Localization of the discrete one-dimensional quasi-periodic Schrödinger operators

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Abstract

In this paper we study the spectral properties of a family of discrete one-dimensional quasi-periodic Schrödinger operators (depending on a phase theta). In large disorder, under some suitable conditions on v and a diophantine rotation number, we prove using basically K.A.M theory that the spectrum of this operator is pure point for all $\theta[?][0, 2\pi)$ with exponential decaying eigenfunctions.

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Abstract

In this paper we study the spectral properties of a family of discrete one-dimensional quasi-periodic Schrödinger operators (depending on a phase theta). In large disorder, under some suitable conditions on v and a diophantine rotation number, we prove using basically K.A.M theory that the spectrum of this operator is pure point for all $\theta \in [0, 2\pi)$ with exponential decaying eigenfunctions.

1 Introduction and statements

The discrete one-dimensional Schrodinger operator with quasi-periodic potential is the selfadjoint bounded operator H_θ on $\ell^2(\mathbb{Z})$ defined by,

$$(H_\theta u)_n =: -\varepsilon(u_{n+1} + u_{n-1}) + v(\theta + n\omega)u_n, \quad n \in \mathbb{Z}, \qquad (1.1)$$

where ω is a real number and v is o smooth function on $[0, 2\pi)$.

We may assume the following on the data:

-Diophantine condition on the frequency ω: That is:

$$||n\omega|| := \inf_{m \in \mathbb{Z}} |n\omega - 2\pi m| \geqslant \frac{\kappa}{|n|^\tau} \quad \forall\, n \in \mathbb{Z} \setminus \{0\}, \qquad (1.2)$$

for some constants $\kappa > 0$ and $\tau > 1$.

- v is a function of class $\mathcal{C}^1$, satisfies:

$$0 < \alpha \leqslant |\partial_\theta v(\theta)| \leqslant c < \infty, \quad \forall \theta. \tag{1.3}$$

Under these two assumptions, we prove the following theorem:

Theorem 1. *Assume that ω and v are as above, then there exists a constant $\varepsilon_0 = \varepsilon_0(\alpha, \kappa, \tau)$ such that:*
If $|\varepsilon| < \varepsilon_0$ then H_θ is pure point with a set of exponential decaying eigenfunctions which form an orthonormal basis of $\ell^2(\mathbb{Z})$ for all θ.

Remark 1.
● *In 1997, L.H. Eliasson considered (see [1]) the operator H_θ given by (1.1) with frequency ω satisfying a Diophantine condition and the function v satisfying a Gevrey-class regularity and a transversality condition. Under these assumptions, he proved using KAM methods that for $|\varepsilon| < \varepsilon_0$ where ε_0 depends on the function v and on the Diophantine condition on ω the operator H_θ has pure point spectrum for a.e. $\theta \in \mathbb{T}$. Moreover, this implies, using Kotani's theory (see [11]) that the Lyapunov exponent is nonzero for a.e. energy E. The author has also suggested that the argument could be modified to obtain exponential decay of the eigenfunctions, but he has not provided a proof of it.*
● *In 2000, J. Bourgain and M. Goldstein considered (see [3]) the operator H_θ given by (1.1) where ω satisfies a Diophantine condition and v is a nonconstant analytic function. Assuming also that the Lyapunov exponent is positive for a.e. ω and for all E .The authors prove that the operator H_θ satisfies Anderson localization -with exponential decay of the eigenfunctions at almost Lyapunov rate for every θ and for a.e. ω. Their result is nonperturbative -the constant ε_0 depends only on the potential v. In this paper we use the K.A.M approach which is a perturbative method -the constant ε_0 depends on v and ω- with different conditions on v, also we prove the Dynamical localization which is stronger than Anderson localization.*
● *For the quasi-periodic model, and unlike Anderson's case, there were less results that were found for this kind of localization. However, several results on the (D.L.) were published for the random model, for more references see [5, 7].*
In the case of quasi-periodic models, this localization phenomenon (D.L) implies Anderson localization, and which also implies by the RAGE theorem that the spectrum is purely punctual (see [4]). In view of this, these models are natural candidates for (D.L). In this context, F. Germinet and S. Jitomirskaya (see [9]), have improved the results of [6] and [8], by proving the

strong (D.L) of the operator $-\Delta + \lambda\cos(2\pi(\theta + n\omega))$, for all $\lambda > 2$ and diophantine ω. Later, in 2004, J. Bourgain and S. Jitomirskaya have announced (without demonstration) this result for the quasi-periodic Schrödinger operators, see [10] for more details.

