Symmetry Algebra Classification of Scalar n th Order Ordinary Differential Equations

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Abstract

We obtain a complete classification of scalar nth order ordinary differential equations for all subalgebras of vector fields in the real plane. While softwares like Maple can compute invariants of a given order; our results are for a general n. The n=1 ,2 ,3 cases are well-known in the literature. Further, it is known that there are three types of nth order equations depending upon the point symmetry algebra they possess, viz. first-order equations which admit an infinite dimensional Lie algebra of point symmetries, second-order equations possessing the maximum eight point symmetries and higher-order, n[?]3, admitting the maximum n+4 dimensional point symmetry algebra. We show that scalar nth order equations for n>5 do not admit maximally an n+3 dimensional real Lie algebra of point symmetries. Moreover, we prove that for n>4 equations can admit two types of n+2 dimensional real Lie algebra of point symmetries: one type resulting in nonlinear equations which are not linearizable via a point transformation and the second type yielding linearizable (via point transformation) equations. Furthermore, we present the types of maximal real n dimensional and higher than n dimensional point symmetry algebras admissible for equations of order n[?]4 and their canonical forms. The types of lower dimensional point symmetry algebras which can be admitted are shown and the equations are constructible as well. We state the relevant results in tabular form and in theorems.

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Abstract: We obtain a complete classification of scalar nth order ordinary differential equations for all subalgebras of vector fields in the real plane. While softwares like Maple can compute invariants of a given order; our results are for a general n. The $n = 1, 2, 3$ cases are well-known in the literature. Further, it is known that there are three types of nth order equations depending upon the point symmetry algebra they possess, viz. first-order equations which admit an infinite dimensional Lie algebra of point symmetries, second-order equations possessing the maximum eight point symmetries and higher-order, $n \geq 3$, admitting the maximum $n+4$ dimensional point symmetry algebra. We show that scalar nth order equations for $n > 5$ do not admit maximally an $n + 3$ dimensional real Lie algebra of point symmetries. Moreover, we prove that for $n > 4$ equations can admit two types of $n + 2$ dimensional real Lie algebra of point symmetries: one type resulting in nonlinear equations which are not linearizable via a point transformation and the second type yielding linearizable (via point transformation) equations. Furthermore, we present the types of maximal real n dimensional and higher than n dimensional point symmetry algebras admissible for equations of order $n \geq 4$ and their canonical forms. The types of lower dimensional point symmetry algebras which can be admitted are shown and the equations are constructible as well. We state the relevant results in tabular form and in theorems.

Keywords: Lie Symmetry Classification of ODEs, Symmetry Lie Algebras, Invariants.

1 Introduction

We obtain a complete classification of scalar nth order ordinary differential equations for all subalgebras of vector fields in the real plane. While softwares like Maple can compute invariants of a given order; our results are for a general n. The precise results are given in section 7 as theorems (7.1 to 7.5).

Symmetry Lie algebras of scalar nth order ordinary differential equations (ODEs) have been extensively studied over several years since the initial ground breaking works of Lie [1, 2, 3]. Lie [3], inter alia, provided all continuous groups of transformations in the complex plane. He emphasized that this can form the basis of classification as well as reduction of scalar nth order ODEs which he implicitly performed.

The classification of Lie algebras in terms of vector fields are essential in the algebraic study of scalar nth order ODEs which possess infinitesimal symmetries, both symmetry classification and reduction algorithms. After the works of Lie, there have been much interest in this area. Lie algebras of vector fields in the real plane are completely classified in González-López et. al. [4]. Recently, a proof of Lie's classification of solvable Lie algebras of vector fields in the plane is presented (see [5]). Of essence also, is the contribution [6] which refers to contemporary works on Lie algebra realizations as well as equivalence of realizations.

In recent years there have been much focus on the Lie algebra classification of ODEs in several works as we refer to henceforth. Equations of order one, have infinite Lie point symmetries and are equivalent to each other via point transformation. For scalar higher order ODEs, Lie [1] proved that the maximum dimension of the point symmetry Lie algebra for a scalar second order ODE is eight dimensional and occurs for linear and linearizable by point transformation equations. Lie [2] obtained a complex classification of second order ODEs in terms of their point symmetry Lie algebras. Mahomed and Leach [7] derived the real classification and showed that a second order equation can admit 0, 1, 2, 3 or the maximum 8 dimension real point symmetry Lie algebra. The original Lie classification and the classification in the real domain [7] for second order ODEs are, inter alia, compared in [8]. Algebraic linearizability criteria were initiated by Lie himself [1], who showed that such second order equations possessing a Lie algebra of dimension 2 and of rank one, are linearizable via point transformations. This falls under Type II and Type IV Lie canonical forms in Lie's classification. The Types I and III cases with focus on linearizability were achieved in [9] and [10]. The reader is also referred to the survey [11].

In the study of scalar linear ODEs of order n, $n \geq 3$, Mahomed and Leach [12] (see also [8], [11] as well as the contribution by Krause and Michel [13]) demonstrated that the point symmetry algebra can be $n+1, n+2$ or $n+4$. Thus, for $n \geq 3$, scalar linear ODEs are not necessarily equivalent to each other via point transformation. Moreover, for $n \geq 3$, there exist linear as well as nonlinearizable ODEs with $n+2$ and $n+3$ symmetry algebras [12]. It is important to remark that second order ODEs are quite different to higher order equations $n \geq 3$ as per the point symmetry algebras they admit. Apart, from the maximum dimension of the point symmetry algebra being 2+6 for second order ODEs and that for higher order, $n \geq 3$, equations $n + 4$ (see Lie [1, 2]), there are two more notable differences. Secondly, that all linear second order ODEs are equivalent to the free particle equation whereas a linear higher order $n \geq 3$ ODE has three equivalence classes depending upon whether it has $n + 1, n + 2$ or the maximum number $n + 4$ of point symmetries [12]. Thirdly, the complete or full algebra of point symmetries of a second order ODE is a subalgebra of its maximum algebra $s1(3, R)$, whereas the full algebra of a higher order $n \geq 3$ ODE is not necessarily a subalgebra of its maximum Lie point symmetry algebra [12].

The point symmetry Lie algebra classification of third order ODEs as well as linearization by point transformation have been investigated in a number of relevant publications (see [12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22]). Furthermore, integrability and reductions for third order ODEs were investigated in [23].

Scalar fourth order ODEs were considered in recent works from the point of view of Lie point symmetry classification in terms of four-dimensional algebras, canonical forms as well as integrability (see [24, 25, 26]). A complete Lie point symmetry classification and algebraic linearization were also attempted [27]. Linearization criteria, by point transformation, for such ODEs were found as well [28].

The classification of scalar nth order ODEs which possess nontrivial irreducible contact Lie symmetry algebras was completed in the work Wafo et al [29]. For scalar third order ODEs, the contact symmetry algebra is a subalgebra of the ten dimensional contact symmetry algebra of $y''' = 0$ except for linear equations that admit four and five dimensional point symmetry algebras [29]. It is shown in this work that fourth order scalar ODEs do not admit irreducible contact symmetry algebras. Further it was proved in this paper [29] that there are three types of contact symmetry algebras (of dimensions 6, 7 and 10) admissible for nth order ODEs for $n > 4$, up to local contact transformations. The present work is a natural generalization of previous contributions on the classification of scalar 2nd and 3rd order ODEs. Here we present the complete classification of nth order, $n \geq 4$, ODEs according to the Lie point symmetry algebras they admit. We explicitly present the maximal n dimensional point symmetry Lie algebra admissible and the representative or canonical equations as well as higher dimensional point symmetry algebras and canonical ODEs. We also have shown how one can obtain lower dimensional point symmetry algebras and determine the corresponding ODEs.

The second section deals with notation which are well-known in books (see e.g. [30, 31, 32, 33, 34]) which we have utilised in [27] as well as provide a overview of the methods used herein. In the third and fourth sections we classify all equations of order n which admit $n + 1$ and $n + 2$ dimensional Lie symmetry algebras, respectively, including those that are linearizable by point transformation. Then in section five we find the $n - 1$ and n dimensional fundamental invariants of algebras of dimension n. In section 6 we discuss admissibility for $n + 3$ dimensional algebras. Thereafter, in section 7, we obtain a complete classification for $n \geq 4$ for the maximal n, lower as well as higher symmetry algebra cases. We finally present concluding remarks.

2 Notation and Methodology

We utilize the vector fields in the plane presented in the important work [4] for the real Lie symmetry classification of scalar nth order ODEs performed in this seminal study.

By (m, n) we denote the real algebra realization, where m is the type of algebra in [4] and n is the dimension of the Lie algebra. One writes a general vector field or generator in the real plane as

$$X_i = \xi_i(x, y)\partial_x + \eta_i(x, y)\partial_y, \qquad i = 1, ..., n.$$

Here $(x, y) \in \mathbb{R}^2$ and ∂_x denotes $\partial/\partial x$. Note also that n is the dimension of a Lie algebra of which X_is are the generators. Other notations will be introduced as they arise in the sequel.

Now let L be an m dimensional Lie subalgebra of vector fields in the real plane. To find an invariant equation of order n we as usual consider the normal form of an nth order ODE

$$y^{(n)} = H(x, y, y', y'', ..., y^{(n-1)}). \tag{2.1}$$

For a generator X to be a symmetry of (2.1), the symmetry condition

$$X^{[n]}(y^{(n)} - H)|_{y^{(n)}=H} = 0 \tag{2.2}$$

must hold on the equation, where $X^{[n]}$ denotes the nth prolongation. If the condition (2.2) is satisfied for every X_i, $i = 1, 2, ..., m$, then the resulting nth order equation is said to be invariant under L and L is called the symmetry Lie algebra of the equation.

Further, let L be an m dimensional Lie subalgebra of vector fields defined on a subspace $\mathbb{D} \subset \mathbb{R}^2$. Then the nth order prolonged Lie algebra is defined on a subspace $\mathbb{D}^{(n)} \subset \mathbb{R}^{n+2}$. Suppose that r_0 is the rank of L in $\mathbb{D}$. Then r_n will be the rank of the prolonged L in $\mathbb{D}^{(n)}$. The rank here means the rank of the matrix whose rows are coefficients of the m generating vector fields of L.

If we denote d_n to be the number of differential invariants of order n, then we have

$$d_n = n + 2 - r_n, \quad n \geq 0$$

Example 2.1. *Consider simply $X = \partial_x$. Here $n = 0$ and $r_o = 1$. Thus $d_0 = 0+2-1 = 1$, so we have one zeroth order differential invariant which is $u = y$.*

Example 2.2. *Now we take $X_1 = \partial_x$, $X_2 = \partial_y$. Again $n = 0$, $r_o = 2$ and $d_0 = 0+2-2 = 0$, and there is no zeroth order invariant. Now for the first prolongations of X_1 and X_2, $n = 1$, $r_1 = 2$ and $d_1 = 1 + 2 - 2 = 1$. Thus, there is a first order differential invariant $u = y'$.*

We now mention invariant differentiation and its operator. If u, v are invariants, then by Lie's theorem, D_xv/D_xu is also an invariant. This process is also called invariant differentiation (see e.g. [31]). Recall that D_x is the total differentiation operator. We can write this as

$$\begin{aligned}\frac{D_xv}{D_xu} &= (D_xu)^{-1}D_xv \\ &= \lambda D_xv\end{aligned}$$

We call $\lambda D_x = \mathcal{D}$ the invariant differentiation operator once we know λ.

