On the distributional fractional derivative: from unidimensional to multidimensional

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In this paper, we use the generalized notions of Riemann-Liouville (fractional calculus with respect to a regular function σ) to extend the definitions of fractional integration and derivative from the functional sense to the distributional sense. First, we give some properties of fractional integral and derivative for the functions infinitely differentiable with compact support. Then, we define the weak derivative, as well as the integral and derivative of a distribution with compact support, the integral and derivative of a distribution using the convolution product. Then, we generalize those concepts from the unidimensional to the multidimensional case. Finally, we propose the definitions of some usual differential operators.

On the distributional fractional derivative: from unidimensional to multidimensional

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In this paper, we use the generalized notions of Riemann-Liouville (fractional calculus with respect to a regular function σ) to extend the definitions of fractional integration and derivative from the functional sense to the distributional sense. First, we give some properties of fractional integral and derivative for the functions infinitely differentiable with compact support. Then, we define the weak derivative, as well as the integral and derivative of a distribution with compact support, the integral and derivative of a distribution using the convolution product. Then, we generalize those concepts from the unidimensional to the multidimensional case. Finally, we propose the definitions of some usual differential operators.

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Introduction

In 1823, Abel used fractional calculation rules to solve the following integral equation:

$$\int_a^x \frac{y(t)}{(x-t)^\alpha} dt = f(x),$$

presenting the current definition of fractional integration, known as Riemann - Liouville:

$$I^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)}{(x-t)^{1-\alpha}} dt, \quad x > 0.$$

This was an introduction to the concept of a Riemann-Liouville fractional-order derivative, which was followed by Hadamard in 1892 [5], who presented another definition using the logarithmic function and the related Stieltjes integral:

$$I^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x f(t).\left(\ln\frac{x}{t}\right)^{\alpha-1}d\ln t=\frac{1}{\Gamma(\alpha)}\int_a^x f(t).\left(\ln\frac{x}{t}\right)^{\alpha-1}\frac{dt}{t},\quad x>0.$$

The Riemann-Liouville type fractional calculus theory has recently been developed. Many authors introduced various properties and established new definitions that generalize those approaches (see for instance [1, 4, 6, 9, 12, 13, 15, 16, 17, 18]).
The concept of distribution, on the other hand, is considered relatively new in the history of derivative. First, Heaviside (1893, 1894) and Dirac (1926) introduced generalized functions, with derivative and integration operations isolates of the mathematical rules that must be available (this problem solved by Schwarts in the early 1950s [19]). The concept of "quasi-derivative" is then introduced in Lery's work on the motion of a liquid in $\mathbb{R}^3$ (1934) [11]. Sobolev introduced the concept of a generalized solution of the wave equation using compactly supported auxiliary functions in 1934 and 1935, paving the way for a new concept, the weak derivative, and laying the groundwork for what were later known as Sobolev spaces [2]. Schwartz developed the theory of distribution. He begins with articles published between 1945 and 1950 and concludes with his famous book, in which he lays the theoretical foundations of the concept of distributions, specifically derivatives and various operations [19].
It is natural to seek a fractional notion of derivative in the sense of distribution. To this, Samko et al. introduced preliminary ideas to present the notion of fractional derivative of generalized functions, as well as fractional derivatives of multivariable functions defined on rectangular sets $[a_1,b_1]\times\ldots[a_n,b_n]$, where $(a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n)\in\mathbb{R}^n$ (see [9, 18]). Next, Mincheva-Kaminska [14] established the notions of fractional integrals and derivatives of functions and distributions in the cone $\overline{\mathbb{R}_+}^n$. According to these concepts, we generalized the notion of fractional derivative from classical to general form in our work, and we also defined fractional derivatives for functions with multiple variables. We performed demonstrations based on distribution theory.
Our work is divided into five sections: In the first section, we introduced the fundamental concepts of fractional calculus that will be used throughout the article. We presented the weak fractional derivative in the second section, which generalizes what are presented in [7]. We tried to present the integral and fractional derivative of distributions with compact support in the third section. We used the convolution product in the fourth section to established the derivatives and fractional integrates of the distributions defined on the entire $\mathbb{R}$. We extended everything presented above to the dimension superior in the fifth section. Finally, we provided a conclusion and some definitions concerning usual differential operators (gradient, divergence, Laplacian).

1 Preliminaries

In this section, we will discuss some derivative and fractional integration concepts and properties with respect to a sufficiently regular function.

Let $(a,b)\,(-\infty \le a < b \le +\infty)$ be an interval of the real axis $\mathbb{R}$ and p such that $1 \le p < +\infty$. All functions in this paper considered real- valued functions. From now on, let α be positive real number and $\eta = [\alpha]+1$. We denote by σ a function in $C^1(a,b)$ at least such that $\sigma' > 0$. Therefore, σ is one by one function from (a,b) to (σ_a, σ_b) where, $\sigma_a = \lim\limits_{x \overset{>}{\longrightarrow} a} \sigma(x), \sigma_b = \lim\limits_{x \overset{<}{\longrightarrow} b} \sigma(x)$.

Definition 1.1. [10] *We denote by $X^p_\sigma(a,b)$ the space of measurable functions f on (a,b) such that $\|f\|_{X^p_\sigma} < \infty$, where*

$$\|f\|_{X^p_\sigma(a,b)} = \left(\int_a^b |f(t)|^p \sigma'(t) dt\right)^{\frac{1}{p}}. \tag{1.1}$$

Note that

$$\|f\|_{X^p_\sigma(a,b)} = \left(\int_{\sigma_a}^{\sigma_b} |f \circ \sigma^{-1}(y)|^p dy\right)^{\frac{1}{p}} = \|f \circ \sigma^{-1}\|_{L^p(\sigma_a,\sigma_b)}, \tag{1.2}$$

so, $f \in X^p_\sigma(a,b)$ if and only if $f \circ \sigma^{-1} \in L^p(\sigma_a, \sigma_b)$.
If $0 < \inf\limits_{x\in(a,b)} \sigma'(x) < \sup\limits_{x\in(a,b)} \sigma'(x) < +\infty$, then, $X^p_\sigma(a,b)$ is the same as $L^p(a,b)$, the usual Lebesgue space on (a,b).

Definition 1.2. [9] *Let $f \in L^p_\sigma(a,b)$.*

i) *The generalized (left and right) fractional integral operators of the function f with respect to the function σ are given by*

$$\mathcal{I}^{\alpha,\sigma}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\sigma(x) - \sigma(t))^{\alpha-1} f(t) \sigma'(t) dt, \tag{1.3}$$

$$\mathcal{I}^{\alpha,\sigma}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (\sigma(t) - \sigma(x))^{\alpha-1} f(t) \sigma'(t) dt, \tag{1.4}$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-t} t^{\alpha-1} dt$, is the Euler gamma function.

ii) *The generalized (left and right) fractional derivative of the function f with respect to the function $\sigma \in C^\eta(a,b)$ are introduced by*

$$\begin{aligned} \mathcal{D}^{\alpha,\sigma}_{a^+} f(x) &= (\gamma_\sigma)^\eta (\mathcal{I}^{\eta-\alpha,\sigma}_{a^+} f)(x) \\ &= \frac{1}{\Gamma(\eta-\alpha)} (\gamma_\sigma)^\eta \int_a^x (\sigma(x) - \sigma(t))^{\eta-\alpha-1} f(t) \sigma'(t) dt, \end{aligned} \tag{1.5}$$

$$\begin{aligned} \mathcal{D}^{\alpha,\sigma}_{b^-} f(x) &= (-\gamma_\sigma)^\eta (\mathcal{I}^{\eta-\alpha,\sigma}_{b^-} f)(x) \\ &= \frac{1}{\Gamma(\eta-\alpha)} (-\gamma_\sigma)^\eta \int_x^b (\sigma(t) - \sigma(x))^{\eta-\alpha-1} f(t) \sigma'(t) dt, \end{aligned} \tag{1.6}$$

where $\gamma_\sigma(x) = \dfrac{1}{\sigma'(x)} \dfrac{d}{dx}$

Notation: For $a = -\infty$, we denote respectively $\mathcal{I}^{\alpha,\sigma}_{a^+}, \mathcal{D}^{\alpha,\sigma}_{a^+}$, by $\mathcal{I}^{\alpha,\sigma}_{+}, \mathcal{D}^{\alpha,\sigma}_{+}$. For $b = +\infty$, we denote respectively $\mathcal{I}^{\alpha,\sigma}_{b^-}, \mathcal{D}^{\alpha,\sigma}_{b^-}$, by $\mathcal{I}^{\alpha,\sigma}_{-}, \mathcal{D}^{\alpha,\sigma}_{-}$.