Idea of proof: The method of proof is a refinement of an already refined K.A.M method developed by Eliasson in a series of fundamental papers in the theory of quasi-periodic Schrodinger operators (especially[1]). The method consists of an infinite sequence of transformations aiming at conjugating the infinite dimensional matrix defined by the operator on $\ell^2(\mathbb{Z})$:

$$D(\theta) + \varepsilon F(\theta) = \begin{pmatrix} \ddots & & & & 0 \\ & v(\theta - \omega) & -\varepsilon & & \\ & -\varepsilon & v(\theta) & -\varepsilon & \\ & & -\varepsilon & v(\theta + \omega) & \\ 0 & & & & \ddots \end{pmatrix}$$

to a diagonal matrix $D_\infty(\theta, \varepsilon)$, by an orthogonal matrix made up of a complete set of eigenvectors. An iterative procedure that permits us to construct a such matrix,

$$U_j^* \cdots U_1^* (D + \varepsilon F) U_1 \cdots U_j = D_{j+1} + F_{j+1}$$

that conjugate $D + \varepsilon F$ closer and closer to a diagonal matrix $D_j = \mathrm{diag}(v_j(\theta + k\omega))$.

2   Iterative study

This section is organized in the following way:
- A first part devoted to the study of the first step of the iteration described in the previous paragraph. Under some conditions on v and ω we construct the matrices U_1, F_2 and D_1 which satisfy the estimates of the lemma 1.
- In the second part and after a suitable choice of parameters, an inductive proposition 1 is introduced in order to prove the theorem 1, which is a simple consequence of the lemma 2.
Consider now the symmetric infinite-dimensional matrix that depends on the parameter θ, $D(\theta) + F(\theta)$ with,

$$D(\theta) = \begin{pmatrix} \ddots & & & & 0 \\ & v(\theta - \omega) & & & \\ & & v(\theta) & & \\ & & & v(\theta + \omega) & \\ 0 & & & & \ddots \end{pmatrix}$$

For the formulation of the first step of iteration we shall assume the following:

- The rotation number ω and the potential v satisfy (1.2) and (1.3).

$$
- \quad \begin{cases} |F_i^j| \leqslant \varepsilon e^{-|i-j|\varrho} \text{ with } \varrho > 0 \\ |\partial_\theta F_i^j| \leqslant \sqrt{\varepsilon}. \end{cases}
$$

-Consider the equation

$$e^{-X}(D + F)e^X = D' + F' \tag{2.1}$$

where the matrices X, D' and F' are defined in the following way:
Let $N := \frac{1}{\varepsilon^a \varrho}$ for $0 < a < \frac{1}{4\tau}$.

1. The matrix X is defined by $\begin{cases} X_i^j = 0 \text{ if } i = j \text{ or } |i-j| > N \\ X_i^j = -\frac{F_i^j}{v_i - v_j} \text{ otherwise} \end{cases}$ and satisfies the equation

$$[D, X] = F^N - D' + D \tag{2.2}$$

$$\text{where } (F^N)_i^j = \begin{cases} F_i^j \text{ if } |i-j| \leqslant N \\ 0 \text{ otherwise} \end{cases}$$

2. $(D' - D)_i^i = F_i^i$

3. $F'(\theta) = e^{-X(\theta)}(D(\theta) + F(\theta))e^{X(\theta)} - D'(\theta)$.

Lemma 1. *Let* $0 < a\tau + b < \dfrac{1}{4}$ *and* $\sigma = \varepsilon^b$.

$$\text{If } \varepsilon \leqslant (\frac{\kappa}{2^6}\alpha\varrho^{\tau+1})^{\frac{1}{1-(a\tau+b)}} \text{ then}$$

1. *(a)* $|X_i^j| \leqslant \varepsilon\dfrac{N^\tau}{\alpha\kappa}e^{-|i-j|\varrho} := Ae^{-|i-j|\varrho}$

   (b) $\|X\| \leqslant \varepsilon\dfrac{N^\tau}{\alpha\kappa}\dfrac{2}{1 - e^{-\varrho}}$