Suppose we have an unknown $\lambda(x, y, y', ..., y^{(n)})$. We require λD_xv to be invariant, and therefore we need

$$X^{[n]}(\lambda D_xv) = 0. \tag{2.3}$$

If we refer to $X^{[n]}$ as X by ignoring the prolongation sign, then we can simply write

$$X = \overline{X} + \xi D_x,$$

where $\overline{X} = w\partial_y + D_xw\partial_{y'} + D_x^2w\partial_{y''} + ... + D_x^nw\partial_{y^{(n)}}$ is called the canonical operator and $w = \eta - y'\xi$. The equation (2.3) then becomes

$$\begin{gathered}(\overline{X} + \xi D_x)(\lambda D_xv) = 0 \\ \Rightarrow \xi D_x(\lambda D_xv) + \overline{X}(\lambda)D_xv + \lambda\overline{X}(D_xv) = 0 \\ \Rightarrow \xi D_x\lambda D_xv + \xi\lambda D_x^2v + (X\lambda - \xi D_x\lambda)D_xv + \lambda\overline{X}(D_xv) = 0 \\ \Rightarrow \xi D_x\lambda D_xv + \lambda\xi D_x^2v + X(\lambda)D_xv - \xi D_x\lambda D_xv + \lambda\overline{X}(D_xv) = 0 \\ \Rightarrow \xi\lambda D_x^2v + X(\lambda)D_xv + \lambda D_x(Xv - \xi D_xv) = 0 \\ \Rightarrow (X(\lambda) - \lambda D_x\xi)D_xv = 0\end{gathered}$$

This yields the known result (see [31])

$$X(\lambda) = \lambda D_x\xi. \tag{2.4}$$

Hence, λ satisfies the non-homogenous linear PDE (2.4). One only requires one nontrivial solution for λ which can be a constant as well.

Example 2.3. *Let $X_1 = \partial_x$ and $X_2 = \partial_y$. We know that $u = y'$ is a first order differential invariant of X_1 and X_2. Applying the condition* (2.4) *we have*

$$X_1\lambda = 0\ ,\ X_2\lambda = 0$$

This clearly shows that we can set $\lambda = 1$ and therefore the invariant differentiation operator can be taken as $\mathcal{D} = (1)D_x = D_x$. Thus

$$\mathcal{D}y' = D_xy' = y''$$

is a second order differential invariant.

We briefly consider Lie determinants.

Definition 2.1. *Consider an m-dimensional Lie subalgebra of vector fields whose generators are given as*

$$X_k = \xi_k \partial_x + \eta_k \partial_y,\ k = 1, 2, ..., m.$$

Then the determinant of the following matrix

$$\begin{pmatrix} \xi_1 & \eta_1 & \eta_1^{[1]} & \eta_1^{[2]} & \cdots & \eta_1^{[m-2]} \\ \xi_2 & \eta_2 & \eta_2^{[1]} & \eta_2^{[2]} & \cdots & \eta_2^{[m-2]} \\ \vdots & \vdots & \ddots & \vdots & & \\ \xi_m & \eta_m & \eta_m^{[1]} & \eta_m^{[2]} & \cdots & \eta_m^{[m-2]} \end{pmatrix}$$

is called the Lie determinant which corresponds to the m dimensional Lie algebra for $m \geq 2$. We denote the Lie determinant by Λ_L.

Lie proved that for an m dimensional Lie subalgebra of vector fields L, the Lie determinant gives rise to all the invariant equations of order $\leq m-2$, [2]. Similarly it can be noticed that the rank of the prolonged algebra is m, i.e. maximal unless the Lie determinant vanishes in which case the rank of L diminishes. Here the algebra is prolonged up to order $m-2$. These invariant equations are called the singular invariant equations of L. The fundamental differential invariants which are not singular must be of order $m-1$ and m and the higher order differential invariants can be then be found from the fundamental invariants.

Note that here, once we find the $(m-1)$th order differential invariant say ϕ and the invariant differentiation operator $\mathcal{D} = \lambda D_x$, we can determine the mth order differential invariant as $\mathcal{D}\phi$.

3 $n+1$ Dimensional Algebras and ODEs

3.1 Nonlinear Equations

It is easy to observe that $(20, n+1)$ is not admitted by an equation as there can only be n solution symmetries. The linearizable case $(21, n+1)$ is considered in subsection 3.2. Also, $(22, n+1)$ and $(23, n+1)$ are not possessed as maximal Lie algebras. We consider the other $n+1$ dimensional algebras. These are as below.

$(24, n+1), r = n-2$, $n \geq 3$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x + \alpha y \partial_y$, $X_4 = x\partial_y$, $X_5 = x^2\partial_y$, ..., $X_{n+1} = x^{n-2}\partial_y$.

The generators, except X_3, imply that an nth order equation of the form (2.1) admitting these generators must be of the form $y^{(n)} = H(y^{(n-1)})$. The nth prolongation of X_3 is: $x\partial_x + \alpha y \partial_y + (\alpha-1)y'\partial_{y'} +, ..., +(\alpha-n)y^{(n)}\partial_{y^{(n)}}$. By applying this to the resultant equation and solving we find the general form of an nth order equation admitting this algebra to be

$$y^{(n)} = K(y^{(n-1)})^{\frac{\alpha-n}{\alpha-n+1}},\ K \neq 0, \tag{3.1}$$

where $K \neq 0$ is an arbitrary constant and $\alpha \neq n-1$. For $K = 0$ or $\alpha = n-1$, it is easy to see that the general form of such an equation admitting this algebra must be $y^{(n)} = 0$, and we know from Lie that this simplest equation admits the maximal $n+4$ dimensional algebra of which such an algebra is a subalgebra.

$(25, n+1), r = n-1,\ n \geq 2$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_y$, $X_4 = x^2\partial_y$, ..., $X_n = x^{n-2}\partial_y$, $X_{n+1} = x\partial_x + (ry + x^r)\partial_y$.

The nth prolongation of X_{n+1} is: $x\partial_x + (ry + x^r)\partial_y + (rx^{r-1} + (r-1)y')\partial_{y'} +, ..., +((r(r-1)...(r-n+1))x^{r-n} + (r-n)y^{(n)})\partial_{y^{(n)}}$.

The invariant equation is

$$y^{(n)} = K \exp\left(\frac{-y^{(n-1)}}{(n-1)!}\right),\ K \neq 0. \tag{3.2}$$

$(26, n+1), r = n-3,\ n \geq 4$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x\partial_y$,..., $X_{n+1} = x^{n-3}\partial_y$.

Here the equation turns out to be (K is constant)

$$y^{(n)} = K\frac{(y^{(n-1)})^2}{y^{(n-2)}},\ K \neq 0, n/(n-1). \tag{3.3}$$

If $K = n/(n-1)$, then there is one more symmetry $X = x^2\partial x + (n-3)xy\partial y$ as discussed in Section 4 as the type (28, $n+2$).

$(27, n+1), r = n-3,\ n \geq 4$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = 2x\partial_x + ry\partial_y$, $X_4 = x^2\partial_x + rxy\partial_y$, $X_5 = x\partial_y$,..., $X_{n+1} = x^{n-3}\partial_y$.

The nth prolongation of X_3 and X_4 are: $X_3 + \sum_{k=1}^{n}(r-2k)y^{(k)}\partial_{y^{(k)}}$ and $X_4 + \sum_{k=1}^{n}(k(r-k+1)y^{(k-1)} + x(r-2k)y^{(k)})\partial_{y^{(k)}}$, respectively.

The equation is (K is constant)

$$y^{(n)} = \frac{n}{n-1}\frac{(y^{(n-1)})^2}{y^{(n-2)}} + K(y^{(n-2)})^{\frac{n+3}{n-1}},\ K \neq 0. \tag{3.4}$$

If $K = 0$, then there is one more symmetry $X = y\partial y$ as pursued in section 4.

$(28, n+1), r = n-4$, $n \geq 5$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x^2\partial_x + (n-4)xy\partial_y$, $X_6 = x\partial_y$, ..., $X_{n+1} = x^{n-4}\partial_y$.

The nth order invariant equation is

$$y^{(n)} = (y^{(n-2)})^3(y^{(n-3)})^{-2}\left[\frac{n(3(n-2)K_1 - 2n+2)}{(n-2)^2} + K((n-2)K_1 - (n-1))^{\frac{3}{2}}\right], \quad (3.5)$$

where $K_1 = y^{(n-3)}y^{(n-1)}(y^{(n-2)})^{-2}$ and K constant.

3.2 Linearizable Equations

Here we consider linearization for higher order $n \geq 3$ equations. We demonstrate how one obtains the linear form. The reader is referred to [12] for details.

We have the algebra with realization $(21, n+1), r = n-1$: $X_1 = \partial_y$, $X_2 = y\partial_y$, $X_3 = \xi_1(x)\partial_y$, ..., $X_{n+1} = \xi_r(x)\partial_y$ which can be simplified by introducing coordinates: $\overline{x} = \xi_1(x)$, $\overline{y} = y$. Ignoring the bars, the generators of this algebra can be transformed to $X_1 = \partial_y$, $X_2 = y\partial_y$, $X_3 = x\partial_y$, $X_4 = \xi_2(x)\partial_y$,..., $X_{n+1} = \xi_r(x)\partial_y$.

Thus we consider

$(21, n+1), r = n-2$: $X_1 = \partial_y$, $X_2 = y\partial_y$, $X_3 = x\partial_y$,$X_4 = \xi_1(x)\partial_y$, ..., $X_{n+1} = \xi_r(x)\partial_y$.

We have the result from [12] which we state as follows.

Proposition 3.1. *(see [12])* $(21, n+1)$ *is the symmetry algebra of the nth,* $n \geq 3$, *order linear homogenous equation*

$$y^{(n)} = \sum_{i=2}^{n-2} A_i(x)y^{(i+1)}, \quad (3.6)$$

such that each ξ_i *for* $i = 1, 2, ..., n-2$, *form independent solutions of this equation, i.e. the* ξ_i*s satisfy the system of homogenous equations*

$$\xi_k^{(n)} = \sum_{i=1}^{n-2} A_i(x)\xi_k^{(i+1)}, \qquad k = 1, ..., n-2. \quad (3.7)$$

Note that if A_is are constant, then a further symmetry ∂_x arises and one gets the algebra (23, $n+2$) as discussed in the next section 4.2. Furthermore, if the A_is satisfy the conditions (3.20) and (3.21) in [12], then one has the maximal algebra (28, $n+4$) admitted as also looked at in section 4.2.

4 $n+2$ Dimensional Algebras and Representative ODEs

4.1 Nonlinear Equations

For the $n+2$ dimensional algebras, $(20, n+2)$, $(21, n+2)$ and $(22, n+2)$ are clearly not admissible algebras. The linearization case $(23, n+2)$ is looked at in section 4.2.

$(24, n+2), r = n-1$, $n \geq 2$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x + \alpha y\partial_y$, $X_4 = x\partial_y$, $X_5 = x^2\partial_y$, ..., $X_{n+2} = x^{n-1}\partial_y$.

$$\Lambda_L = 1 \cdot 2! \cdot 3! ... (n-1)!(\alpha - n)y^{(n)}.$$

$(25, n+2), r = n$, $n \geq 1$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_y$, $X_4 = x^2\partial_y$, ..., $X_{n+1} = x^{n-1}\partial_y$, $X_{n+2} = x\partial_x + (ry + x^r)\partial_y$.

$$\Lambda_L = 1 \cdot 2! \cdot 3! .... (n-2)! \cdot (n-1)! \cdot n!$$

$(26, n+2), r = n-2$, $n \geq 3$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x\partial_y$,..., $X_{n+2} = x^{n-2}\partial_y$.

$$\Lambda_L = 1 \cdot 2! \cdot 3! ... (n-3)! \cdot (n-2)! y^{(n-1)} y^{(n)}.$$

$(27, n+2), r = n-2$, $n \geq 3$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = 2x\partial_x + ry\partial_y$, $X_4 = x^2\partial_x + rxy\partial_y$, $X_5 = x\partial_y$,..., $X_{n+2} = x^{n-2}\partial_y$.

$$\Lambda_L = 1 \cdot 2! \cdot 3! ... (n-2)! \cdot n^2 (y^{(n-1)})^2.$$

$(28, n+2), r = n-3$, $n \geq 4$: $X_1 = \partial_x$, $X_2 = x\partial_x$, $X_3 = y\partial_y$, $X_4 = x^2\partial_x + rxy\partial_y$, $X_5 = \partial_y$, $X_6 = x\partial_y$,..., $X_{n+2} = x^{n-3}\partial_y$.

