p -th mean pseudo almost automorphic solutions of class r under the light of measure theory

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Abstract

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Abstract. The objective in this work is to present a new concept of p-th mean pseudo almost automorphic by use of the measure theory. We use the (μ,ν)-ergodic process to define the spaces of (μ,ν)-pseudo almost automorphic process of class r in the p-th sense. To do this, firstly we show some interesting results regarding the completeness and composition theorems. Secondly we study the existence, uniqueness of the p-th mean (μ,ν)-pseudo almost automorphic solution of class r for the stochastic evolution equation.

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1. Introduction

In this work, we study some properties of the p-th mean (μ,ν)-pseudo almost automorphic process using the measure theory and we use those results to study the following stochastic evolution equations in a Hilbert space H,

$$dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \text{ for } t \in \mathbb{R} \tag{1.1}$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H such that

$$\|T(t)\| \leq Me^{-\omega t}, \text{ for } \ t \geq 0,$$

for some $M, \omega > 0$, $f : \mathbb{R} \to L^p(\Omega, H)$ and $g : \mathbb{R} \to L^p(\Omega, H)$ are appropriate functions specified later, and $W(t)$ is a two-sided standard Brownian motion with values in H.
$\mathcal{C} = C([-r, 0], L^p(\Omega, H))$ denotes the space of continuous functions from $[-r, 0]$ to $L^p(\Omega, H)$ endowed with the uniform topology norm. For every $t \geq 0$, x_t denotes the history function of $\mathcal{C}$ defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

We assume $(H, \|,\|)$ is a real separable Hilbert space and $L^p(\Omega, H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}\|x\|^p = \int_\Omega \|x\|^p dP < \infty.$$

The concept of almost automorphiy is a generalization of the clasical periodicity. It was indroduced in literrature by Bochner This work is an extension of [11] whose authors had studied equation (1.1) in the deterministic case. Some recent contributions concerning p-th mean pseudo

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almost automorphic for abstract differential equations similar to equation (1.1) have been made. For example [12, 25] the authors studied equation (1.1) without operator L. They showed that equation has unique p-th mean μ-pseudo almost periodic and μ-pseudo almost automorphic solutions on $\mathbb{R}$ when f g are p-th mean pseudo almost periodic or p-th mean pseudo almost automophic functions.
This work is organized as follows, in section 2, we give the spectral decomposition of the phase space, in section 3, we study p-th mean (μ,ν)-ergodic process of class r, in section 4, we study p-th mean (μ,ν)-pseudo almost automorphic process and we discuss the existence and uniqueness of p-th mean (μ,ν)-pseudo almost automorphic solution of class r, the last section is devoted to application.

2. **Spectral decomposition**

To equation (1.1), associate the following initial value problem

$$\begin{cases} du_t = [Au_t + Lu_t + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in C = C([-r,0], L^p(\Omega, H)), \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to L^p(\Omega, H)$ and $g : \mathbb{R}^+ \to L^p(\Omega, H)$ are two stochastic processes continuous.

Definition 2.1. *We say that a continuous function $u : [-r,+\infty[\to L^p(\Omega, H)$ is an integral solution of equation, if the following conditions hold :*

(1) $\displaystyle\int_0^t u(s)ds \in D(A)$ *for* $t \geq 0$,

(2) $\displaystyle u(t) + A\int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds + \int_0^t g(s)dW(s)$ *for* $t \geq 0$,

(3) $u_0 = \varphi$.

If $\overline{D(A)} = L^p(\Omega, H)$, the integral solution coincide with the know mild solutions. One can se that if $u(t)$ is an integral solution of equation (2.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$. Let us introduce the part A_0 of the operator A which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ \\ A_0 x = Ax \text{ for } x \in D(A_0). \end{cases}$$

We make the following assumption.
$\mathbf{H}_0$ A satisfies the Hille-Yosida condition.

Proposition 2.2. *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. The phase C_0 of equation* (2.1) *is defined by*

$$C_0 = \{\varphi \in C : \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t) = v_t(.,\varphi),$$

where $v(.,\varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = Av(t) + L(v_t) \text{for } t \geq 0 \\ \\ v_0 = \varphi \in \mathcal{C} \end{cases}$$

Proposition 2.3. *$(\mathcal{U}(t))_{t\geq 0}$ is strongly continuous semigroup of linear operators on C_0. Moreover $(\mathcal{U}(\sqcup))_{t\geq 0}$ satisfies for $t \geq 0$ and $\theta \in [-r, 0]$ the following translation property*

$$(\mathcal{U}(t))_{\geq 0} = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \textit{ for } t+\theta \geq 0. \\ \\ \varphi(t+\theta) \textit{ for } t+\theta \leq 0. \end{cases}$$

Proposition 2.4. [23] *Let $\mathcal{A}_{\mathcal{U}}$ defined on C_0 by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \{\varphi \in C^1([-r,0];X), \varphi(0) \in (D(A), \varphi(0)' \in \overline{D(A)} \textit{ and } \varphi(0)' = A\varphi(0) + L(\varphi)\} \\ \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \in D(\mathcal{A}_{\mathcal{U}}) \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_t \geq 0$ on C_0.
Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 \text{ if } \theta \in [-r, 0[\\ \\ c \text{ if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0 c|_{\mathcal{C}} = |\phi|_{\mathcal{C}} + |c|$ for $(\phi, c) \in C_0 \times X$ is a Banach space. Consider the extension $\widetilde{\mathcal{A}_{\mathcal{U}}}$ defined on $C_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \left\{\varphi \in C^1([-r,0],X) : \varphi(0) \in D(A) \textit{ and } \varphi(0)' \in \overline{D(A)}\right\} \\ \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\varphi = X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 2.5. [24] *Assume that $(\boldsymbol{H}_0)$ holds. Then, $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hile-Yosida condition on $C_0 \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \textit{ for } n \in \mathbb{N} \textit{ and } \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}}_{\mathcal{U}}$ on $D(\widetilde{\mathcal{A}}_{\mathcal{U}}) = C_0$ is exactly the operator $\widetilde{\mathcal{A}}_{\mathcal{U}}$.

Definition 2.6. *We say a semigroup, $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \varnothing.$$

For the sequel, we make the following assumption :
$(\mathbf{H}_1)$ $(T(t))_{\geq 0}$ is compact on $\overline{D(A)}$ for $t > 0$.

Proposition 2.7. *Assume that $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is compact for $t > r$.*

We get the following result on the spectral decomposition of the phase space C_0.

Proposition 2.8. *Assume that $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. If the semigroup $\mathcal{U}(t)_{t\geq 0}$ is hyperbolic,then the space C_0 is decomposed as a direct sum*

$$C_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t\geq 0}$ on U is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \;\; \textit{for } t \geq 0 \;\; \textit{and} \;\; \varphi \in S,$$

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \;\; for \;\; t \leq 0 \;\; and \;\; \varphi \in U,$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

3. (μ, ν)-ergodic process in p-th mean sense of class r

Let $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$. Let $p \geq 2$. $L^p(\Omega, H)$ is a Hilbert space with the following norm

$$\|x\|_{L^p} = \Big(\int_{\Omega} \|x\|^p dP \Big)^{\frac{1}{p}}$$

Definition 3.1. [20] *Let $x : \mathbb{R} \to L^p(\Omega, H)$ be a stochastic process.*

(1) x said to be stochastically bounded in p-th mean sense, if there exists $M > 0$ such that

$$\mathbb{E}\|x(t)\|^p \leq M \; for \; all \, t \in \mathbb{R}.$$

(2) x said to be stochastically continuous in p-th mean sense if

$$\lim_{t \to s} \mathbb{E}\|x(t) - x(s)\|^p \leq M \; for \; all \;\; t, s \in \mathbb{R}.$$

Let $BC(\mathbb{R}, L^p(\Omega, H))$ denote the space of all the stochastically bounded continuous processes.

Remark 3.2. [20] *$(BC(\mathbb{R}, L^p(\Omega, H)), \|.\|_{\infty})$ is a Banach space, where*

$$\|x\|_{\infty} = \sup_{t \in \mathbb{R}} (\mathbb{E}(\|x(t)\|^p))^{\frac{1}{p}}$$

Definition 3.3. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be (μ, ν)-ergodic in p-th $(p \geq 2)$ mean sense, if $f \in BC(\mathbb{R}, L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|^p d\mu(t) = 0.$$

We denote by $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$, the space of all such process.

Proposition 3.4. *Let $\mu, \nu \in \mathcal{M}$. Then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$ is a Banach space with the supremum norm $\|.\|_{\infty}$.*

Definition 3.5. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be p-th mean (μ, ν)-ergodic of class r if $f \in BC(\mathbb{R}, L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r, t]} \mathbb{E}\|f(\theta)\|^p d\mu(t) = 0.$$

We denote by $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, the space of all such process.
For $\mu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote μ_a the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$\mu_a(A) = \mu([a + b : b \in A]) \text{ for } A \in \mathcal{N} \tag{3.1}$$

From μ, $\nu \in \mathcal{M}$, we formulate the following hypotheses.

$(\mathbf{H}_2)$ Let μ,$\nu \in \mathcal{M}$ be such that

$$\limsup_{\tau \to +\infty} \frac{\mu([-\tau, \tau])}{\nu([-\tau, \tau])} = \delta < \infty.$$

$(\mathbf{H}_3)$ For all a, b and $c \in \mathbb{R}$ such that $0 \leq a < b < c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \Rightarrow \mu(a + \delta, b + \delta) \geq \alpha_0 \mu(\delta, c + \delta).$$

$(\mathbf{H}_4)$ For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval I such that

$$\mu\Big(\{a+\tau : a \in A\}\Big) \leq \beta\mu(A) \text{ when } A \in \mathcal{N} \text{ and satisfies } A \cap I = \varnothing.$$

Proposition 3.6. *Assume that* $(\boldsymbol{H}_2)$ *holds. Then* $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ *is a Banach space with the norm* $\|.\|_\infty$

Proof. We can see that $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is a vector subspace of $BC(\mathbb{R}, L^p(\Omega, H))$. To complete the proof is enough to prove that $(\mathbf{H}_2)$ holds. Then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is closed in $BC(\mathbb{R}, L^p(\Omega, H))$. Let $(f_n)_n$ be a sequence in $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ such that $\lim\limits_{n\to+\infty} f_n = f$ uniformly in $BC(\mathbb{R}, L^p(\Omega, H))$. From $\nu(\mathbb{R}) = +\infty$, it follows that $\nu([-\tau, \tau]) > 0$ for τ sufficiently large. Let $n_0 \in \mathbb{N}$ such taht for all $n \geq n_0$, $\|f_n - f\|_\infty < \varepsilon$. Let $n \geq n_0$, then we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(t)\|^p\Big)d\mu(t) \quad &\leq \quad \frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)-f(t)\|^p\Big)d\mu(t)\\
&\quad +\frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)\|^p\Big)d\mu(t)\\
&\leq \quad \frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}\mathbb{E}\|f_n(t)-f(t)\|^p\Big)d\mu(t)\\
&\quad +\frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)\|^p\Big)d\mu(t)\\
&\leq \quad 2^{p-1}\|f_n-f\|_\infty^p\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)\|^p\Big)d\mu(t).
\end{aligned}$$

We deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(t)\|^p\Big)d\mu(t) \leq 2^{p-1}\delta\varepsilon \;\text{ for any }\; \varepsilon > 0.$$

■

Next result is a characterisation of p-th mean (μ, ν)-ergodic processes of class r.