Remark 1.1. *For $\sigma(x) = x$, the notions of Definition 1.2 coincide with the notions of Riemann-Liouville integrals and derivatives $\mathcal{I}^{\alpha}_{a^+}, \mathcal{I}^{\alpha}_{b^-}, \mathcal{D}^{\alpha}_{a^+}, \mathcal{D}^{\alpha}_{b^-}$ (see for example [9]). So, by using the change of variable $s = \sigma(t)$, we get*

$$\mathcal{I}^{\alpha,\sigma}_{a^+} f(x) = \mathcal{I}^{\alpha}_{\sigma_a^+}(f \circ \sigma^{-1})(\sigma(x)), \qquad \mathcal{I}^{\alpha,\sigma}_{b^-} f(x) = \mathcal{I}^{\alpha}_{\sigma_b^-}(f \circ \sigma^{-1})(\sigma(x)). \tag{1.7}$$

Hence, for $\sigma_a = -\infty$ or $\sigma_b = +\infty$ we need $1 \leq p < \dfrac{1}{\alpha}$ (for the definition of $\mathcal{I}^{\alpha,\sigma}_{a^+}, \mathcal{I}^{\alpha,\sigma}_{b^-}$ we need $0 < \alpha < 1$. For more details, we refer to [9](Lemma 2.11).

Now, we will present some properties of fractional operators of functions of infinitely differentiable functions, with compact support. In what remains, we consider that $\sigma \in C^\infty([a, b])$.

Theorem 1.1. *For all $\varphi \in C^\infty_c(a, b)$ we have*

$$\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi \in C^\infty(a, b), \qquad \mathcal{I}^{\alpha,\sigma}_{b^-} \varphi \in C^\infty(a, b),$$

where $C^\infty_c(a, b)$ is the space of infinitely differentiable functions, with compact support in (a, b). Moreover, if $\operatorname{supp} \varphi \subset [a_0, b_0] \subset (a, b)$ then, $\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi$ vanishes in the interval (a, a_0) and $\mathcal{I}^{\alpha,\sigma}_{b^-} \varphi$ vanishes in the interval (b_0, b).

Proof. Let $\varphi \in C^\infty_c(a, b)$. Then,

$$\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (\sigma(x) - \sigma(t))^{\alpha-1} \varphi(t)\sigma'(t)dt = \frac{1}{\alpha\Gamma(\alpha)} \int_a^x (\sigma(x) - \sigma(t))^{\alpha} \varphi'(t)dt.$$

Since $(\sigma(x) - \sigma(t))^\alpha \varphi'(t) \in C(a, b)$, then,

$$\frac{d}{dx}\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi(x) = \frac{\sigma'(x)}{\Gamma(\alpha)} \int_a^x (\sigma(x) - \sigma(t))^{\alpha-1} \varphi(t)dt = \sigma'(x)\mathcal{I}^{\alpha,\sigma}_{a^+}(\gamma_\sigma \varphi)(x).$$

Since $\sigma'.\varphi \in C([a, b])$ then $\sigma'(x)\mathcal{I}^{\alpha,\sigma}_{a^+}(\gamma_\sigma \varphi) \in C(a, b)$. Hence $\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi \in C^1(a, b)$.

We will proceed in the same manner for $\dfrac{d^k}{dx^k}\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi \, (k \in \mathbb{N})$, keeping in mind that

$$\frac{d^k}{dx^k}\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi = \sum_{i=0}^{k} P_i(\sigma', \sigma'', \ldots \sigma^{(i)}).\mathcal{I}^{\alpha,\sigma}_{a^+} \gamma^i_\sigma \varphi, \tag{1.8}$$

where P_i is a polynomial of degree i.
Now, let $x \in (a, a_0)$, then $\varphi(t) = 0$ for all $t \in (x, b)$. Hence, $\mathcal{I}^{\alpha,\sigma}_{a^+} \varphi(x) = 0$.
We follow the same steps for $\mathcal{I}^{\alpha,\sigma}_{b^-} \varphi$. $\square$

Remark 1.2. *It is not always true that* $\mathcal{I}_{a^+}^{\alpha,\sigma}\varphi, \mathcal{I}_{b^-}^{\alpha,\sigma}\varphi \in C_c^\infty(a,b)$ *for* $\varphi \in C_c^\infty(a,b)$. *For example, let* $\varphi \in C_c^\infty(a,b)$ *such that* $\varphi' > 0$ *on* (a,b) *and* $\operatorname{supp}\varphi \subset [a_0,b_0] \subset (a,b)$. *Then, for* $x \in (b_0,b)$ *we have*

$$\mathcal{I}_{a^+}^{\alpha,\sigma}\varphi(x) = \frac{1}{\Gamma(\alpha+1)}\int_{a_0}^{b_0}(\sigma(x)-\sigma(t))^\alpha\varphi'(t)dt > 0.$$

Remark 1.3. *We only introduce the proofs for the fractional operators* $\mathcal{I}_{a^+}^{\alpha,\sigma}$ *and* $\mathcal{D}_{a^+}^{\alpha,\sigma}$ *in the sections that follow. The operator* $\mathcal{I}_{b^-}^{\alpha,\sigma}$ *and operator* $\mathcal{D}_{b^-}^{\alpha,\sigma}$ *proofs follow suit.*

The following theorem ensures the continuity of the operators $\mathcal{I}_{a^+}^{\alpha,\sigma}$ and $\mathcal{I}_{b^-}^{\alpha,\sigma}$.

Theorem 1.2. *For all* $\sigma \in C^\infty(a,b)$, *the operators* $\mathcal{I}_{a^+}^{\alpha,\sigma}$ *and* $\mathcal{I}_{b^-}^{\alpha,\sigma}$ *are continuous from* $C_c^\infty(a,b)$ *to* $C^\infty(a,b)$.

Proof. Let $\sigma \in C^\infty(a,b), K$ be a compact in (a,b) (we can always choose $K=[a_0,b_0]$), $k \in \mathbb{N}\cup\{0\}$ and $1 \le i \le k$. Then, for $\varphi \in C_c^\infty(a,b)$ such that $\operatorname{supp}\varphi \subset [a_1,b_1] \subset (a,b)$, we have

$$(\gamma_\sigma)^i\varphi(x) = \sum_{j,l=0}^{i} a_{j,l}\left(\frac{1}{\sigma'}\right)^{(l)}(x).\varphi^{(j)}(x).$$

Hence, $(\gamma_\sigma)^i\varphi$ is bounded on all compact in (a,b). So, there exists $M_i > 0$ such that

$$\sup_{a_0\le x\le b_0}|(\gamma_\sigma)^i\varphi(x)| \le M_i \sup_{a_0\le x\le b_0, 0\le j\le i}|\varphi^{(j)}(x)|$$

Then,

$$\begin{aligned}\sup_{a_0\le x\le b_0}|\mathcal{I}_{a^+}^{\alpha,\sigma}(\gamma_\sigma)^i\varphi(x)(x)| &\le M_i\frac{(\sigma(b_0)-\sigma(a_0))^\alpha}{\Gamma(\alpha+1)}\sup_{a_0\le x\le b_0, 0\le j\le i}|\varphi^{(j)}(x)|,\\ &= M_i'\sup_{a_0\le x\le b_0, 0\le j\le i}|\varphi^{(j)}(x)|,\end{aligned}$$

which leads to

$$\begin{aligned}\sup_{a_0\le x\le b_0}\left|\frac{d^k}{dx^k}\mathcal{I}_{a^+}^{\alpha,\sigma}\varphi(x)\right| &\le \sum_{i=0}^{k}M_i'\sup_{a_0\le x\le b_0}P_i(\sigma',\sigma'',\dots\sigma^{(i)}).\sup_{a_0\le x\le b_0, 0\le j\le i}|\varphi^{(j)}(x)|,\\ &\le M\sup_{a_0\le x\le b_0, 0\le i\le k}|\varphi^{(i)}(x)|.\end{aligned}$$