The equation is

$$y^{(n)} = \frac{n}{n-1} \frac{(y^{(n-1)})^2}{y^{(n-2)}}. \tag{4.1}$$

Here we have only one case, the last, which constitute an invariant equation with maximal $n+2$ dimensional real symmetry algebra. The rest do not form an equation or are singular equations with the maximal $n+4$ dimension algebra.

Next we need to again deal with linearization. We also mention (28, $n+4$) as it results in linearization. This is stated at the end of section 4.2.

4.2 Linearizable Equations for $n+2$ Symmetries

We obtain the form for the reduced linear equation that is a consequence of linearizability (see [12]).

$(23, n+2), r = n$: $X_1 = \eta_1(x)\partial_y$, $X_2 = \eta_2(x)\partial_y$, ..., $X_n = \eta_n(x)\partial_y$, $X_{n+1} = y\partial_y$, $X_{n+2} = \partial_x$.

We sate the following proposition (see [12] for details).

Proposition 4.1. *(see [12]) The generators given in* $(23, n+2)$ *form a Lie symmetry algebra of the homogenous constant coefficient equation*

$$y^{(n)} = \sum_{i=0}^{n-1} A_i y^{(i)}. \tag{4.2}$$

if the η_i, $i = 1, \ldots, n$, *form a fundamental set of solutions of the constant coefficient equation itself.*

We conclude by considering the following algebra which is maximal.

$(28,\ n+4)$, $r = n-1$, $n \geq 3$: $X_1 = \partial_x$, $X_2 = x\partial_x$, $X_3 = y\partial_y$, $X_4 = x^2\partial_x + rxy\partial_y$, $X_5 = \partial_y$, $X_6 = x\partial_y$,..., $X_{n+4} = x^{n-1}\partial_y$.

This algebra has as representative, the simplest, nth, $n \geq 3$, order ODE

$$y^{(n)} = 0. \tag{4.3}$$

which admits the maximal $n+4$ dimensional algebra as is well-known from the landmark works of Lie. Further, if (4.2) has A_is satisfying the conditions (3.20) and (3.21) as in [12], then the maximal algebra $(28,\ n+4)$ is admitted by the equation.

5 n Dimensional Algebras and Corresponding Equations

In this section we determine all the fundamental invariants (invariants of order n and $n-1$) for five and higher dimensions as well as present the representative ODEs. Note that the algebras $(20, n)$ and $(21, n)$ are not admissible as maximal symmetry algebras.

$(5, 5)$: The generators of this algebra are $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x - y\partial_y$, $X_4 = y\partial_x$, $X_5 = x\partial_y$.

This is a five-dimensional algebra, hence $r = 5$. We find the 3rd prolongations of the generators of this algebra. Then the Lie determinant, which is the determinant of the matrix

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 \\ x & -y & -2y' & -3y'' & -4y''' \\ y & 0 & -y'^2 & -3y'y'' & -(3y''^2 + 4y'y'''), \end{pmatrix}$$

is $\Lambda_L(M) = 9y''^2$. This shows that $y'' = 0$ is the only equation of order ≤ 3 invariant under this algebra.

The fundamental differential invariants can be found in the usual way. For the given algebra we already know a fourth order invariant

$$\phi_1 = 3y^{(4)}y''^{\frac{-5}{3}} - 5y'''^2y''^{-8/3}.$$

Thus, a fourth order ODE $\phi_1 = 3K$, K a constant, has (5,5).

To find a 5th order differential invariant we can solve the determining system of linear partial differential equations $X_i^{[5]}(y^{(5)} - H)|_{eq} = 0, \quad$ for $i = 1, 2, \cdots, 5$. The solution easily results in the 5th order equation

$$y^{(5)} = \frac{-40}{9}y'''^3y''^{-2} + 5y'''y^{(4)}y''^{-1} + y''^2H(\phi_1) \tag{5.1}$$

invariant under this algebra, where ϕ_1 is the fourth order invariant of the algebra and H is an arbitrary function of its argument. We can thus write the fifth order differential invariant to be

$$\phi_2 = (y''^2y^{(5)} + \tfrac{40}{9}y'''^3 - 5y''y'''y^{(4)})/y''^4.$$

Here ϕ_1 and ϕ_2 are the fundamental differential invariants of this algebra. All higher order differential invariants can be deduced by Lie's invariant differentiation $\phi_{n+1} = D_x(\phi_n)/D_x(\phi_{n-1})$ or by using the invariant derivative operator $\mathcal{D} = y''^{-1/3}D_x$ so that higher order equations possessing this algebra are

$$\mathcal{D}^{n-4}\phi_1 = H(\phi_1, \ldots, \mathcal{D}^{n-5}\phi_1),\ n \geq 5, \tag{5.2}$$

where D_x is the total derivative operator.

$(15, 5)$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x^2\partial_x$.

The generators X_1 to X_3 and X_5 gives rise to the third order invariant

$$K_1 = y'^{-3}y''' - \frac{3}{2}y'^{-4}y''^2$$

and the invariant differentiation operator is $y'^{-1}D_x$.

Now writing X_4 in terms of K_1, $K_2 = y'^{-1}D_xK_1$ and $K_3 = y'^{-1}D_xK_2$ results in

$$\tilde{X}_4 = -2K_1\partial/\partial K_1 - 3K_2\partial/\partial K_2 - 4K_3\partial/\partial K_3.$$

This provides the invariants

$$\phi_1 = K_2K_1^{-3/2}, \quad \phi_2 = y'^{-1}K_1^{-2}D_xK_2.$$

We can evaluate K_2 as

$$K_2 = y'^{-4}y^{(4)} - 6y'^{-5}y''y''' + 6y'^{-6}y''^3.$$

Also, similarly D_xK_2. Thus, one has the fundamental invariants

$$\phi_1 = \frac{y'^2y^{(4)} - 6(y'y''y''' - y''^3)}{(y'y''' - \frac{3}{2}y''^2)^{\frac{3}{2}}}$$

and

$$\phi_2 = \frac{y'^3y^{(5)} - 10y'^2y''y^{(4)} - 6y'^2y'''^2 + 48y'y''^2y''' - 36y''^4}{(y'y''' - \frac{3}{2}y''^2)^2}$$

with the 5th order equation as

$$\frac{y'^3y^{(5)} - 10y'^2y''y^{(4)} - 6y'^2y'''^2 + 48y'y''^2y''' - 36y''^4}{(y'y''' - \frac{3}{2}y''^2)^2} = H(\phi_1) \tag{5.3}$$

and with $\mathcal{D} = y'^{-1}K_1^{-1/2}D_x$ we can invoke (5.2) for higher order invariant equations.

Note that a 4th order ODE admitting (15,5) is $\phi_1 = K$, K constant.

$(6,6)$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_x$, $X_5 = x\partial_y$, $X_6 = y\partial_y$.

The generators X_1, X_2, X_5 and X_6 result in the invariants

$$K_1 = 3y''^{-1}y^{(4)} - 5y'''^2y''^{-2},$$

$$K_2 = 9y''^{-1}y^{(5)} - 45y''^{-2}y'''y^{(4)} + 40y''^{-3}y'''^3.$$

Writing X_4 in terms of K_1 and K_2 we find (up to scaling)

$$\tilde{X}_4 = 2K_1\partial_{K_1} + 3K_2\partial_{K_2}$$

which provides the 5th order invariant of $(6,6)$

$$\begin{aligned} \phi_1 &= K_1^{-3/2}K_2, \\ &= \frac{9y''^2y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3}{(3y''y^{(4)} - 5y'''^2)^{\frac{3}{2}}}. \end{aligned}$$

The invariant differentiation operator is $\mathcal{D} = K_1^{-1/2}D_x$ and hence one can derive the sixth order invariant $K_1^{-3}(-\frac{3}{2}K_2D_xK_1 + K_1D_xK_2)$.

The 5th order invariant equation is

$$K_1^{-3/2}K_2 = \frac{9y''^2y^{(5)} - 45y''y'''y^{(4)} + 40y'''^3}{(3y''y^{(4)} - 5y'''^2)^{\frac{3}{2}}} = K, \tag{5.4}$$

where K is constant and with $\mathcal{D} = y''(3y''y^{(4)} - 5y'''^2)^{-1/2}D_x$, we have

$$\mathcal{D}^{n-5}\phi_1 = H(\phi_1, \ldots, \mathcal{D}^{n-6}\phi_1),\ n \geq 6, \tag{5.5}$$

for sixth and higher order equations. Thus a sixth order invariant ODE is

$$K_1^{-3}(-\frac{3}{2}K_2D_xK_1 + K_1D_xK_2) = H(K_1^{-3/2}K_2). \tag{5.6}$$

Note that $K_1 = 0$ is the fourth order singular invariant equation having this algebra.

$(16,6)$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x^2\partial_x$, $X_6 = y^2\partial_y$.

Here the invariants, using X_1, X_2, X_4 and X_6, are

$$K_1 = y'''y^{-1} - \frac{3}{2}y'^{-2}y''^2,$$

as well as $K_2 = D_xK_1$ and $K_3 = D_xK_2$.

Utilising X_5 in (K_1, K_2, K_3) space, we end up with, up to scaling,

$$\tilde{X}_5 = 2xK_1\partial_{K_1} + (3xK_2 + 2K_1)\partial_{K_2} + (4xK_3 + 5K_2)\partial_{K_3}.$$

Hence, a 5th order invariant transpires as $\phi_1 = K_1^{-3}(5(D_xK_1)^2 - 4K_1D_x^2K_1)$
and invariant 5th order ODE is (K constant)

$$K_1^{-3}(5(D_xK_1)^2 - 4K_1D_x^2K_1) = K. \tag{5.7}$$

The invariant differentiation operator is $\mathcal{D} = K_1^{-1/2}D_x$ and one can deduce the 6th order invariant. Higher order invariant equations are obtained as in (5.2). The 6th order invariant ODE is therefore

$$K_1^{-1/2}D_x(K_1^{-3}(5(D_xK_1)^2 - 4K_1D_x^2K_1)) = H(K_1^{-3}(5(D_xK_1)^2 - 4K_1D_x^2K_1). \tag{5.8}$$

We remark that $K_1 = 0$ is the third order singular invariant equation admitting this algebra.

$(7,6)$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x + y\partial_y$, $X_4 = y\partial_x - x\partial_y$, $X_5 = (x^2 - y^2)\partial_x + 2xy\partial_y$, $X_6 = 2xy\partial_x + (y^2 - x^2)\partial_y$.

