STABILITY OF STEADY-STATE SOLUTIONS OF A CLASS OF KELLER-SEGEL MODELS WITH MIXED BOUNDARY CONDITIONS

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Abstract

In this paper, we investigate the the existence and stability of non-trivial steady state solutions of a class of chemotaxis models with zero-flux boundary conditions and Dirichlet boundary conditions on one-dimensional bounded interval. By using upper-lower solution and the monotone iteration scheme method, we get the existence of the steady-state solution of the chemotaxis model. Moreover, by adopting the "inverse derivative" technique and the weighted energy method to obtain the stability of the steady-state solution of this chemotaxis model.

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1. INTRODUCTION

Chemotaxis phenomenon is an essential and basic property of the living cells which responses to the environments changes, it describes a directional movement of the cells to the gradient of chemical concentrations, such as aggregation of bacteria, slime mould formation, fish pigmentation, tumor angiogenesis, blood vessel formation, wound healing/inflammation, and cancer metastasis[2, 3, 33]. The chemotaxis model was first proposed by Patlak [18] when studying the random motion of particles, and later Keller and Segel [19] proposed a chemotaxis model when studying amoeba aggregation effects. The general mathematical Keller-Segel model reads as:

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \phi(w)), & x \in \Omega, \\ w_t = D\Delta w + g(u, w), & x \in \Omega, \end{cases} \tag{1.1}$$

where u and w represent the bacterial density and oxygen concentration at position x and time t, respectively. The constant $D > 0$ represent the chemical diffusion coefficient. The functional $\phi(w)$ is called the chemotaxis sensitivity function accounting for the signal response mechanism and the reaction term $g(u, w)$ is the chemical kinetics (growth and degradation). The chemotactic sensitivity function $\phi(w)$ usually has two prototypes: $\phi(w) = \ln w$ (logarithmic sensitivity) and $\phi(w) = w$ (linear sensitivity). Logarithmic sensitivity was originally used in [19], based on Weber-Fechner's law (the sensory response to stimuli is logarithmic), and it has various biological applications (cf. [9, 16, 21]). The reaction function usually can be chosen as $g(u, w) = u^\gamma w^m$ with nonnegative constant parameters $\gamma \geq 1$ and $m \geq 0$, when $\gamma = 1$ and for any $0 \leq m \leq 1$, the existence of the traveling wave solution of (1.1) for logarithmic sensitivity was reported in [17, 30, 34]. The stability of the travel wavefront $(m = 1)$ was investigated in [6, 7, 23, 24, 25, 29]. The instability of pulsation waves $(m = 0)$ was studied in [8, 26] and the stability of boundary layers (see [14, 15]). We also refer to [10, 31, 32] for the global existence and large-time behavior of solution to (1.1) with $\gamma = m = 1$ and linear sensitivity under the Neumman boundary conditions $\partial_\nu u|_{\partial\Omega} = \partial_\nu w|_{\partial\Omega} = 0$. When the physical boundary conditions, namely, the zero-flux boundary condition and Dirichlet boundary condition are imposed to cell density u and chemical concentration w

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(see also the experiment in [1]):

$$\partial_\nu u - u\partial_\nu w = 0, w = w_*, x \in \partial\Omega, \tag{1.2}$$

where ν is the outward unit normal vector of $\partial\Omega$ and $w_* > 0$ represents the boundary date of w on the $\partial\Omega$. Until very recently, significant progress has been made for (1.1)-(1.2), the authors of [5] showed that under the boundary conditions (1.2) and the parametric restrictions $\gamma = 1$, $m \geq 0$ and $\chi > |m-1|$, system (1.1)-(1.2) with $\phi(w) = \ln w$ generates the so-called boundary spike layer (BSL) solutions that exist on the half-line:$[0,\infty)$ under smallness assumptions on the initial perturbations. When $\gamma = 1$, $m = 1$ and $\phi(w) = w$, Braukhoff and Lankeit in [4] proved the stationary solution under the no-flux boundary conditions for u and the physically meaningful condition for w in bounded domain $\Omega \subset \mathbb{R}^n, n \geq 1$. Later, Lee et.al [20] proved the existence and uniqueness of the steady-state solution of equations (1.1)-(1.2) in space $\Omega \subset R^n (n \geq 1)$; in [35], Winkler consider the existence of globally weak solutions in one-dimensional bounded interval; later, Hong and Wang [13] improved the result of [20] to obtain the asymptotic stability of the steady-state solution in the bounded domain $\Omega = (0,1)$. When $\gamma > 1, m = 0$, we refer the readers to [36, 12] where the global existence and large time behavior of the one dimensional Cauchy problem solution of equation and dynamic boundary condition problem were proved, respectively. In [11], the author also extended the result [5] to the case $1 < \gamma < 2$ on the half line. However, the stability of steady-state solution to (1.1)-(1.2) with linear sensitivity function is a open problem, which has more biological significance. Motivated by the ideas of [20] and [13], we are devoted to investigating the following system of reaction-diffusion-advection equations:

$$\begin{cases} u_t = u_{xx} - \chi(uw_x)_x, & \text{in } \mathcal{I}, \\ w_t = Dw_{xx} - u^\gamma, & \text{in } \mathcal{I}, \\ (u,w)(x,0) = (u_0,w_0)(x), & \text{in } \mathcal{I}, \end{cases} \tag{1.3}$$

with physical mixed boundary condition

$$(u_x - \chi u w_x)|_{x=0,1} = 0, w(0,t) = w(1,t) = w_*. \tag{1.4}$$

where $(x,t) \in \mathcal{I} \times \mathbb{R}^+$ and $1 < \gamma \leq 2$.

With the zero-flux boundary condition (1.4) on u, by integrating the first equation (1.3), we immediately find that the cell mass is preserved in time, namely :

$$\int_{\mathcal{I}} u(x,t)dx = \int_{\mathcal{I}} u_0(x)dx := M,$$

where $M > 0$ denotes the initial cell mass. Then the stationary solution $(\bar{u}, \bar{w})$ of (1.3) subject to boundary condition satisfies

$$\begin{cases} \bar{u}_{xx} - \chi(\bar{u}\bar{w}_x)_x = 0, & x \in \mathcal{I}, \\ D\bar{w}_{xx} - \bar{u}^\gamma = 0, & x \in \mathcal{I}, \\ \int_{\mathcal{I}} \bar{u}(x,t)dx = M, \\ (\bar{u}_x - \chi\bar{u}\bar{w}_x)|_{x=0,1} = 0, \bar{w}|_{x=0,1} = w_*. \end{cases} \tag{1.5}$$

The goal of this paper is twofold. First, we can use a similar method as in [20] to prove the existence of the classical solution to equation (1.5). The second goal of this paper is to show the nonlinear stability of the stationary solution $(\bar{u}, \bar{w})$, we can show that the system (1.3) with (1.4) admits a unique solution (u, w) satisfying:

$$||(u,w) - (\bar{u}, \bar{w})||_{L^\infty} \to 0 \text{ as } t \to \infty.$$

The rationale underlining the studies of such a model is that monomials are the building blocks of genuinely nonlinear functions. Hence, the study of the model will help shed light on how to advance fundamental research of chemotaxis models with nonlinear rates of chemical production/consumption. Our intention of this paper is to continue exploring the mathematical properties of the model by investigating the existence and stability of steady state solutions under the same boundary conditions as specified in (1.4).

The rest of this paper is organized as follows: In Sec. 2, we state our main results on the existence and stability of stationary solution to the (1.3)– (1.4). Then we prove the existence results and stability results in Sec. 3.