Since $\operatorname{supp}\varphi \subset [a_1,b_1]$, we get

$$\sup_{a_0\le x\le b_0 1, 0\le i\le k}|\varphi^{(i)}(x)| \le \sup_{a_1\le x\le b_1, 0\le i\le k}|\varphi^{(i)}(x)|$$

Then,

$$\sup_{a_0\le x\le b_0, 0\le i\le k}\left|\frac{d^k}{dx^k}\mathcal{I}_{a^+}^{\alpha,\sigma}\varphi(x)\right| \le M.\sup_{a_1\le x\le b_1, 0\le i\le k}|\varphi^{(i)}(x)|.$$

So, the continuity of $\mathcal{I}_{a^+}^{\alpha,\sigma}$. $\square$

2 The weak derivative

In this section, we present a first version of distributional fractional derivative, which is the weak derivative of functions. For this we consider $\alpha > 0, \eta = [\alpha]+1$ and $u \in L^p_\sigma(a,b)$. If $\mathcal{D}^{\alpha,\sigma}_{a^+}u, \mathcal{D}^{\alpha,\sigma}_{b^-}u$ exist in the sense of Definition 1.2 and verify properties of Lemma 2.7 in [9] then, for all $\varphi \in C^\infty_c(a,b)$ we have [10]

$$\langle \mathcal{D}^{\alpha,\sigma}_{a^+}u, \varphi\rangle = \int_a^b \mathcal{D}^{\alpha,\sigma}_{a^+}u(x).\varphi(x)\sigma'(x)dx = \int_a^b u(x).\mathcal{D}^{\alpha,\sigma}_{b^-}\varphi(x)\sigma'(x)dx$$

$$\langle \mathcal{D}^{\alpha,\sigma}_{b^-}u, \varphi\rangle = \int_a^b \mathcal{D}^{\alpha,\sigma}_{b^-}u(x).\varphi(x)\sigma'(x)dx = \int_a^b u(x).\mathcal{D}^{\alpha,\sigma}_{a^+}\varphi(x)\sigma'(x)dx$$

Using this characteristic, we extend definition 1.2 in the manner described below.

Definition 2.1. *let $u \in L^p_\sigma(a,b)$. The weak derivatives ${}^w\mathcal{D}^{\alpha,\sigma}_{a^+}u$ and ${}^w\mathcal{D}^{\alpha,\sigma}_{b^-}u$ of the function u are given as follows*

i) *If $\mathcal{I}^{\alpha,\sigma}_{a^+}|u|$ exists then,*

$$\langle {}^w\mathcal{D}^{\alpha,\sigma}_{a^+}u, \varphi\rangle = \int_a^b u(x).\mathcal{D}^{\alpha,\sigma}_{b^-}\varphi(x)\sigma'(x)dx \tag{2.1}$$

ii) *If $\mathcal{I}^{\alpha,\sigma}_{b^-}|u|$ exists then,*

$$\langle {}^w\mathcal{D}^{\alpha,\sigma}_{b^-}u, \varphi\rangle = \int_a^b u(x).\mathcal{D}^{\alpha,\sigma}_{a^+}\varphi(x)\sigma'(x)dx \tag{2.2}$$

Theorem 2.1. *${}^w\mathcal{D}^{\alpha,\sigma}_{a^+}u, {}^w\mathcal{D}^{\alpha,\sigma}_{b^-}u$ above define distributions on (a,b).*

Proof. Let $[a_0,b_0] \subset (a,b)$ and $\varphi \in C^\infty_c(a,b)$ such that $\operatorname{supp}\varphi \subset [a_0,b_0]$. Then, we have

$$\begin{aligned}
|\mathcal{D}^{\alpha,\sigma}_{a^+}\varphi(x)| &= \frac{1}{\Gamma(\eta-\alpha)}\left|\int_x^b (\sigma(t)-\sigma(x))^{\eta-\alpha-1}(-\gamma_\sigma)^\eta\varphi(t)\sigma'(t)dt\right| \\
&= \frac{1}{\Gamma(\eta-\alpha)}\left|\int_x^{b_0} (\sigma(t)-\sigma(x))^{\eta-\alpha-1}(-\gamma_\sigma)^\eta\varphi(t)\sigma'(t)dt\right| \\
&\leq \frac{1}{\Gamma(\eta-\alpha)}\int_x^{b_0} (\sigma(t)-\sigma(x))^{\eta-\alpha-1}|(-\gamma_\sigma)^\eta\varphi(t)|\sigma'(t)dt \\
&\leq \frac{1}{\Gamma(\eta-\alpha)}\sup_{x\leq t\leq b_0}|(-\gamma_\sigma)^\eta\varphi(t)|\int_x^{b_0} (\sigma(t)-\sigma(x))^{\eta-\alpha-1}\sigma'(t)dt \\
&= \frac{\eta-\alpha}{\Gamma(\eta-\alpha)}\sup_{x\leq t\leq b_0}|(-\gamma_\sigma)^\eta\varphi(t)|(\sigma(b_0)-\sigma(x))^{\eta-\alpha} \\
&= \frac{(\eta-\alpha)(\sigma(b_0)-\sigma(x))}{\Gamma(\eta-\alpha)}\sup_{x\leq t\leq b_0}|(-\gamma_\sigma)^\eta\varphi(t)|(\sigma(b_0)-\sigma(x))^{\eta-\alpha-1}.
\end{aligned}$$

So,

$$\begin{aligned}
|\mathcal{D}_{a^+}^{\alpha,\sigma}\varphi(x)| &\leq \frac{(\eta-\alpha)(\sigma(b_0)-\sigma(a_0))}{\Gamma(\eta-\alpha)} \sup_{a_0\leq x\leq b_0} |(-\gamma_\sigma)^\eta \varphi(x)|(\sigma(b_0)-\sigma(x))^{\eta-\alpha-1}.\\
&= \frac{A}{\Gamma(\eta-\alpha)} \sup_{a_0\leq x\leq b_0} |(-\gamma_\sigma)^\eta \varphi(x)|(\sigma(b_0)-\sigma(x))^{\eta-\alpha-1}.
\end{aligned}$$

Hence,

$$\begin{aligned}
|\langle {}^w\mathcal{D}_{a^+}^{\alpha,\sigma}u,\varphi\rangle| &= \left|\int_a^b u(x).\mathcal{D}_{a^+}^{\alpha,\sigma}\varphi(x)\sigma'(x)dx\right|\\
&= \left|\int_a^{b_0} u(x).\mathcal{D}_{a^+}^{\alpha,\sigma}\varphi(x)\sigma'(x)dx\right|\\
&\leq \frac{A\sup\limits_{a_0\leq x\leq b_0}|(-\gamma_\sigma)^\eta\varphi(x)|}{\Gamma(\eta-\alpha)}\int_a^{b_0}|u(x)|.(\sigma(b_0)-\sigma(x))^{\eta-\alpha-1}\sigma'(x)dx\\
&= \left[(\eta-\alpha)(\sigma(b_0)-\sigma(a_0))\mathcal{I}_{a^+}^{\eta-\alpha,\sigma}|u|(b_0)\right].\sup_{a_0\leq x\leq b_0}|(-\gamma_\sigma)^\eta\varphi(x)|.
\end{aligned}$$

Note that $(-\gamma_\sigma)^\eta\varphi(x) = \sum\limits_{i,j=0}^{\eta} a_{i,j}\left(\frac{1}{\sigma'}\right)^{(i)}(x).\varphi^{(j)}(x)$.