The generators X_1 to X_4 gives rise to the invariant

$$K_1 = (1 + y'^2)y''^{-2}y''' - 3y',$$

as well as the 4th and 5th order invariants

$$K_2 = y''^{-1}(1 + y'^2)D_xK_1, \quad K_3 = y''^{-1}(1 + y'^2)D_xK_2.$$

Now X_5 in (K_1, K_2, K_3) space yields

$$\tilde{X}_5 = (1+y'^2)y''^{-1}[-4y'K_1\partial_{K_1} + (4y'K_1^2 - 6y'K_2 - 4K_1)\partial_{K_2}$$

$$+(14y'K_1K_2 + 4y'K_1 - 4y'K_1^3 - 8y'K_3 - 10K_2 + 8K_1^2)\partial_{K_3}].$$

The invariant deduced is of order five (which is admitted by X_6 as well since $[X_4, X_5] = X_6$) given by $\phi_1 = K_1^{-3}(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2)$. The fifth order invariant ODE hence is

$$K_1^{-3}(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2) = K. \tag{5.9}$$

The operator of invariant differentiation is $\mathcal{D} = (1+y'^2)y''^{-1}K_1^{-1/2}D_x$ and sixth and higher order equations are given by (5.5). One has the 6th order invariant ODE

$$(1+y'^2)y''^{-1}K_1^{-1/2}D_x(K_1^{-3}(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2))$$

$$= H(K_1^{-3}(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2). \tag{5.10}$$

Here $K_1 = 0$ is the third order singular invariant equation.

$(8,8)$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_y$, $X_4 = y\partial_y$, $X_5 = y\partial_x$, $X_6 = x\partial_x$, $X_7 = x^2\partial_x + xy\partial_y$, $X_8 = xy\partial_x + y^2\partial_y$.

The operators X_1 to X_6 result in the 5th order invariant of the algebra (6,6), viz.

$$J_1 = K_1^{-3/2}K_2,$$

where K_1 and K_2 are as in (6,6).

The operator X_7 in terms of J_1 and its invariant derivatives $J_2 = K_1^{-1/2}D_xJ_1$ and $J_3 = K_1^{-1/2}D_xJ_2$ turns out to be

$$\tilde{X}_7 = -9K_1^{-1}y''^{-1}y'''J_1\partial/\partial_{J_1} + (-12K_1^{-1}J_2y''^{-1}y''' + 3J_1^2K_1^{-1}y''^{-1}y''' - 3J_1K_1^{-1/2})\partial/\partial_{J_2}$$

$$+(10J_1J_2K_1^{-1}y''^{-1}y''' - J_1^3K_1^{-1}y''^{-1}y''' + J_1K_1^{-1}y''^{-1}y'''$$

$$-15J_3K_1^{-1}y''^{-1}y''' - 7J_2K_1^{-1/2} + \frac{3}{2}J_1^2K_1^{-1/2})\partial/\partial_{J_3}$$

which yields the invariant of order seven of this algebra (8,8) (note $[X_5, X_7] = X_8$) as $\phi_1 = 12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3}$ and thus the 7th order invariant ODE is (K is a constant)

$$12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3} = K \tag{5.11}$$

The invariant differential operator of this algebra is $\mathcal{D} = K_2^{-1/3}D_x$.

Therefore, eight order ODEs admitting this eight dimensional algebra is given by

$$K_2^{-1/3} D_x(12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3}) = H(\phi_1) \qquad (5.12)$$

and higher order by invariant differentiation as $\mathcal{D}^{n-7}\phi_1 = H(\phi_1, \ldots, \mathcal{D}^{n-8}\phi_1)$, $n \geq 8$.

For this algebra the singular invariant equation is $K_2 = 0$ which is fifth order. This is also given in section 6.

$(22, n)$: $r = n-1$, $n \geq 2$: $X_1 = \partial_x$, $X_2 = \eta_1(x)\partial_y$, $\ldots$, $X_n = \eta_{n-1}(x)\partial_y$.

The nth order equation is

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1}y' = H(y^{(n-1)} + a_1 y^{(n-2)} + \cdots + a_{n-1}y), \qquad (5.13)$$

where the a_is are constants and the η_is satisfy the linear $(n-1)$th order equation

$$\eta_i^{(n-1)} + a_1\eta_i^{(n-2)} + \cdots + a_{n-1}\eta_i = 0,\ i = 1, \ldots, n-1.$$

$(23, n)$: $r = n-2$, $n \geq 3$: $X_1 = \partial_x$, $X_2 = y\partial_y$, $X_3 = \eta_1(x)\partial_y$,...,$X_n = \eta_{n-2}(x)\partial_y$.

The nth order equation that admits this algebra is

$$\frac{D_x^2(y^{(n-2)} + a_1 y^{(n-3)} + \cdots + a_{n-2}y)}{y^{(n-2)} + a_1 y^{(n-3)} + \cdots + a_{n-2}y} = H(D_x \ln|y^{(n-2)} + a_1 y^{(n-3)} + \cdots + a_{n-2}y|), \qquad (5.14)$$

where the a_is are constants and the η_is satisfy the linear $(n-2)$th order equation

$$\eta_i^{(n-2)} + a_1\eta_i^{(n-3)} + \cdots + a_{n-2}\eta_i = 0,\ i = 1, \ldots, n-2.$$

$(24, n)$: $r = n-3$, $n \geq 4$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x + \alpha y\partial_y$, $X_4 = x\partial_y$,..., $X_n = x^{n-3}\partial_y$.

Fundamental invariants, if $\alpha \neq n-1$, are

$$\phi_1 = (y^{(n-1)})^{\alpha-n+2}(y^{(n-2)})^{-(\alpha-n+1)}, \qquad \phi_2 = y^{(n)}(y^{(n-1)})^{-(\frac{\alpha-n}{\alpha-n+1})},$$

and for $\alpha = n-1$

$$\phi_1 = y^{(n-1)}, \qquad \phi_2 = y^{(n)}y^{(n-2)}.$$

Thus the invariant equation is

$$y^{(n)}(y^{(n-1)})^{-(\frac{\alpha-n}{\alpha-n+1})} = H((y^{(n-1)})^{\alpha-n+2}(y^{(n-2)})^{-(\alpha-n+1)}),\ \alpha \neq n-1 \qquad (5.15)$$

If $\alpha = n-1$, the invariant equation simply is

$$y^{(n)} = (y^{(n-2)})^{-1}H(y^{(n-1)}). \qquad (5.16)$$

$(25, n)$: $r = n-2$, $n \geq 3$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_y$, $X_4 = x^2\partial_y, \cdots, X_{n-1} = x^{n-3}\partial_y$, $X_n = x\partial_x + ((n-2)y + x^{(n-2)})\partial_y$.

The fundamental differential invariants are

$$\phi_1 = y^{(n-1)} \exp \frac{y^{(n-2)}}{(n-2)!}, \ \phi_2 = y^{(n)} \exp \frac{2y^{(n-2)}}{(n-2)!}$$

and invariant equation

$$y^{(n)} \exp \frac{2y^{(n-2)}}{(n-2)!} = H(y^{(n-1)} \exp \frac{y^{(n-2)}}{(n-2)!}). \tag{5.17}$$

$(26, n)$: $r = n-4$, $n \geq 5$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x\partial_y, \cdots, X_n = x^{n-4}\partial_y$.

The fundamental differential invariants:

$$\phi_1 = \frac{y^{(n-3)}y^{(n-1)}}{(y^{(n-2)})^2}, \quad \phi_2 = \frac{(y^{(n-3)})^2 y^{(n)}}{(y^{(n-2)})^3}$$

and invariant ODE

$$\frac{(y^{(n-3)})^2 y^{(n)}}{(y^{(n-2)})^3} = H(\frac{y^{(n-3)}y^{(n-1)}}{(y^{(n-2)})^2}). \tag{5.18}$$

$(27, n)$: $r = n-4$, $n \geq 5$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = 2x\partial_x + ry\partial_y$, $X_4 = x^2\partial_x + rxy\partial_y$, $X_5 = x\partial_y, \cdots, X_n = x^{n-4}\partial_y$.

The fundamental invariants:

$$\phi_1 = y^{(n-1)}(y^{(n-3)})^{\frac{n+2}{2-n}} - \frac{n-1}{n-2}(y^{(n-2)})^2(y^{(n-3)})^{\frac{-2n}{n-2}}$$

$$\phi_2 = y^{(n)}(y^{(n-3)})^{\frac{n+4}{2-n}} + \frac{2n(n-1)}{(n-2)^2}(y^{(n-2)})^3(y^{(n-3)})^{\frac{3n}{2-n}} + \frac{3n}{2-n}y^{(n-1)}y^{(n-2)}(y^{(n-3)})^{\frac{2(n+1)}{2-n}}.$$

The invariant equation here is

$$y^{(n)}(y^{(n-3)})^{\frac{n+4}{2-n}} + \frac{2n(n-1)}{(n-2)^2}(y^{(n-2)})^3(y^{(n-3)})^{\frac{3n}{2-n}}$$
$$+\frac{3n}{2-n}y^{(n-1)}y^{(n-2)}(y^{(n-3)})^{\frac{2(n+1)}{2-n}} = H(\phi_1). \tag{5.19}$$

$(28, n)$: $r = n-5$, $n \geq 6$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_y, \cdots, X_{n-3} = x^{n-5}\partial_y$, $X_{n-2} = x\partial_x$, $X_{n-1} = y\partial_y$, $X_n = x^2\partial_x + rxy\partial_y$.

The fundamental invariants:

$$\phi_1 = \frac{(n-3)^2K_2 - 3(n-1)(n-3)K_1 + 2(n-1)(n-2)}{(n-3)^2((n-3)K_1-(n-2))^{\frac{3}{2}}},$$

$$\phi_2 = \frac{(n-3)^3K_3 - 4n(n-3)^2K_2 + 6n(n-1)(n-3)K_1 - 3n(n-1(n-2)}{(n-3)^3((n-3)K_1-(n-2))^2}$$

where

$$K_i = \frac{y^{(n-3+i)}(y^{(n-4)})^i}{(y^{(n-3)})^{i+1}},\ i=1,2,3.$$

The invariant ODE is

$$\frac{(n-3)^3K_3 - 4n(n-3)^2K_2 + 6n(n-1)(n-3)K_1 - 3n(n-1(n-2)}{(n-3)^3((n-3)K_1-(n-2))^2} = H(\phi_1). \quad (5.20)$$

6 $n+3$ Dimensional Algebras and Equations

Scalar first order equations admit infinite number of point symmetries. A second order equation does not admit a five dimensional symmetry algebra, as is well-known. The only third order equations that admit 6 dimensional algebras are

$$y''' = \frac{3y''^2}{2y'} \text{ and } y''' = \frac{3y'y''^2}{1+y'^2}, \quad (6.1)$$

where the algebras are $\mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R)$ and $\mathfrak{so}(3,1)$, respectively. These are the algebras (16,6) and (7,6) as stated in the previous section 5. These are also known from the initial seminal works of Lie.

In [27], it was shown that a fourth order equation does not admit a maximal 7 dimensional algebra.

For fifth order equations, the only equation which admits an 8 dimensional algebra is

$$y^{(5)} = \frac{5y'''y^{(4)}}{y''} - \frac{40y'''^3}{9y''^2}, \quad (6.2)$$

whose symmetry algebra is $\mathfrak{sl}(3,R)$ (also referred to as (8,8)) as stated in the previous section.

We check all the possible algebras of dimension $n+3$ for $n \geq 5$. Since an nth order linear equation cannot have more than n independent solutions, the algebras $(m, n+3)$ for $m = 20, \cdots, 25$ are not admitted. We discuss the remaining possible algebras:

$(26, n+3)$: $X_1 = \partial_x$, $X_2 = \partial_y$, $X_3 = x\partial_x$, $X_4 = y\partial_y$, $X_5 = x\partial_y, \cdots, X_{n+3} = x^{n-1}\partial_y$.

The generators X_1, X_2 and $X_5, \cdots, X_{n+3}$ implies that the equation must be of the form $y^{(n)} = K$. The X_4 then implies that K must vanish. Then X_3 is automatically admitted by this equation. However, the maximal symmetry algebra of this equation is $n+4$. In the same way it can be shown easily that the algebras $(27, n+3)$ and $(28, n+3)$ are admitted by an nth order equation if and only if the equation is equivalent to $y^{(n)} = 0$ and hence these algebras are admitted by the equation but are not maximal. We state the general result on admission of $(n+3)$ dimensional algebra in the next section.