Theorem 3.7. *Asume that* $(\boldsymbol{H}_2)$ *holds and let* $\mu, \nu \in \mathcal{M}$ *and* I *be a bounded interval (eventually* $I = \varnothing$*). Assume that* $f \in BC(\mathbb{R}, L^p(\Omega, H))$*. The following assertions are equivalent*

i) $f \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$

ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t) = 0.$

iii) For any $\varepsilon > 0$, $\displaystyle\lim_{\tau\to+\infty}\frac{\mu\Big(\{t\in[-\tau,\tau]\setminus I : \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\nu([-\tau,\tau]\setminus I)} = 0$

Proof. The proof uses the same arguments of the proof of Theorem 2.22 in [28].
i) ⇔ *ii)*. Denote By $A = \mu(I)$ and $B = \displaystyle\int_I\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)$.
Since the interval I is bounded and the process f is stochastically bounded continuous. Then A, B and C are finite.
For $\tau > 0$, such that $I \subset [-\tau, \tau]$ and $\nu([-\tau, \tau] \setminus I) > 0$, we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])\setminus I}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta\|^p\Big)d\mu(t) &= \frac{1}{\nu[-\tau,\tau]-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)-B\Big]\\ &= \frac{\nu([-\tau,\tau])}{\nu[-\tau,\tau]-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}|f(\theta)\|^p\Big)d\mu(t)\\ &\quad -\frac{B}{\nu([-\tau,\tau])}\Big]\end{aligned}$$

From above equalities and the fact $\nu(\mathbb{R})=+\infty$, we deduce $ii)$ is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)=0,$$

that $i)$. $iii)\Rightarrow ii)$ Denote by A_τ^ε and B_τ^ε the following sets

$$A_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p>\varepsilon\Big\}\text{ and }B_\tau^\varepsilon=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\le\varepsilon\Big\}.$$

Assume that $ii)$ holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau])\setminus I)}=0. \tag{3.2}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)=\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}\|f(\theta)\|^p\Big)d\mu(t)+\int_{B_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)$$

we deduce that for τ sufficient large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\quad\le\quad\|f\|_\infty\times\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}$$

Since $\mu(\mathbb{R})=\nu(\mathbb{R})=\infty$ and by using $(\mathbf{H}_2)$ then for all $\varepsilon>0$ we have

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\le\delta\varepsilon$$

Consequently $ii)$ holds.
$ii)\Rightarrow iii)$

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ge\int_{A_\tau^\varepsilon}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)$$

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ge\varepsilon\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)}$$

$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ge\frac{\mu(A_\tau^\varepsilon)}{\nu([-\tau,\tau]\setminus I)},$$

for τ sufficiently large, we obtain equation (3.2), that is $iii)$. ■

Definition 3.8. *Let $\mu, \nu \in \mathcal{M}$. A function $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be (μ,ν)-ergodic in p-th mean sense in $t \in \mathbb{R}$ uniformly with the respect to $x \in \mathcal{K}$, if $f \in BC(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t,x)\|^p d\mu(t) = 0,$$

where $\mathcal{K} \subset L^p(\Omega, H)$ is compact.

We denote $\mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$ the set of all such functions.

Definition 3.9. *Let $\mu, \nu \in \mathcal{M}$. A function $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be p-th mean (μ,ν)-ergodic of class r in $t \in \mathbb{R}$ uniformly with the respect to $x \in \mathcal{K}$, if $f \in BC(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta,x)\|^p d\mu(t) = 0,$$

where $\mathcal{K} \subset L^p(\Omega, H)$ is compact.

We denote $\mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$ the set of all such functions.

Definition 3.10. *Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2 , denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha\mu_1(A) \le \mu_2(A) \le \beta\mu_1(A)$, when $A \in \mathcal{N}$ satisfies $A \cap I = \varnothing$.*

Remark 3.11. *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 3.12. *Let $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, there exists some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha_1\mu_1(A) \le \mu_2(A) \le \beta_1\mu_2(A)$ and $\alpha_2\nu_1(A) \le \nu_2(A) \le \beta_1\nu_1(A)$ for each $A \in \mathcal{N}$ satisfies $A \cap I = \varnothing$, i.e

$$\frac{1}{\beta_2\nu_1(A)} \le \frac{1}{\nu_2(A)} \le \frac{1}{\alpha_2\nu_1(A)}$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field for τ sufficiently large,

$$\begin{aligned}
\frac{\alpha_1\mu_1\Big(\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\beta_2\mu_2([-\tau,\tau] \setminus I)} &\le \frac{\mu_2\Big(\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\nu_2([-\tau,\tau] \setminus I)} \\
&\le \frac{\beta_1\mu_1\Big(\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\alpha_2\nu([-\tau,\tau] \setminus I)}.
\end{aligned}$$

By using Theorem 3.7, we deduce that $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.

Let $\mu, \nu \in \mathcal{M}$, we denote by

$$cl(\mu,\nu) = \Big\{\overline{\omega}_1, \overline{\omega}_2 \in \mathcal{M} : \mu_1 \sim \mu_2, \nu_1 \sim \nu_2\Big\}$$

Lemma 3.13. [14] *Let $\mu \in \mathcal{M}$ satisfy $(\boldsymbol{H}_4)$. Then the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 3.14. [14] ($\boldsymbol{H}_4$) *implies*

$$\text{for all } \sigma > 0, \quad \limsup_{\tau \to +\infty} \frac{\mu([-\tau - \sigma, \tau + \sigma])}{\nu([-\tau, \tau])} < \infty.$$

Theorem 3.15. *Assume that* ($\boldsymbol{H}_4$) *holds. Then* $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ *is invariant by translation.*

Proof. The proof is inspired by Theorem 3.5 in [13].
Let $f \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$, there exists $a_0 > 0$ such that $\nu([-\tau - |a|, \tau + |a|]) > 0$ for $|a| > a_0$. Denote

$$M_a(\tau) = \frac{1}{\nu_a([\tau, \tau])} \int_{-\tau}^{\tau} \big(\sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|^p \big) d\mu_a(t) \ \ \forall \tau > 0 \text{ and } a \in \mathbb{R},$$

where ν_a is the positive measure define by equation (3.1) By using Lemma (3.13), it follows that ν and ν_a are equivalent, μ and μ_a are equivalent and by Theorem (3.12), we have $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_a, \nu_a, r) = \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, therefore $f \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_a, \nu_a, r)$ that is $\lim\limits_{t \to +\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.
For all $A \in \mathcal{N}$, we denote χ_A the characteristic function of A. By using definition of the μ_a, we obtain that

$$\int_{[-\tau,\tau]} \chi_A(t) d\mu_a(t) = \int_{[-\tau,\tau]} \chi_A(t) d\mu_a(t + a) = \int_{[-\tau+a,\tau+a]} \chi_A(t) d\mu_a(t).$$

Since $t \mapsto \sup\limits_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|^p$ is the pointwise limit of an increasing sequence of function see([19, Theorem 1.17, p.15]), we deduce that

$$\int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta)\|^p d\mu_a(t) = \int_{[-\tau+a,\tau+a]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t).$$

We denote by $a^+ = \max(a, 0)$ and $a^- = \max(-a, 0)$. Then we have $|a| + a = 2a^+$, $|a| - a = 2a^-$ and $[-\tau + a - |a|, \tau + a + |a|] = [-\tau - 2a^-, \tau + 2a^+]$. Therefore we obtain

$$M_a(\tau + |a|) = \frac{1}{\nu([-\tau - 2a^-, \tau + 2a^+])} \int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \tag{3.3}$$

From (3.3) and the following inequality

$$\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \leq \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)|^p d\mu(t),$$

we obtain

$$\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \leq \frac{\nu([-\tau - 2a^-, \tau + 2a^+])}{\nu([-\tau, \tau])} \times M_a(\tau + |a|).$$

This implies

$$\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \leq \frac{\nu([-\tau - 2|a|, \tau + 2|a|])}{\nu([-\tau, \tau])} \times M_a(\tau + |a|). \tag{3.4}$$

From equation (3.3) and equation (3.4) and using Lemma3.14, we deduce that

$$\frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) = 0,$$

which equivalent to

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta-a)\|^p d\mu(t)=0,$$

that is $f_a\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$. We have proved that $f\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ then $f_{-a}\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ for all $a\in\mathbb{R}$, that is $\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ invariant by translation. ■

Proposition 3.16. *The space $PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ is invariant by translation, that is for all $a\in\mathbb{R}$ and $f\in PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$, $f_a\in PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$.*

4. p-th mean (μ,ν)-pseudo almost automorphic processes

In this section, we define p-th mean (μ,ν) -pseudo almost automorphic and their properties.

Definition 4.1. [4] *A continuous function stochastic process $f:\mathbb{R}\to L^p(\Omega,H)$ is said to be almost automorphic process in the p-th mean sense if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g:\mathbb{R}\to L^p(\Omega,H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n)-g(t)\|^p=0$$

is well defined for each $t\in\mathbb{R}$ and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n)-f(t)\|^p=0$$

for each $t\in\mathbb{R}$.