Since $\left(\frac{1}{\sigma'}\right)^{(i)}$, $\varphi^{(j)}$ are bounded in $[a_0,b_0]$ there exists $M_{\sigma',\eta}>0$ such that

$$\sup_{a_0\leq x\leq b_0}|(-\gamma_\sigma)^\eta\varphi(x)| = \sup_{a_0\leq x\leq b_0}|\sum_{i,j=0}^{\eta} a_{i,j}\left(\frac{1}{\sigma'}\right)^{(i)}(x).\varphi^{(j)}(x)| \leq M_{\sigma',\eta}\sup_{a_0\leq x\leq b_0;\,k\leq\eta}|\varphi^{(k)}(x)|.$$

Hence,

$$\left|\int_a^b u(x).\mathcal{D}_{a^+}^{\alpha,\sigma}\varphi(x)\sigma'(x)dx\right| \leq M_{u,\sigma',\eta}\sup_{a_0\leq x\leq b_0;\,k\leq\eta}|\varphi^{(k)}(x)|,$$

where $= M_{u,\sigma',\eta} = (\eta-\alpha)((\sigma(b_0)-\sigma(a_0))\mathcal{I}_{a^+}^{\eta-\alpha,\sigma}|u|(b_0).M_{\sigma',\eta}$,
which is required. ☐

Theorem 2.2. *If ${}^w\mathcal{D}_{a^+}^{\alpha,\sigma}u$ (${}^w\mathcal{D}_{b^-}^{\alpha,\sigma}u$) exists, it is unique.*

Proof. Let ${}^{w,1}\mathcal{D}_{a^+}^{\alpha,\sigma}u, {}^{w,2}\mathcal{D}_{a^+}^{\alpha,\sigma}u$ be two weak derivatives of u, then,

$$\langle {}^{w,1}\mathcal{D}_{a^+}^{\alpha,\sigma}u - {}^{w,2}\mathcal{D}_{a^+}^{\alpha,\sigma}u,\varphi\rangle = 0, \text{ for all } \varphi\in C_c^\infty(a,b).$$

Hence, ${}^{w,1}\mathcal{D}_{a^+}^{\alpha,\sigma}u = {}^{w,2}\mathcal{D}_{a^+}^{\alpha,\sigma}u$ in the distributional sense. ☐

Corollary 2.1. *If $\mathcal{D}_{a^+}^{\alpha,\sigma}u$ exists in the ordinary sense then, ${}^w\mathcal{D}_{a^+}^{\alpha,\sigma}u = \mathcal{D}_{a^+}^{\alpha,\sigma}u$. Similarly, we put ${}^w\mathcal{D}_{b^-}^{\alpha,\sigma}u = \mathcal{D}_{b^-}^{\alpha,\sigma}u$.*

From now on, we will use the symbol $\mathcal{D}^{\alpha,\sigma}_{a^+}u, \mathcal{D}^{\alpha,\sigma}_{b^-}u$ for either an ordinary derivative or a derivative in the concept of distributions.

Example 2.1. *Let* $(a,b) = (-\infty, +\infty)$ *and* $0 < \alpha < 1$. *We will calculate the fractional derivative of the Heaviside function, defined by*

$$H(x) = \begin{cases} 0 & : \quad x \leq 0 \\ 1 & : \quad x > 0 \end{cases} .$$

Let $\varphi \in C^\infty_c(\mathbb{R})$. *Then,*

$$\begin{aligned}
\langle \mathcal{D}^{\alpha,\sigma}_{+} H, \varphi \rangle &= \int_{-\infty}^{+\infty} H(x) \mathcal{D}^{\alpha,\sigma}_{-} \varphi(x) \sigma'(x) dx \\
&= \int_{0}^{+\infty} -\frac{d}{dx} \mathcal{I}^{1-\alpha,\sigma}_{-} \varphi(x) dx \\
&= \lim_{A\to+\infty} \int_0^A -\frac{d}{dx} \mathcal{I}^{1-\alpha,\sigma}_{-} \varphi(x) dx \\
&= \mathcal{I}^{1-\alpha,\sigma}_{-} \varphi(0) - \lim_{A\to+\infty} \mathcal{I}^{1-\alpha,\sigma}_{-} \varphi(A) \\
&= \mathcal{I}^{1-\alpha,\sigma}_{-} \varphi(0).
\end{aligned}$$

$$\begin{aligned}
\langle \mathcal{D}^{\alpha,\sigma}_{-} H, \varphi \rangle &= \int_{-\infty}^{+\infty} H(x) \mathcal{D}^{\alpha,\sigma}_{+} \varphi(x) dx \\
&= \int_{0}^{+\infty} \frac{d}{dx} \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(x) dx \\
&= \lim_{A\to+\infty} \int_0^A \frac{d}{dx} \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(x) dx \\
&= \lim_{A\to+\infty} [\mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(A)] - \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(0).
\end{aligned}$$

If $\lim\limits_{A\to+\infty} \sigma(A) = +\infty$ *then,* $\lim\limits_{A\to+\infty} \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(A) = +\infty$. *Hence,* $\mathcal{D}^{\alpha,\sigma}_{-} H$ *does not exist. Considering now that* $\lim\limits_{A\to+\infty} \sigma(A) = \sigma_+ < +\infty$. *Then,*

$$\lim_{A\to+\infty} \mathcal{I}^{1-\alpha,\sigma}_{-} \varphi(A) = \frac{1}{\Gamma(1-\alpha)} \int_0^B (\sigma_+ - \sigma(t))^{-\alpha} \varphi(t) \sigma'(t) dt = \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(\sigma_+),$$

where B *is chosen in a way that* $\operatorname{supp} \varphi \in [-B, B]$.
So, $\langle \mathcal{D}^{\alpha,\sigma}_{-} H, \varphi \rangle = \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(\sigma_+) - \mathcal{I}^{1-\alpha,\sigma}_{+} \varphi(0)$.

3 Fractional derivative of distributions with compact support

Let T be a distribution with compact support on (a,b), denoted by $\operatorname{supp} T$. First, we will define $\mathcal{I}^{\alpha,\sigma}_{a^+}$ and $\mathcal{I}^{\alpha,\sigma}_{b^-}$. Next, we introduce the notions $\mathcal{D}^{\alpha,\sigma}_{a^+}$ and $\mathcal{D}^{\alpha,\sigma}_{b^-}$. It is known that the space $\xi(a,b)$ of distributions with compact support is the dual of the space $C^\infty(a,b)$. According to Urysohn's Lemma, there exists $\chi \in C_c^\infty(a,b)$ such that $\chi = 1$ in the neighbourhood of $\operatorname{supp} T$. Using this important property to give the following notions

Definition 3.1. *The integrals $\mathcal{I}_{a^+}^{\alpha,\sigma}T, \mathcal{I}_{b^-}^{\alpha,\sigma}T$ are given by the following formulas*

$$\langle \mathcal{I}_{a^+}^{\alpha,\sigma}T, \varphi\rangle = \langle T, \chi.\mathcal{I}_{b^-}^{\alpha,\sigma}\varphi\rangle, \tag{3.1}$$

$$\langle \mathcal{I}_{b^-}^{\alpha,\sigma}T, \varphi\rangle = \langle T, \chi.\mathcal{I}_{a^+}^{\alpha,\sigma}\varphi\rangle. \tag{3.2}$$

Note that the choice of χ has no impact on the above formulas.