In the following section we review the classification for $n = 4$ and discuss $n \geq 5$ in some detail. Then we present the results on how one can obtain a complete classification for $n \geq 4$. We state relevant theorems of our main results.

7 Classification of Higher , $n \geq 4$, Order ODEs

In this section, we review the main aspects on the classification of 4th order ODEs and then discuss $n \geq 5$. We thus consider the Lie algebraic classification of ODEs of any high order.

In general we can classify scalar ODEs into 3 subclasses as follows:

Subclass (1): nth order equations admitting $n+1$, $n+2$, $n+3$ and the maximal $n+4$ dimensional algebras. These are higher symmetries admitted by a scalar ODE.

Subclass (2): nth order equations admitting n dimensional algebras. For this we have found the fundamental invariants of n dimensional algebras which are of order $n-1$ and n.

Subclass (3): nth order equations admitting algebras of dimension lower than n.

All these subclasses (1), (2) and (3) are completed herein for $n \geq 4$ and the algebra of dimension $n+4$ is already a well-known algebra since the initial work of Lie with corresponding equation $y^{(n)} = 0$ that possesses this maximal dimension Lie algebra. This is the algebra (28,$n+4$) as stated in section 4.2. Also for all subclasses we provide the procedure to obtain the representative ODEs. These are presented in tabular forms and the main results in theorems at the end of this section.

For algebras of dimensions 1 and 2, the algebras and canonical equation are given in Table 1.

The nth order equations admitting 1 and 2 dimensional algebras for $n \geq 3$ are easy to determine as in Table 1.
For any 3 dimensional algebra we already know the invariants of orders 2, 3 and 4 from previous works [7, 8, 15, 16] as well as [27]. Then by invariant differentiation we can find

Algebra	Generators	nth Order Invariant ODE
(9, 1)	∂_x	$y^{(n)} = H(y, y', y'', ..., y^{(n-1)})$
(10, 2)	$\partial_x,\ x\partial_x$	$y^{(n)} = y'^n H(y, y''y'^{-2}, ..., y^{(n-1)}y'^{1-n})$
(20, 2)	$\partial_y,\ x\partial_y$	$y^{(n)} = H(x, y'', ...y^{(n-1)})$
(22, 2)	$\partial_x,\ \partial_y$	$y^{(n)} = H(y', y'', ..., y^{(n-1)})$
$A_{2,1}$	$\partial_x,\ x\partial_x + y\partial_y$	$y^{(n)} = y^{1-n}H(y', y''y, y'''y^2, ..., y^{(n-1)}y^{n-2})$

Table 1: The nth Order Equations Admitting 1 and 2 Dimensional Algebras for $n \geq 3$

higher order invariants up to the required order. Lie's recursive formula may also be used.

We provide the types of 3 D algebras that are possessed by 4th order ODEs. They are
$(1,3):\ \alpha \geq 0,\ \partial_x,\ \partial_y,\ (\alpha x + y)\partial_x + (\alpha y - x)\partial_y,$
$(2,3):\ \partial_y,\ x\partial_x + y\partial_y,\ 2xyp_x + (y^2 - x^2)\partial_y,$
$(3,3):\ y\partial_x - x\partial_y,\ (1 + x^2 - y^2)\partial_x + 2xy\partial_y,\ 2xy\partial_x + (1 + y^2 - x^2)\partial_y,$
$(11,3):\ \partial_x,\ x\partial_x,\ x^2\partial_x,$
$(12,3):\ 0 < \alpha \leq 1,\ \partial_x,\ \partial_y,\ x\partial_x + \alpha y\partial_y,$
$(17,3):\ \partial_y,\ x\partial_x + y\partial_y,\ 2xy\partial_x + (x^2 + y^2)\partial_y,$
$(18,3):\ \partial_y,\ x\partial_x + y\partial_y,\ 2xyp_x + y^2\partial_y,$
$(20,3):\ \partial_y,\ x\partial_y,\ \xi(x)\partial_y,$
$(21,3):\ \partial_y,\ y\partial_y,\ x\partial_y,$
$(22,3):\ \partial_x,\ \eta_1(x)\partial_y,\ \eta_2(x)\partial_y,$
$(23,3):\ \partial_x,\ y\partial_y,\ \partial_y,$
$(25,3):\ \partial_x,\ \partial_y,\ x\partial_x + (x + y)\partial_y,$

The algebras (2,3), (17,3) and (18,3) above are equivalent to the realizations as given in [4] via the transformations $\bar{x} = y, \bar{y} = x$, $\bar{x} = x + y, \bar{y} = y - x$ and $\bar{x} = y, \bar{y} = |x|^{1/2}$, respectively.

We compactly review scalar fourth order ODEs which admit real 3 dimensional Lie algebras as alluded to above. The reader is also referred to [27] for further details. This is important for higher than fourth order symmetry classification of scalar ODEs. We provide the main results in Table 2.

Remark. In Table 2, the ODEs for 4 types of algebras are as follows.

$$
\begin{gathered}
(1,3)\text{: } \phi_4 = y^{(4)}a^2y''^{-3} - 2a^2y''^{-4}y'''^2 + 2ay'y''^{-2}y''' - 3a, \\
(2,3)\text{: } xa^{-1/2}D_x\phi_3 = x^2a^{-5/2}(2y''' + xy^{(4)} - 10xy'y''y'''a^{-1} \\
-6y'y''^2a^{-1} - 3xy''^3a^{-1} + 18xy'^2y''^3a^{-2}), \\
(3,3)\text{: } ra^{-1/2}D_x\phi_3 = r^2a^{-5/2}(4(x + yy')y''' - 10ra^{-1}y'y''y''' + ry^{(4)} \\
-3ra^{-1}y''^3 - 12a^{-1}y'y''^2(x + yy') + 18ra^{-2}y'^2y''^3),
\end{gathered}
$$

Algebra	λ	Invariants ϕ_i	4th Order ODE
(1, 3)	$a^{-1/2}\exp(-\alpha\arctan y')$	$\phi_2 = y''a^{-3/2}\exp(-\alpha\arctan y')$	$\phi_4 = H(\phi_2,\phi_3)$
$\alpha \geq 0$	$a = 1 + y'^2$	$\phi_3 = y''^{-2}y'''a - 3y'$	
(2, 3)	$xa^{-1/2}$	$\phi_2 = a^{-3/2}(xy'' - ay')$	$xa^{-1/2}D_x\phi_3 = H(\phi_2,\phi_3)$
	$a = 1 + y'^2$	$\phi_3 = a^{-3}x^2(ay''' - 3y'y''^2)$	
(3, 3)	$ra^{-1/2}$, $a = 1 + y'^2$	$\phi_2 = a^{-3/2}(ry'' + 2a(y - xy'))$	$ra^{-1/2}D_x\phi_3$
	$r = 1 + x^2 + y^2$	$\phi_3 = r^2a^{-3}(ay''' - 3y'y''^2)$	$= H(\phi_2,\phi_3)$
(11,3)	y'^{-1}	$\phi_0 = y$	$y^{(4)}y'^{-4} + 6y'^{-6}y''^3$
		$\phi_3 = y'^{-4}(2y'y''' - 3y''^2)$	$-6y'^{-5}y''y''' = H(\phi_0,\phi_3)$
(12,3)	$y'^{1/(\alpha-1)}$	$\phi_2 = y''y'^{\frac{2-\alpha}{\alpha-1}}$	$y^{(4)} = y'''^{\frac{\alpha-4}{\alpha-3}}H(\phi_2,\phi_3)$
$\|\alpha\| < 1$	$\alpha \neq 0$	$\phi_3 = y'''y''^{\frac{3-\alpha}{\alpha-2}}$	
$\alpha = 1$	y''^{-1}	$\phi_1 = y', \phi_3 = y''^{-2}y'''$	$y^{(4)} = y'''^{3/2}H(\phi_1,\phi_3)$
(17,3)	$xb^{-1/2}$	$\phi_2 = (xy'' - by')b^{-3/2}$	$xb^{-1/2}D_x\phi_3 = H(\phi_2,\phi_3)$
	$b = 1 - y'^2$	$\phi_3 = x^2b^{-3}(by''' + 3y'y''^2)$	
(18,3)	y^2	$\phi_2 = y^3y''$	$y^7y^{(4)} + 8y'y^6y'''$
		$\phi_3 = y^5y''' + 3y'y''y^4$	$+12y^5y'^2y'' = H(\phi_2,\phi_3)$
(20,3)	1	$\phi_0 = x, \phi_3 = y''' - y''\frac{\xi'''}{\xi''}$	$y^{(4)} = y'''\frac{\xi^{(4)}}{\xi'''} + H(\phi_0,\phi_3)$
(21,3)	1	$\phi_0 = x, \phi_3 = y'''/y''$	$y^{(4)} = y'''H(\phi_0,\phi_3)$
(22,3)	1	$\phi_2 = E(y) = y'' + a_1y' + a_2y$	$D_x\phi_3 = H(\phi_2,\phi_3)$
		$\phi_3 = D_x\phi_2$, η_i satisfy	
		$E(\eta_i) = 0, i = 1,2$, a_i const	
(23,3)	1	$\phi_2 = y''/y', \phi_3 = y'''/y''$	$y^{(4)} = y'''H(\phi_2,\phi_3)$
(25,3)	$\exp y'$	$\phi_2 = y''\exp y', \phi_3 = y''^{-2}y'''$	$y^{(4)} = y'''^{3/2}H(\phi_2,\phi_3)$

Table 2: Scalar 4th Order Equations Admitting Real 3 Dimensional Algebras

$$(17,3):\ xb^{-1/2}D_x\phi_3 = x^2b^{-5/2}(2y''' + xy^{(4)} + 10xb^{-1}y'y''y''' \\ +6y'y''^2b^{-1} + 3xy''^3b^{-1} + 18xb^{-2}y'^2y''^3)$$

The a, b and r are as in Table 2. There are 12 types of real 3 dimensional Lie algebras admitted by scalar 4th order ODEs as listed in Table 2. One can utilise the operator of invariant differentiation $\mathcal{D} = \lambda D_x$ for fourth and higher order ODEs possessing 3 dimensional algebras. Therefore, nth order, $n \geq 4$, scalar ODEs with 3 dimensional symmetry algebras, have the form $\mathcal{D}^{n-3}\phi_3 = H(\phi, \phi_3, \ldots, \mathcal{D}^{n-4}\phi_3)$, $n \geq 4$, where ϕ is ϕ_0, ϕ_1 or ϕ_2 as given in Table 2. Hence, for $n = 4$, we have the 4th order ODE given by $\mathcal{D}\phi_3 = H(\phi, \phi_3)$ as in Table 2 with two arguments in H. In general for higher order ODEs there are $n - 2$ arguments.

We present the list of 4 dimensional algebra types as used in [27]. These are
$(4,4):\ \partial_x,\ \partial_y,\ x\partial_x + y\partial_y,\ y\partial_x - x\partial_y,$
$(13,4):\ \partial_x,\ \partial_y,\ x\partial_x,\ y\partial_y,$
$(14,4):\ \partial_x,\ \partial_y,\ x\partial_x,\ x^2\partial_x,$
$(19,4):\ \partial_x,\ x\partial_x,\ y\partial_y,\ x^2\partial_x + xy\partial_y,$

Also the algebras (22,4), (23,4), (24,4) and (25,4) are easily found by setting $n = 4$ in the general n dimensional cases discussed in section 5.

We now briefly review fourth order equations that possess real 4 dimensional algebras in a compact way than discussed in [27]. This is presented in Table 3.