We denote the space of all such stochastic processes by $AA(\mathbb{R},L^p(\Omega,H))$

Lemma 4.2. [4] *The space $AA(\mathbb{R},L^p(\Omega,H))$ of p-th mean almost automorphic stochastic processes equipped with the norm $\|.\|_\infty$ is a Banach space.*

Definition 4.3. [4] *A continuous function stochastic process $f:\mathbb{R}\times L^p(\Omega,H)\to L^p(\Omega,H)$, $(t,x)\mapsto f(t,x)$ is said to be almost automorphic process in the p-th mean sense in $t\in\mathbb{R}$ uniformly with respect to $x\in K$, if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g:\mathbb{R}\times L^p(\Omega,H)\to L^p(\Omega,H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n,x)-g(t,x)\|^p=0$$

is well defined for each $t\in\mathbb{R}$ and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n,x)-f(t,x)\|^p=0$$

for each $t\in\mathbb{R}$, where $K\subset L^p(\Omega,H)$ is compact.

We denote the space of all such stochastic processes by $AA(\mathbb{R}\times L^p(\Omega,H),L^p(\Omega,H))$.

Lemma 4.4. [4] *If x and y are two automorphic processes in p-th mean sense, then*

(1) $x+y$ is almost automorphic in p-th mean sense ;

(2) for every scalar λ, λx is almost automorphic in p-th mean sense ;

(3) there exists a constant $M>0$ such that

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|x(t)\|^p\le M,$$

that is, x is bounded in $L^p(\Omega,H)$.

We now introduce some new spaces used in the sequel.

Definition 4.5. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Definition 4.6. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Proposition 4.7. [28] *Assume that ($\boldsymbol{H}_3$) holds. Then the decomposition of (μ,ν)-pseudo almost automorphic function in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, X)$ and $\phi \in \mathscr{E}(\mathbb{R}, X, \mu, \nu)$ is unique.*

Remark 4.8. *Let $X = L^p(\Omega, H)$. Then the Proposition 4.7 always holds.*

Proposition 4.9. [11] *Assume that ($\boldsymbol{H}_3$) holds. Then the decomposition of (μ,ν)-pseudo almost automorphic function of class r in the form $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}, X)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}, X, \mu, \nu, r)$ is unique.*

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Definition 4.10. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic of class r in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

We denote by $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ the space of all such stochastic processes.

Proposition 4.11. *Assume that ($\boldsymbol{H}_2$) holds. Let $\mu, \nu \in \mathcal{M}$. The space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.*

Proof. This Proposition is the consequence of Lemma 4.2 and Proposition 3.6 ■

Definition 4.12. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic of class r in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

Proposition 4.13. *Let μ_1, μ_2, ν_1 and $\nu_2 \in \mathcal{M}$ if $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAA(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = PAA(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.*

This Proposition is just a consequence of Theorem 3.12.

Theorem 4.14. *Assume that ($\boldsymbol{H}_3$) holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then the function $t \to \phi_t$ belongs to $PAA((C[-r, 0], L^p(\Omega, H), \mu, \nu, r)$.*

Proof. Assume that $\phi = g + h$, where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $h \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Then we ca see that $\phi_t = g_t + h_t$ and g_t is p-th mean almost automorphic process. Let us denote

$$M_a = \frac{1}{\nu_a([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p d\mu_a(t),$$

where μ_a and ν_a are the positive measures defined by equation (3.1). By using Lemma 3.13 it follows that μ and μ_a are equivalent, ν and ν_a are equivalent by using theorem 3.12 $\mathscr{E}_p(\mathbb{R}L^p(\Omega, H), \mu, \nu, r) = \mathscr{E}_p(\mathbb{R}L^p(\Omega, H), \mu_a, \nu_a, r)$ therefore $f \in \mathscr{E}_p(\mathbb{R}L^p(\Omega, H), \mu_a, \nu_a, r)$ that is $\lim_{\tau \to \infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.

On the other hand for $\tau > 0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \Big(\sup_{\theta \in [-r,0]} \mathbb{E}\|h(\theta + \xi)\|^p a \Big) d\mu(t) \leq \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in [t-2r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in [t-2r,t-r]} \mathbb{E}\|h(\theta)\|^p + \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta \in [t-2r,t-r]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau-r}^{\tau-r} \Big(\sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau, \tau])} \int_{-\tau-r}^{\tau+} \Big(\sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t+r) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{\mu([-\tau - r, \tau + r])}{\nu([-\tau, \tau])} \Big(\frac{1}{\mu([-\tau - r, \tau + r])} \int_{-\tau-r}^{\tau} \Big(\sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t+r) \Big) + \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t).
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \Big(\sup_{\theta \in [-r,0]} \mathbb{E}\|h(\theta + \xi)\|^p \Big) d\mu(t) \quad &\leq \quad \frac{\mu([-\tau - r, \tau + r])}{\nu([-\tau, \tau])} \times M_\delta(\tau + r) \\
&+ \frac{1}{\nu([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t),
\end{aligned}$$

which shows usind Lemma 3.13 and Lemma 3.14 that ϕ_t belongs to $PAA(C[-r, 0], \mu, \nu, r)$. Thus we obtain the desired result. ■

Next, we study the composition of (μ, ν)-pseudo almost automorphic process in p-th mean sense.

Theorem 4.15. [5] *Let $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$, $(t, x) \mapsto f(t, x)$ be almost automorphic in p-th sense in $t \in \mathbb{R}$, for each $x \in L^p(\Omega, H)$ and assume that f satisfies the lipschitz condition in the following sense*

$$\mathbb{E}\|f(t, x) - f(t, y)\|^p \leq L\|x - y\|^p \ \forall x, y \in L^p(\Omega, H),$$

where L is positive number. Then $t \mapsto f(t, x(t)) \in AA(\mathbb{R}, L^p(\Omega, H))$ for any $x \in AA(\mathbb{R}, L^p(\Omega, H))$.

Theorem 4.16. *Let $(\boldsymbol{H}_2)$ holds and $\mu, \nu \in \mathcal{M}$ satisfy $(\boldsymbol{H}_4)$. Suppose that $f \in PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$ satisfies the Lipschitz condition in the second variable that is, there exists a positive number L such that for any $x, y \in L^p(\Omega, H)$,*

$$\mathbb{E}\|f(t, x) - f(t, y)\|^p \leq L\|x - y\|^p, \ t \in \mathbb{R}.$$

Then $t \mapsto f(t, x(t)) \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for any $x \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

Proof. Since $x \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, then we can decompose $x = x_1 + x_2$, where $x_1 \in AA(\mathbb{R}, L^p(\Omega, H))$ and $x_2 \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Otherwise, since $f \in PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$

then $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{R} \times L^p(\Omega, H))$ and $f_2 \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. Then the function f can be decomposed as follows

$$\begin{aligned} f(t, x(t)) &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1)] + [f(t, x_1(t)) - f_1(t, x_1(t))] \\ &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + f_2(t, x_1(t)). \end{aligned}$$

Using Theorem4.15, we have $t \mapsto f_1(t, x_1) \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$. It remains to show that the both functions $t \mapsto [f(t, x_1(t)) - f_1(t, x_1(t))]$ and $t \mapsto +f_2(t, x_1(t))$ belong to $\mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.
We have

$$\mathbb{E}\|f(t, x(t)) - f(t, x_1(t))\|^p \leq L\|x(t) - x_1(t)\|^p$$

$$\sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p \leq L \sup_{\theta \in [t-r,t]} \|x(\theta) - x_1(\theta)\|^p.$$

It follows that

$$\begin{aligned} \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p d\mu(t) &\leq \frac{L}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|x(\theta) - x_1(\theta)\|^p d\mu(t) \\ &\leq \frac{L}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|x_2(\theta)\|^p d\mu(t) \end{aligned}$$

Since $x_2 \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then

$$\lim_{\tau \to +\infty} \frac{L}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|x_2(\theta)\|^p d\mu(t) = 0.$$

We deduce that

$$\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p d\mu(t) = 0,$$

therefore $[f(t, x(t)) - f(t, x_1(t))] \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. Now to complete the proof it is enough to prove that $t \mapsto f_2(t, x_1(t)) \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$

In fact for each $t \in \mathbb{R}$, we have

$$\begin{aligned} \|f_2(t, x) - f_2(t, y)\|^p &= \|f(t, x) - f_1(t, x) - f_1(t, y) + f(t, y)\|^p \\ &\leq 2^{p-1}\|f(t, x) - f(t, y)\|^p + 2^{p-1}\|f_1(t, x) - f_1(t, y)\|^p. \end{aligned}$$

By using the Lipschitz condition, we have

$$\begin{aligned} \mathbb{E}\|f_2(t, x) - f_2(t, y)\|^p &\leq 2^{p-1}\mathbb{E}\|f(t, x) - f(t, y)\|^p + 2^{p-1}\mathbb{E}\|f_1(t, x) - f_1(t, y)\|^p \\ &\leq 2^p\|x - y\|^p \end{aligned}$$

Since $K = \overline{\{x_1(t) : t \in \mathbb{R}\}}$ is compact. Then for $\varepsilon > 0$, there exists a finite number $x_1, \ldots, x_m$ such that

$$K \subset \bigcup_{i=1}^{m} B\Big(x_i, \frac{\varepsilon}{2^{2p-1}L}\Big),$$

where $B\Big(x_i, \frac{\varepsilon}{2^{2p-1}L}\Big) = \{x \in K, \|x_i - x\|^p \leq \frac{\varepsilon}{2^{2p-1}L}\}$. Its implies that

$$K \subset \bigcup_{i=1}^{m} \Big\{x \in K, \forall\, t \in \mathbb{R}, \|f_2(t, x) - f_2(t, x_i)\|^p \leq \frac{\varepsilon}{2^{p-1}}\Big\}$$

Let $t \in \mathbb{R}$ and $x \in K$, there exists $i_0 \in \{1, \ldots, m\}$ such that

$$\mathbb{E}\|f_2(t,x) - f_2(t, x_{i_0})\|^p \leq \frac{\varepsilon}{2^{p-1}},$$

therefore

$$\begin{aligned}
\mathbb{E}\|f_2(t, x_1(t))\|^p &\leq 2^{p-1}\|f_2(t, x_1(t)) - f_2(t, x_{i_o}(t))\|^p + 2^{p-1}\mathbb{E}\|f_2(t, x_{i_o}(t))\|^p \\
&\leq \varepsilon + 2^{p-1}\mathbb{E}\|f_2(t, x_{i_o}(t))\|^p \\
&\leq \varepsilon + 2^{p-1}\sum_{i=1}^{m}\mathbb{E}\|f_2(t, x_{i_o}(t))\|^p.
\end{aligned}$$