Theorem 3.1. *$\mathcal{I}_{a^+}^{\alpha,\sigma}T$ and $\mathcal{I}_{b^-}^{\alpha,\sigma}T$ define distributions on (a,b).*

Proof. Assume that $\operatorname{supp} T \subset [a_0, b_0] \subset (a,b)$ and let $\chi \in C_c^\infty(a,b)$ such that $\chi = 1$ on $(a_0 - \varepsilon, b_0 + \varepsilon)$, where $\varepsilon > 0$ is small enough. Let $\varphi \in C_c^\infty(a,b)$ such that $\operatorname{supp}\varphi \subset [a_1, b_1] \subset (a,b)$. Then, there exists $k \in \mathbb{N} \cup \{0\}$ and $M, M' > 0$ such that

$$\begin{aligned}
|\langle \mathcal{I}_{a^+}^{\alpha,\sigma}T, \varphi\rangle| &= |\langle T, \chi\mathcal{I}_{b^-}^{\alpha,\sigma}\varphi\rangle|, \\
&\leq M \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k} \left| \frac{d^m}{dx^m}(\chi\mathcal{I}_{b^-}^{\alpha,\sigma}\varphi) \right|, \\
&= M \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k} \left| \frac{d^m}{dx^m}(\mathcal{I}_{b^-}^{\alpha,\sigma}\varphi) \right|, \\
&\leq M'. \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k} |\varphi^{(m)}(x)|, \\
&\leq M'. \sup_{a_1 \leq x \leq b_1, 0 \leq m \leq k} |\varphi^{(m)}(x)|.
\end{aligned}$$

□

Example 3.1. *Let $(a,b) = (-\infty, +\infty)$. We will calculate the Dirac mass' fractional integral. Let $\varphi \in C_c^\infty(\mathbb{R})$. Then,*

$$\langle \mathcal{I}_{+}^{\alpha,\sigma}\delta, \varphi\rangle = \langle \delta, \chi\mathcal{I}_{-}^{\alpha,\sigma}\varphi\rangle = \chi(0)\mathcal{I}_{-}^{\alpha,\sigma}\varphi(0) = \mathcal{I}_{-}^{\alpha,\sigma}\varphi(0).$$

Hence, $\mathcal{I}_{+}^{\alpha,\sigma}\delta = \mathcal{D}_{+}^{1-\alpha,\sigma}H$.

$$\langle \mathcal{I}_{-}^{\alpha,\sigma}\delta, \varphi\rangle = \langle \delta, \chi\mathcal{I}_{+}^{\alpha,\sigma}\varphi\rangle = \chi(0).\mathcal{I}_{+}^{\alpha,\sigma}\varphi(0) = \mathcal{I}_{+}^{\alpha,\sigma}\varphi(0).$$

Definition 3.2. *The derivatives $\mathcal{D}_{a^+}^{\alpha,\sigma}T, \mathcal{D}_{b^-}^{\alpha,\sigma}T$ are given by the following formulas*

$$\langle \mathcal{D}_{a^+}^{\alpha,\sigma}T, \varphi\rangle = \langle T, \chi.\mathcal{D}_{b^-}^{\alpha,\sigma}\varphi\rangle, \tag{3.3}$$

$$\langle \mathcal{D}_{b^-}^{\alpha,\sigma}T, \varphi\rangle = \langle T, \chi.\mathcal{D}_{a^+}^{\alpha,\sigma}\varphi\rangle. \tag{3.4}$$

Theorem 3.2. *$\mathcal{D}_{a^+}^{\alpha,\sigma}T$ and $\mathcal{D}_{b^-}^{\alpha,\sigma}T$ define distributions on (a,b).*

Proof. Considering that $\operatorname{supp} T \subset [a_0, b_0]$ and let $\chi \in C_c^\infty(a,b)$ such that $\chi = 1$ on $(a_0 - \varepsilon, b_0 + \varepsilon)$, include in (a,b). Let $\varphi \in C_c^\infty(a,b)$ such that $\operatorname{supp}\varphi \subset [a_1, b_1] \subset (a,b)$. Then, there exists $k \in \mathbb{N} \cup \{0\}$ and $M, M' > 0$ such that

$$
\begin{aligned}
|\langle \mathcal{D}_{a^+}^{\alpha,\sigma} T, \varphi\rangle| &= |\langle T, \chi \mathcal{D}_{b^-}^{\alpha,\sigma} \varphi\rangle|, \\
&\leq M \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k} \left| \frac{d^m}{dx^m} (\chi \mathcal{D}_{b^-}^{\alpha,\sigma} \varphi) \right|, \\
&= M \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k} \left| \frac{d^m}{dx^m} (\mathcal{D}_{b^-}^{\alpha,\sigma} \varphi) \right|, \\
&= M \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k} \left| \frac{d^m}{dx^m} (\mathcal{I}_{b^-}^{\eta-\alpha,\sigma} (-\gamma_\sigma)^\eta \varphi) \right|,
\end{aligned}
$$

Using the same reasoning as before, it can be demonstrated that

$$
\begin{aligned}
|\langle \mathcal{D}_{a^+}^{\alpha,\sigma} T, \varphi\rangle| &\leq M'. \sup_{a_0 \leq x \leq b_0, 0 \leq m \leq k+\eta} |\varphi^{(m)}(x)|, \\
&\leq M'. \sup_{a_1 \leq x \leq b_1, 0 \leq m \leq k+\eta} |\varphi^{(m)}(x)|.
\end{aligned}
$$

$\square$

Example 3.2. *Let $(a,b) = (-\infty, +\infty)$. We will calculate the Dirac mass' fractional derivative. Let $\varphi \in C_c^\infty(\mathbb{R})$. Then,*

$$\langle \mathcal{D}_{+}^{\alpha,\sigma} \delta, \varphi\rangle = \langle \delta, \chi \mathcal{D}_{-}^{\alpha,\sigma} \varphi\rangle = \chi(0) \mathcal{D}_{-}^{\alpha,\sigma} \varphi(0) = \mathcal{D}_{-}^{\alpha,\sigma} \varphi(0).$$

$$\langle \mathcal{D}_{-}^{\alpha,\sigma} \delta, \varphi\rangle = \langle \delta, \chi \mathcal{D}_{+}^{\alpha,\sigma} \varphi\rangle = \chi(0). \mathcal{D}_{+}^{\alpha,\sigma} \varphi(0) = \mathcal{D}_{+}^{\alpha,\sigma} \varphi(0).$$

Remark 3.1. *If $T \circ \sigma^{-1}$ has a compact support in $(\sigma(a), \sigma(b))$, we can extend the Definitions 3.1, 3.2 as fellows:*

$$\langle \mathcal{I}_{a^+}^{\alpha,\sigma} T, \varphi\rangle = \langle T \circ \sigma^{-1}, \chi. \mathcal{I}_{b^-}^{\alpha} (\varphi \circ \sigma^{-1})\rangle, \tag{3.5}$$

$$\langle \mathcal{I}_{b^-}^{\alpha,\sigma} T, \varphi\rangle = \langle T \circ \sigma^{-1}, \chi. \mathcal{I}_{a^+}^{\alpha} (\varphi \circ \sigma^{-1})\rangle. \tag{3.6}$$

$$\langle \mathcal{D}_{a^+}^{\alpha,\sigma} T, \varphi\rangle = \langle T \circ \sigma^{-1}, \chi. \mathcal{D}_{b^-}^{\alpha} (\varphi \circ \sigma^{-1})\rangle, \tag{3.7}$$

$$\langle \mathcal{D}_{b^-}^{\alpha,\sigma} T, \varphi\rangle = \langle T \circ \sigma^{-1}, \chi. \mathcal{D}_{a^+}^{\alpha} (\varphi \circ \sigma^{-1})\rangle. \tag{3.8}$$

where, $\chi \in C_c^\infty(\sigma(a), \sigma(b))$ such that $\chi = 1$ in a neighbourhood of $\operatorname{supp}(T \circ \sigma^{-1})$ and $\mathcal{I}_{a^+}^{\alpha}, \mathcal{I}_{b^-}^{\alpha}, \mathcal{D}_{a^+}^{\alpha}, \mathcal{D}_{b^-}^{\alpha}$ are the integrals and derivatives in the sense of Riemann - Liouville.