Notations. Throughout the paper, we denote by L^∞, L^2, H_0^1 and H^k the standard function spaces $L^\infty(\mathcal{I}), L^2(\mathcal{I}), H_0^1(\mathcal{I})$ and $H^k(\mathcal{I})$, respectively. We denote $\bar{\mathcal{I}}$ by the closure of $\mathcal{I}$ and by C a generic time-independent constant which may take different values in different places. In the sequel, we often omit $\mathcal{I}$ without ambiguity.

2. MAIN RESULTS

In this section, we introduce the existence of the stationary solution of the equation (1.5) and state our main results on the stability of stationary solutions. The first results on the existence of steady state solutions are stated below.

Theorem 2.1. *For any $M \in (0,\infty)$, the system (1.5) admits a unique classical non-constant solution $(\bar{u},\bar{w}) \in C^1(\bar{\mathcal{I}}) \cap C^\infty(\mathcal{I})$ such that*

$$\bar{u} = \frac{M}{\int_{\mathcal{I}} e^{\chi\bar{w}}dx}e^{\chi\bar{w}}, \bar{u} > 0, 0 < \bar{w} \leq w_* \text{ for any } x \in \bar{\mathcal{I}}. \tag{2.1}$$

The second result is the nonlinear local stability of stationary solutions obtained in Theorem 2.1 for the initial-boundary value problem (1.3)-(1.4) as time goes to infinity.

Theorem 2.2. *Suppose that $u_0 \in H^1$ and $w_0 \in H^2$ with $u_0 \geq 0, w_0 \geq 0$ such that $\int_{\mathcal{I}} u_0 dx = M$. Let $(\bar{u},\bar{w})$ be the stationary solution given in Theorem 2.1 with $\int_{\mathcal{I}} \bar{u}dx = M$ and define*

$$\varphi_0(x) = \int_0^x (u_0(y) - \bar{u}(y))dy.$$

Then there exists a constant $\delta_0 > 0$ such that if

$$||\varphi_0||_{H^2}^2 + ||w_0 - \bar{w}||_{H^2}^2 \leq \delta_0,$$

then the initial-boundary value problem (1.3)–(1.4) *admits a unique global solution (u, w) satisfying*

$$u \in C([0,\infty); H^1) \cap L^2([0,\infty); H^2), w \in C([0,\infty); H^2) \cap L^2([0,\infty); H^3),$$

and the following large time behavior:

$$||(u - \bar{u}, w - \bar{w})(\cdot, t)||_{L^\infty} \to 0. \tag{2.2}$$

3. Existence and stability for the case $D > 0$

3.1. **Existence of stationary solution.** In this section,we first prove the existence of steady state of (1.5). We start the proof by considering the following auxiliary problem

$$\begin{cases} D\bar{w}_{\kappa,xx} = \left(\kappa M e^{\chi \bar{w}_\kappa}\right)^\gamma, & x \in \mathcal{I}, \\ \bar{w}_\kappa = w_*, & x = 0, 1, \end{cases} \tag{3.1}$$

where κ is an arbitrary positive constant. Since $w_* > 0$, by maximal principle we have

$$0 \leq \bar{w}_\kappa \leq w_*, x \in \mathcal{I}.$$

Then it is not difficult to see that $\bar{w}_\kappa = w_*$ is a super-solution of (3.1), while $\bar{w}_\kappa = 0$ provides a sub-solution. Therefore, following the standard monotone iteration scheme and the fact that $f(x) = e^x$ is increasing function for x positive, we immediately know that (3.1) has a unique classical solution $\bar{w}_\kappa$ satisfying

$$0 \leq \bar{w}_\kappa \leq w_*.$$

Now we claim that there exists κ_m such that

$$\kappa_m \int_{\mathcal{I}} e^{\chi \bar{w}_{\kappa_m}} dx = 1. \tag{3.2}$$

We postpone the proof of (3.2) in lemma 3.1. Using this claim we can easily see that $\bar{w} = \bar{w}_{\kappa_m}$ is a solution of (1.5).

In order to prove the claim (3.2), we give the following lemma.

Lemma 3.1. *Let $\kappa_1 \geq \kappa_2 > 0$ and $\bar{w}_{\kappa_i}, i = 1, 2$ be the solutions of (3.1) with $\kappa = \kappa_i, i = 1, 2$ respectively. Then*

$$0 \leq \bar{w}_{\kappa_2} - \bar{w}_{\kappa_1} \leq \left(\left(\frac{\kappa_1}{\kappa_2}\right)^\gamma - 1\right) \frac{e^{\gamma\chi w_*}}{\gamma\chi}. \tag{3.3}$$

Moreover, there exists a constant κ_m such that

$$\kappa_m \int_{\mathcal{I}} e^{\chi \bar{w}_{\kappa_m}} dx = 1. \tag{3.4}$$

Proof. The left-hand side inequality follows from the standard comparison theorem directly. We only prove the inequality for the right-hand side. Due to the fact $\kappa_1 \geq \kappa_2 > 0$ and $\bar{w}_{\kappa_1} > 0$, it's easy to see that

$$\begin{aligned} D\left(\bar{w}_{\kappa_1,xx} - \bar{w}_{\kappa_2,xx}\right) &= \left(\kappa_1 M e^{\chi \bar{w}_{\kappa_1}}\right)^\gamma - \left(\kappa_2 M e^{\chi \bar{w}_{\kappa_2}}\right)^\gamma \\ &= (\kappa_2 M)^\gamma \left(e^{\gamma\chi \bar{w}_{\kappa_1}} - e^{\gamma\chi \bar{w}_{\kappa_2}}\right) + \left((\kappa_1 M)^\gamma - (\kappa_2 M)^\gamma\right) e^{\gamma\chi \bar{w}_{\kappa_1}} \\ &\leq (\kappa_2 M)^\gamma F(\gamma\chi\bar{w}_{\kappa_1}, \gamma\chi\bar{w}_{\kappa_2})(\gamma\chi\bar{w}_{\kappa_1} - \gamma\chi\bar{w}_{\kappa_2}) \\ &\quad + \left((\kappa_1 M)^\gamma - (\kappa_2 M)^\gamma\right) e^{\gamma\chi w_*} \end{aligned} \tag{3.5}$$

where

$$F(a, b) = \begin{cases} \frac{e^a - e^b}{a - b}, & \text{if } a \neq b, \\ e^a, & \text{if } a = b. \end{cases}$$

From the fact

$$0 < \bar{w}_{\kappa_1} \leq \bar{w}_{\kappa_2} \leq w_*, \tag{3.6}$$

we have

$$1 < F(\gamma\chi\bar{w}_{\kappa_1}, \gamma\chi\bar{w}_{\kappa_2}) \leq e^{\gamma\chi w_*}. \tag{3.7}$$

By (3.5) and (3.7), we have

$$\begin{aligned}\bar{w}_{\kappa_2}-\bar{w}_{\kappa_1} &\leq \frac{\left((\kappa_1 M)^\gamma-(\kappa_2 M)^\gamma\right)e^{\gamma\chi\bar{w}_{\kappa_1}}}{(\kappa_2 M)^\gamma F(\gamma\chi\bar{w}_{\kappa_1},\gamma\chi\bar{w}_{\kappa_2})\gamma\chi}\\ &\leq \left(\left(\frac{\kappa_1}{\kappa_2}\right)^\gamma-1\right)\frac{e^{\gamma\chi w_*}}{\gamma\chi}.\end{aligned}$$

Thus, we prove the right-hand side of (3.3). It implies that the continuity of $\bar{w}_\kappa$ with respect to κ. On the other hand, we have

$$\lim_{\kappa\to 0^+}\kappa\int_{\mathcal{I}} e^{\chi\bar{w}_\kappa}dx=0 \text{ and } \lim_{\kappa\to\infty}\kappa\int_{\mathcal{I}} e^{\chi\bar{w}_\kappa}dx=\infty.$$

Then we can find κ_m such that $\kappa_m\int_{\mathcal{I}} e^{\chi\bar{w}_{\kappa_m}}dx=1$ and it completes the proof of lemma 3.1. □

3.2. **Nonlinear asymptotic stability.** In this section, we shall focus on attention to investigate the global well-posedness and stability of the steady state of (1.3)-(1.4) for $D>0$ by the method of energy estimates when the initial data (u_0,w_0) is a small perturbation of $(\bar{u},\bar{w})$. Before proceeding, we present an well-known inequality that will be frequently used in the sequel.