It follows that

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta, x_1(\theta))\|^p d\mu(t) \leq \Big(\frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])} + \sum_{i=1}^{m}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta, x_i(\theta))\|^p d\mu(t)\Big).$$

By the fact $\forall i \in \{1, \ldots, m\}$, $\lim_{\tau\to+\infty}\sum_{i=1}^{m}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta, x_i(\theta))\|^p d\mu(t) = 0$, we deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta, x_1(\theta))\|^p d\mu(t) \leq \varepsilon\delta.$$

Therefore $t \mapsto f_2(t, x_1(t)) \in \mathscr{E}_p(\mathbb{R}\times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. ■

Proposition 4.17. *Assume that ($\boldsymbol{H}_4$) holds. Then the space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is invariant by translation, that is $f \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ implies $f_\alpha \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for all $\alpha \in \mathbb{R}$.*

Lemma 4.18. [6] *Let $G : [0, T]\times\Omega \to \mathscr{L}(L^p(\Omega, H))$ be an $\mathcal{F}_t$-adapted measurable stochastic process satisfying*

$$\int_0^T \mathbb{E}\|G(t)\|^2 < \infty$$

almost surely, where $\mathscr{L}(L^p(\Omega, H))$ denote the space of all linear operators from $L^p(\Omega, H)$ to itself. Then for any $p \geq 1$, there exists a constant $C_p > 0$ such that

$$\mathbb{E}\sup_{0\leq s\leq T}\Big\|\int_0^T G(s)dW(s)\Big\|^p \leq C_p\mathbb{E}\Big(\int_0^T\|G(s)\|^2 ds\Big)^{p/2}, T > 0.$$

We make the following assumption

($\mathbf{H}_5$) g is a stochastically bounded process in p-th mean sense.

Proposition 4.19. *Assume that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_5$) hold and the semigroup $(U(t))_{t\geq0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then there exists a unique bounded solution u of equation (1.1) on $\mathbb{R}$, given by*

$$\begin{aligned}
u_t &= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)
\end{aligned}$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_{\mathcal{U}})^{-1}$ for$\lambda > \widetilde{\omega}$, Π^s and Π^u are projections of C_0 onto the stable and unstable subspaces respectively.

Proof. Let

$$u_t = v(t) + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s),$$

where

$$v(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [23]. Now we show that the limit $\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds$ exists.
For each $t\in\mathbb{R}$ and by Lemma 4.18, we have

$$\begin{aligned}
\mathbb{E}\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\leq C_p\mathbb{E}\Big(\int_{-\infty}^{t}\|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\|^2 ds\Big)^{p/2}\\
&\leq C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\mathbb{E}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}\|g(s)\|^2 ds\Big)^{p/2}\\
&\leq C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\mathbb{E}\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}\|g(s)\|^2 ds\Big)^{p/2}\\
&\leq C_p(\overline{M}\widetilde{M})|\pi^s|)^p\sum_{n=1}^{+\infty}\mathbb{E}\Big(\int_{t-n}^{t-n+1}e^{-2(\frac{p-2}{p})\omega(t-s)}\\
&\quad\times e^{-\frac{4}{p}\omega(t-s)}\|g(s)\|^2 ds\Big)^{p/2}.
\end{aligned}$$

By using Hölder inequality, we obtain

$$\begin{aligned}
\mathbb{E}\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\leq C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big[\Big(\int_{t-n}^{t-n+1}\big(e^{-2(\frac{p-2}{p})\omega(t-s)}\big)^{\frac{p}{p-2}}ds\Big)^{\frac{p}{p-2}}\Big]^{p/2}\\
&\quad\times\mathbb{E}\Big[\Big(\int_{t-n}^{t-n+1}\big(e^{-\frac{4}{p}\omega(t-s)}\|g(s)\|^2\big)^{\frac{p}{2}}ds\Big)^{\frac{2}{p}}\Big]^{p/2}\\
&\leq C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\\
&\quad\times\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}\mathbb{E}\|g(s)\|^p ds\Big).
\end{aligned}$$

Since g stochastic bounded process in p-th mean sense, then there exists, $M>0$ such that $\mathbb{E}\|g(s)\|^P\leq M$.

It follows that

$$
\begin{aligned}
\mathbb{E}\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\leq C_p M(\overline{M}\widetilde{M})|\Pi^s|)^p \sum_{n=1}^{+\infty} \Big(\int_{t-n}^{t-n+1} e^{-2\omega(t-s)}ds\Big)^{p/2} \\
&\leq C_p M(\overline{M}\widetilde{M})|\Pi^s|)^p \sum_{n=1}^{+\infty} \frac{1}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2} \times e^{-\omega pn} \\
&\leq C_p M(\overline{M}\widetilde{M})|\Pi^s|)^p \frac{1}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2} \times \sum_{n=1}^{+\infty} e^{-\omega pn}.
\end{aligned}
$$

Since the serie $\displaystyle\sum_{n=1}^{+\infty} e^{-\omega pn} = 1 - \frac{1}{1-e^{-\omega p}} = \frac{e^{-\omega p}}{1-e^{-\omega p}} < \infty.$
It follows that

$$
\mathbb{E}\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \leq \gamma, \tag{4.1}
$$

where

$$
\gamma = \frac{C_p M(\overline{M}\widetilde{M})|\Pi^s|)^p}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2} \times \frac{e^{-\omega p}}{1-e^{-\omega p}}.
$$

Set

$$
F(n,s,t) = \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s)) \text{ for } n \in \mathbb{N} \text{ for } s \leq t.
$$

For n is sufficiently large and $\sigma \leq t$, we have

$$
\begin{aligned}
\mathbb{E}\| \int_{-\infty}^{\sigma} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\leq C_p(\overline{M}\widetilde{M})|\Pi^s|)^p \sum_{n=1}^{+\infty} \Big(\int_{\sigma-n}^{\sigma-n+1} e^{-2\omega(t-s)}ds\Big)^{\frac{p-2}{2}} \\
&\quad \times \Big(\int_{\sigma-n}^{\sigma-n+1} e^{-2\omega(t-s)}\mathbb{E}\|g(s)\|^p ds\Big) \\
&\leq C_p M(\overline{M}\widetilde{M})|\Pi^s|)^p \sum_{n=1}^{+\infty} \Big(\int_{\sigma-n}^{\sigma-n+1} e^{-2\omega(t-s)}ds\Big)^{p/2} \\
&\leq C_p M(\overline{M}\widetilde{M})|\Pi^s|)^p \frac{1}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2} e^{-\omega p(t-\sigma)} \times \sum_{n=1}^{+\infty} e^{-\omega pn} \\
&\leq \gamma e^{-\omega p(t-\sigma)}.
\end{aligned}
$$

It follow that for n and m sufficiently large and $\sigma \le t$, we have

$$\begin{aligned}
\mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^p &\le \mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(n,s,t)dW(s) + \int_{\sigma}^{t} F(n,s,t)dW(s) \\
&\quad - \int_{-\infty}^{\sigma} F(m,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^p \\
&\le 3^{p-1}\mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big\|^p + 3^{p-1}\mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(m,s,t)dW(s)\Big\|^p \\
&\quad +3^{p-1}\mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^p \\
&\le 2\times 3^{p-1}\gamma e^{-\omega p(t-\sigma)} \\
&\quad +3^{p-1}\mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^p
\end{aligned}$$

Since $\lim_{n\to+\infty} \mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s)\Big\|^p$ exists, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^p \le 2\times 3^{p-1}\gamma e^{-\omega p(t-\sigma)}.$$

If $\sigma \to -\infty$, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^p = 0.$$

We deduce that

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s)\|^p = \lim_{n\to\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)\Big\|^p$$

exists.

Therefore the limit $\lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)$ exists. In addition, one can see from the equation (4.1) that the function

$$\eta_1 : t \to \lim_{n\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)\|^p$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$\eta_2 : t \to \lim_{n\to+\infty} \mathbb{E}\Big\| \int_{t}^{+\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_n X_0 g(s))dW(s)\Big\|^p$$

is well defined and bounded on $\mathbb{R}$. ■

Proposition 4.20. *Assume that ($\boldsymbol{H}_5$) holds. Let $f, g \in AA(\mathbb{R}, X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by*

$$\begin{aligned}
\Gamma(f,g)(t) &= \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big](0).
\end{aligned}$$

Then $\Gamma(f,g) \in AA(\mathbb{R}, L^p(\Omega, H))$.

Proof. The proof of this Proposition will be in two steps.

Step 1 : We will show that $\Gamma(f,g)$ is continuous. For $t_0 \in \mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\|\Gamma(f,g)(t)-\Gamma(f,g)(t_0)\|^p &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s) \\
&- \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&- \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big\|^p \\
&\leq 4^{p-1}\mathbb{E}\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p \\
&+4^{p-1}\mathbb{E}\| \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\|^p \\
&+4^{p-1}\mathbb{E}\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \\
&+4^{p-1}\mathbb{E}\| \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \\
&= 4^{p-1}(I_1+I_2+I_3+I_4).
\end{aligned}$$

We have

$$I_1 = \mathbb{E}\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p.$$

Let $\sigma = s-t+t_0$ and by Hölder inequality, we have

$$\begin{aligned}
I_1 &= \mathbb{E}\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-\sigma)\Pi^s(\widetilde{B}_\lambda X_0 f(\sigma+t-t_0))d\sigma - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p \\
&\leq \mathbb{E}\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s+t-t_0)-f(s)])ds\|^p \\
&\leq \mathbb{E}\Big(\overline{M}\widetilde{M}\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}|\Pi^s|\|f(s+t-t_0)-f(s)\|ds\Big)^p \\
&\leq \mathbb{E}\Big(\overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t_0}e^{-\frac{\omega(p-1)(t_0-s)}{p}}\times e^{-\frac{\omega(t-s)}{p}}\|f(s+t-t_0)-f(s)\|ds\Big)^p \\
&\leq (\overline{M}\widetilde{M}|\Pi^s|)^p\mathbb{E}\Bigg[\Big(\int_{-\infty}^{t}\Big(e^{-\frac{\omega(p-1)(t_0-s)}{p}}\Big)^{\frac{p}{p-1}}ds\Big)^{\frac{p-1}{p}}\times\Big(\int_{-\infty}^{t}\Big(e^{-\frac{\omega(t-s)}{p}}\|f(s+t-t_0)-f(s)\|\Big)^p ds\Big)^{\frac{1}{p}}\Bigg]^p \\
&\leq (\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}ds\Big)^{p-1}\times\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds \\
&\leq \frac{(\overline{M}\widetilde{M}|\Pi^s|)^p}{\omega^{p-1}}\times\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds.
\end{aligned}$$