4 Fractional calculus using the convolution product

In this section, using the convolution product of distributions, we present a conception of fractional integrals and derivatives of distributions on the entire line $\mathbb{R}$. Considering that $a = -\infty, b = +\infty, \sigma \in C^{\infty}(\mathbb{R}), \lim\limits_{x\to-\infty} \sigma(x) = -\infty$ and $\lim\limits_{x\to+\infty} \sigma(x) = +\infty$. Let $f \in L^p_\sigma(\mathbb{R})$. Then, for all $\varphi \in C^\infty_c(\mathbb{R})$ [10]

$$\begin{aligned}\int_{-\infty}^{+\infty} I_+^{\alpha,\sigma} f(x)\varphi(x)\sigma'(x)dx &= \int_{-\infty}^{+\infty} f(x) I_-^{\alpha,\sigma}\varphi(x)\sigma'(x)dx\\ &= \int_{-\infty}^{+\infty} f(x)\sigma'(x)dx \int_x^{+\infty} \frac{(\sigma(t)-\sigma(x))^{\alpha-1}}{\Gamma(\alpha)}\varphi(t)\sigma'(t)dt.\end{aligned}$$

Using the change of variable $y = \sigma(t) - \sigma(x)$, we get

$$\int_{-\infty}^{+\infty} I_+^{\alpha,\sigma} f(x)\varphi(x)\sigma'(x)dx = \int_{-\infty}^{+\infty} f(x)\sigma'(x)dx \int_0^{+\infty} \frac{y^{\alpha-1}}{\Gamma(\alpha)}(\varphi\circ\sigma^{-1})(y+\sigma(x))dy.$$

Using the change of variable $z = \sigma(x)$, we obtain

$$\int_{-\infty}^{+\infty} I_+^{\alpha,\sigma} f(x)\varphi(x)\sigma'(x)dx = \int_{-\infty}^{+\infty}\int_0^{+\infty} \frac{y^{\alpha-1}}{\Gamma(\alpha)}(f\circ\sigma^{-1})(z)(\varphi\circ\sigma^{-1})(y+z)dydz.$$

which lead to

$$\int_{-\infty}^{+\infty} I_+^{\alpha,\sigma} f(x)\varphi(x)\sigma'(x)dx = \langle (f\circ\sigma^{-1}) * Y_\alpha, \psi\rangle, \tag{4.1}$$

where $Y_\alpha(x) = \begin{cases} 0 & : \ x \le 0 \\ \dfrac{x^{\alpha-1}}{\Gamma(\alpha)} & : \ x > 0 \end{cases}$, $\psi = \varphi\circ\sigma^{-1} \in C^\infty_c(\mathbb{R})$ and $*$ denotes the convolution product of distributions on $\mathbb{R}$.

In a similar argument, we can write

$$\int_{-\infty}^{+\infty} I_-^{\alpha,\sigma} f(x)\varphi(x)\sigma'(x)dx = \langle (f\circ\sigma^{-1}) * \check{Y}_\alpha, \psi\rangle, \tag{4.2}$$

where $\check{Y}_\alpha(x) = \begin{cases} \dfrac{(-x)^{\alpha-1}}{\Gamma(\alpha)} & : \ x < 0 \\ 0 & : \ x \ge 0 \end{cases}$, $\psi = \varphi\circ\sigma^{-1} \in C^\infty_c(\mathbb{R})$.

We know that in order for the convolution product to be well defined, $\operatorname{supp} f$ and $\operatorname{supp} Y_\alpha$ (respectively $\operatorname{supp} f$ and $\operatorname{supp} \check{Y}_\alpha$) must be permitted convolution at the following sense

Definition 4.1. *[8] The closed sets $A_1, \ldots, A_k \subset \mathbb{R}^n$ $(n \in \mathbb{N})$ are called permitted convolution if for all compact $K \subset \mathbb{R}^n$ the set $E = \{(x_1,\ldots,x_k) \in \prod_{i=1}^k A_i : x_1 + \cdots + x_k \in K\}$ is compact in $\mathbb{R}^{nk}$.*

Remark 4.1. *The following requirements must be fulfilled for the set E to be compact:*

For all $r > 0$, there exists $\rho > 0$ such that for all $x_i \in A_i$: if $\lvert x_1 + \cdots + x_k \rvert \leq r$ then, $\max\limits_{1 \leq i \leq k} \lvert x_i \rvert \leq \rho$.

Taking into account that $\operatorname{supp} Y_\alpha = [0, +\infty), \operatorname{supp} \check{Y}_\alpha = (-\infty, 0]$, we obtain the following result

Proposition 4.1. *Let T be a distribution on $\mathbb{R}$. Then,*

i) $\operatorname{supp} T$ *and* $\operatorname{supp} Y_\alpha$ *are permitted convolution if and only if* $\operatorname{supp} T \subset [A, +\infty)$ *where* $A \in \mathbb{R}$,

ii) $\operatorname{supp} T$ *and* $\operatorname{supp} \check{Y}_\alpha$ *are permitted convolution if and only if* $\operatorname{supp} T \subset (-\infty, B]$ *where* $B \in \mathbb{R}$.

Proof. **i)** Assume that $\operatorname{supp} T \subset [A, +\infty)$ and let $R > 0$. Then, for $x \in \operatorname{supp} T$ and $y \in \operatorname{supp} Y_\alpha$ such that $\lvert x + y \rvert \leq R$ we obtain $A \leq x \leq R$ and $0 \leq y \leq R - A$. So, $\lvert x \rvert \leq R + \lvert A \rvert$ and $\lvert y \rvert = y \leq R + \lvert A \rvert$. Hence, $\operatorname{supp} T$ and $\operatorname{supp} Y_\alpha$ are permitted convolution.
Otherwise, there exists $C \in \mathbb{R}$ such that $(-\infty, B] \subset \operatorname{supp} T$. Then, for $m \in \mathbb{N}$ large enough we have $-m \in \operatorname{supp} T$ and $m \in \operatorname{supp} Y_\alpha$. But we have $\lvert (-m) + m \rvert < R$ for some R and $\lim\limits_{m \to +\infty} \lvert -m \rvert = +\infty$. Hence, $\operatorname{supp} T$ and $\operatorname{supp} Y_\alpha$ are not permitted convolution.

ii) The same argument is made for both $\operatorname{supp} T$ and $\operatorname{supp} \check{Y}_\alpha$.

□

So, we will give the definition of left and right fractional integral and derivative of a distribution of the type of proposition 4.1:

Definition 4.2. *Let T be a distribution on $\mathbb{R}$. Then,*

i) *If* $\operatorname{supp} T \subset [A, +\infty)$ *with* $A \in \mathbb{R}$ *then, we put*

$$I_+^{\alpha,\sigma} T(x) = ((T \circ \sigma^{-1}) * Y_\alpha) \circ \sigma, \qquad D_+^{\alpha,\sigma} T (\gamma_\sigma)^\eta I_+^{\alpha,\sigma} T.$$

ii) *If* $\operatorname{supp} T \subset (-\infty, B]$ *with* $B \in \mathbb{R}$ *then, we put*

$$I_-^{\alpha,\sigma} T(x) = ((T \circ \sigma^{-1}) * \check{Y}_\alpha) \circ \sigma, \qquad D_-^{\alpha,\sigma} T = (-\gamma_\sigma)^\eta I_-^{\alpha,\sigma} T.$$

5 Extension to multidimensional case

Let $n \in \mathbb{N} \setminus \{1\}$. We use the notations of [9, 18] and [14] which

$\boldsymbol{a} = (a_1, a_2, \ldots, a_n), \boldsymbol{b} = (b_1, b_2, \ldots, b_n)\ (-\infty \leq a_i < b_i \leq +\infty), (\boldsymbol{a}, \boldsymbol{b}) = \prod\limits_{i=1}^{n} (a_i, b_i),$

$\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_N), \alpha_i > 0, \boldsymbol{\eta} = (\eta_1, \eta_2, \ldots, \eta_n)$,
$\eta_i = [\alpha_i] + 1$, $\Gamma(\boldsymbol{\alpha}) = \prod_{i=1}^{n} \Gamma(\alpha_i), x^{\alpha} = \prod_{i=1}^{n} x_i^{\alpha_i}, \frac{\partial}{\partial x} = \frac{\partial}{\partial x_1} \frac{\partial}{\partial x_2} \cdots \frac{\partial}{\partial x_n}$.
Let $\boldsymbol{\sigma}(\boldsymbol{x}) = (\sigma_1(x_1), \sigma_2(x_2), \ldots, \sigma_n(x_n))$, where σ_i are increasing functions in $C^{\infty}(a_i, b_i)$.
Finally, let $\boldsymbol{p} = (p_1, p_2, \ldots, p_n)\,(1 \leq p_i < +\infty)$.
Following the definitions of Riemann-Liouville integrals and derivatives in [9, 18], we present some definitions of fractional integrals and derivatives.
First, as well as [3], we give the definition of anisotropic space $X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\boldsymbol{a}, \boldsymbol{b})$.