Lemma 3.2. *(cf.[27]). For any $f\in H^1(\mathcal{I})$, there exists a constant $c_1>0$ such that*

$$||f||_{L^\infty}\leq c_1(||f||_{L^2}^{\frac{1}{2}}||f_x||_{L^2}^{\frac{1}{2}}+||f||_{L^2}). \tag{3.8}$$

Furthermore, if $f\in H_0^1(\mathcal{I})$, then it holds that

$$||f||_{L^\infty}\leq c_2||f||_{L^2}^{\frac{1}{2}}||f_x||_{L^2}^{\frac{1}{2}} \text{ and } ||f||_{L^\infty}\leq c_3||f_x||_{L^2} \tag{3.9}$$

for some constants $c_2,c_3>0$.

3.2.1. *Reformulation of the problem.* Now let us use energy methods and anti-derivatives technique to prove Theorem 2.2. Integrating the equation $(1.5)_1$ over $(0,x)$, we obtain that the stationary solution $(\bar{u},\bar{w})$ satisfies

$$\begin{cases}\bar{u}_x-\chi(\bar{u}\bar{w})_x=0,\\ D\bar{w}_{xx}-\bar{u}^\gamma=0,\\ \bar{w}(0)=\bar{w}(1)=w_*.\end{cases} \tag{3.10}$$

With the zero-flux boundary condition of u in (1.4), which combined with the facts $\int_{\mathcal{I}}\bar{u}dx=\int_{\mathcal{I}}u_0dx=M$ implies that

$$\int_{\mathcal{I}}(u(x,t)-\bar{u}(x))dx=0$$

for any $t\geq 0$. This enables us to define anti-derivatives of the perturbed functions and more importantly, carry out stability analysis in the framework of Sobolev spaces. Define

$$\varphi(x,t)=\int_0^x(u(y,t)-\bar{u}(y))dy,\psi=w-\bar{w},$$

which implies

$$u=\varphi_x+\bar{u}, w=\psi+\bar{w}. \tag{3.11}$$

Substituting (3.11) into (1.3), by integrating the first resulting equation with the spatial variable from 0 to x, then using (3.10), we can show that:

$$\begin{cases}\varphi_t=\varphi_{xx}-\chi(\varphi_x\psi_x+\bar{u}\psi_x+\varphi_x\bar{w}_x),\\ \psi_t=D\psi_{xx}+\bar{u}^\gamma-(\varphi_x+\bar{u})^\gamma,\end{cases} \tag{3.12}$$

with the initial data

$$(\varphi,\psi)(x,0)=(\varphi_0,\psi_0)=\left(\int_0^x(u_0(y)-\bar{u}(y))dy, w_0-\bar{w}\right) \tag{3.13}$$

and the boundary conditions

$$(\varphi,\psi)(0,t)=(\varphi,\psi)(1,t)=0. \tag{3.14}$$

By standard approaches, such as iteration scheme and fixed point theorems (cf. [22, 28]), one can prove the local existence of solutions to the initial-boundary value problem (3.12)-(3.14). To simplify the presentation, we only record the result here without producing the proof. Precisely, for any $T>0$, if we define

$$X(0,T):=\{(\varphi,\psi)|\varphi\in\mathcal{C}([0,T];H_0^1\cap H^2)\cap L^2([0,T];H^3),$$
$$\psi\in\mathcal{C}([0,T];H_0^1\cap H^2)\cap L^2([0,T];H^3)\}$$

and denote

$$N(T):=\sup_{0\le t\le T}(||\varphi||_{H^2}^2+||\psi||_{H^2}^2),$$

then we have the following local existence result.

Proposition 3.3. *(Local existence). Let $\varphi_0\in H_0^1\cap H^2$ and $\psi_0\in H_0^1\cap H^2$ such that*

$$\varphi_{0,x}+\bar{u}\ge 0,\psi_0+\bar{w}\ge 0$$

for any $x\in\mathcal{I}$. Then there exists a positive constant T_0 depending on N_0,$\bar{u}$ and $\bar{w}$ such that the initial-boundary value problem (3.12)-(3.14) admits a unique solution $(\varphi,\psi)\in X(0,T_0)$ satisfying $N(T_0)\le 2N(0)$ and

$$\varphi_x+\bar{u}\ge 0,\psi+\bar{w}\ge 0$$

for any $(x,t)\in\mathcal{I}\times[0,T_0)$.

Next, we shall derive the *a priori* estimates of the local solution, in order to extend it to a global one and to study the stability of stationary solutions to the initial-boundary value problem (1.3)-(1.4).

Proposition 3.4. *Assume the conditions of Proposition 3.3 hold. Then there exists a positive constant δ_1, such that if $||\varphi_0||_{H^2}^2+||\psi_0||_{H^2}^2\le\delta_1$, then the problem (3.12)-(3.14) admits a unique global solution $(\varphi,\psi)\in X(0,\infty)$ which satisfies that for all $t\ge 0$,*

$$||\varphi(\cdot,t)||_{H^2}^2+||\psi(\cdot,t)||_{H^2}^2+\int_0^t\left(||\varphi||_{H^3}^2+||\psi||_{H^3}^2+||\varphi_\tau||_{H^1}^2+||\psi_\tau||_{H^1}^2\right)d\tau\le CN^2(0). \tag{3.15}$$

for some constants $C>0$ independent of $t\in(0,\infty)$.

By the local existence result and the standard continuation argument, to prove the Proposition 3.4, we only need to prove the requisite *a priori* estimates below.

Proposition 3.5. *(A priori estimates). For any $T>0$ and any solution $(\varphi,\psi)\in X(0,T)$ to the problem (3.12)-(3.14) with $(\varphi_0,\psi_0)\in H_0^1\cap H^2$, there exists a suitably small $\delta_2>0$ independent of T such that if $||\varphi||_{H^2}^2+||\psi||_{H^2}^2\le\delta_2$, then (φ,ψ) satisfies* (3.15) *for any $t\in[0,T]$.*

Before proceeding to the estimate of (φ,ψ), we collect some technical lemmas, which will often be used in the proof of Proposition 3.5.

Lemma 3.6. *Let $(\bar{u}, \bar{w})$ be the stationary solution of (1.3), (1.4) stated in Proposition 2.1. Then it holds that*

$$0 < C_1^{-1} \leq \bar{u}, \bar{w} \leq C_1 \tag{3.16}$$

for some constant $C_1 > 0$, and that

$$D\bar{w}_x^2 \leq \frac{2}{\gamma\chi}\bar{u}^\gamma. \tag{3.17}$$

Proof. According to Proposition 3.3, it is trivial to obtain (3.16) by comparison principle, so we only need to prove (3.17). For any $x \in \bar{\mathcal{I}}$, when $0 < \bar{w}(x) \leq w_*$ there exists an $x_0 \in (0, 1)$ such that

$$0 < \bar{w}(x_0) = \min_{x \in \bar{\mathcal{I}}} \bar{w}(x) \text{ and } \bar{w}_x(x_0) = 0.$$