For an arbitrary sequence of real $\{t_n\}$ with $t_n \to t$ as $n \to +\infty$. By Lemma 4.4 and the defiition of $AA(\mathbb{R}, L^p(\Omega, H))$ we deduce that $f \in BC(\mathbb{R}, L^p(\Omega, H))$. So

$$e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p \to 0 \text{ as }, \to +\infty.$$

Hence

$$e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p \leq 2^p e^{-\omega(t_0-s)}\|f\|_\infty^p,$$

for every n sufficiently large. Note that

$$\int_{-\infty}^{t_0} 2^p e^{-\omega(t_0-s)}\|f\|_\infty^p ds < \infty.$$

Then according to Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{-\infty}^{t_0} e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds = 0.$$

Since the arbitrariness of $\{t_n\}$, we deduce that

$$\lim_{t\to t_0}\int_{-\infty}^{t_0} e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds = 0,$$

which implies that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p = 0 \tag{4.2}$$

Similarly, we can se that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\|^p = 0. \tag{4.3}$$

Let $\widetilde{W}(\tau) = W(\tau+t-t_0) - W(t-t_0)$. One can see that $\widetilde{W}$ is a Winner process and has the same distribution as W. Let $\tau = s-t+t_0$. Then by Lemma 4.18 and Hölder inequality, we have

$$\begin{aligned}
I_3 &= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \\
&= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(\tau+t-t_0))dW(\tau+t-t_0) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \\
&= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(\tau+t-t_0))d\widetilde{W}(\tau) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))d\widetilde{W}(s)\|^p \\
&= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s+t-t_0)-g(s)])d\widetilde{W}(s) \\
&\leq C_p(\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t_0} e^{-2\omega(t_0-s)}ds\Big)^{\frac{P-2}{2}} \times \int_{-\infty}^{t_0} e^{-2\omega(t_0-s)}\mathbb{E}\|g(s+t-t_0)-g(s)\|^p ds.
\end{aligned}$$

By the similar arguments as above, we obtain

$$\lim_{t\to t_0}\int_{-\infty}^{t_0} e^{-2\omega(t_0-s)}\mathbb{E}\|g(s+t-t_0)-g(s)\|^p ds = 0,$$

which implies that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p=0. \tag{4.4}$$

Similarly, we can see that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p=0. \tag{4.5}$$

From equations (4.2), (4.3), (4.4) and (4.5), we deduce that

$$\lim_{t\to t_0}\mathbb{E}\|\Gamma(f,g)(t)-\Gamma(f,g)(t_0)\|^p=0$$

and yield the continuity of $\Gamma(f,g)$.

Step 2 : Since $f,g\in AA(\mathbb{R},L^p(\Omega,H))$. Thus, for every sequence of real numbers $(s_m)m\in\mathbb{N}$, there exists a subsequence $(s_n)n\in\mathbb{N}$ and stochastic processes $\widetilde{f},\widetilde{g}:\mathbb{R}\to L^p(\Omega,H)$ which each $t\in\mathbb{R}$ such that

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n)-\widetilde{f}(t)\|^p=0,\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{f}(t-s_n)-f(t)\|^p=0$$

and

$$\lim_{n\to\infty}\mathbb{E}\|g(t+s_n)-\widetilde{g}(t)\|^p=0,\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{g}(t-s_n)-g(t)\|^p=0.$$

Let

$$\begin{aligned} w(t+s_n) &= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s+s_n))ds \\ &\quad +\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s+s_n) \\ &\quad +\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s+s_n)\Big]\end{aligned}$$

and

$$\begin{aligned} \widetilde{w}(t) &= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\ &\quad +\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big].\end{aligned}$$

Then we have

$$\begin{aligned}
\mathbb{E}\|w(t+s_n)-\widetilde{w}(t)\|^p \ \leq\ & 4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s+s_n)-f(s)]ds\|^p \\
& +4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0[f(s+s_n)-f(s)]ds\|^p \\
& 4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s+s_n)-g(s)]d\widetilde{W}(s)\|^p \\
& +4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0[g(s+s_n)-g(s)]d\widetilde{W}(s)\|^p,
\end{aligned}$$

where $\widetilde{W}(s)=W(s+s_n)-W(s)$. Note that W and $\widetilde{W}$ are two Wiener processes and have the same distribution. Then we have

$$\begin{aligned}
\mathbb{E}\|w(t+s_n)-\widetilde{w}(t)\|^p \ \leq\ & 4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t}e^{-\omega(t-s)}ds\Big)^{p-1}\times\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds \\
& +4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p\Big(\int_{+\infty}^{t}e^{\omega(t-s)}ds\Big)^{p-1}\times\int_{+\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds \\
& +4^{p-1}C_p(\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds \\
& +4^{p-1}C_p(\overline{M}\widetilde{M}|\Pi^u|)^p\Big(\int_{-\infty}^{t}e^{2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{+\infty}^{t}e^{2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds.
\end{aligned}$$

By similarly arguments as above, we have
$\lim_{n\to+\infty}\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds=0$, $\lim_{n\to+\infty}\int_{+\infty}^{t}e^{\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds=0$,
$\lim_{n\to+\infty}\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds=0$ and $\lim_{n\to+\infty}\int_{+\infty}^{t}e^{2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds=0$.
Thus

$$\lim_{n\to+\infty}\mathbb{E}\|w(t+s_n)-\widetilde{w}(t)\|^p=0.$$

Similarly, we have

$$\lim_{n\to+\infty}\mathbb{E}\|\widetilde{w}(t-s_n)-w(t)\|^p=0.$$

Therefore by **Steps 1** and **2**, we proved that $\Gamma(f,g)\in AA(\mathbb{R},L^p(\Omega,H))$.

Theorem 4.21. *Assume that* ($\boldsymbol{H}_3$) *and* ($\boldsymbol{H}_5$) *hold. Let* $f,g\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$, *then* $\Gamma(f,g)\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$.

Proof. We have

$$\begin{aligned}
\Gamma(f,g)(t) \ =\ & \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0f(s))ds \\
& +\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0g(s))dW(s).
\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\|\Gamma(f,g)(\theta)\|^p &= \mathbb{E}\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\ &\quad+\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\\ &\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big\|^p.\end{aligned}$$

Then for $\tau>0$, using Lemma 4.18 we have

$$\begin{aligned}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}\mathbb{E}\Big(\overline{M}\widetilde{M}\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}|\Pi^s|\|f(s)\|ds\Big)^p d\mu(t)\\ &\quad+\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}\mathbb{E}\Big(\overline{M}\widetilde{M}\int_{+\infty}^{\theta}e^{\omega(\theta-s)}|\Pi^u|\|f(s)\|ds\Big)^p d\mu(t)\\ &\quad+\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}C_p\mathbb{E}\Big(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}|\Pi^s|^2\|g(s)\|^2ds\Big)^{p/2}d\mu(t)\\ &\quad+\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}C_p\mathbb{E}\Big(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}|\Pi^u|^2\|g(s)\|^2ds\Big)^{p/2}d\mu(t)\end{aligned}$$

By using Höder inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Gamma(f,g)(\theta)\|^p &= \mathbb{E}\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\ &\quad+\lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\\ &\quad+\lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big\|^p .vspace*0.25cm\end{aligned}$$

Then for $\tau>0$, using Lemma 4.18 again, we have

$$\begin{aligned}&\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t)\\ &\leq\ 4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}ds\Big)^{p-1}\times\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big]d\mu(t)\\ &\quad+4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{+\infty}^{\theta}e^{\omega(\theta-s)}ds\Big)^{p-1}\times\int_{+\infty}^{\theta}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big]d\mu(t)\\ &\quad+4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}\mathbb{E}\|g(s)\|^p ds\Big]d\mu(t)\\ &\quad+4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{+\infty}^{\theta}e^{2\omega(\theta-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{+\infty}^{\theta}e^{2\omega(\theta-s)}\mathbb{E}\|g(s)\|^p ds\Big]d\mu(t).\end{aligned}$$

Thus, we have

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t)$$

$$\begin{aligned}
&\le \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p}{\omega^{p-1}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\le \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p}{\omega^{p-1}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\le \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p}{(2\omega)^{\frac{p-2}{2}}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{2\omega r}\int_{-\infty}^{\theta}e^{-2\omega(t-s)}\mathbb{E}\|g(s)\|^p ds\Big)d\mu(t)\\
&\le \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p}{(2\omega)^{\frac{p-2}{2}}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}\mathbb{E}\|g(s)\|^p ds\Big)d\mu(t).
\end{aligned}$$

On the one hand using Fubini's theorem, we have

$$\begin{aligned}
\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t) &\le \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{\omega r}\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\le e^{\omega r}\int_{-\tau}^{\tau}\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds d\mu(t)\\
&\le e^{\omega r}\int_{-\tau}^{\tau}\int_{0}^{\infty}e^{-\omega s}\mathbb{E}\|f(t-s)\|^p ds d\mu(t)\\
&\le e^{\omega r}\int_{0}^{\infty}e^{-\omega s}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t) ds.
\end{aligned}$$

By Theorem 3.15, we deduce that

$$\lim_{\tau\to+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)\to 0 \text{ for all } s\in\mathbb{R}_+$$

and

$$\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)\le\frac{e^{-\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|f\|_\infty^p.$$

Similarly, we have

$$\lim_{\tau\to+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)\to 0 \text{ for all } s\in\mathbb{R}_+$$

and

$$\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)\le\frac{e^{-\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|g\|_\infty^p.$$

Since f and g are two bounded functions, then the functions $s \mapsto \dfrac{e^{-\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|f\|_\infty^p$ and $s \mapsto \dfrac{e^{-2\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|g\|_\infty^p$ belong to $L^1(]0,\infty[)$ in view of the Lebesgue dominated convergence theorem, it follows that

$$e^{\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds = 0$$

and

$$e^{2\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)ds = 0.$$

On the other hand, we have

$$\begin{aligned}
\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{t-r}^{+\infty}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{t-r}^{+\infty}e^{\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\leq \int_{-\tau}^{\tau}\int_{-\infty}^{r}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds d\mu(t)\\
&\leq \int_0^{+\infty}e^{\omega s}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds
\end{aligned}$$

