-axis is constant. Thus, from Eq. (25), we have the following ordinary differential equation

$$\mathbf{X}^{'''}+\widetilde{\kappa}^2\mathbf{X}'=\mathbf{0}. \tag{3.26}$$

Then without loss of generality, we may assume $\mathbf{X}^{'}(0)$=$(0,1,0)$. Under such initial condition, a spacelike dual unit vector $\mathbf{X}^{'}$ is given by

$$\mathbf{X}^{'}(\widehat{s})\text{=}A_1\sin\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_1+\left(\cos\left(\widetilde{\kappa}\widehat{s}\right)+A_2\sin\left(\widetilde{\kappa}\widehat{s}\right)\right)\mathbf{F}_2+A_3\sin\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_3,$$

where A_1, A_2, and A_3 are some dual constants satisfying $A_1^2-A_3^2=1$, and $A_2=0$. From this, we can obtain

$$\mathbf{X}(\widehat{s})\text{=}\left(-\widetilde{\rho}A_1\cos\left(\widetilde{\kappa}\widehat{s}\right)+D_1\right)\mathbf{F}_1+\widetilde{\rho}\sin\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_2+\left(-\widetilde{\rho}A_3\cos\left(\widetilde{\kappa}\widehat{s}\right)+D_3\right)\mathbf{F}_3,$$

where D_1, D_3, are some dual constants satisfying $A_3D_3-A_1D_1=0$, and $D_3^2-D_1^2=\widetilde{\rho}^2+1$. If we adopt the dual coordinates transformation such that

$$\begin{pmatrix}\overline{X_1}\\ \overline{X_2}\\ \overline{X_3}\end{pmatrix}=\begin{pmatrix}A_1 & 0 & -A_3\\ 0 & 1 & 0\\ -A_3 & 0 & A_1\end{pmatrix}\begin{pmatrix}X_1\\ X_2\\ X_3\end{pmatrix},$$

with respect to the new coordinates $\overline{X_i}$, the dual vector $\mathbf{X}(\widehat{s})$ becomes

$$\mathbf{X}(\widehat{s})\text{=}\widetilde{\rho}\cos\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_1\text{+}\widetilde{\rho}\sin\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_2+D\mathbf{F}_3, \tag{3.27}$$

for a dual constant $D=A_1D_3-A_3D_1$, with $D=\mp\cosh\Psi$. It is noted that $\mathbf{X}(\widehat{s})$ does not depend on the choice of the lower sign or upper sign of $\mp$. Therefore through the paper we choice upper sign, that is,

$$\mathbf{X}(\Phi)\text{=}\sinh\Psi\cos\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_1\text{+}\sinh\Psi\sin\left(\widetilde{\kappa}\widehat{s}\right)\mathbf{F}_2-\cosh\Psi\mathbf{F}_3. \tag{3.28}$$

where $\Theta=\vartheta+\varepsilon\vartheta^*=\widetilde{\kappa}\widehat{s}$. This means that the timelike lines $\mathbf{B}$ and $\mathbf{F}_3$ are coincident, and

$$\psi=f_1(real\ const.),\ \psi^*=f_2(real\ const.). \tag{3.29}$$

Since ϑ, and ϑ^* are two-independent parameters, we can say that (X) is, in generally, a timelike line congruence in $\mathbb{L}_f-$space.

Now we locate the equation of this timelike line congruence in terms of the Plücker coordinates. By separating the real and dual parts of Eq. (34), respectively, we have

$$\mathbf{x}(\vartheta,\vartheta^*)=(\sinh\psi\cos\vartheta,\sinh\psi\sin\vartheta,-\cosh\psi)\,, \tag{3.30}$$

and

$$\mathbf{x}^*(\vartheta,\vartheta^*)=\begin{pmatrix} x_1^* \\ x_2^* \\ x_3^* \end{pmatrix}=\begin{pmatrix} \psi^*\cos\vartheta\cosh\psi-\vartheta^*\sin\vartheta\sinh\psi \\ \psi^*\sin\vartheta\cosh\psi+\vartheta^*\cos\vartheta\sinh\psi \\ -\psi^*\sinh\psi \end{pmatrix}. \tag{3.31}$$