   (c) $|(e^{\pm X} - I)_i^j| \leqslant \dfrac{2^7 A}{\sigma\varrho}e^{-|i-j|\varrho'}$ *where* $\varrho' = \varrho - \dfrac{\sigma}{2}\varrho$

2. *(a)* $|F'^j_i| \leqslant 16(\dfrac{2^5 A}{\sigma\varrho})^2 e^{-|i-j|\varrho'}$

   (b) $\|F'\| \leqslant 16(\dfrac{2^5 A}{\sigma\varrho})^2\dfrac{2}{1 - e^{-\varrho'}} := \varepsilon'\dfrac{2}{1 - e^{-\varrho'}}$

3.   (a) $|\partial_\theta v'(\theta)| \geqslant \alpha - \sqrt{\varepsilon} := \alpha'$ where $D'(\theta) = diag(v'(\theta + n\omega))$

    (b) $|\partial_\theta F'^j_i(\theta)| \leqslant \sqrt{\varepsilon'}$

    (c) $|\partial_\theta v'(\theta)| < c + \sqrt{\varepsilon} := c'$

4. $|(D' - D)^i_i| \leqslant \varepsilon.$

Proof.

1.(a): Let $i \neq j$ and $|i - j| \leqslant N$, we have:

$$X^j_i = -\frac{F^j_i}{v_i - v_j} \text{ therefore } |X^j_i| \leqslant \frac{\varepsilon}{|v_i - v_j|} e^{-|i-j|\varrho}$$

since $|v_i - v_j| = |v(\theta + i\omega) - v(\theta + j\omega)| \geqslant \inf |\partial_\theta v||(i - j)\omega| \geqslant \frac{\alpha\kappa}{|i-j|^\tau}$, then it follows that

$$|X^j_i| \leqslant \varepsilon \frac{N^\tau}{\alpha\kappa} e^{-|i-j|\varrho}.$$

1.(b): Using 1.(a) we obtain

$$\sum_{i\in\mathbb{Z}} |X^j_i| \leqslant A \sum_{i\in\mathbb{Z}} e^{-|i-j|\varrho} \leqslant A \sum_{i\in\mathbb{Z}} e^{-|i|\varrho} \leqslant \frac{2A}{1 - e^{-\varrho}}$$

thus applying Young theorem the result follows.

1.(c): By lemma A_8(Eliasson [2]) and for all $n \in \mathbb{N}$ we deduce that:

$$|(\underbrace{X \cdots X}_{n \text{ times}})^j_i| \leqslant (\frac{2^5 A}{\sigma\varrho})^n e^{-|i-j|\varrho'},$$

hence $|(e^{\pm X} - I)^j_i| \leqslant \dfrac{2^7 A}{\sigma\varrho} e^{-|i-j|\varrho'}.$

2.(a): Let $\widetilde{F}_N := F - F^N$ then we have

$$\begin{aligned}
F' &= e^{-X}(D + F)e^X - D' \\
&= e^{-X}(D + F^N)e^X + e^{-X}\widetilde{F}_N e^X - D' \\
&= [X, F^N] - XDX - XF^N X + \sum_{\substack{m + n \geqslant 2 \\ (m,n) \neq (1,1)}} \frac{(-X)^n}{n!}(D + F^N)\frac{X^m}{m!}
\end{aligned}$$

$$+ \sum_{m+n\geqslant 0} \frac{(-X)^n}{n!}\widetilde{F}_N \frac{X^m}{m!}.$$

Now we have to estimate the elements of all matrices which constitute the matrix F'.

$(*)_1$

$$|([X, F^N])_i^j| \leqslant 2 \sum_{k \in I} |X_i^k||(F^N)_k^j|$$

$$\leqslant 2\varepsilon A \sum_{k \in I} e^{-(|k-i|+|k-j|)\varrho}$$

$$\leqslant 2\varepsilon A \sum_{k \in I} e^{-(|k-i|+|k-j|)\varrho'} e^{-(|k-i|+|k-j|)\frac{\sigma\varrho}{2}}$$

$$\leqslant \varepsilon \frac{2^5 A}{\sigma\varrho} e^{-|i-j|\varrho'}$$

where $I = \{k \in \mathbb{Z};\ |k-i| \leqslant N$ and $|k-j| \leqslant N\}$.