Algebra	λ	Invariant/s	4th Order ODE
(4, 4)	$(1+y'^2)y''^{-1}$	$y''^{-2}y'''(1+y'^2) - 3y'$	$y^{(4)} = y''^3(1+y'^2)^{-2}(15y'^2$ $+10\phi y' + H(\phi))$
(13, 4)	$y'y''^{-1}$	$y'y''^{-2}y'''$	$y^{(4)} = y'^{-2}y''^3 H(\phi)$, $H \neq 0$
(14, 4)	y'^{-1}	$y'''y'^{-3} - \frac{3}{2}y''^2 y'^{-4}$	$y^{(4)} = y'^4 H(\phi) + 6y'^2 y''\phi$ $+3y'^{-2}y''^3$
(19, 4)	$y^{1/2}y''^{-1/2}$	$y^{1/2}y''^{-3/2}y''' + 3y'y^{-1/2}y''^{-1/2}$	$y^{(4)} = \frac{4}{3}y''^{-1}y'''^2$ $+y^{-1}y''^2 H(\phi)$, $H \neq 0$
(22,4)	1	$E(y) \equiv y''' + a_1 y'' + a_2 y' + a_3 y$ η_i solves $E(\eta_i) = 0$ for a_is const.	$D_x E(y) = H(E(y))$, $H \neq 0$
(23,4)	1	$D_x \ln \lvert E(y) \rvert \equiv D_x \ln \lvert y'' + a_1 y' + a_2 y \rvert$ η_i satisfy $E(\eta_i) = 0$ for a_i const.	$D_x^2 E(y) = E(y)H(\phi)$, $H \neq 0$
(24,4)	$y''^{1/(\alpha-2)}$,	$y'''^{\alpha-2}y''^{3-\alpha}$, $\alpha \neq 3$	$y^{(4)} = y'''^{\frac{\alpha-4}{\alpha-3}} H(\phi)$,$H \neq 0$
	y''	y''', $\alpha = 3$	$y^{(4)} = y''^{-1}H(\phi)$, $H \neq 0$
	y'''^{-1}	$\phi = y''$, $y^{(4)}/y'''^2$, $\alpha = 2$	$y^{(4)} = y'''^2 H(\phi)$, $H \neq 0$
(25,4)	$\exp(y''/2)$	$y''' \exp(y''/2)$	$y^{(4)} = \exp(-y'')H(\phi)$, $H \neq 0$

Table 3: The 4th Order Equations Admitting 4 Dimensional Algebras

Remark. There are multiple cases of (22,4) in [27]. These are considered as a single case in Table 3. We observe that there are 8 types of four dimensional algebras admissible by 4th order ODEs and also there are 7 types of 5 dimensional algebras as listed in Table 4, viz., (5,5), (15,5) (section 5), (21,5) (linear homogeneous ODE, Proposition 3.1), (24,5), (25,5), (26,5) and (27,5) (these 4 are given in section 3.1 for $n = 4$). There are as well 3 types of 6 dimensional algebras (as in Table 4) which are (6,6) (singular invariant ODE stated in section 5), (23,6) (linear constant coefficient ODE, Proposition 4.1) and (28,6) (section 4.1). Further, there is the maximum 8 dimensional case (28,8) (simplest linear ODE, section 4.2 end). Thus, altogether there are 11 types of higher than 4 dimensional algebra types. All these 11 types are mentioned in the previous sections 3, 4 and 5 as indicated and summarised in Table 4.

Note that invariant differentiation of the invariants in the Table 3 will give fifth and higher order invariants. It is also important to include the algebras $(20,m)$ and $(21,m)$ for higher than mth order ODEs. We now focus on these two algebra types. The algebras (20,3) and (21,3) are stated in Table 2 and included here.

Higher Algebra	4th Order ODE
(5, 5)	$y^{(4)} = Ky''^{\frac{5}{3}} + \frac{5}{3}y'''^2y''^{-1}$, $K \neq 0$,
(15, 5)	$\dfrac{y'^2y^{(4)} - 6(y'y''y''' - y''^3)}{(y'y''' - \frac{3}{2}y''^2)^{\frac{3}{2}}} = K$,
(21,5)	$y^{(4)} = \sum_{i=1}^{2} A_i(x)y^{(i+1)}$, ξ_k satisfy $\xi_k^{(4)} = \sum_{i=1}^{2} A_i(x)\xi_k^{(i+1)}$, $k = 1, 2$,
(24, 5)	$y^{(4)} = Ky'''^{\frac{\alpha-4}{\alpha-3}}$, $\alpha \neq 3$ $K \neq 0$,
(25, 5)	$y^{(4)} = K\exp(-y'''/6)$, $K \neq 0$,
(26, 5)	$y^{(4)} = Ky'''^2y''^{-1}$, $K \neq 0, 4/3, 5/3$,
(27, 5)	$y^{(4)} = Ky''^{7/3} + \frac{4}{3}y'''^2y''^{-1}$, $K \neq 0$
(6, 6)	$y^{(4)} = \frac{5}{3}y'''^2y''^{-1}$
(23, 6)	$y^{(4)} = \sum_{i=0}^{3} A_iy^{(i)}$, A_i const., η_i satisfy same equation
(28, 6)	$y^{(4)} = \frac{4}{3}y'''^2y''^{-1}$
(28, 8)	$y^{(4)} = 0$

Table 4: The 4th order invariant ODEs that correspond to their 5 and higher dimensional symmetry algebras. Note that K is a constant and for (21,5) as well as (23, 6), the A_is do not satisfy the maximal symmetry conditions of [12] as mentioned in sections 3.2 and 4.2.

For (20,m): $\partial/\partial y, x\partial/\partial y, \xi_1\partial/\partial y, \ldots, \xi_{m-2}\partial/\partial y$, $3 \leq m < n$, one has $\lambda = 1$ and the invariants $\phi = x$ and $\phi_m = y^{(m)} - a_1y^{(m-1)} - \ldots - a_{m-2}y''$, where ξ_i, $i = 1, \ldots, m-2$ satisfy $\xi_i^{(m)} - a_1\xi_i^{(m-1)} - \ldots - a_{m-2}\xi_i'' = 0$. The nth order ODE with (20,m) is

$$\mathcal{D}^{(n-m)}\phi_m = H(\phi, \phi_m, \ldots, \mathcal{D}^{(n-m-1)}\phi_m) \tag{7.1}$$

with $\mathcal{D} = D_x$.
In the case (21,m): $\partial/\partial y, y\partial/\partial y, x\partial/\partial y, \xi_1\partial/\partial y, \ldots, \xi_{m-3}\partial/\partial y$, $3 \leq m < n$, we have $\lambda = 1$ and invariants $\phi = x$, $\phi_m = D_x \ln|\phi_{m-1}|$, where $\phi_{m-1} = y^{(m-1)} - a_1y^{(m-2)} - \ldots - a_{m-3}y''$ and ξ_i, $i = 1, \ldots, m-3$ satisfy $\xi_i^{(m-1)} - a_1\xi_i^{(m-2)} - \ldots - a_{m-3}\xi_i'' = 0$. The nth order ODE with (21,m) is of the form (7.1) with appropriate ϕ, ϕ_m and $\mathcal{D} = D_x$.

In the case of higher than 4th order ODEs admitting 4 dimensional symmetry algebras we resort to Table 3 and also have $(20, 4)$ and $(21, 4)$. For (20,4), the invariants as above are $\phi = x$, $\phi_4 \equiv E(y) = y^{(4)} - a_1y''' - a_2y''$, where ξ_i, $i = 1, 2$ satisfy $E(\xi_i) = 0$ for a_i constants. One thus obtains the 5th order ODE $D_x\phi_4 = H(\phi, \phi_4)$ which has (20,4). In the case (21,4), the invariants are $\phi = x$ and $\phi_4 = D_x \ln|y''' - y''\xi'''/\xi''|$ and one has again the form $D_x\phi_4 = H(\phi, \phi_4)$. For higher than 4, nth order ODEs with 4 dimensional algebra, we have the form (7.1) with $m = 4$. Therefore for higher nth, $n \geq 5$, order ODEs with 4 dimensional algebras there are 10 types. These follow from Table 3 and inclusion of (20,4) and (21,4). One deduces these by invariant differentiation as (H has $n-3$ arguments) $\mathcal{D}^{n-4}\phi_4 = H(\phi, \phi_4, \ldots, \mathcal{D}^{n-5}\phi_4)$, $n \geq 5$.

We now consider fifth order invariant equations admitting one or more dimensional algebras. The one (single type) and 2 symmetry algebras (4 types) are known as in Table

1. In the case of 3 dimensional algebras, there are 12 types as a consequence of Table 2 and the ensuing discussions. Thus, a 5th order ODE possessing 3 D algebras has the form $\mathcal{D}^2\phi_3 = H(\phi, \phi_3, \mathcal{D}\phi_3)$, where $\mathcal{D} = \lambda D_x$ for each of the λs as stated in Table 2.

For 4 dimensional algebras, there are 10 types as in Table 3 and immediate deliberations which included (20,4) and (21,4). Hence, a 5th order ODE which admits a 4 D algebra is of the form $\mathcal{D}\phi_4 = H(\phi, \phi_4)$ for each λ in Table 3. One can use each of the λs in Table 3 to find $\mathcal{D}\phi_4$ for all the types.

Now we focus on algebras of dimension 5 and greater. These are discussed and follow from our deliberations in the previous sections as well.

The 5th order ODEs admitting a five dimensional algebra are presented in Table 5. Higher symmetries admitted by such ODEs are listed in Table 6.

Algebra	5th Order ODE
(5, 5)	$y^{(5)} = \frac{-40}{9}y'''^3y''^{-2} + 5y'''y^{(4)}y''^{-1} + y''^2H(\phi)$, $H \neq 0$,
	$\phi = 3y^{(4)}y''^{\frac{-5}{3}} - 5y'''^2y''^{-8/3}$,
(15, 5)	$\dfrac{y'^3y^{(5)} - 10y'^2y''y^{(4)} - 6y'^2y'''^2 + 48y'y''^2y''' - 36y''^4}{(y'y''' - \frac{3}{2}y''^2)^2} = H(\phi)$
	$\phi = \dfrac{y'^2y^{(4)} - 6(y'y''y''' - y''^3)}{(y'y''' - \frac{3}{2}y''^2)^{\frac{3}{2}}}$,
(22,5)	$D_x(E_4(y)) = H(E_4(y))$, $H \neq 0$,
	η_i is solution of $E_4(\eta_i) = 0$, $i = 1, \ldots, 4$ for a_i consts.
(23,5)	$D_x^2(E_3(y)) = E_3(y)H(D_x \ln \lvert E_3(y)\rvert)$, $H \neq 0$,
	η_i is solution of $E_3(\eta_i) = 0$, $i = 1, \ldots, 3$ for a_i consts.
(24, 5)	$y^{(5)} = (y^{(4)})^{\frac{\alpha-5}{\alpha-4}}H((y^{(4)})^{\alpha-3}, (y^{(3)})^{4-\alpha})$, $\alpha \neq 4$, $H \neq 0$,
	$y^{(5)} = (y''')^{-1}H(y^{(4)})$, $\alpha = 4$, $H \neq 0$,
(25, 5)	$y^{(5)} = \exp(-\frac{2y'''}{3!})H(y^{(4)}\exp(y'''/3!))$, $H \neq 0$,
(26, 5)	$y^{(5)}y''^2y'''^{-3} = H(y''y'''^{-2}y^{(4)})$, $H \neq 0$,
(27, 5)	$y^{(5)}y''^{-3} + \frac{40}{9}y'''^3y''^{-5} - 5y^{(4)}y'''y''^{-4} = H(\phi)$, $H \neq 0$,
	$\phi = y^{(4)}y''^{-7/3} - \frac{4}{3}y'''^2y''^{-10/3}$, $H \neq C\phi^{3/2}$, C const.