Multiplying the second equation in (1.5) by $\bar{w}_x$, and integrating the resulting equation from x_0 to x, we get

$$\frac{D}{2}\bar{w}_x^2 = \int_{x_0}^{x} \bar{u}^\gamma \bar{w}_y dy = \int_{\bar{w}(x_0)}^{\bar{w}(x)} \bar{u}^\gamma d\bar{w} = \lambda^\gamma \int_{\bar{w}(x_0)}^{\bar{w}(x)} e^{\gamma\chi s} ds \leq \lambda^\gamma \int_0^{\bar{w}(x)} e^{\gamma\chi s} ds$$

with $\lambda = \frac{M}{\int_{\mathcal{I}} e^{\chi\bar{w}} dx}$, where we have used the following identity

$$\bar{u} = \frac{M}{\int_{\mathcal{I}} e^{\chi\bar{w}} dx} e^{\chi\bar{w}} =: \lambda e^{\chi\bar{w}} \tag{3.18}$$

from (2.1). Thanks to (3.18),we can get

$$\frac{D}{2}\bar{w}_x^2 \leq \frac{1}{\gamma\chi}\lambda^\gamma \left(e^{\gamma\chi\bar{w}} - 1\right) \leq \frac{1}{\gamma\chi}\lambda^\gamma e^{\gamma\chi\bar{w}} = \frac{1}{\gamma\chi}\bar{u}^\gamma.$$

The proof is completed. □

Lemma 3.7 ([36])**.** *For any $a \geq -1$ and $1 < \gamma < 2$, it holds that*

$$(a+1)^\gamma - 1 - \gamma a \leq a^2.$$

Lemma 3.8 ([36])**.** *For any $a \geq 0$ and $0 < \gamma < 1$, it holds that*

$$|a^\gamma - 1| \leq |a - 1|.$$

In the following part, we will consider the global existence of (3.12)-(3.14). Let us begin with the estimate of the zeroth and first order frequencies of the perturbation.

Lemma 3.9. *For any solution $(\varphi, \psi) \in X(0, T)$ to the problem (3.12)-(3.14) satisfying $N(t) \ll 1$, it holds that*

$$\int_{\mathcal{I}} \frac{\varphi^2}{\bar{u}} + \bar{u}^{1-\gamma}\psi^2 dx + \int_0^t \left(||\psi||_{L^2}^2 + ||\varphi_x||_{L^2}^2 + ||\psi_x||_{L^2}^2\right) d\tau \leq C\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{L^2}^2\right) \tag{3.19}$$

for any $t \in [0, T]$, where $C > 0$ is a constant independent of t.

Proof. We Multiply the first equation in (3.12) by $\gamma\frac{\varphi}{\bar{u}}$, then integrating by parts with respect to x, we can show that

$$\begin{aligned}
&\frac{\gamma}{2}\frac{d}{dt}\int_{\mathcal{I}} \frac{\varphi^2}{\bar{u}} dx + \gamma\int_{\mathcal{I}} \frac{\varphi_x^2}{\bar{u}} dx \\
&= -\gamma\int_{\mathcal{I}} \varphi\varphi_x \left[\left(\frac{1}{\bar{u}}\right)_x + \chi\frac{\bar{w}_x}{\bar{u}}\right] dx - \gamma\chi\int_{\mathcal{I}} \frac{\varphi\varphi_x\psi_x}{\bar{u}} dx - \gamma\chi\int_{\mathcal{I}} \varphi\psi_x dx.
\end{aligned} \tag{3.20}$$

By using (3.10), we can show that

$$\left(\frac{1}{\bar{u}}\right)_x + \chi\frac{\bar{w}_x}{\bar{u}} = -\frac{1}{\bar{u}^2}(\bar{u}_x - \chi\bar{u}\bar{w}_x) = 0,$$

thus we get

$$-\gamma\int_{\mathcal{I}}\varphi\varphi_x\left[\left(\frac{1}{\bar{u}}\right)_x + \chi\frac{\bar{w}_x}{\bar{u}}\right]dx = 0. \tag{3.21}$$

Using the fact that $\|\varphi\|_{L^\infty} \le N(t)$ and the Cauchy-Schwarz inequality, we can show that

$$\gamma\chi\int_{\mathcal{I}}\frac{\varphi\varphi_x\psi_x}{\bar{u}}dx \le C||\varphi||_{L^\infty}||\psi_x||_{L^2}||\varphi_x||_{L^2} \le CN(t)(||\varphi_x||_{L^2}^2 + ||\psi_x||_{L^2}^2). \tag{3.22}$$

Substituting (3.21) and (3.22) into (3.20), we have

$$\frac{\gamma}{2}\frac{d}{dt}\int_{\mathcal{I}}\frac{\varphi^2}{\bar{u}}dx + \gamma\int_{\mathcal{I}}\frac{\varphi_x^2}{\bar{u}}dx \le -\gamma\chi\int_{\mathcal{I}}\varphi\psi_x dx + CN(t)(||\varphi_x||_{L^2}^2 + ||\psi_x||_{L^2}^2). \tag{3.23}$$

We Multiply the first equation in (3.12) by $\chi\bar{u}^{1-\gamma}\psi$ and then integrating by parts with respect to x, we get

$$\begin{aligned}\frac{\chi}{2}\frac{d}{dt}\int_{\mathcal{I}}\bar{u}^{1-\gamma}\psi^2dx + \chi D\int_{\mathcal{I}}\bar{u}^{1-\gamma}\psi_x^2dx =& -\chi D\int_{\mathcal{I}}(\bar{u}^{1-\gamma})_x\psi\psi_x dx\\ &- \chi\int_{\mathcal{I}}\bar{u}\psi\left[\left(\frac{\varphi_x}{\bar{u}}+1\right)^\gamma - 1 - \gamma\frac{\varphi_x}{\bar{u}}\right]dx\\ &- \gamma\chi\int_{\mathcal{I}}\psi\varphi_x dx,\end{aligned} \tag{3.24}$$

For the first term on the right hand side of (3.24), by (3.14) and then integrating by parts, we can show that

$$-\chi D\int_{\mathcal{I}}\left(\bar{u}^{1-\gamma}\right)_x\psi\psi_x dx = \frac{1}{2}\chi D\int_{\mathcal{I}}\left(\bar{u}^{1-\gamma}\right)_{xx}\psi^2dx, \tag{3.25}$$

by (3.10), (3.16) and (3.17), we have

$$\begin{aligned}\frac{D\chi}{2}\left(\bar{u}^{1-\gamma}\right)_{xx} &= -\frac{D\chi}{2}\gamma(1-\gamma)\bar{u}^{-\gamma-1}\bar{u}_x^2 + \frac{D\chi}{2}(1-\gamma)\bar{u}^{-\gamma}\bar{u}_{xx}\\ &= \frac{D\chi}{2}(1-\gamma)^2\bar{u}^{1-\gamma}\chi^2\bar{w}_x^2 + \frac{D\chi}{2}(1-\gamma)\bar{u}^{1-\gamma}\chi\bar{w}_{xx}\\ &\le [\frac{(1-\gamma)^2}{\gamma} + \frac{1-\gamma}{2}]\chi^2\bar{u}\end{aligned}$$