In the same way we get:

$(*)_2$

$$|(XDX)_i^j| \leqslant \sum_{k \in I} |X_i^k|(DX)_k^j| \leqslant \left(\frac{2^5 A}{\sigma\varrho}\right)^2 e^{-|i-j|\varrho'}$$

$(*)_3$

$$|(XFX)_i^j| \leqslant \left(\frac{2^5 A}{\sigma\varrho}\right)^2 e^{-|i-j|\varrho'}$$

$(*)_4$

$$|(\sum_{\substack{m+n \geqslant 2 \\ (m,n) \neq (1,1)}} \frac{(-X)^n}{n!}(D + F^N)\frac{X^m}{m!})_i^j| \leqslant 3\left(\frac{2^5 A}{\sigma\varrho}\right)^2 (1+\varepsilon) e^{-|i-j|\varrho'}$$

$(*)_5$

$$|(\sum_{m+n \geqslant 0} \frac{(-X)^n}{n!}\widetilde{F}_N \frac{X^m}{m!})_i^j| \leqslant \sum_{m+n \geqslant 0} \frac{1}{n!m!} \sum_{k \in \mathbb{Z}} |(X^n)_i^k||(\widetilde{F}_N X^m)_k^j|$$

$$\leqslant \sum_{m+n \geqslant 0} \frac{1}{n!m!} \sum_{k \in \mathbb{Z}} |(X^n)_i^k| \sum_{|k-\ell| > N} |(\widetilde{F}_N)_k^\ell||(X^m)_\ell^j|$$

$$\leqslant \varepsilon e^{-\frac{1}{\varepsilon^a}} \sum_{m+n \geqslant 0} \frac{1}{n!m!} \sum_{k \in \mathbb{Z}} |(X^n)_i^k| \sum_{|k-\ell| > N} |(X^m)_\ell^j|$$

$$\leqslant \frac{3\varepsilon e^{-\frac{1}{\varepsilon^a}}}{(\sigma\varrho)^2} e^{-|i-j|\varrho'}.$$

This gives

$$|F'^j_i| \leqslant 16(\frac{2^5 A}{\sigma \varrho})^2 e^{-|i-j|\varrho'}.$$

2.(b):

$$\sum_{i \in \mathbb{Z}} |F'^j_i| \leqslant 16(\frac{2^5 A}{\sigma \varrho})^2 \sum_{i \in \mathbb{Z}} e^{-|i-j|\varrho'}$$

$$\leqslant 16(\frac{2^5 A}{\sigma \varrho})^2 \sum_{i \in \mathbb{Z}} e^{-|i|\varrho'}$$

$$\leqslant 16(\frac{2^5 A}{\sigma \varrho})^2 \frac{2}{1 - e^{-\varrho'}}$$

and we obtain the estimate from Young theorem.

3.(a): Since $(D' - D)^i_i = F^i_i$ then $v'_i = v_i + F^i_i$ therefore $\partial_\theta v'_i = \partial_\theta v_i + \partial_\theta F^i_i$ thus

$$|\partial_\theta v'_i| \geqslant |\partial_\theta v_i| - |\partial_\theta F^i_i|$$

$$\geqslant \alpha - \sqrt{\varepsilon} := \alpha'.$$

3.(b): In order to estimate $|\partial_\theta F'^j_i|$, we have to find an upper bound of $|\partial_\theta X^j_i|$.

We have $\partial_\theta X^j_i = -\dfrac{\partial_\theta F^j_i}{v_i - v_j} + F^j_i \dfrac{\partial_\theta v_i - \partial_\theta v_j}{(v_i - v_j)^2}$ then

$$|\partial_\theta X^j_i| \leqslant \sqrt{\varepsilon} \frac{N^\tau}{\alpha \kappa} + \varepsilon (\frac{N^\tau}{\alpha \kappa})^2 |\partial_\theta v_i - \partial_\theta v_j|$$

$$\leqslant \sqrt{\varepsilon} C_1.$$

which implies
$(*)_1$

$$|\partial_\theta([X, F^N])^j_i| = \left| \partial_\theta(\sum_{k \in \mathbb{Z}} X^k_i (F^N)^j_k - (F^N)^k_i X^j_k) \right|$$

$$\leqslant \sum_{k \in \mathbb{Z}} |\partial_\theta X^k_i||(F^N)^j_k| + |X^k_i||\partial_\theta (F^N)^j_k| + |\partial_\theta (F^N)^k_i||X^j_k| + |(F^N)^k_i||\partial_\theta X^j_k|$$