Let $\alpha(\alpha_1,\alpha_2,\alpha_3)$ denote a point on $\mathbf{X}$. Since $\alpha\times\mathbf{x}=\mathbf{x}^*$ we have the system of linear equations in α_1, α_2, and α_3:

$$\left.\begin{array}{c} -\alpha_3\sin\vartheta\sinh\psi-\alpha_2\cosh\psi=x_1^*, \\ \alpha_3\cos\vartheta\sinh\psi+\alpha_1\cosh\psi=x_2^*, \\ -\alpha_1\sin\vartheta\sinh\psi+\alpha_2\cos\vartheta\sinh\psi=x_3^*. \end{array}\right\}$$

The matrix of coefficients of unknowns α_1, α_2, and α_3 is

$$\begin{pmatrix} 0 & -\cosh\psi & -\sin\vartheta\sinh\psi \\ \cosh\psi & 0 & \cos\vartheta\sinh\psi \\ -\sin\vartheta\sinh\psi & \cos\vartheta\sinh\psi & 0 \end{pmatrix},$$

and therefore its rank is 2 with $\psi\neq 0$, and $\vartheta\neq 0$. In addition the rank of the augmented matrix

$$\begin{pmatrix} 0 & \cosh\psi & -\sin\vartheta\sinh\psi & x_1^* \\ -\cosh\psi & 0 & \cos\vartheta\sinh\psi & x_2^* \\ -\sin\vartheta\sinh\psi & \cos\vartheta\sinh\psi & 0 & x_3^* \end{pmatrix},$$

is 2. Hence this system has infinitely many solutions represented with

$$\begin{array}{c} \alpha_1=\psi^*\sin\vartheta+(\vartheta^*-\alpha_3)\tanh\psi\cos\vartheta, \\ \alpha_2=-\psi^*\cos\vartheta+(\vartheta^*-\alpha_3)\tanh\psi\sin\vartheta, \\ -\alpha_1\sin\vartheta\sinh\psi+\alpha_2\cos\vartheta\sinh\psi=x_3^*. \end{array} \tag{3.32}$$

Since α_3 is taken at random, then we may take $\vartheta^*-\alpha_3=0$. In this case, Eq. (38) reduces to

$$\alpha_1(\vartheta)=\psi^*\sin\vartheta,\ \alpha_2(\vartheta)=-\psi^*\cos\vartheta,\ \alpha_3(\vartheta)=-\vartheta^*. \tag{3.33}$$

Thus, the director surface of this timelike line congruence is given by

$$\alpha(\vartheta, \vartheta^*) = (\psi^* \sin\vartheta, -\psi^* \cos\vartheta, -\vartheta^*)\,. \tag{3.34}$$

Let $\mathbf{m}(m_1,\ m_2,\ m_3)$ denote a point on this timelike line congruence. Hence, we obtain:

$$\mathbf{m}(\vartheta, \vartheta^*, v) = \alpha(\vartheta, \vartheta^*) + v\mathbf{x}(\vartheta, \vartheta^*),\ v \in \mathbb{R}, \tag{3.35}$$

which consists of family of timelike ruled surfaces $\mathbf{m}(\vartheta, \vartheta_0^*, v)$, $\mathbf{m}(\vartheta_0, \vartheta^*, v)$, and $\mathbf{m}(\vartheta(t), \vartheta^*(t), v)$. Here ϑ_0^*, ϑ_0, and t, respectively, are real constants. By means of Eqs. (36), (39), and (41) we simply find that

$$\left.\begin{array}{c} m_1 = \psi^* \sin\vartheta + v \sinh\psi \cos\vartheta, \\ m_2 = -\psi^* \cos\vartheta + v \sinh\psi \sin\vartheta, \\ m_3 = -\vartheta^* - v\cosh\psi, \end{array}\right\} \tag{3.36}$$

or by eliminating ϑ, we have

$$(X) : \frac{m_1^2}{\psi^{*2}} + \frac{m_2^2}{\psi^{*2}} - \frac{M_3^2}{n^2} = 1, \tag{3.37}$$

where $n = \psi^* \coth\psi$, and $M_3 = m_3 + \vartheta^*$. Then (X) is two-parameter Lorentzian spheres. The intersection of each Lorentzian sphere, and the corresponding spacelike plane $M_3 := m_3 + \vartheta^* = 0$ is $m_1^2 + m_2^2 = \psi^{*2}$. Therefore the envelope of (X) is the timelike cylinder $m_1^2 + m_2^2 = \psi^{*2}$. Notice that if $\vartheta^* = 0$, then

$$(X) : \frac{m_1^2}{\psi^{*2}} + \frac{m_2^2}{\psi^{*2}} - \frac{m_3^2}{n^2} = 1. \tag{3.38}$$