Table 5: The 5th order invariant ODEs that correspond to their 5 dimensional symmetry algebras. Note that $E_n(z) \equiv z^{(n)} + a_1z^{(n-1)} + \cdots + a_nz$ and in (27,5) if $H = C\phi^{3/2}$ for C constant, then the algebra (28, 6) occurs.

The 5th order ODEs for the algebras (5,5) and (15,5) are the equations (5.1) and (5.3). Those for the types (22,5) to (27,5) are the 7 equations (5.13) to (5.19).

The 5th order equations for (6,6), (16,6) and (7,6) are (5.4), (5.7) and (5.9). Also for (24,6) to (28,6) the ODEs are (3.1) to (3.5) for $n = 5$. The linear case for (21,6) follows from Proposition 3.1 by setting $n = 5$. Moreover, the ODE for (8,8) is the singular

Higher Algebra	5th Order Equation
(6, 6)	$\dfrac{9y''^2y^{(5)}-45y''y'''y^{(4)}+40y'''^3}{(3y''y^{(4)}-5y'''^2)^{\frac{3}{2}}}=K, K\neq 0,$
(16,6)	$K_1^{-3}(5(D_xK_1)^2-4K_1D_x^2K_1)=K,$
	$K_1=y^{-1}y'''-\frac{3}{2}y'^{-2}y''^2,$
(7,6)	$K_1^{-3}(2K_1K_3+4K_1^2K_2-\frac{5}{2}K_2^2+2K_1^4-2K_1^2)=K,$
	$K_1=(1+y'^2)y''^{-2}y'''-3y',\ K_2=y''^{-1}(1+y'^2)D_xK_1,$
	$K_3=y''^{-1}(1+y'^2)D_xK_2,$
(21,6)	$y^{(5)}=\sum_{i=1}^3 A_i(x)y^{(i+1)}$, ξ_k satisfy $\xi_k^{(5)}=\sum_{i=1}^3 A_i(x)\xi_k^{(i+1)}$, $k=1,2,3,$
(24,6)	$y^{(5)}=K(y^{(4)})^{\frac{\alpha-5}{\alpha-4}},\ \alpha\neq 4, K\neq 0,$
(25, 6)	$y^{(5)}=K\exp(-y^{(4)}/4!),\ K\neq 0,$
(26, 6)	$y^{(5)}=Ky'''^{-1}(y^{(4)})^2,\ K\neq 0, 5/4,$
(27, 6)	$y^{(5)}=K(y^{(3)})^2+\frac{5}{4}(y^{(4)})^2y'''^{-1},\ K\neq 0,$
(28, 6)	$(\frac{y^{(5)}y''^2}{y'''^3}-\frac{5(9K_1-8)}{9})/(3K_1-4)^{3/2}=K,\ K\neq 0,$
	$K_1=y^{(4)}y''y'''^{-2}$
(23,7)	$y^{(5)}=\sum_{i=0}^4 A_iy^{(i)}$, A_i const., η_i satisfy the same equation.
(28, 7)	$y^{(5)}=\frac{5}{4}(y^{(4)})^2y'''^{-1}$
(8,8)	$9y''^2y^{(5)}-45y''y'''y^{(4)}+40y'''^3=0$
(28, 9)	$y^{(5)}=0$

Table 6: The 5th order invariant ODEs that correspond to their higher symmetry algebras. Note that K is a constant and in (21,6) (A_is not constant) and (23,7) the A_is do not satisfy the maximal symmetry condition as in [12], see sections 3.2, 4.2.

invariant equation $K_2=0$; (23,7) is the linear ODE (4.2); for (28,7) the ODE is (4.1) and for (28,9) one has the simplest equation (4.3).

One can proceed to classify 6th order ODEs as follows. For 1 and 2 dimensions, these are in Table 1. As for 3 dimensional algebras, there are 12 types and equations are of the form $\mathcal{D}^3\phi_3=H(\phi,\phi_3,\mathcal{D}\phi_3,\mathcal{D}^2\phi_3)$. In the case of 4 dimensions we have 10 types with ODEs $\mathcal{D}^2\phi_4=H(\phi,\phi_4,\mathcal{D}\phi_4)$. Five dimensions result in 10 types with equations $\mathcal{D}\phi_5=H(\phi,\phi_5)$ (Table 5 and (20,5), (21,5)). Note that here $\lambda=1/D_x\phi$ in $\mathcal{D}=\lambda D_x$.

The algebras for dimension 6 and ODEs are as follows. One has the algebra (6,6) with equation given by (5.6); (16,6) with equation (5.8); (7,6) with representative ODE (5.10). Each of these three types are presented in section 5.

Also one can easily deduce the algebras and forms for the equations for the 6 dimensional algebras (22,6) to (28,6) from section 5 by setting $n=6$ for each type. The equations are (5.13) to (5.20) (there are 2 equations for (24.6)). Altogether, there are hence 10 maximal algebra types for 6th order ODEs admitting 6 dimensional algebras. Moreover, for higher symmetries $n+1, n+2, n+4$ for $n=6$, there are 9 types given by (21,7) (linear ODE in Proposition 3.1 with $n=6$), (24,7) to (28,7) (section 3.1 for $n=6$ with corresponding equations (3.1) to (3.5)) as well as (23,8) (linear equation (4.2)), (28,8) ((4.1) for $n = 6$) and (28,10) simplest linear ODE with maximal $n + 4 = 10$ symmetries, as in (4.3).

Likewise, if we investigate 7th order ODEs, one has the same number of types of lower dimensional algebras as for 6th order ODEs, viz. 1, 2, 3, 4 and 5 dimensional algebras. For 6 dimensions, we have 12 types.

The number of types of maximal algebras of dimension 7 are 7 given by (22,7) to (28,7) in section 5 with ODEs (5.13) to (5.20) with $n = 7$. Moreover, for higher symmetries, we end up with 10 types, viz. (21,8) (Proposition 3.1), (24,8) to (28,8) with equations (3.1) to (3.5) with $n = 7$, (23,9) (linear, Proposition 4.1), (28,9) having (4.1), (28,11) with maximal symmetry equation and (8,8) given by the ODE (5.11), viz.

$$12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3} = K,$$

where K is constant.

Now proceeding to 8th order ODEs, we have similar 1 to 6 lower dimension types as for 7th order ODEs. For dimension 7 we have 9 types. One also has the 8 maximal algebra types (22,8) to (28,8) with corresponding ODEs (5.13) to (5.20) for $n = 8$ as well as (8,8) given by equation (5.12).

There are also 9 types of higher symmetries, viz. (21,9) (linear), (24,9) to (28,9), (23,10) (linear), (28,10) and (28,12) (linear maximal case).

We have the generalization to order 9 and greater as follows:

For $n \geq 9$, there are the lower dimension types similar to 8th order ODEs with dimension 8 as lower dimension algebra having 10 types. Moreover, 7 maximal types for nth order ODEs admitting n symmetries. These are $(22, n)$ to $(28, n)$ with representative ODEs (5.13) to (5.20) with 2 cases for (24,n).

Also, for higher symmetries for nth order equations, $n \geq 9$, there are 9 types, viz. $(21, n+1)$ (Proposition 3.1, linear), $(24, n+1)$ to $(28, n+1)$ with ODEs (3.1) to (3.5), $(23, n+2)$ (Proposition 4.1, linear), $(28, n+2)$ with ODE (4.1) and $(28, n+4)$ which is the maximal algebra with simplest ODE $y^{(n)} = 0$.

We now state the following important results on maximal n dimensional and higher dimensional symmetry algebras admitted by a scalar higher order ODEs for $n \geq 4$. These are consequences of our prior discussions.

Theorem 7.1. *The number of types of maximal n dimensional Lie symmetry algebras admitted by scalar nth order ODEs for $n \geq 4$ are as follows:*

8 for $n = 4, 5$ *with algebras and canonical ODEs given in Tables 3 and 5,*

10 for $n = 6$ *with algebras and representative equations*

Algebra	6th Order Equation
(6,6)	$K_1^{-3}(-\frac{3}{2}K_2D_xK_1 + K_1D_xK_2) = H(K_1^{-3/2}K_2)$
	$K_1 = 3y''^{-1}y^{(4)} - 5y'''^2y''^{-2}, K_2 = 9y''^{-1}y^{(5)} - 45y''^{-2}y'''y^{(4)} + 40y''^{-3}y'''^3$
(16,6)	$K_1^{-1/2}D_x\phi = H(\phi),\ \phi = K_1^{-3}(5(D_xK_1)^2 - 4K_1D_x^2K_1)$
	$K_1 = y'''y^{-1} - \frac{3}{2}y'^{-2}y''^2$
(7,6)	$(1+y'^2)y''^{-1}K_1^{-1/2}D_x\phi = H(\phi),\ \phi = K_1^{-3}(2K_1K_3 + 4K_1^2K_2 - \frac{5}{2}K_2^2 + 2K_1^4 - 2K_1^2),$
	$K_1 = (1+y'^2)y''^{-2}y''' - 3y',\ K_2 = y''^{-1}(1+y'^2)D_xK_1,\ K_3 = y''^{-1}(1+y'^2)D_xK_2$
(22,6)	$D_xE_5(y) = H\left(E_5(y)\right),\ E_5(\eta_i) = 0,\ i = 1,\ldots,5,\ H \neq 0,$
(23,6)	$D_x^2E_4(y) = E_4(y)H\left(D_x \ln\|E_4(y)\|\right),\ E_4(\eta_i) = 0,\ i = 1,\ldots,4,\ H \neq 0$
(24,6)	$y^{(6)} = (y^{(5)})^{\frac{\alpha-6}{\alpha-4}}H\left((y^{(5)})^{\alpha-4}(y^{(4)})^{5-\alpha}\right),\ \alpha \neq 5,\ H \neq 0,$
	$y^{(6)} = (y^{(4)})^{-1}H(y^{(5)}), \alpha = 5\ ,\ H \neq 0,$
(25,6)	$y^{(6)} = \exp(-y^{(4)}/12)H\left(y^{(5)}\exp(y^{(4)}/24)\right),\ H \neq 0,$
(26,6)	$y^{(6)} = (y^{(4)})^3(y^{(3)})^{-2}H\left(y^{(5)}y^{(3)}(y^{(4)})^{-2}\right),\ H \neq 0,$
(27,6)	$y^{(6)} + \frac{15}{4}(y^{(4)})^3(y^{(3)})^{-2} - \frac{9}{2}(y^{(5)}y^{(4)}(y^{(3)})^{-1} = (y^{(3)})^{5/2}H(\phi)$
	$\phi = y^{(5)}(y^{(3)})^{-2} - \frac{5}{4}(y^{(4)})^2(y^{(3)})^{-3},\ H \neq C\phi^{3/2},$
(28,6)	$K_3 - 8K_2 + 20K_1 - \dfrac{40}{3} = (3K_1-4)^2H\left(\dfrac{9K_2 - 45K_1 + 40}{9(3K_1-4)^{3/2}}\right)$
	$K_1 = y^{(4)}y^{(2)}(y^{(3)})^{-2},\ K_2 = y^{(5)}(y^{(2)})^2(y^{(3)})^{-3},\ K_3 = y^{(6)}(y^{(2)})^3(y^{(3)})^{-4}$

7 for $n = 7$ with algebras and canonical ODEs given below in (7.2) by letting $n = 7$

8 for $n = 8$ with algebras and equations

$$K_2^{-1/3}D_x(12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3}) = H(\phi),$$

where $\phi = 12J_1^{-2/3}J_2 - 28J_1^{-8/3}J_2^2 + 24J_1^{-5/3}J_3 + J_1^{4/3} - 4J_1^{-2/3}$

as well as by letting $n = 8$ in the 7 cases (7.2).