By the same arguments, we have

$$\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t) \leq \int_0^{+\infty}e^{2\omega s}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)ds$$

Similarly as above, we have the functions $s \mapsto \dfrac{e^{\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|f\|_\infty^p$ and $s \mapsto \dfrac{e^{2\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|g\|_\infty^p$ belong to $L^1(]0,\infty[)$ in view of the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r}\frac{e^{\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds = 0$$

and

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r}\frac{e^{2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)ds = 0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t) = 0.$$

Thus, we obtain the desired result. ■

Our next objective is to show the existence of p-th ($p \geq 2$) mean (μ,ν)-pseudo almost automorphic solution of class r for the following problem

$$dx(t) = [Ax(t) + L(x_t) + f(t, x_t)]dt + g(t, x_t)dW(t), \text{ for } t \in \mathbb{R}, \tag{4.6}$$

where $f : \mathbb{R} \times \mathcal{C} \to L^p(\Omega, H)$ and $g : \mathbb{R} \times \mathcal{C} \to L^p(\Omega, H)$ are two processes.
For the sequel we make the following assumptions.
($\mathbf{H}_6$) Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \times C(([-r, 0], L^p(\Omega, H)) \to L^p(\Omega, H)$ p-th mean $cl(\mu, \nu)$-pseudo almost automorphic of class r such that there exists a positive constant L_f such that

$$\mathbb{E}\|f(t, \phi_1) - f(t, \Phi_2)\|^p \leq L_f \mathbb{E}\|\phi_1 - \phi_2\|^p \text{ for all } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in C(([-r, 0], L^p(\Omega, H)).$$

($\mathbf{H}_7$) Let $\mu, \nu \in \mathcal{M}$ and $g : \mathbb{R} \times C(([-r, 0], L^p(\Omega, H)) \to L^p(\Omega, H)$ p-th mean $cl(\mu, \nu)$ pseudo almost automorphic of class r such that there exists a positive constant L_g such that

$$\mathbb{E}\|g(t, \phi_1) - g(t, \Phi_2)\|^p \leq L_g \mathbb{E}\|\phi_1 - \phi_2\|^p \text{ for all } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in C(([-r, 0], L^p(\Omega, H)).$$

($\mathbf{H}_8$) The instable space $U \equiv \{0\}$.

Theorem 4.22. *Let $p \geq 2$, assume that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$), ($\boldsymbol{H}_4$), ($\boldsymbol{H}_6$) ($\boldsymbol{H}_7$) and ($\boldsymbol{H}_8$) hold. If*

$$(\overline{M}\widetilde{M}|\Pi|^s)^p \Big[\frac{L_f}{\omega^p} + \frac{L_g C_p}{(2\omega)^{\frac{p}{2}}}\Big] < \frac{1}{2^{p-1}},$$

then equation (4.6) *has a unique p-th mean $cl(\mu, \nu)$-pseudo almost automorphic solution of class r.*

Proof. Let x be a function in $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. From Theorem 4.14 the function $t \to x_t$ belongs to $PAA(C([-r, 0]); L^p(\Omega, H), \mu, \nu, r)$. Hence Theorem 4.16 implies that $g(.) = f(., x)$ is in $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Since the unstable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider the following mapping $H : PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r) \to PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) \;=\; \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s, x_s))ds + \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s, x_s))dW(s)\Big](0).$$

Let $x_1, x_2 \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, we have

$$\begin{aligned}
\mathbb{E}\|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|^p &= \mathbb{E}\Big\|\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s, x_{1s}) - f(s, x_{2s})])ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s, x_{1s}) - g(s, x_{2s})])dW(s)\Big\|^p \\
&\leq 2^{p-1}\mathbb{E}\Big\|\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s, x_{1s}) - f(s, x_{2s})])ds\Big\| \\
&\quad + 2^{p-1}\mathbb{E}\Big\|\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s, x_{1s}) - g(s, x_{2s})])dW(s)\Big\|^p \\
&\leq 2^{p-1}(I_1 + I_2).
\end{aligned}$$

By Hölder inequality, it follows that

$$
\begin{aligned}
I_1 &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s,x_{1s})-f(s,x_{2s})])ds\Big\| \\
&\le \mathbb{E}\Big[(\overline{M}\widetilde{M}|\Pi|^s)^p\Big(\int_{-\infty}^{t}e^{-\omega(t-s)}ds\Big)^{p-1}\times\int_{-\infty}^{t}e^{-\omega(t-s)}\|f(s,x_{1s})-f(s,x_{2s})\|^p\Big] \\
&\le (\overline{M}\widetilde{M}|\Pi|^s)^p\Big(\int_{-\infty}^{t}e^{-\omega(t-s)}ds\Big)^{p-1}\times\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s,x_{1s})-f(s,x_{2s})\|^p ds \\
&\le \frac{(\overline{M}\widetilde{M}|\Pi|^s)^p}{\omega^{p-1}}\int_{-\infty}^{t}e^{-\omega(t-s)}L_f\mathbb{E}\|x_{1s}-x_{2s}\|^p ds \\
&\le \frac{(\overline{M}\widetilde{M}|\Pi|^s)^p}{\omega^{p-1}}L_f\sup_{t\in\mathbb{R}}\|x_1(t)-x_2(t)\|^p\Big(\int_{-\infty}^{t}e^{-\omega(t-s)}ds\Big) \\
&\le \frac{(\overline{M}\widetilde{M}|\Pi|^s)^p}{\omega^{p}}L_f\|x_1-x_2\|_\infty^p.
\end{aligned}
$$

By Hölder inequality and by Lemma 4.18, we have

$$
\begin{aligned}
I_2 &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s,x_{1s})-g(s,x_{2s})])dW(s)\Big\|^p \\
&\le C_p\mathbb{E}\Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\Big\|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s,x_{1s})-g(s,x_{2s})])\|^2 ds\Big]^{p/2} \\
&\le (\overline{M}\widetilde{M}|\Pi|^s)^p C_p\Big(\int_{-\infty}^{t}e^{-\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\|g(s,x_{1s})-g(s,x_{2s})\|^p ds \\
&\le \frac{C_p(\overline{M}\widetilde{M}|\Pi|^s)^p}{(2\omega)^{\frac{p-2}{2}}}\int_{-\infty}^{t}e^{-2\omega(t-s)}L_g\mathbb{E}\|x_{1s}-x_{2s}\|^p ds \\
&\le \frac{C_p(\overline{M}\widetilde{M}|\Pi|^s)^p}{(2\omega)^{\frac{p-2}{2}}}L_g\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t)-x_2(t)\|^p\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}ds\Big) \\
&\le \frac{C_p(\overline{M}\widetilde{M}|\Pi|^s)^p}{(2\omega)^{\frac{p}{2}}}L_g\|x_1-x_2\|_\infty^p.
\end{aligned}
$$

Thus we have

$$
\mathbb{E}\|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)\|^p \;\le\; 2^{p-1}(\overline{M}\widetilde{M}|\Pi|^s)^p\Big[\frac{L_f}{\omega^p}+\frac{L_gC_p}{(2\omega)^{\frac{p}{2}}}\Big]\|x_1-x_2\|_\infty^p
$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAA(\mathbb{R};L^p(\Omega,H),\mu,\nu,r)$. We conclude that equation (4.6), has one and only one p-th mean $cl(\mu,\nu)$-pseudo almost automorphic solution of class r. ■

Proposition 4.23. *Let $p\ge 2$, assume that $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_4)$ hold, f, g are lipschitz continuous with respect the second argument. If*

$$
Lip(f)=Lip(g)<\frac{1}{2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big[\frac{1}{\omega^p}+\frac{1}{(2\omega)^{\frac{p}{2}}}\Big]},
$$

then(4.6) *has a unique p-th mean* $cl(\mu,\nu)$*-pseudo almost automorphic of class* r*, where* $Lip(f)$ *and* $Lip(g)$ *are respectively the lipschitz constants of* f *and* g*.*

Proof. Let us pose $k = Lip(f) = Lip(g)$, we have

$$\begin{aligned}\mathbb{E}\|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)\|^p &\leq 2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big[\frac{k}{\omega^p}+\frac{k}{(2\omega)^{\frac{p}{2}}}\Big]\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t)-x_2(t)\|^p\\ &\leq 2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p k\Big[\frac{1}{\omega^p}+\frac{1}{(2\omega)^{\frac{p}{2}}}\Big]\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t)-x_2(t)\|^p\end{aligned}$$

Consequently $\mathcal{H}$ is a strict contraction if

$$k<\frac{1}{2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big[\frac{1}{\omega^p}+\frac{1}{(2\omega)^{\frac{p}{2}}}\Big]}$$

■

5. **Application**

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} dz(t,x) = -\dfrac{\partial^2}{\partial x^2}z(t,x)dt+\Big[\displaystyle\int_{-r}^{0}G(\theta)z(t+\theta,x)d\theta+\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)+\arctan(t)+\int_{-r}^{0}h(\theta,z(t+\theta,x))d\theta\Big]dt\\ \qquad +\Big[\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{3}t}\Big)+\arctan(t)+\displaystyle\int_{-r}^{0}h(\theta,z(t+\theta,x))d\theta\Big]dW(t) \text{ for } t\in\mathbb{R}, \text{ and } x\in[0,\pi],\\ z(t,0)=z(t,\pi)=0 \text{ for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \end{cases} \tag{5.1}$$

where $G:[-r,0]\to\mathbb{R}$ is continuous function and $h:[-r,0]\to\mathbb{R}$ is lipschitz continuous with the respect of the second argument. $W(t)$ is a two-sided standard Brownian motion with values in separable Hilbert space H. To rewrite equation (5.1) in abstract form, we introduce the space $H=L^2((0,\pi))$. Let $A:D(A)\to L^2((0,\pi))$ defined by

$$\begin{cases} D(A)=H^2(0,\pi)\cap H^1(0,\pi)\\ Ay(t)=y''(t) \text{ for } t\in(0,\pi) \text{ and } y\in D(A). \end{cases}$$