Hence, when $1 < \gamma \le 2$, we update (3.25) as

$$-\chi D\int_{\mathcal{I}}\left(\bar{u}^{1-\gamma}\right)_x\psi\psi_x dx \le \left(\frac{(1-\gamma)^2}{\gamma} + \frac{1-\gamma}{2}\right)\chi^2\int_{\mathcal{I}}\bar{u}\psi^2dx := -\mathcal{K}\int_{\mathcal{I}}\bar{u}\psi^2dx, \tag{3.26}$$

where $\mathcal{K} > 0$ when $1 < \gamma < 2$ and $\mathcal{K} = 0$ when $\gamma = 2$. For the second term on the right hand side of (3.24), using (3.9), (3.16), Lemma3.7 and the Hölder inequality, we have

$$-\chi\int_{\mathcal{I}}\bar{u}\psi\left[\left(\frac{\varphi_x}{\bar{u}}+1\right)^\gamma - 1 - \gamma\frac{\varphi_x}{\bar{u}}\right]dx \le C||\psi||_{L^\infty}||\varphi_x||_{L^2}^2 \le CN(t)||\varphi_x||_{L^2}^2, \tag{3.27}$$

then substituting (3.26) and (3.27) into (3.24), we get

$$\begin{aligned}&\frac{\chi}{2}\frac{d}{dt}\int_{\mathcal{I}}\bar{u}^{1-\gamma}\psi^2dx+D\chi\int_{\mathcal{I}}\bar{u}^{1-\gamma}\psi_x^2dx+\mathcal{K}\int_{\mathcal{I}}\bar{u}\psi^2dx\\&\le-\gamma\chi\int_{\mathcal{I}}\psi\varphi_xdx+CN(t)||\varphi_x||_{L^2}^2.\end{aligned}\tag{3.28}$$

Adding (3.28) to (3.23), we can show that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{I}}\gamma\frac{\varphi^2}{\bar{u}}+\chi\bar{u}^{1-\gamma}\psi^2dx+\int_{\mathcal{I}}\mathcal{K}\bar{u}\psi^2+\gamma\frac{\varphi_x^2}{\bar{u}}+D\chi\bar{u}^{1-\gamma}\psi_x^2dx\le CN(t)(||\varphi_x||_{L^2}^2+||\psi_x||_{L^2}^2).$$

By (3.16), we obtain

$$\begin{aligned}&\frac{1}{2}\min\{\gamma,\chi\}\frac{d}{dt}\int_{\mathcal{I}}\frac{\varphi^2}{\bar{u}}+\bar{u}^{1-\gamma}\psi^2dx+\min\{C_1\mathcal{K},\frac{\gamma}{C_1},D\chi C_1^{1-\gamma}\}\int_{\mathcal{I}}\psi^2+\varphi_x^2+\psi_x^2dx\\&\quad\le CN(t)(||\varphi_x||_{L^2}^2+||\psi_x||_{L^2}^2),\end{aligned}$$

which implies that

$$\frac{d}{dt}\int_{\mathcal{I}}\frac{\varphi^2}{\bar{u}}+\bar{u}^{1-\gamma}\psi^2dx+\beta\left(||\psi||_{L^2}^2+||\varphi_x||_{L^2}^2+||\psi_x||_{L^2}^2\right)\le0\tag{3.29}$$

with

$$CN(t)\le\frac{\min\{\frac{\gamma}{C_1},D\chi C_1^{1-\gamma},C_1\mathcal{K}\}}{2\min\{\gamma,\chi\}}=:\frac{\beta}{2}.$$

Integrating (3.29) over $(0,t)$, we get (3.19) and hence completes the proof of the lemma. □

We are ready to derive the estimate on φ_x and ψ_x in the next lemma.

Lemma 3.10. *For any solution $(\varphi,\psi)\in X(0,T)$ to the problem (3.12)-(3.14) satisfying $N(t)\ll1$, it holds that for any $t\in[0,T]$ that*

$$||\varphi_x||_{L^2}^2+||\psi_x||_{L^2}^2+\int_0^t||\varphi_\tau||_{L^2}^2+||\psi_\tau||_{L^2}^2d\tau\le C\left(||\varphi_0||_{H^1}^2+||\psi_0||_{H^1}^2\right).\tag{3.30}$$

.

Proof. Multiplying the first equation of (3.12) by φ_t, then integrating by parts, we can show that

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{I}}\varphi_x^2dx+\int_{\mathcal{I}}\varphi_t^2dx=-\chi\int_{\mathcal{I}}\bar{u}\varphi_t\psi_xdx-\chi\int_{\mathcal{I}}\bar{w}_x\varphi_t\varphi_xdx-\chi\int_{\mathcal{I}}\varphi_t\varphi_x\psi_xdx,\tag{3.31}$$

Using (3.16), (3.17) and the Cauchy-Schwarz inequality, we have

$$-\chi\int_{\mathcal{I}}\bar{w}_x\varphi_t\varphi_xdx\le\chi||\bar{w}_x||_{L^\infty}||\varphi_t||_{L^2}||\varphi_x||_{L^2}\le\varepsilon||\varphi_t||_{L^2}^2+C_\varepsilon||\varphi_x||_{L^2}^2\tag{3.32}$$

and

$$-\chi\int_{\mathcal{I}}\bar{u}\varphi_t\psi_xdx\le\chi||\bar{u}||_{L^\infty}||\varphi_t||_{L^2}||\psi_x||_{L^2}\le\varepsilon||\varphi_t||_{L^2}^2+C_\varepsilon||\psi_x||_{L^2}^2,\tag{3.33}$$

for any $\varepsilon>0$. Using $\|\varphi_x\|_{L^\infty}\le N(t)$ and the Cauchy-Schwarz inequality, we can show that

$$-\chi\int_{\mathcal{I}}\varphi_t\varphi_x\psi_xdx\le\chi||\varphi_x||_{L^\infty}||\varphi_t||_{L^2}||\psi_x||_{L^2}\le CN(t)\left(||\varphi_t||_{L^2}^2+||\psi_x||_{L^2}^2\right).\tag{3.34}$$

Then we substitute (3.32)-(3.34) into (3.31), by choosing ε suitably small and letting $N(t)$ small enough such that

$$CN(t)\le\frac{1}{4},$$

we obtain

$$\frac{d}{dt}\int_{\mathcal{I}}\varphi_x^2dx+\int_{\mathcal{I}}\varphi_t^2dx\le C||\psi_x||_{L^2}^2. \tag{3.35}$$

Using (3.19) and integrating by parts, we update (3.35) as

$$\begin{aligned}||\varphi_x||_{L^2}^2+\int_0^t||\varphi_\tau||_{L^2}^2d\tau&\le||\varphi_{0,x}||_{L^2}^2+C\int_0^t||\psi_x||_{L^2}^2d\tau\\&\le||\varphi_{0,x}||_{L^2}^2+C\left(||\varphi_0||_{L^2}^2+||\psi_0||_{L^2}^2\right)\\&\le C\left(||\varphi_0||_{H^1}^2+||\psi_0||_{L^2}^2\right).\end{aligned} \tag{3.36}$$

Multiplying the second equation of (3.12) by ψ_t and then integrating the resultant equation with respect to x, we can show that

$$\frac{1}{2}\frac{d}{dt}D\int_{\mathcal{I}}\psi_x^2dx+\int_{\mathcal{I}}\psi_t^2dx=-\int_{\mathcal{I}}\bar{u}^\gamma\psi_t\left[\left(\frac{\varphi_x}{\bar{u}}+1\right)^\gamma-1-\gamma\frac{\varphi_x}{\bar{u}}\right]dx-\gamma\int_{\mathcal{I}}\bar{u}^{\gamma-1}\psi_t\varphi_xdx. \tag{3.37}$$

For the two terms on the right side of (3.37), by (3.9), (3.16), Lemma 3.7 and the Cauchy-Schwarz inequality, we can show that

$$\begin{aligned}-\int_{\mathcal{I}}\bar{u}^\gamma\psi_t\left[\left(\frac{\varphi_x}{\bar{u}}+1\right)^\gamma-1-\gamma\frac{\varphi_x}{\bar{u}}\right]dx&\le\int_{\mathcal{I}}\bar{u}^{\gamma-2}|\psi_t|\varphi_x^2dx\\&\le C||\varphi_x||_{L^\infty}||\psi_t||_{L^2}||\varphi_x||_{L^2}\\&\le CN(t)(||\psi_t||_{L^2}^2+||\varphi_x||_{L^2}^2)\end{aligned}$$

and

$$\begin{aligned}-\gamma\int_{\mathcal{I}}\bar{u}^{\gamma-1}\psi_t\varphi_xdx&\le\gamma||\bar{u}^{\gamma-1}||_{L^\infty}||\psi_t||_{L^2}||\varphi_x||_{L^2}\le C||\psi_t||_{L^2}||\varphi_x||_{L^2}\\&\le\varepsilon||\psi_t||_{L^2}^2+C_\varepsilon||\varphi_x||_{L^2}^2\end{aligned}$$

for any $\varepsilon>0$. Choosing ε and $N(t)$ small enough, we thus update (3.37) as

$$D\frac{d}{dt}\int_{\mathcal{I}}\psi_x^2dx+\int_{\mathcal{I}}\psi_t^2dx\le(CN(t)+C_\varepsilon)||\varphi_x||_{L^2}^2. \tag{3.38}$$