3.3 Constant parameter timelike ruled surfaces

Based on the properties of the Disteli-axis is constant, we can discuss the constant parameter timelike ruled surfaces. For this aim, a relation such as $F(\vartheta, \vartheta^*) = 0$, between the parameters restricts Eq. (34) (resp. (41)) to a one-parameter set of timelike lines, that is, a timelike ruled surface in the congruence. Therefore, if we select $\vartheta^* = h\vartheta$, h indicating to the pitch of the motion $\mathbb{H}^2_{+m}/\mathbb{H}^2_{+f}$, and ϑ as the motion parameter, then Eq. (34) (resp. (41)) performs a timelike ruled surface in $\mathbb{L}_f-$space. Thus,

$$\begin{pmatrix} \mathbf{X} \\ \mathbf{T} \\ \mathbf{G} \end{pmatrix} = \begin{pmatrix} \sinh\Psi\cos\Theta & \sinh\Psi\sin\Theta & -\cosh\Psi \\ -\sin\Theta & \cos\Theta & 0 \\ \cosh\Psi\cos\Theta & \cosh\Psi\sin\Theta & -\sinh\Psi \end{pmatrix} \begin{pmatrix} \mathbf{F}_1 \\ \mathbf{F}_2 \\ \mathbf{F}_3 \end{pmatrix}.$$

In this case we get:

$$\frac{d}{d\vartheta}\begin{pmatrix}\mathbf{X}\\ \mathbf{T}\\ \mathbf{G}\end{pmatrix}=\begin{pmatrix}0 & (1+\varepsilon h)\sinh\Psi & 0\\ (1+\varepsilon h)\sinh\Psi & 0 & (1+\varepsilon h)\cosh\Psi\\ 0 & -(1+\varepsilon h)\cosh\Psi & 0\end{pmatrix}\begin{pmatrix}\mathbf{X}\\ \mathbf{T}\\ \mathbf{G}\end{pmatrix}.$$

Thereby, the Blaschke invariants are

$$P=(1+\varepsilon h)\sinh\Psi,\ Q=(1+\varepsilon h)\cosh\Psi, \text{ and } \coth\Psi=\frac{Q}{P}. \tag{3.39}$$

By means of the real and dual parts of Eq. (45), we obtain

$$\mu=h+\psi^{*}\coth\psi,\ \Gamma=h+\psi^{*}\tanh\psi,\ \gamma=\coth\psi, \tag{3.40}$$

where μ, Γ, and γ are constants. Hence, as a direct consequence of Definition 2, (X) is a constant parameter timelike ruled surface. We simply find the base curve as;

$$\alpha(\vartheta)=\left(\psi^{*}\sin\vartheta,-\psi^{*}\cos\vartheta,-h\vartheta\right). \tag{3.41}$$

It can be show that $<\frac{d\alpha}{d\vartheta},\frac{d\mathbf{x}}{d\vartheta}>=0$; so the base curve of (X) is its striction curve. Moreover, it can be show that $\alpha(\vartheta)$ is a spacelike (resp. a timelike) if and only if $|\psi^{*}|>|h|$ (resp. $|\psi^{*}|<|h|$). For the curvature κ, and the torsion τ, we can find the following calculations simply;

$$\kappa(\vartheta)=\frac{\psi^{*}}{\psi^{*2}-h^{2}}, \text{ and } \tau(\vartheta)=\frac{h}{\psi^{*2}-h^{2}}.$$