Then A generates a C_0-semigroup $(\mathcal{U}(t))_{\geq 0}$ on $L^2((0,\pi))$ given by

$$(\mathcal{U}(t)x)(r)=\sum_{n=1}^{+\infty}e^{-n^2\pi^2 t}<x,e_n>_{L^2}e_n(r),$$

where $e_n(r)=\sqrt{2}\sin(n\pi r)$ for $n=1,2\ldots$ and $\|\mathcal{U}(t)\|\leq e^{-\pi^2 t}$ for all $t\geq 0$. Thus $\overline{M}=1$ and $\omega=\pi^2$. Then A satisfies the Hille-Yosida conditions in $L^2(0,\pi)$. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of compact semigroup. It follows that $(\mathbf{H}_0)$ and $(\mathbf{H}_1)$ are satisfied.
We define $f:\mathbb{R}\times\mathcal{C}\to L^2((0,\pi))$ and $L:\mathcal{C}\to L^2(\Omega,H)$ as follows

$$f(t,\phi)(x)=\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big)+\arctan(t)+\int_{-r}^{0}h(\theta,\phi(\theta)(x))d\theta \text{ for } x\in(0,\pi) \text{ and } t\in\mathbb{R},$$

$$g(t,\phi)(x) = \sin\Big(\frac{1}{2+\cos t + \cos\sqrt{3}t}\Big) + \arctan(t) + \int_{-r}^{0} h(\theta,\phi(\theta)(x))d\theta \text{ for } x \in (0,\pi) \text{ and } t \in \mathbb{R},$$

and

$$L(\phi)(x) = \int_{-r}^{0} G(\theta)(\phi(\theta)(x) \text{ for } -r \leq \theta \text{ and } x \in (0,\phi).$$

Let us pose $v(t) = z(t,x)$. Then equation (5.1) takes the following abstract form

$$dv(t) = [Av(t) + L(v_t) + f(t,v_t)]dt + g(t,v_t)dW(t) \text{ for } t \in \mathbb{R}. \tag{5.2}$$

Consider the measure μ and ν where its Randon-Nikodym derivates are respectively ρ_1 and ρ_2

$$\rho_1(t) = \begin{cases} 1 & \text{for } t > 0 \\ e^t & \text{for } t \leq 0 \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t \in \mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\mu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \rho_2(t)dt \text{ for } A \in \mathcal{N}$$

From[14] $\mu,\nu \in \mathcal{M}$ satisfies Hypothesis $(\mathbf{H}_4)$.

$$\lim_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty} \frac{\displaystyle\int_{-r}^{0} e^t dt + \int_0^\tau dt}{2\displaystyle\int_0^\tau t dt} = \limsup_{\tau\to+\infty} \frac{1+e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H}_2)$ is satisfied.

For $t \in \mathbb{R}$, $-\frac{\pi}{2} \leq \arctan(\theta) \leq \frac{\pi}{2}$, therefore for all $\theta \in [t-r,t]$ $\arctan(t-r) \leq \arctan(t)$. It follows that $|\arctan\theta - \frac{\pi}{2}| = \frac{\pi}{2} - \arctan\theta \leq |\arctan(t-r) - \frac{\pi}{2}| = \frac{\pi}{2} - \arctan(t-r)$ which implies that $|\arctan\theta - \frac{\pi}{2}|^p \leq |\arctan(t-r) - \frac{\pi}{2}|^p$, hence, we have

$$\sup_{\theta\in[t-r,t]} \mathbb{E}|\arctan\theta - \frac{\pi}{2}|^p \leq \mathbb{E}|\arctan(t-r) - \frac{\pi}{2}|^p.$$

On one hand, we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_0^\tau \mathbb{E}\sup_{\theta\in[t-r,t]} \mathbb{E}|\arctan\theta - \frac{\pi}{2}|^p dt &\leq \frac{1}{\nu([-\tau,\tau])}\int_0^\tau \mathbb{E}|\arctan(t-r) - \frac{\pi}{2}|^p dt \\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_0^\tau \mathbb{E}\big(\frac{\pi}{2} - \arctan(t-r)\big)^p dt \\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_0^\tau \frac{\pi^p}{2^p} dt \\
&\leq \frac{\pi^p}{2^{p+1}\tau} \to 0, \text{ as } \tau \to +\infty.
\end{aligned}$$

On other hand, we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{0}\mathbb{E}\sup_{\theta\in[t-r,t]}\mathbb{E}|\arctan\theta-\frac{\pi}{2}|^{p}dt &\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{0}\frac{\pi^{p}}{2^{p}}e^{t}dt\\ &\leq \frac{\pi^{p}(1-e^{-\tau})}{2^{p+1}\tau}\to 0,\ \text{as}\ \tau\to+\infty.\end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\sup_{\theta\in[t-r,t]}\mathbb{E}|\arctan\theta-\frac{\pi}{2}|^{p}\,d\mu(t)=0.$$

It follows that $t\mapsto\arctan(t)-\dfrac{\pi}{2}$ is p-th mean (μ,ν)-ergodic of class r, consequently f is uniformly p-th mean (μ,ν)-pseudo almost automorphic of class r. Moreover L is bounded linear operator from $\mathcal{C}$ to $L^2(\Omega,H)$.

Let k be the lipschitz constant of h. Then by using Hölder-inequality for every $\varphi_1,\ \varphi_2\in\mathcal{C}$ and $t\geq 0$, we have

$$\begin{aligned}\mathbb{E}\|f(t,\varphi_1)(x)-f(t,\varphi_2)(x)\|^{p} &= \mathbb{E}\Big\|\int_{-r}^{0}\Big(h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\Big)d\theta\Big\|^{p}\\ &\leq \Big[\int_{-r}^{p}\|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\|d\theta\Big]^{p}\\ &\leq \mathbb{E}\Big[\Big(\int_{-r}^{0}d\theta\Big)^{\frac{p-1}{p}}\times\Big(\int_{-r}^{0}\|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\|^{p}\Big)^{\frac{1}{p}}\Big]^{p}\\ &\leq r^{p-1}\int_{-r}^{0}\mathbb{E}\|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\|^{p}d\theta\\ &\leq r^{p-1}k\int_{-r}^{0}\mathbb{E}\|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)\|^{p}d\theta\\ &\leq r^{p}k\sup_{-r\leq\theta\leq 0}\mathbb{E}\|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)\|^{p}\\ &\leq r^{p}k\alpha\sup_{-r\leq\theta\leq 0}\mathbb{E}\|\varphi_1(x)-\varphi_2(x)\|^{p}\ \text{for a certain}\ \alpha\in\mathbb{R}_{+}.\end{aligned}$$

Consequently, we conclude that f and g are Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost automorphic in p-th mean sense. Moreover, since h is stochastically bounded in p-th mean sense, $i.e$ $\mathbb{E}\|h(t,\phi(t))\|^p \leq \beta$. By Hölder inequality, we have

$$\begin{aligned}
\mathbb{E}\|g(t,\varphi(x))\|^p &= 1+\frac{\pi}{2}+\mathbb{E}\Big\|\int_{-r}^{0} h(\theta,\varphi(\theta)(x))\,d\theta\Big\|^p \\
&\leq \frac{2+\pi}{2}+\Big(\int_{-r}^{0} d\theta\Big)^{p-1}\times\int_{-r}^{0}\mathbb{E}\|h(\theta,\varphi(\theta)(x))\|^p d\theta \\
&\leq \frac{2+\pi}{2}+r^{p-1}\int_{-r}^{0}\beta d\theta \\
&\leq \frac{2+\pi}{2}+r^p\beta \\
&\leq \beta_1, \text{ with } \beta_1=\frac{2+\pi}{2}+r^p\beta,
\end{aligned}$$

which implies that g satisfies $(\mathbf{H}_5)$.

For hyperbolicity, we suppose that

$(\mathbf{H}_8)$ $\displaystyle\int_{-r}^{0}|G(\theta)|d\theta < 1.$

Proposition 5.1. [18] *Assume that $(\boldsymbol{H}_6)$ and $(\boldsymbol{H}_7)$ hold. The the seùigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic.*

Then by Proposition 4.23, we deduce the following result

Theorem 5.2. *Under above assumptions, if Lip(h) is small enough large, then equation* (5.1) *has unique p-th mean cl(μ,ν)-pseudo almost automorphic solution of class r.*

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p-th mean pseudo almost automorphic solutions of class r under the light of measure theory

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Abstract. The objective in this work is to present a new concept of p-th mean pseudo almost automorphic by use of the measure theory. We use the (μ,ν)-ergodic process to define the spaces of (μ,ν)-pseudo almost automorphic process of class r in the p-th sense. To do this, firstly we show some interesting results regarding the completeness and composition theorems. Secondly we study the existence, uniqueness of the p-th mean (μ,ν)-pseudo almost automorphic solution of class r for the stochastic evolution equation.

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1. **Introduction**

In this work, we study some properties of the p-th mean (μ,ν)-pseudo almost automorphic process using the measure theory and we use those results to study the following stochastic evolution equations in a Hilbert space H,

$$dx(t) = [Ax(t) + L(x_t) + f(t)]dt + g(t)dW(t), \text{ for } t \in \mathbb{R} \tag{1.1}$$

where $A : D(A) \subset H$ is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on H such that

$$\|T(t)\| \leq Me^{-\omega t}, \text{ for } \ t \geq 0,$$

for some $M, \omega > 0$, $f : \mathbb{R} \to L^p(\Omega, H)$ and $g : \mathbb{R} \to L^p(\Omega, H)$ are appropriate functions specified later, and $W(t)$ is a two-sided standard Brownian motion with values in H.
$\mathcal{C} = C([-r, 0], L^p(\Omega, H))$ denotes the space of continuous functions from $[-r, 0]$ to $L^p(\Omega, H)$ endowed with the uniform topology norm. For every $t \geq 0$, x_t denotes the history function of $\mathcal{C}$ defined by $x_t(\theta) = x(t + \theta)$ for $-r \leq \theta \leq 0$.

We assume $(H, \|, \|)$ is a real separable Hilbert space and $L^p(\Omega, H)$ is the space of all H-valued random variables x such that

$$\mathbb{E}\|x\|^p = \int_\Omega \|x\|^p dP < \infty.$$

The concept of almost automorphiy is a generalization of the clasical periodicity. It was indroduced in literrature by Bochner This work is an extension of [11] whose authors had studied equation (1.1) in the deterministic case. Some recent contributions concerning p-th mean pseudo

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almost automorphic for abstract differential equations similar to equation (1.1) have been made. For example [12, 25] the authors studied equation (1.1) without operator L. They showed that equation has unique p-th mean μ-pseudo almost periodic and μ-pseudo almost automorphic solutions on $\mathbb{R}$ when f g are p-th mean pseudo almost periodic or p-th mean pseudo almost automophic functions.
This work is organized as follows, in section 2, we give the spectral decomposition of the phase space, in section 3, we study p-th mean (μ,ν)-ergodic process of class r, in section 4, we study p-th mean (μ,ν)-pseudo almost automorphic process and we discuss the existence and uniqueness of p-th mean (μ,ν)-pseudo almost automorphic solution of class r, the last section is devoted to application.