Using (3.19) and integration by parts, for any $t\in[0,T]$, it implies that

$$\begin{aligned}D\int_{\mathcal{I}}\psi_x^2dx+\int_0^t||\psi_\tau||_{L^2}^2d\tau&\le||\psi_{0,x}||_{L^2}^2+(CN(t)+C_\varepsilon)\int_0^t||\varphi_x||_{L^2}^2d\tau\\&\le C||\psi_{0,x}||_{L^2}^2+C\left(||\varphi_0||_{L^2}^2+||\psi_0||_{L^2}^2\right),\end{aligned}$$

thus

$$||\psi_x||_{L^2}^2+\int_0^t||\psi_\tau||_{L^2}^2d\tau\le C\left(||\varphi_0||_{L^2}^2+||\psi_0||_{H^1}^2\right). \tag{3.39}$$

Adding (3.39) to (3.36), we complete the proof of Lemma 3.9. □

In the following, we establish the estimates of the second order derivatives of (φ,ψ).

Lemma 3.11. *Let $(\varphi,\psi)\in X(0,T)$ be a solution to the initial-boundary value problem (3.12)-(3.14) and assume the conditions of Lemma3.9 hold. Then it holds for any $t\in[0,T]$ that*

$$||\varphi_{xx}||_{L^2}^2+||\psi_{xx}||_{L^2}^2+\int_0^t\int_{\mathcal{I}}\varphi_{xxx}^2+\psi_{xxx}^2dxd\tau \tag{3.40}$$

$$\le C(\|\varphi_0\|_{H^2}^2+\|\psi_0\|_{H^2}^2) \tag{3.41}$$

.

Proof. By differentiating (3.12) with respect to t to get

$$\begin{cases} \varphi_{tt} = \varphi_{xxt} - \chi(\bar{w}_x\varphi_{xt} + \bar{u}\psi_{xt} + \varphi_{xt}\psi_x + \varphi_x\psi_{xt}), \\ \psi_{tt} = D\psi_{xxt} - \gamma(\varphi_x + \bar{u})^{\gamma-1}\varphi_{xt}. \end{cases} \tag{3.42}$$

Multiplying the first equation in (3.42) by φ_t and the second one by ψ_t, then integrate by parts over $\mathcal{I}$, and adding the results to show that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathcal{I}} \varphi_t^2 + \psi_t^2 dx + \int_{\mathcal{I}} \varphi_{xt}^2 + D\psi_{xt}^2 dx = \\ &\quad - \chi\int_{\mathcal{I}} \bar{w}_x\varphi_t\varphi_{xt}dx - \chi\int_{\mathcal{I}} \bar{u}\varphi_t\psi_{xt}dx - \chi\int_{\mathcal{I}} \varphi_t\varphi_{xt}\psi_x dx - \chi\int_{\mathcal{I}} \varphi_t\varphi_x\psi_{xt}dx \\ &\quad - \gamma\int_{\mathcal{I}} (\varphi_x + \bar{u})^{\gamma-1}\psi_t\varphi_{xt}dx, \end{aligned} \tag{3.43}$$

where we have used (3.14) to get $\varphi_t(0,t) = \varphi_t(1,t) = 0$. Now, we estimate the right hand side of (3.43). Using (3.16), (3.17) and the Cauchy-Schwarz inequality, we can show that

$$-\chi\int_{\mathcal{I}} \bar{w}_x\varphi_t\varphi_{xt}dx - \chi\int_{\mathcal{I}} \bar{u}\varphi_t\psi_{xt}dx \le C_\varepsilon||\varphi_t||_{L^2}^2 + \varepsilon\left(||\varphi_{xt}||_{L^2}^2 + ||\psi_{xt}||_{L^2}^2\right) \tag{3.44}$$

for any $\varepsilon > 0$. Again, by using $\|\varphi_x\|_{L^\infty}, \|\psi_x\|_{L^\infty} \le N(t)$ and the Cauchy-Schwarz inequality, we have

$$\begin{aligned} -\chi\int_{\mathcal{I}} \varphi_t\varphi_{xt}\psi_x dx - \chi\int_{\mathcal{I}} \varphi_t\varphi_x\psi_{xt}dx &\le \chi||\psi_x||_{L^\infty}||\varphi_t||_{L^2}||\varphi_{xt}||_{L^2} + \chi||\varphi_x||_{L^\infty}||\varphi_t||_{L^2}||\psi_{xt}||_{L^2} \\ &\le CN(t)(||\varphi_t||_{L^2}^2 + ||\varphi_{xt}||_{L^2}^2 + ||\psi_{xt}||_{L^2}^2). \end{aligned} \tag{3.45}$$

For the last term on the right hand side of (3.43), using (3.9), Lemma 3.8 and the Cauchy-Schwarz inequality we can show that

$$\begin{aligned} -\gamma\int_{\mathcal{I}} (\varphi_x + \bar{u})^{\gamma-1}\psi_t\varphi_{xt}dx &= -\gamma\int_{\mathcal{I}} \bar{u}^{\gamma-1}(\frac{\varphi_x}{\bar{u}} + 1)^{\gamma-1}\psi_t\varphi_{xt}dx \\ &\le \gamma\int_{\mathcal{I}} \bar{u}^{\gamma-1}|(\frac{\varphi_x}{\bar{u}} + 1)^{\gamma-1} - 1||\psi_t||\varphi_{xt}|dx + \gamma\int_{\mathcal{I}} \bar{u}^{\gamma-1}|\psi_t||\varphi_{xt}|dx \\ &\le \gamma\int_{\mathcal{I}} \bar{u}^{\gamma-2}|\varphi_x||\psi_t||\varphi_{xt}|dx + \gamma\int_{\mathcal{I}} \bar{u}^{\gamma-1}|\psi_t||\varphi_{xt}|dx \\ &\le C||\varphi_x||_{L^\infty}||\psi_t||_{L^2}||\varphi_{xt}||_{L^2} + C_\varepsilon||\psi_t||_{L^2}^2 + \varepsilon||\varphi_{xt}||_{L^2}^2 \\ &\le CN(t)\left(||\psi_t||_{L^2}^2 + ||\varphi_{xt}||_{L^2}^2\right) + C_\varepsilon||\psi_t||_{L^2}^2 + \varepsilon||\varphi_{xt}||_{L^2}^2 \\ &\le (C_\varepsilon + CN(t))||\psi_t||_{L^2}^2 + (\varepsilon + CN(t))||\varphi_{xt}||_{L^2}^2. \end{aligned} \tag{3.46}$$

Substituting (3.44)-(3.46) into (3.43), we update (3.43) as

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_{\mathcal{I}} \varphi_t^2 + \psi_t^2 dx + \int_{\mathcal{I}} \varphi_{xt}^2 + D\psi_{xt}^2 dx \\ &\quad \le (C_\varepsilon + CN(t))\left(||\varphi_t||_{L^2}^2 + ||\psi_t||_{L^2}^2\right) + (\varepsilon + CN(t))\left(||\varphi_{xt}||_{L^2}^2 + ||\psi_{xt}||_{L^2}^2\right). \end{aligned}$$