$$\leqslant \sum_{k \in \mathbb{Z}} \sqrt{\varepsilon} C_1 |(F^N)^j_k| + \sqrt{\varepsilon}|X^k_i| + \sqrt{\varepsilon}|X^j_k| + \sqrt{\varepsilon} C_1 |(F^N)^k_i|$$

$$\leqslant \varepsilon^{3/2} M_1.$$

In the same way we get
$(*)_2$

$$|\partial_\theta(XDX)^j_i| \leqslant \varepsilon^{3/2} M_2.$$

$(*)_3$
$$|\partial_\theta(XFX)_i^j| \leqslant \varepsilon^{3/2} M_3.$$

$(*)_4$
$$(X^n)_i^j = \sum_{\ell_{n-1}\in\mathbb{Z}} \sum_{\ell_{n-2}\in\mathbb{Z}} \cdots \sum_{\ell_1\in\mathbb{Z}} X_i^{\ell_1} X_{\ell_1}^{\ell_2} \cdots X_{\ell_{n-1}}^j$$

Since $|(X^k)_i^j|$ is bounded for all $k < n$ then $|\partial_\theta(X^n)_i^j| \leqslant \varepsilon^{3/2} C_2$ therefore

$$\left|\partial_\theta\Big(\sum_{\substack{m+n\geqslant 2 \\ (m,n)\neq(1,1)}} \frac{(-X)^n}{n!}(D+F^N)\frac{X^m}{m!}\Big)_i^j\right| \leqslant \varepsilon^{3/2} M_4$$

and

$$\left|\partial_\theta\Big(\sum_{m+n\geqslant 0} \frac{(-X)^n}{n!}\widetilde{F}_N \frac{X^m}{m!}\Big)_i^j\right| \leqslant \varepsilon^{3/2} M_5$$

where all constants M_i and C_i depend on N, α, κ and τ. It follows that

$$|\partial_\theta(F'{}_i^j)| \leqslant 4\varepsilon^{3/2} \max(M_1, \cdots, M_5) \leqslant \sqrt{\varepsilon'}.$$

3.(c):
$$|\partial_\theta v'| \leqslant |\partial_\theta v| + \sqrt{\varepsilon} \leqslant c + \sqrt{\varepsilon}.$$

4. By construction of D' the result follows immediately. $\qquad\square$

3   Induction

Let a, b such that $0 < a\tau + b < \dfrac{1}{4}$ and consider $\varepsilon_1 = \varepsilon$, $\varrho_1 = \varrho$, $\alpha_1 = \alpha$, $A_1 = A$, $D^1 = D$, $F^1 = F$ and for all $n \geqslant 1$ we define the sequences

$$
\begin{aligned}
A_{n+1} &= \frac{(N_{n+1})^\tau}{\kappa\alpha_{n+1}}\varepsilon_{n+1} & N_n &= \frac{1}{(\varepsilon_n)^a \varrho_n} \\
\varepsilon_{n+1} &= 16\big(\frac{2^5 A_n}{\sigma_n \varrho_n}\big)^2 & \sigma_n &= (\varepsilon_n)^b \\
\varrho_{n+1} &= \varrho_n - \frac{\sigma_n \varrho_n}{2} & \alpha_{n+1} &= \alpha_n - \sqrt{\varepsilon_n}
\end{aligned}
$$

These parameters are defined in an iterative way and it is with which we will be able to define the matrices X_n, F^{n+1} and D^{n+1} satisfying

$$e^{-X_n}(D^n + F^n)e^{X_n} = D^{n+1} + F^{n+1} \tag{3.1}$$

where the matrices X_n, D^{n+1} and F^{n+1} are defined in the following way:

1. The matrix X_n is defined by $\begin{cases} (X_n)_i^j = 0 \text{ if } i = j \text{ or } |i-j| > N_n \\ (X_n)_i^j = -\frac{(F_n)_i^j}{v_i^n - v_j^n} \text{ otherwise} \end{cases}$ and

    satisfies the equation

$$[D^n, X_n] = (F^n)^{N_n} - D^{n+1} + D^n \tag{3.2}$$

$$\text{where } ((F^n)^{N_n})_i^j = \begin{cases} (F^n)_i^j \text{ if } |i-j| \leqslant N_n \\ 0 \text{ otherwise} \end{cases}$$