Hence, $\alpha(\vartheta)$ is a spacelike (resp. a timelike) helix if and only if $|\psi^{*}|>|h|$ (resp. $|\psi^{*}|<|h|$). Furthermore, we have

$$(X):\mathbf{m}(\vartheta,v)=\left(\psi^{*}\sin\vartheta+vc_{1}\cos\vartheta,-\psi^{*}\cos\vartheta+vc_{1}\sin\vartheta,-h\vartheta-c_{2}v\right), \tag{3.42}$$

where $c_1=\sinh\psi$, and $c_2=\cosh\psi;.\psi=0.7$, $\vartheta\in[0,2\pi]$, $v\in[-4,4]$. According to Eq. (48), we have the following types;
(1) Timelike general helicoid: For $h=0.5$(resp. $h=1$), and $\psi^{*}=1$ (resp. $\psi^{*}=0.5$), respectively, the graph of the surfaces are shown in Figs. 4, and 5;
(2) Lorentzian sphere: For $h=0$, $\psi^{*}=1$, the graph of the surface is shown in Fig. 6;
(3) Timelike Cone: Figure 7 shows the surface (X) with $\psi^{*}=h=0$.

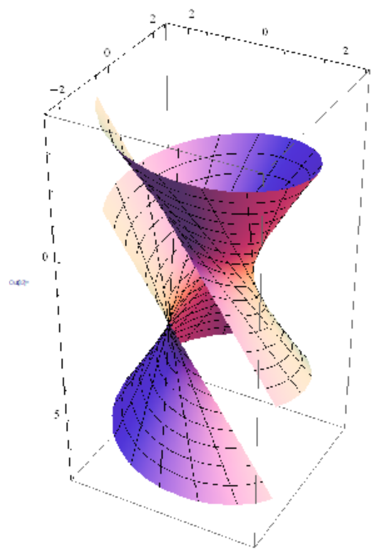

Figure 4: (X) with $h = 0$, and $\psi^* = 0.5$

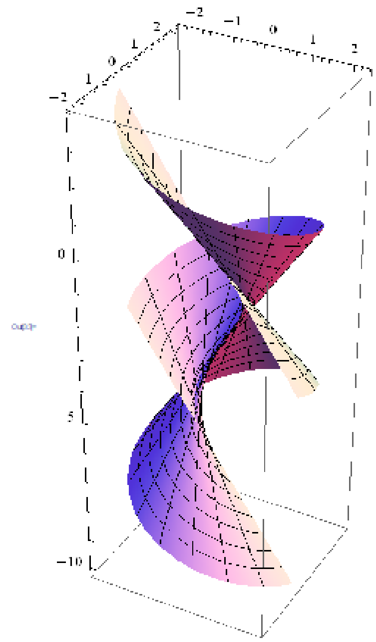

Figure 5: (X) with $h = 0.5$, and $\psi^* = 1$

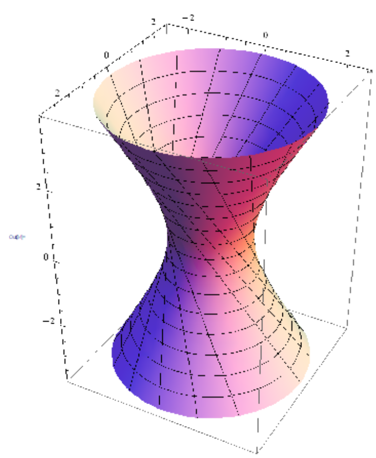

Figure 6: (X) with $h = 0$, and $\psi^* = 1$

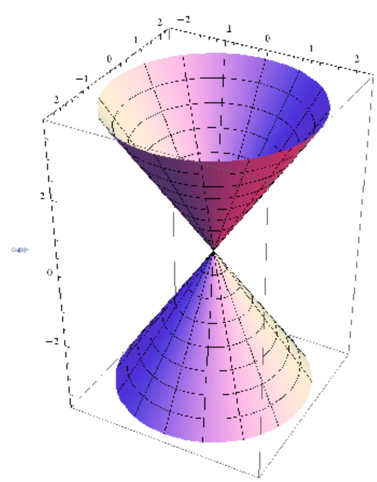

Figure 7: (X) with $h = 0 = \psi^* = 0$

4 Conclusion

In this work, we analyze a certain class of timelike ruled with constant Disteli-axis in Minkowski 3-space $\mathbb{E}_1^3$. As a result, the timelike ruled surface generated by a timelike line undergoing a Lorentzian screw motion is examined in detail. We believe that the study of spatial kinematics in Minkowski 3-space $\mathbb{E}^3$ via line geometry may shed some light on current research problems and perhaps suggest new ones.

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