Then we choose ε and δ small enough such that

$$\varepsilon + CN(t) \le \frac{1}{2}, \tag{3.47}$$

thus

$$\frac{d}{dt}\int_{\mathcal{I}}\varphi_t^2+\psi_t^2dx+\int_{\mathcal{I}}\varphi_{xt}^2+D\psi_{xt}^2dx\le(C_\varepsilon+C\delta)\left(||\varphi_t||_{L^2}^2+||\psi_t||_{L^2}^2\right). \tag{3.48}$$

Combine (3.39), (3.36) and (3.48), we can show that

$$\begin{aligned}&\int_{\mathcal{I}}\varphi_t^2+\psi_t^2dx+\int_0^t\int_{\mathcal{I}}\varphi_{x\tau}^2+\psi_{x\tau}^2dxd\tau\\&\le C\int_{\mathcal{I}}\left(\varphi_{0,xx}^2+\varphi_{0,x}^2+\psi_{0,xx}^2+\psi_{0,x}^2+\varphi_{0,x}^4+\psi_{0,x}^4\right)dx\\&\quad+(\varepsilon+CN(t))\int_0^t||\varphi_\tau||_{L^2}^2+||\psi_\tau||_{L^2}^2d\tau\\&\le C\left(||\varphi_{0,x}||_{H^1}^2+||\psi_{0,x}||_{H^1}^2+\left(||\varphi_{0,x}||_{L^2}^2+||\psi_{0,x}||_{L^2}^2\right)^2\right)\\&\quad+C\left(||\varphi_0||_{H^1}^2+||\psi_0||_{H^1}^2\right),\end{aligned} \tag{3.49}$$

where we have used the following inequality

$$\begin{aligned}\int_{\mathcal{I}}\varphi_t^2dx|_{t=0}&\le C\int_{\mathcal{I}}\left(\varphi_{0,xx}^2+\varphi_{0,x}^2+\psi_{0,x}^2+\varphi_{0,x}^2\psi_{0,x}^2\right)dx\\&\le C\int_{\mathcal{I}}\left(\varphi_{0,xx}^2+\varphi_{0,x}^2+\psi_{0,x}^2+\varphi_{0,x}^4+\psi_{0,x}^4\right)dx\end{aligned} \tag{3.50}$$

and

$$\begin{aligned}\int_{\mathcal{I}}\psi_t^2dx|_{t=0}&\le\int_{\mathcal{I}}(D\psi_{0,xx}^2+\bar u^\gamma-(\varphi_{0,x}+\bar u)^\gamma)^2dx\\&\le C\int_{\mathcal{I}}\psi_{0,xx}^2dx+C\int_{\mathcal{I}}\left((\varphi_{0,x}+\bar u)^\gamma-\bar u^\gamma\right)^2dx\\&\le C\int_{\mathcal{I}}\psi_{0,xx}^2dx+C\int_{\mathcal{I}}\left(\gamma\bar u^{\gamma-1}\varphi_{0,x}+\circ(\varphi_{0,x})\right)^2dx\\&\le C\left(||\psi_{0,xx}||_{L^2}^2+||\varphi_{0,x}||_{L^2}^2\right),\end{aligned} \tag{3.51}$$

in which we have also used $\varphi_t|_{t=0}=\varphi_{0,xx}-\chi(\varphi_{0,x}\psi_{0,x}+\bar u\psi_{0,x}+\bar w_x\varphi_{0,x})$ and $\psi_t|_{t=0}=D\psi_{0,xx}+\bar u^\gamma-(\varphi_{0,x}+\bar u)^\gamma$ from (3.12). For the first equation in (3.12) and using the Cauchy-Schwarz inequality, we get from (3.8), (3.16) and (3.17) that

$$\begin{aligned}\int_{\mathcal{I}}\varphi_{xx}^2dx&\le C\int_{\mathcal{I}}\left(\varphi_t^2+\varphi_x^2+\psi_x^2\right)dx+\chi||\varphi_x||_{L^\infty}^2\int_{\mathcal{I}}\psi_x^2dx\\&\le C\int_{\mathcal{I}}\left(\varphi_t^2+\varphi_x^2+\psi_x^2\right)dx+C\left(||\varphi_x||_{L^2}||\varphi_{xx}||_{L^2}+||\varphi_x||_{L^2}^2\right)\int_{\mathcal{I}}\psi_x^2dx\\&\le C\int_{\mathcal{I}}\left(\varphi_t^2+\varphi_x^2+\psi_x^2\right)dx+\frac{1}{2}||\varphi_{xx}||_{L^2}^2+C||\varphi_x||_{L^2}^2\left(||\psi_x||_{L^2}^2+||\psi_x||_{L^2}^4\right),\end{aligned}$$

thus

$$\int_{\mathcal{I}}\varphi_{xx}^2dx\le C\int_{\mathcal{I}}\left(\varphi_t^2+\varphi_x^2+\psi_x^2\right)dx. \tag{3.52}$$

This together with (3.19), (3.36), (3.39) and (3.49) can obtain that

$$\begin{aligned}
\int_{\mathcal{I}} \varphi_{xx}^2 dx \leq & C\left(||\varphi_{0,x}||_{H^1}^2 + ||\psi_{0,x}||_{H^1}^2 + \left(||\varphi_{0,x}||_{L^2}^2 + ||\psi_{0,x}||_{L^2}^2\right)^2\right) + C\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{H^1}^2\right) \\
& + C\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{L^2}^2\right)\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2 + \left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right)^2\right) \\
\leq & C\left(||\varphi_{0,x}||_{H^1}^2 + ||\psi_{0,x}||_{H^1}^2 + \left(||\varphi_{0,x}||_{L^2}^2 + ||\psi_{0,x}||_{L^2}^2\right)^2\right) \\
& + C\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{H^1}^2\right)\left(1 + ||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2 + \left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right)^2\right) \\
\leq & C\left(||\varphi_{0,x}||_{H^1}^2 + ||\psi_{0,x}||_{H^1}^2 + \left(||\varphi_{0,x}||_{L^2}^2 + ||\psi_{0,x}||_{L^2}^2\right)^2\right) \\
& + C\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{H^1}^2\right)\left(1 + ||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right)^2
\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}
\int_0^t \int_{\mathcal{I}} \varphi_{xx}^2 dx d\tau \leq & C\int_0^t \int_{\mathcal{I}} \left(\varphi_\tau^2 + \varphi_x^2 + \psi_x^2\right) dx d\tau + C\int_0^t ||\varphi_x||_{L^2}^2 \left(||\psi_x||_{L^2}^2 + ||\psi_x||_{L^2}^4\right) d\tau \\
\leq & C\int_0^t \int_{\mathcal{I}} \left(\varphi_\tau^2 + \varphi_x^2 + \psi_x^2\right) dx d\tau + C\sup_{\tau\in[0,t]} \left(||\psi_x||_{L^2}^2 + ||\psi_x||_{L^2}^4\right) \int_0^t ||\varphi_x||_{L^2}^2 d\tau \\
\leq & C\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{L^2}^2\right) \\
& + C\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2 + \left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right)^2\right)\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{L^2}^2\right) \\
\leq & C\left(1 + ||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2 + \left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right)^2\right)\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{L^2}^2\right) \\
\leq & C\left(1 + ||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right)^2\left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{L^2}^2\right).
\end{aligned} \tag{3.54}$$