7 for $n \geq 9$ with algebras and representative equations

Algebra	nth Order Equation
(22,n)	$D_x E_{n-1}(y) = H\left(E_{n-1}(y)\right),\ E_{n-1}(\eta_i) = 0, i = 1, \ldots, n-1,\ H \neq 0,$
(23,n)	$D_x^2 E_{n-2}(y) = E_{n-2}(y) H\left(D_x \ln \lvert E_{n-2}(y)\rvert\right),\ E_{n-2}(\eta_i) = 0, i = 1, \ldots, n-2,\ H \neq 0$
(24,n)	$y^{(n)} = (y^{(n-1)})^{\frac{\alpha-n}{\alpha-n+1}} H\left((y^{(n-1)})^{\alpha-n+2}(y^{(n-2)})^{-(\alpha-n+1)}\right),\ \alpha \neq n-1,\ H \neq 0,$ $y^{(n)} = (y^{(n-2)})^{-1} H(y^{(n-1)}),\ \alpha = n-1,\ H \neq 0,$
(25,n)	$y^{(n)} \exp \dfrac{2y^{(n-2)}}{(n-2)!} = H\left(y^{(n-1)} \exp \dfrac{y^{(n-2)}}{(n-2)!}\right),\ H \neq 0$
(26,n)	$\dfrac{(y^{(n-3)})^2 y^{(n)}}{(y^{(n-2)})^3} = H\left(\dfrac{y^{(n-3)} y^{(n-1)}}{(y^{(n-2)})^2}\right),\ H \neq 0,$
(27,n)	$y^{(n)} + \dfrac{2n(n-1)}{(n-2)^2}(y^{(n-2)})^3 (y^{(n-3)})^{-2} + \dfrac{3n}{2-n} y^{(n-1)} y^{(n-2)} (y^{(n-3)})^{-1}$ $= (y^{(n-3)})^{\frac{n+4}{n-2}} H(\phi),\ H \neq C\phi^{3/2},$ $\phi = y^{(n-1)} (y^{(n-3)})^{\frac{n+2}{2-n}} - \dfrac{n-1}{n-2}(y^{(n-2)})^2 (y^{(n-3)})^{\frac{-2n}{n-2}}$
(28,n)	$K_3 - \dfrac{4n}{n-3} K_2 + \dfrac{6n(n-1)}{(n-3)^2} K_1 - \dfrac{3n(n-1)(n-2)}{(n-3)^3}$ $= ((n-3)K_1 - (n-2))^2) H(\phi)$ $\phi = \dfrac{(n-3)^2 K_2 - 3(n-1)(n-3)K_1 + 2(n-1)(n-2)}{(n-3)^2((n-3)K_1 - (n-2))^{\frac{3}{2}}}$ $K_i = \dfrac{y^{(n-3+i)}(y^{(n-4)})^i}{(y^{(n-3)})^{i+1}},\ i = 1, 2, 3,$ (7.2)

Remark: Note that in the above $E_n(z) \equiv z^{(n)} + a_1 z^{(n-1)} + \cdots + a_n z$.
If $H = C\phi^{3/2}$ then (28,7) arises for 6th order ODEs or (28,$n+1$) for (27, n).

Theorem 7.2. *The number of types of maximal $n+1$ dimensional Lie symmetry algebras admitted by scalar nth order ODEs for $n \geq 4$ are:*

7 for $n = 4$, with algebras and canonical ODEs given in Table 4

9 for $n = 5$, with algebras and representative equations in Table 6

6 for $n = 6$, with algebras and equations given in (7.3) by setting $n = 6$

7 for $n = 7$, with algebras and canonical ODEs

(8,8), $12J_1^{-2/3} J_2 - 28 J_1^{-8/3} J_2^2 + 24 J_1^{-5/3} J_3 + J_1^{4/3} - 4J_1^{-2/3} = K,$

where $J_1 = K_1^{-3/2} K_2$, with K_1 and K_2 as in (6,6), $J_2 = K_1^{-1/2} D_x J_1$ and $J_3 = K_1^{-1/2} D_x J_2$

and the 6 algebras in (7.3) by setting $n = 7$.

6 for $n \geq 8$, with algebras and representative ODEs

Higher Algebra	nth Order Equation
$(24,\, n+1)$	$y^{(n)} = K(y^{(n-1)})^{\frac{\alpha-n}{\alpha-n+1}},\ K \neq 0$
$(25,\, n+1)$	$y^{(n)} = K\exp\left(\frac{-y^{(n-1)}}{(n-1)!}\right),\ K \neq 0$
$(26,\, n+1)$	$y^{(n)} = K\frac{(y^{(n-1)})^2}{y^{(n-2)}},\ K \neq 0, n/(n-1)$
$(27,\, n+1)$	$y^{(n)} = \frac{n}{n-1}\frac{(y^{(n-1)})^2}{y^{(n-2)}} + K(y^{(n-2)})^{\frac{n+3}{n-1}},\ K \neq 0$
$(28,\, n+1)$	$y^{(n)} = (y^{(n-2)})^3(y^{(n-3)})^{-2}\left[\frac{n(3(n-2)K_1-2n+2)}{(n-2)^2} + K((n-2)K_1-(n-1))^{\frac{3}{2}}\right]$
$(21,\, n+1)$	$y^{(n)} = \sum_{i=2}^{n-2} A_i(x)y^{(i+1)}$,
	ξ_k, satisfy $\xi_k^{(n)} = \sum_{i=1}^{n-2} A_i(x)\xi_k^{(i+1)}, \qquad k = 1, ..., n-2. \qquad (7.3)$

Remark. For $(21, n+1)$, the A_is are not constant or satisfy the maximal conditions as stated in section 3.2.

Theorem 7.3. *The number of types of $n+2$ dimensional maximal symmetry algebras admitted by an nth order ODEs for $n \geq 5$ are two, one for linear class of equations, viz.* $(23, n+2)$*, with equation* (4.2) *and one for nonlinear equations* (4.1) *with algebra* $(28, n+2)$*. For $n = 4$ there are three types, (23,6) results in a linear equation as well as (6,6) and (28,6) which give rise to nonlinear classes of equations.*

Theorem 7.4. *The only higher order $n \geq 3$, ODEs which admit real $n+3$ dimensional algebras are the two ODEs for $n = 3$, viz.* (6.1) *possessing* $\mathfrak{sl}(2,R) \oplus \mathfrak{sl}(2,R$ *and* $\mathfrak{so}(3,1)$*, respectively and the 5th order equation which is* (6.2) *having* $\mathfrak{sl}(3,R)$ *as maximal algebra.*

Note that the maximum real Lie algebra admissible is $(28, n+4)$ as is well-known from Lie for $y^{(n)} = 0$, $n \geq 3$ and all scalar linear higher order ODEs (3.20) and (3.21) as shown in [12].

In order to recall, for lower dimensional algebras admitted by higher order ODEs $n \geq 4$, one can refer to the one and two dimensional algebras listed in Table 1 as well as the 12 types of real 3 D algebra realizations given in Table 2. Moreover, one should note that for lower dimensions, there also appears the two non-maximal Lie algebras $(20, m)$ and $(21, m)$, when one classifies scalar ODEs $n \geq 4$ as discussed. Therefore, we can state the following theorem for lower dimensions.

Theorem 7.5. *The number of types of lower $m < n$ dimensional Lie algebras possessed by scalar nth order ODEs for $n \geq 4$ are given by*

1,4 for $m = 1, 2$ dimensional algebras with the algebras and ODEs as in Table 1

12 for $m = 3$ *dimension algebras with algebras and ODEs given in Table 2 for 4th order equations and for higher* $n > 4$ *order equations obtained by (with algebras as in Table 2) invariant differentiation from* (7.1) *with* $m = 3$

10 for dimensions $m = 4, 5$ *with 8 algebras in Tables 3, 5 as well as the 2 algebras* $(20, m)$ *and* $(21, m)$ *with ODEs deduced from* (7.1) *via invariant derivatives. The equations are* $\phi_4 = H(\phi)$ *and* $\phi_5 = H(\phi)$ *in Tables 3, 5.*

12 for dimension $m = 6$ *with 10 algebras as in Theorem 7.1 together with* $(20, m)$ *and* $(21, m)$ *and equations determined by* (7.1). *The 6th order ODEs* $\phi_6 = H(\phi)$ *are stated in Theorem 7.1*

9 for dimension $m = 7$ *with 7 algebras in Theorem 7.1 as well as the 2 algebras* $(20, m)$ *and* $(21, m)$ *and ODEs derived by* (7.1). *The equations of the form* $\phi_7 = H(\phi)$ *are known.*

10 for dimension $m = 8$ *with 8 algebras in Theorem 7.1 and the 2 algebras* $(20, m)$ *and* $(21, m)$ *and ODEs deduced by* (7.1). *ODEs of the form* $\phi_8 = H(\phi)$ *are stated.*

9 for dimension $m \geq 9$ *with 7 algebras in Theorem 7.1 as well as the algebras* $(20, m)$ *and* $(21, m)$ *and ODEs deduced by* (7.1)*;* n*th order equations are given.*

Thus, we have a full understanding of the real symmetry Lie algebras admissible by scalar nth order ODEs for $n \geq 2$ keeping in mind that the second and third order ODEs are well-known from the literature as referenced.

8 Concluding Remarks

We have completely classified scalar nth, $n \geq 4$ order ODEs according to the real Lie algebras they possess by using the classification of Lie algebras in the plane as in the seminal contribution [4].

The classification for second and third order equations are well-known in a number of influential works [2, 7, 8, 15, 16, 23].

In the case of 4th order equations, there have been progress in a number of papers, viz. [24, 25, 26, 27]. Here we have re-looked at the maximal four dimensional algebras in a compact form to enhance further classification of higher order ODEs which admit four dimensional algebras as subalgebras.

We have shown that a higher nth order equation cannot admit maximally $n + 3$ point symmetries with the exception of 3rd and 5th order ODEs. For 3rd order equations, 6 symmetries occur for two classes having algebras $so(3, 1)$ and $so(2, 2)$ which are famous from the pioneering contribution of Lie [2]. In the case of 5th order ODEs, one has

$sl(3, R)$ admitted for one class of equations which is quite interesting as usually this algebra is often alluded to as the maximal Lie algebra of scalar second order ODEs.

The results on the number of $n + 1$ dimensional symmetry algebras for scalar higher order ODEs are provided with theorems on higher symmetries.

Further, it is important to mention that $n + 2$ dimensional algebras occur for 3rd order equations only in the linear class having 3+2=5 symmetries (see e.g. [8]), for 4th order there are two further classes apart from the linear class as discussed (see also [27]) and for 5th and higher order ODEs, there this is one more class besides the linear class. These can be deduced from the discussions and results stated relating to 4th and higher order ODEs.

We have derived all the maximal n dimensional symmetry algebras and their representative equations for $n \geq 4$. Furthermore, one can easily extract the higher symmetry classification as well as pointed out. These are presented as main results. For lower dimensional algebras one can continue by considering the further two types $(20, m)$ and $(21, m)$ for $m < n$, where n is the order of the ODE, as well as the familiar two and three dimensional algebra types which are recalled in tabular form. This is stated as a theorem as well on when higher order ODEs admit lower dimensional Lie symmetry algebras and the ODEs that result.

The theorems on linearization appears in two cases which are mentioned here as propositions with reference to the initial works [11, 12].

We have therefore achieved a complete classification of scalar nth, $n \geq 4$, order ODEs in terms of the real Lie algebras of point symmetries they admit.

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