Definition 5.1. *We denote by $X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\boldsymbol{a}, \boldsymbol{b})$ the space of measurable functions f on $(\boldsymbol{a}, \boldsymbol{b})$ such that $\|f\|_{X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\boldsymbol{a},\boldsymbol{b})} < \infty$, where*

$$\|f\|_{X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\boldsymbol{a},\boldsymbol{b})} = \left(\int_{a_n}^{b_n} \left(\int_{a_{n-1}}^{b_{n-1}} \cdots \left(\int_{a_1}^{b_1} |f(\boldsymbol{x})|^{p_1} \sigma_1' dx_1 \right)^{\frac{p_2}{p_1}} \cdots \sigma_{n-1}' dx_{n-1} \right)^{\frac{p_n}{p_{n-1}}} \sigma_n' dx_n \right)^{\frac{1}{p_n}},$$

which is a Banach space, with respect to the norm $\|.\|_{X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\boldsymbol{a},\boldsymbol{b})}$.

Remark 5.1. *Let Ω be a domain of $\mathbb{R}^n$. Then, there exists $(\boldsymbol{a}, \boldsymbol{b}) = \prod_{i=1}^{n}(a_i, b_i)\,(-\infty \leq a_i < b_i \leq +\infty)$ such that $\Omega \subset (\boldsymbol{a}, \boldsymbol{b})$. Hence, we can define the space $X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\Omega)$ by extending a function f by 0 from Ω, always denoting by f. So, we set $\|f\|_{X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\Omega)} = \|f\|_{X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\boldsymbol{a},\boldsymbol{b})}$.*

Let $f \in X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\Omega)$, where $\Omega \subset (\boldsymbol{a}, \boldsymbol{b}) = \prod_{i=1}^{n}(a_i, b_i)$. Following the definitions of paragraph (§2.9) in [9], we present those definitions

Definition 5.2. *The generalized (left and right) fractional partial integral and derivative operators with respect to the function σ_i are given by*

$$\mathcal{I}^{\alpha_i,\sigma_i}_{i^+} f(\boldsymbol{x}) = \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{x_i} (\sigma_i(x_i) - \sigma_i(t_i))^{\alpha_i - 1} f(\boldsymbol{t}) \sigma_i'(t_i) dt_i, \tag{5.1}$$

$$\mathcal{I}^{\alpha_i,\sigma_i}_{i^-} f(\boldsymbol{x}) = \frac{1}{\Gamma(\alpha_i)} \int_{x_i}^{b_i} (\sigma_i(t_i) - \sigma_i(x_i))^{\alpha_i - 1} f(\boldsymbol{t}) \sigma_i'(t_i) dt. \tag{5.2}$$

$$\begin{aligned} \mathcal{D}^{\alpha_i,\sigma_i}_{i^+} f(\boldsymbol{x}) &= (\gamma_{\sigma_i})^{\eta_i} (\mathcal{I}^{\eta_i - \alpha_i,\sigma_i}_{i^+} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(\eta_i - \alpha_i)} (\gamma_{\sigma_i})^{\eta_i} \int_{a_i}^{x_i} (\sigma_i(x_i) - \sigma_i(t_i))^{\eta_i - \alpha_i - 1} f(\boldsymbol{t}) \sigma_i'(t_i) dt_i, \end{aligned} \tag{5.3}$$

$$\begin{aligned} \mathcal{D}^{\alpha_i,\sigma_i}_{i^-} f(\boldsymbol{x}) &= (-\gamma_{\sigma})^{\eta_i} (\mathcal{I}^{\eta_i - \alpha_i,\sigma_i}_{i^-} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(\eta_i - \alpha_i)} (-\gamma_{\sigma_i})^{\eta_i} \int_{x_i}^{b_i} (\sigma_i(t_i) - \sigma_i(x_i))^{\eta_i - \alpha_i - 1} f(\boldsymbol{t}) \sigma_i'(t_i) dt_i, \end{aligned} \tag{5.4}$$

where $\gamma_{\sigma_i}(x_i) = \dfrac{1}{\sigma_i'(x)} \dfrac{\partial}{\partial x_i}$.

Definition 5.3. *The generalized (left and right) fractional integral and derivative operators with respect to the function* $\boldsymbol{\sigma}$ *are represented by*

$$\mathcal{I}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} f(\boldsymbol{x}) = \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} (\boldsymbol{\sigma}(\boldsymbol{x}) - \boldsymbol{\sigma}(\boldsymbol{t}))^{\boldsymbol{\alpha}-\mathbf{1}} f(\boldsymbol{t})(\sigma_1' dt_1) \ldots (\sigma_n' dt_n), \tag{5.5}$$

$$\mathcal{I}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} f(\boldsymbol{x}) = \frac{1}{\Gamma(\boldsymbol{\alpha})} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} (\boldsymbol{\sigma}(\boldsymbol{t}) - \boldsymbol{\sigma}(\boldsymbol{x}))^{\boldsymbol{\alpha}-\mathbf{1}} f(\boldsymbol{t})(\sigma_1' dt_1) \ldots (\sigma_n' dt_n). \tag{5.6}$$

$$\begin{aligned} \mathcal{D}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} f(\boldsymbol{x}) &= (\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}} (\mathcal{I}_{+}^{\boldsymbol{\eta}-\boldsymbol{\alpha},\boldsymbol{\sigma}} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(\boldsymbol{\eta}-\boldsymbol{\alpha})} (\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}} \int_{a_1}^{x_1} \cdots \int_{a_n}^{x_n} (\boldsymbol{\sigma}(\boldsymbol{x}) - \boldsymbol{\sigma}(\boldsymbol{t}))^{\boldsymbol{\eta}-\boldsymbol{\alpha}-\mathbf{1}} f(\boldsymbol{t})(\sigma_1' dt_1) \ldots (\sigma_n' dt_n), \end{aligned} \tag{5.7}$$

$$\begin{aligned} \mathcal{D}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} f(\boldsymbol{x}) &= (-\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}} (\mathcal{I}_{-}^{\boldsymbol{\eta}-\boldsymbol{\alpha},\boldsymbol{\sigma}} f)(\boldsymbol{x}) \\ &= \frac{1}{\Gamma(\boldsymbol{\eta}-\boldsymbol{\alpha})} (-\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}} \int_{x_1}^{b_1} \cdots \int_{x_n}^{b_n} (\boldsymbol{\sigma}(\boldsymbol{t}) - \boldsymbol{\sigma}(\boldsymbol{x}))^{\boldsymbol{\eta}-\boldsymbol{\alpha}-\mathbf{1}} f(\boldsymbol{t})(\sigma_1' dt_1) \ldots (\sigma_n' dt_n), \end{aligned} \tag{5.8}$$

where $\gamma_{\boldsymbol{\sigma}}^{\boldsymbol{\eta}}(\boldsymbol{x}) = \left(\dfrac{1}{\sigma_1'(x_1)} \dfrac{\partial}{\partial x_1}\right)^{\eta_1} \ldots \left(\dfrac{1}{\sigma_n'(x_n)} \dfrac{\partial}{\partial x_n}\right)^{\eta_n}$, $\mathbf{1} = \underbrace{(1, 1, \ldots, 1)}_{n \text{ times}}$.

We introduce also the weak (left or right) derivative of a function $u : \mathbb{R}^n \to \mathbb{R}$ (if there exists) as follow

Definition 5.4. *For all* $\varphi \in C_c^\infty(\boldsymbol{a}, \boldsymbol{b})$

$$\langle \mathcal{D}_{i^+}^{\alpha_i,\sigma_i} u, \varphi \rangle = \int_{a_i}^{b_i} u(\boldsymbol{x}).\mathcal{D}_{i^-}^{\alpha_i,\sigma_i} \varphi(\boldsymbol{x}) \sigma_i'(x_i) dx_i, \tag{5.9}$$

$$\langle \mathcal{D}_{i^-}^{\alpha_i,\sigma_i} u, \varphi \rangle = \int_{a_i}^{b_i} u(\boldsymbol{x}).\mathcal{D}_{i^+}^{\alpha_i,\sigma_i} \varphi(\boldsymbol{x}) \sigma_i'(x_i) dx_i, \tag{5.10}$$

$$\langle \mathcal{D}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} u, \varphi \rangle = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(\boldsymbol{x}).\mathcal{D}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} \varphi(\boldsymbol{x})(\sigma_1'(x_1) dx_1) \ldots (\sigma_n'(x_n) dx_n), \tag{5.11}$$

$$\langle \mathcal{D}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} u, \varphi \rangle = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} u(\boldsymbol{x}).\mathcal{D}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} \varphi(\boldsymbol{x})(\sigma_1'(x_1) dx_1) \ldots (\sigma_n'(x_n) dx_n). \tag{5.12}$$

It follows that the definition of fractional operators for a distribution with compact support should be included.