By using the same method and utilizing (3.14), (3.19), (3.36), (3.39), (3.49) and the second equation in (3.12), we obtain

$$\begin{aligned}
\int_{\mathcal{I}} \psi_{xx}^2 dx = & \frac{1}{D^2}\int_{\mathcal{I}} \left(\psi_t - \bar{u}^\gamma + (\varphi_x + \bar{u})^\gamma\right)^2 dx \\
\leq & C\int_{\mathcal{I}} \psi_t^2 dx + C\int_{\mathcal{I}} \left((\varphi_x + \bar{u})^\gamma - \bar{u}^\gamma\right)^2 dx \\
\leq & C\int_{\mathcal{I}} \psi_t^2 dx + C\int_{\mathcal{I}} \left(\gamma\bar{u}^{\gamma-1}\varphi_x + \circ(\varphi_x)\right)^2 dx \\
\leq & C\int_{\mathcal{I}} \psi_t^2 dx + C\int_{\mathcal{I}} \varphi_x^2 dx
\end{aligned} \tag{3.55}$$

and

$$\begin{aligned}
\int_0^t \int_{\mathcal{I}} \psi_{xx}^2 dx d\tau \leq & C\int_0^t ||\psi_\tau||_{L^2}^2 d\tau + C\int_0^t ||\varphi_x||_{L^2}^2 d\tau \\
\leq & C\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right) + C\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{L^2}^2\right) \\
\leq & C\left(||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2\right).
\end{aligned} \tag{3.56}$$

Combine (3.49), (3.53) and (3.55), we get

$$\begin{aligned}
&||\varphi_{xx}||_{L^2}^2 + ||\psi_{xx}||_{L^2}^2 + \int_0^t \int_{\mathcal{I}} \varphi_{x\tau}^2 + \psi_{x\tau}^2 dx d\tau \\
&\le C \Big(||\varphi_{0,x}||_{H^1}^2 + ||\psi_{0,x}||_{H^1}^2 + \left(||\varphi_{0,x}||_{L^2}^2 + ||\psi_{0,x}||_{L^2}^2 \right)^2 \Big) \\
&\quad + C \left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{H^1}^2 \right) \left(1 + ||\varphi_0||_{L^2}^2 + ||\psi_0||_{H^1}^2 \right)^2 \\
&\quad + C \left(||\varphi_0||_{H^1}^2 + ||\psi_0||_{H^1}^2 \right) \\
&\le \|\varphi_0\|_{H^2}^2 + \|\psi_0\|_{H^2}^2. \qquad (3.57)
\end{aligned}$$

Next we differentiate the first equation in (3.12) with respect to x to obtain

$$\varphi_{xxx} = \varphi_{xt} + \chi(\bar{w}_{xx}\varphi_x + \bar{w}_x\varphi_{xx} + \bar{u}_x\psi_x + \bar{u}\psi_{xx} + \varphi_{xx}\psi_x + \varphi_x\psi_{xx}),$$

then we combine (3.16), (3.17), (3.19), (3.54) and (3.57) the Sobolev inequality (3.8) to show that

$$\int_0^t \int_{\mathcal{I}} \varphi_{xxx}^2 dx d\tau \le C \left(||\varphi_0||_{H^2}^2 + ||\psi_0||_{H^2}^2 \right). \qquad (3.58)$$

Similarly, using (3.9), (3.16), (3.17), (3.19), (3.54) and (3.57) we have

$$\int_0^t \int_{\mathcal{I}} \psi_{xxx}^2 dx d\tau \le C \left(||\varphi_0||_{H^2}^2 + ||\psi_0||_{H^2}^2 \right). \qquad (3.59)$$

Combining (3.54), (3.56), (3.57), (3.58) and (3.59), we then get (3.40) and complete the proof of Lemma 3.11. □

Utilizing the *a priori* estimates established in §3.2, the global well-posedness of the initial and boundary value problem (3.12)–(3.14) can be proved by combining Proposition 3.3 and standard continuation argument. To complete the proof of Theorem 2.2, it remains to derive the energy estimates leading to the long-time behavior of the solution.

3.3. **Long time behavior.** In this section, we are ready to prove (2.2).

Step 1. To prove the convergence of the zeroth order frequency of the perturbation, we note that under the conditions of Theorem 2.2, Lemma 3.9 implies

$$\left(\|\varphi_x(t)\|_{L^2}^2 + \|\psi\|_{L^2}^2 + \|\psi_x(t)\|_{L^2}^2 \right) \in L^1(0, \infty). \qquad (3.60)$$

Applying the arguments in the proof of Lemma 3.9 to (3.29), Lemma 3.10 to (3.35) and (3.38), we can show that

$$\left| \frac{\mathrm{d}}{\mathrm{d}t} \left(\|\varphi_x(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2 + \|\psi_x(t)\|_{L^2}^2 \right) \right| \le C \big(\|\varphi_x\|_{H^1}^2 + \|\psi_x\|_{H^1}^2 \big). \qquad (3.61)$$

Integrating (3.61) with respect to time and applying Lemmas 3.9–3.10 yield

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|\varphi_x(t)\|_{L^2}^2 + \|\psi_x(t)\|_{L^2}^2 \right) \in L^1(0, \infty). \qquad (3.62)$$

Since a positive function $f(t) \in W^{1,1}(0, \infty)$ converges to zero as $t \to \infty$, (3.60) and (3.62) imply

$$\left(\|\varphi_x(t)\|_{L^2}^2 + \|\psi(t)\|_{L^2}^2 + \|\psi_x(t)\|_{L^2}^2 \right) \to 0 \quad \text{as} \quad t \to \infty. \qquad (3.63)$$

Since $\psi = w - \bar{w}$ and $(\varphi_x, \psi_x) = (u - \bar{u}, w_x - \bar{w}_x)$, it follows from (3.63) that

$$\left(\|(u - \bar{u})(t)\|_{L^2}^2 + \|(w - \bar{w})(t)\|_{H^1}^2 \right) \to 0 \quad \text{as} \quad t \to \infty. \qquad (3.64)$$

Step 2. To show the convergence of the first order spatial derivative of the perturbation, we first note that Lemma3.10 imply that

$$\big(\|\varphi_t\|_{L^2}^2 + \|\psi_t\|_{L^2}^2\big) \in L^1(0,\infty). \tag{3.65}$$

Moreover, applying the arguments in the proof of Lemma 3.11 to (3.48), we can show that

$$\begin{aligned}&\left|\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi_t(t)\|_{L^2}^2 + \|\psi_t(t)\|_{L^2}^2\right)\right| \\ &\le C\big(\|\varphi_t\|_{H^1}^2 + \|\psi_t\|_{H^1}^2\big).\end{aligned} \tag{3.66}$$

Integrating (3.66) with respect to time and applying Lemmas 3.10–3.11, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi_t(t)\|_{L^2}^2 + \|\psi_t(t)\|_{L^2}^2\right) \in L^1(0,\infty). \tag{3.67}$$

It follows from (3.65) and (3.67) that

$$\big(\|\varphi_t(t)\|_{L^2}^2 + \|\psi_t(t)\|_{L^2}^2\big) \to 0 \quad \text{as} \quad t\to\infty. \tag{3.68}$$

As a direct consequence of (3.52), (3.63) and (3.68), we have

$$\|\varphi_{xx}(t)\|_{L^2}^2 \to 0 \quad \text{as} \quad t\to\infty, \tag{3.69}$$

which implies

$$\|(u_x - \bar{u}_x)(t)\|_{L^2}^2 \to 0 \quad \text{as} \quad t\to\infty. \tag{3.70}$$

which, together with (3.48), (3.63) and (3.68), implies

$$\|\psi_{xx}(t)\|_{L^2}^2 \to 0 \quad \text{as} \quad t\to\infty. \tag{3.71}$$

It then follows that

$$\|(w_{xx} - \bar{w}_{xx})(t)\|_{L^2}^2 \to 0 \quad \text{as} \quad t\to\infty. \tag{3.72}$$

This finishes the proof of Theorem 2.2.

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