2. $(D^{n+1} - D^n)_i^i = (F^n)_i^i$

3. $F^{n+1}(\theta) = e^{-X_n(\theta)}(D^n(\theta) + F^n(\theta))e^{X_n(\theta)} - D^{n+1}(\theta)$.

and satisfy the property $\mathcal{P}_n$ described in the following proposition

Proposition 1. *Let $n \in \mathbb{N}$. If $\forall m \leqslant n$,*

$$\varepsilon_m \leqslant \left(\frac{\kappa \alpha_m \varrho_m^{1+\tau}}{2^6} \right)^{\frac{1}{1-(a\tau+b)}}$$

then the following property $\mathcal{P}_n$ is holds.

$$\mathcal{P}_n \begin{cases} 1. \, |(X_n)_i^j| \leqslant A_n e^{-|i-j|\varrho_n} \\ 2. \, \|X_n\| \leqslant \frac{2A_n}{1-e^{-\varrho_n}} \\ 3. \, |(e^{\pm X_n} - I)_i^j| \leqslant \frac{2^7 A_n}{\sigma_n \varrho_n} e^{-|i-j|\varrho_{n+1}} \\ 4. \, |(F^{n+1})_i^j| \leqslant \varepsilon_{n+1} e^{-|i-j|\varrho_{n+1}} \\ 5. \, \|F^{n+1}\| \leqslant \frac{2\varepsilon_{n+1}}{1-e^{-\varrho_{n+1}}} \\ 6. \, |\partial_\theta v^{n+1}| \geqslant \alpha_{n+1} \\ 7. \, |\partial_\theta (F^{n+1})_i^j| \leqslant \sqrt{\varepsilon_{n+1}} \\ 8. \, |\partial_\theta v^{n+1}| < c_{n+1} = c_n + \sqrt{\varepsilon_n} \\ 9. \, |(D^{n+1} - D^n)_i^i| \leqslant \varepsilon_n. \end{cases}$$

Proof. A direct application of the lemma1 allows us to obtain the desired result for each n. $\qquad \square$

4   Study of convergence

Now we will deal with the study of the convergence of our iteration. We will therefore look the conditions and the size of ε with which we will have the convergence, this will be the goal of the next lemma. Finally, we conclude with the proof of the theorem1 which is a simple deduction of the proposition1 and the lemma2.

Lemma 2. *Suppose that*

$$(\frac{1}{2^9\alpha^3})^{1-(a\tau+b)} < \alpha\kappa\varrho^{\tau+1} < 2^{\tau+7}3$$

$$(2^6\alpha^2)^b \geqslant 2; \qquad 0 < a\tau + b < \frac{1}{4}.$$

Then for $\varepsilon < \dfrac{1}{2^{37}}(\dfrac{\kappa\varrho^{\tau+1}}{2^\tau 3})^4\alpha$ we have for all n

1. $\varepsilon_n \leqslant (\dfrac{1}{2^6\alpha^2})^{(3/2)^n} \forall \alpha \geqslant \dfrac{11}{2^4}$
   In particular $\lim_{n\to\infty}\varepsilon_n = 0$

2. $\varepsilon_n \leqslant (\dfrac{\kappa\alpha_n\varrho_n^{\tau+1}}{2^6})^{\frac{1}{1-(a\tau+b)}}.$

Proof: 1.) The result is holds for $n = 1$. Suppose that the result remain holds for $1, 2, \cdots, n$ thus $\alpha_1 > \alpha_2 > \cdots > \dfrac{2}{3}\alpha$. Now we shall prove that the result is also true for $n + 1$.
Let

$$M = \sum_{j=1}^{+\infty}(\frac{1}{2(1 - (a\tau + b))})^j$$

we have

$$\varepsilon_{n+1} = 16(\frac{2^5 A_n}{\sigma_n\varrho_n})^2 = 2^{14}(\frac{1}{\kappa\alpha_n\varrho_n^{\tau+1}})^2\varepsilon_n^{2(1-(a\tau+b))} = \cdots =$$

$$= \left[2^{14\sum_{k=1}^n\left(2(1-(a\tau+b))\right)^{-k}}\prod_{k=1}^n(\frac{1}{\kappa\alpha_k\varrho_k^{\tau+1}})^{2\left(2(1-(a\tau+b))\right)^{-k}}\varepsilon\right]^{\left(2(1-(a\tau+b))\right)^n}.$$