Definition 5.5. *Let T be a distribution with compact support in Ω. Then, for all $\varphi \in C_c^\infty(\Omega)$ we put*

$$\langle \mathcal{I}_{i^+}^{\alpha_i,\sigma_i} T, \varphi\rangle = \langle T, \chi.\mathcal{I}_{i^-}^{\alpha_i,\sigma_i}\varphi\rangle, \qquad \mathcal{D}_{i^+}^{\alpha_i,\sigma_i} T = (\gamma_{\sigma_i})^{\eta_i}\mathcal{I}_{i^+}^{\eta_i-\alpha_i,\sigma_i} T \tag{5.13}$$

$$\langle \mathcal{I}_{i^-}^{\alpha_i,\sigma_i} T, \varphi\rangle = \langle T, \chi.\mathcal{I}_{i^+}^{\alpha_i,\sigma_i}\varphi\rangle, \qquad \mathcal{D}_{i^-}^{\alpha_i,\sigma_i} T = (-\gamma_{\sigma_i})^{\eta_i}\mathcal{I}_{i^-}^{\eta_i-\alpha_i,\sigma_i} T \tag{5.14}$$

$$\langle \mathcal{I}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T, \varphi\rangle = \langle T, \chi.\mathcal{I}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}}\varphi\rangle, \qquad \mathcal{D}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T = (\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}}\mathcal{I}_{+}^{\boldsymbol{\eta}-\boldsymbol{\alpha},\boldsymbol{\sigma}} T \tag{5.15}$$

$$\langle \mathcal{I}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T, \varphi\rangle = \langle T, \chi.\mathcal{I}_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}}\varphi\rangle, \qquad \mathcal{D}_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T = (-\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}}\mathcal{I}_{-}^{\boldsymbol{\eta}-\boldsymbol{\alpha},\boldsymbol{\sigma}} T \tag{5.16}$$

where $\chi \in C_c^\infty(\boldsymbol{a},\boldsymbol{b})$ such that $\chi = 1$ in a neighbourhood of $\operatorname{supp} T$.

Remark 5.2. *Setting $\boldsymbol{\sigma}^{-1} = (\sigma_1^{-1}\dots\sigma_i^{-1})$. Then, with respect to distributions T where* $\operatorname{supp}(T\circ\boldsymbol{\sigma}^{-1})$ *is compact, remark 3.1 is always true.*

Finally, we introduce the definitions of left and right fractional integral and derivative of a distribution T using the convolution product. We denote by $\pi_i(E)\,(E\subset\mathbb{R}^n)$ the *ith* projection map of E, $H_{\alpha_i}, \check{H}_{\alpha_i}$ the following functions defined in $\pi_i(\mathbb{R}^n)$ as following

$$H_{\alpha_i}(x_i) = \begin{cases} 0 & : \; x_i < 0 \\ x_i^{\alpha-1} & : \; x_i \geq 0 \end{cases} \qquad \check{H}_{\alpha_i}(x_i) = \begin{cases} (-x_i)^{\alpha-1} & : \; x_i < 0 \\ 0 & : \; x_i \geq 0 \end{cases}$$

So, we have the follows definition

Definition 5.6. *Let T be a distribution on $\mathbb{R}^n$. Then,*

i) *If $\pi_i(\operatorname{supp} T)\subset[A_i,+\infty)(A_i\in\mathbb{R})$ then, we put*

$$I_{i^+}^{\alpha_i,\sigma_i} T(\boldsymbol{x}) = \frac{1}{\Gamma(\alpha_i)}(T(x_1,\dots,\sigma^{-1}(x_i),\dots x_N))*H_{\alpha_i}, \qquad D_{i^+}^{\alpha_i,\sigma_i} T = (\gamma_{\sigma_i})^{\eta_i} I_{i^+}^{\alpha_i,\sigma_i} T$$

ii) *If $\pi_i(\operatorname{supp} T)\subset(-\infty,B_i](B_i\in\mathbb{R})$ then, we put*

$$I_{i^-}^{\alpha_i,\sigma_i} T(\boldsymbol{x}) = \frac{1}{\Gamma(\alpha_i)}(T(x_1,\dots,\sigma^{-1}(x_i),\dots x_N))*\check{H}_{\alpha_i}, \qquad D_{i^-}^{\alpha_i,\sigma_i} T = (\gamma_{\sigma_i})^{\eta_i} I_{i^-}^{\alpha_i,\sigma_i} T$$

iii) *If $\operatorname{supp} T$ and $([0,+\infty))^n$ are permitted convolution then, we put*

$$I_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T(\boldsymbol{x}) = \frac{1}{\Gamma(\alpha)}(T\circ\boldsymbol{\sigma}^{-1}) * H_{\alpha_1} * \dots * H_{\alpha_n}, \qquad D_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T = (\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}} I_{+}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T$$

iv) *If $\operatorname{supp} T$ and $((-\infty,0])^n$ are permitted convolution then, we put*

$$I_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T(\boldsymbol{x}) = \frac{1}{\Gamma(\boldsymbol{\alpha})}(T\circ\boldsymbol{\sigma}^{-1}) * \check{H}_{\alpha_1} * \dots * \check{H}_{\alpha_n}, \qquad D_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T = (\gamma_{\boldsymbol{\sigma}})^{\boldsymbol{\eta}} I_{-}^{\boldsymbol{\alpha},\boldsymbol{\sigma}} T$$

6 Conclusion

In this work, we defined fractional integrals and derivatives in terms of distributions that are compatible with those in terms of functions. We then extended all of this to the multidimensional case, allowing us to provide a fractional version of the usual differential operators (gradient, divergence, Laplace). Let $\Omega \subset \mathbb{R}^n$ be an open. Taking into account Remark 5.1, definitions and notations og the above section, we introduce the follows definitions

Definition 6.1. *Let $f \in X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\Omega)$. Then, the $\boldsymbol{\alpha}-$ gradient of f is given by*

$$\nabla_+^\alpha F = (D_+^{\alpha_1} f, D_+^{\alpha_2} f, \dots, D_+^{\alpha_n} f).$$

$$\nabla_-^\alpha F = (D_-^{\alpha_1} f, D_-^{\alpha_2} f, \dots, D_-^{\alpha_n} f).$$

Definition 6.2. *Let $F \in (X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\Omega))^n$. Then, the $\boldsymbol{\alpha}-$ divergence of F is given as follow*

$$div_+^\alpha f = D_+^{\alpha_1} F_1 + D_+^{\alpha_2} F_2 + \dots + D_+^{\alpha_n} F_n.$$

$$div_-^\alpha f = D_-^{\alpha_1} F_1 + D_-^{\alpha_2} F_2 + \dots + D_-^{\alpha_n} F_n.$$

Definition 6.3. *Let $f \in X^{\boldsymbol{p}}_{\boldsymbol{\sigma}}(\Omega)$. Then, the $\boldsymbol{\alpha}-$ Laplacian of f is defined by*

$$\Delta_-^\alpha f = div_-^\alpha(\nabla_+^\alpha f) = D_-^{\alpha_1} D_+^{\alpha_1} f + D_-^{\alpha_2} D_+^{\alpha_2} f + \dots + D_-^{\alpha_n} D_+^{\alpha_n} f.$$

$$\Delta_+^\alpha f = div_+^\alpha(\nabla_-^\alpha f) = D_+^{\alpha_1} D_-^{\alpha_1} f + D_+^{\alpha_2} D_-^{\alpha_2} f + \dots + D_+^{\alpha_n} D_-^{\alpha_n} f.$$

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