Since

$$\prod_{k=1}^n(\frac{1}{\kappa\alpha_k\varrho_k^{\tau+1}})^{2\left(2(1-(a\tau+b))\right)^{-k}} \leqslant \left(\frac{1}{\kappa\alpha_n\varrho_n^{\tau+1}}\right)^{2\sum_{k=1}^n\left(2(1-(a\tau+b))\right)^{-k}}$$

hence

$$\varepsilon_{n+1} \leqslant \left[2^{14M}\left(\frac{1}{\kappa\alpha_n\varrho_n^{\tau+1}}\right)^{2M}\varepsilon\right]^{\left(2(1-(a\tau+b))\right)^n}$$

$$\leqslant \left[2^{28}(\frac{2^\tau 3}{\kappa\alpha\varrho^{\tau+1}})^4\varepsilon\right]^{(3/2)^n}$$

which proves that $\lim\limits_{n\to+\infty}\varepsilon_n = 0$.

2.) By 1.) we have $\forall n$, $\varepsilon_n \leqslant 1$ then $\varepsilon_{n+1} = 2^2\Big(\dfrac{2^6\varepsilon_n^{1-(a\tau+b)}}{\kappa\alpha_n\varrho_n^{\tau+1}}\Big)^2 \leqslant 1$, thus

$\dfrac{2^6\varepsilon_n^{1-(a\tau+b)}}{\kappa\alpha_n\varrho_n^{\tau+1}} \leqslant 1$ hence $\varepsilon_n \leqslant \Big(\dfrac{\kappa\alpha_n\varrho_n^{\tau+1}}{2^6}\Big)^{\frac{1}{1-(a\tau+b)}}$. $\hfill\square$

Remark 2. *1.One can assume without loss of generality that $\alpha = 1$ and we have the same result, in fact the operators H_θ and αH_θ have the same spectral properties.*
2.The real b exists and satisfying all conditions.

Proof of theorem 1. the operator H_θ is identified to matrix $D + F$ with $|F_i^j| \leqslant (e\varepsilon)e^{-|i-j|\varrho}$. Then for $\varrho = 1$ and $\varepsilon < \dfrac{\kappa\alpha\varrho^{\tau+1}}{3e2^{\tau+37}}$ we have the existence of matrices X_n and D^{n+1} for all $n \in \mathbb{N}$ such that for all θ

$$\Big(e^{X_1(\theta)}\cdots e^{X_n(\theta)}\Big)^*\Big(D(\theta) + F(\theta)\Big)e^{X_1(\theta)}\cdots e^{X_n(\theta)} = D^{n+1}(\theta) + F^{n+1}(\theta)$$

where $D^{n+1}(\theta)$ is a diagonal matrix, $||F^{n+1}|| \leqslant \varepsilon_{n+1}\dfrac{1}{1-e^{-\varrho_{n+1}}}$,

$|(e^{\pm X_n} - I)_i^j| \leqslant \dfrac{2^7 A_n}{\sigma_n\varrho_n}e^{-|i-j|\varrho_{n+1}}$ and $|(D^{n+1} - D^n)_i^i| \leqslant \varepsilon_n$.

Therefore $F^n(\theta) \to 0$ and $D^n(\theta) \to D^\infty(\theta)$ with $D^\infty(\theta)$ is a diagonal matrix. All convergence are fulfilled for all θ.
On the other hand $e^{X_1(\theta)}\cdots e^{X_n(\theta)} \to U(\theta)$ in norm and for all θ with $U(\theta)$ is an orthogonal matrix. In fact: Let $U_j(\theta) = e^{X_j(\theta)}$ we have $\prod\limits_{j\geqslant 1} U_j(\theta)$ converges

iff $\sum\limits_{j\geqslant 1}||U_j(\theta)-I||$ converges, now since $||U_j(\theta)-I|| \leqslant \dfrac{2\sqrt{\varepsilon_j}}{1-e^{-\varrho_{j+1}}}$ then we have

the existence of U for all θ. Moreover from lemma 2 and for $\varepsilon_0 = \dfrac{1}{e2^{37}}(\dfrac{\kappa}{2^\tau}3)^4\alpha$ the matrix $D^\infty(\theta)$ is is pure point with finite-dimensional eigenvectors for all θ and the measure of $\sigma(D)\backslash\sigma(D + F)$ goes to 0 as $\varepsilon \to 0$. The eigenvectors of $D + F$ are formed by the columns of U. $\hfill\square$

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