2. **Spectral decomposition**

To equation (1.1), associate the following initial value problem

$$\begin{cases} du_t = [Au_t + Lu_t + f(t)]dt + g(t)dW(t) \text{ for } t \geq 0 \\ \\ u_0 = \varphi \in C = C([-r,0], L^p(\Omega,H)), \end{cases} \tag{2.1}$$

where $f : \mathbb{R}^+ \to L^p(\Omega,H)$ and $g : \mathbb{R}^+ \to L^p(\Omega,H)$ are two stochastic processes continuous.

Definition 2.1. *We say that a continuous function $u : [-r,+\infty[\to L^p(\Omega,H)$ is an integral solution of equation, if the following conditions hold :*

(1) $\int_0^t u(s)ds \in D(A)$ *for* $t \geq 0$,

(2) $u(t) + A\int_0^t u(s)ds + \int_0^t (L(u_s) + f(s))ds + \int_0^t g(s)dW(s)$ *for* $t \geq 0$,

(3) $u_0 = \varphi$.

If $\overline{D(A)} = L^p(\Omega,H)$, the integral solution coincide with the know mild solutions. One can se that if $u(t)$ is an integral solution of equation (2.1), then $u(t) \in \overline{D(A)}$ for all $t \geq 0$, in particular $\varphi(0) \in \overline{D(A)}$. Let us introduce the part A_0 of the operator A which defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ \\ A_0x = Ax \text{ for } x \in D(A_0). \end{cases}$$

We make the following assumption.
$\mathbf{H}_0$ A satisfies the Hille-Yosida condition.

Proposition 2.2. *A_0 generates a strongly continuous semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$. The phase C_0 of equation (2.1) is defined by*

$$C_0 = \{\varphi \in C : \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $\mathcal{U}(t)$ on C_0 by

$$\mathcal{U}(t) = v_t(.,\varphi),$$

where $v(.,\varphi)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}v(t) = Av(t) + L(v_t) \text{for } t \geq 0 \\ \\ v_0 = \varphi \in \mathcal{C} \end{cases}$$

Proposition 2.3. *$(\mathcal{U}(t))_{t\geq 0}$ is strongly continuous semigroup of linear operators on C_0. Moreover $(\mathcal{U}(\sqcup))_{t\geq 0}$ satisfies for $t \geq 0$ and $\theta \in [-r,0]$ the following translation property*

$$(\mathcal{U}(t))_{\geq 0} = \begin{cases} (\mathcal{U}(t+\theta)\varphi)(0) \text{ for } t+\theta \geq 0. \\ \\ \varphi(t+\theta) \text{ for } t+\theta \leq 0. \end{cases}$$

Proposition 2.4. [23] *Let $\mathcal{A}_{\mathcal{U}}$ defined on C_0 by*

$$\begin{cases} D(\mathcal{A}_{\mathcal{U}}) = \{\varphi \in C^1([-r,0];X), \varphi(0) \in (D(A), \varphi(0)' \in \overline{D(A)} \text{ and } \varphi(0)' = A\varphi(0) + L(\varphi)\} \\ \\ \mathcal{A}_{\mathcal{U}}\varphi = \varphi' \in D(\mathcal{A}_{\mathcal{U}}) \end{cases}$$

Then $\mathcal{A}_{\mathcal{U}}$ is the infinitesimal generator of the semigroup $(\mathcal{U}(t))_t \geq 0$ on C_0.
Let $\langle X_0 \rangle$ be the space defined by

$$\langle X_0 \rangle = \{X_0 c : c \in X\},$$

where the function $X_0 c$ is defined by

$$(X_0 c)(\theta) = \begin{cases} 0 \text{ if } \theta \in [-r, 0[\\ \\ c \text{ if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0 \rangle$ equipped with the norm $|\phi + X_0 c|_{\mathcal{C}} = |\phi|_{\mathcal{C}} + |c|$ for $(\phi, c) \in C_0 \times X$ is a Banach space. Consider the extension $\widetilde{\mathcal{A}_{\mathcal{U}}}$ defined on $C_\alpha \oplus \langle X_0 \rangle$ by

$$\begin{cases} D(\widetilde{\mathcal{A}_{\mathcal{U}}}) = \left\{\varphi \in C^1([-r,0],X) : \varphi(0) \in D(A) \text{ and } \varphi(0)' \in \overline{D(A)}\right\} \\ \\ \widetilde{\mathcal{A}}_{\mathcal{U}}\varphi = X_0(A\varphi(0) + L(\varphi) - \varphi(0)'). \end{cases}$$

Lemma 2.5. [24] *Assume that $(\boldsymbol{H}_0)$ holds. Then, $\widetilde{\mathcal{A}_{\mathcal{U}}}$ satisfies the Hile-Yosida condition on $C_0 \oplus \langle X_0 \rangle$ there exist $\widetilde{M} \geq 0$, $\widetilde{\omega} \in \mathbb{R}$ such that $]\widetilde{\omega}, +\infty[\subset \rho(\widetilde{\mathcal{A}_{\mathcal{U}}})$ and*

$$|(\lambda I - \widetilde{\mathcal{A}_{\mathcal{U}}})^{-n}| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \text{ for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Moreover, the part of $\widetilde{\mathcal{A}}_{\mathcal{U}}$ on $D(\widetilde{\mathcal{A}}_{\mathcal{U}}) = C_0$ is exactly the operator $\widetilde{\mathcal{A}}_{\mathcal{U}}$.

Definition 2.6. *We say a semigroup, $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic if*

$$\sigma(\mathcal{A}_{\mathcal{U}}) \cap i\mathbb{R} = \varnothing.$$

For the sequel, we make the following assumption :
$(\mathbf{H}_1)$ $(T(t))_{\geq 0}$ is compact on $\overline{D(A)}$ for $t > 0$.

Proposition 2.7. *Assume that $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. Then the semigroup $(\mathcal{U}(t))_{t\geq 0}$ is compact for $t > r$.*

We get the following result on the spectral decomposition of the phase space C_0.

Proposition 2.8. *Assume that $(\boldsymbol{H}_0)$ and $(\boldsymbol{H}_1)$ hold. If the semigroup $\mathcal{U}(t)_{t\geq 0}$ is hyperbolic,then the space C_0 is decomposed as a direct sum*

$$C_0 = S \oplus U$$

of two $\mathcal{U}(t)$ invariant closed subspaces S and U such that the restriction of $(\mathcal{U}(t))_{t\geq 0}$ on U is a group and there exist positive constants $\overline{M}$ and ω such that

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \text{ for } t \geq 0 \text{ and } \varphi \in S,$$

$$|\mathcal{U}(t)\varphi| \leq \overline{M}e^{-\omega t}|\varphi| \ \textit{for} \ t \leq 0 \ \textit{and} \ \varphi \in U,$$

where S and U are called respectively the stable and unstable space, Π^s and Π^u denote respectively the projection operator on S and U.

3. (μ,ν)-ergodic process in p-th mean sense of class r

Let $\mathcal{N}$ the Lebesgue σ-field of $\mathbb{R}$ and by $\mathcal{M}$ the set of all positive measures μ on $\mathcal{N}$ satisfying $\mu(\mathbb{R}) = +\infty$ and $\mu([a,b]) < \infty$ for all $a, b \in \mathbb{R}$ $(a \leq b)$. Let $p \geq 2$. $L^p(\Omega, H)$ is a Hilbert space with the following norm

$$\|x\|_{L^p} = \Big(\int_\Omega \|x\|^p dP\Big)^{\frac{1}{p}}$$

Definition 3.1. [20] *Let $x : \mathbb{R} \to L^p(\Omega, H)$ be a stochastic process.*

(1) x said to be stochastically bounded in p-th mean sense, if there exists $M > 0$ such that

$$\mathbb{E}\|x(t)\|^p \leq M \ \textit{for all} \ t \in \mathbb{R}.$$

(2) x said to be stochastically continuous in p-th mean sense if

$$\lim_{t\to s} \mathbb{E}\|x(t) - x(s)\|^p \leq M \ \textit{for all} \ t, s \in \mathbb{R}.$$

Let $BC(\mathbb{R}, L^p(\Omega, H))$ denote the space of all the stochastically bounded continuous processes.

Remark 3.2. [20] *$(BC(\mathbb{R}, L^p(\Omega, H)), \|.\|_\infty)$ is a Banach space, where*

$$\|x\|_\infty = \sup_{t\in\mathbb{R}} (\mathbb{E}(\|x(t)\|^p))^{\frac{1}{p}}$$

Definition 3.3. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be (μ,ν)-ergodic in p-th $(p \geq 2)$ mean sense, if $f \in BC(\mathbb{R}, L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t)\|^p d\mu(t) = 0.$$

We denote by $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$, the space of all such process.

Proposition 3.4. *Let $\mu, \nu \in \mathcal{M}$. Then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$ is a Banach space with the supremum norm $\|.\|_\infty$.*

Definition 3.5. *Let $\mu, \nu \in \mathcal{M}$. A stochastic process f is said to be p-th mean (μ,ν)-ergodic of class r if $f \in BC(\mathbb{R}, L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p d\mu(t) = 0.$$

We denote by $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, the space of all such process.
For $\mu \in \mathcal{M}$ and $a \in \mathbb{R}$, we denote μ_a the positive measure on $(\mathbb{R}, \mathcal{N})$ defined by

$$\mu_a(A) = \mu([a + b : b \in A]) \text{ for } A \in \mathcal{N} \tag{3.1}$$

From μ, $\nu \in \mathcal{M}$, we formulate the following hypotheses.

$(\mathbf{H}_2)$ Let μ,$\nu \in \mathcal{M}$ be such that

$$\limsup_{\tau\to+\infty} \frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \delta < \infty.$$

$(\mathbf{H}_3)$ For all a, b and $c \in \mathbb{R}$ such that $0 \leq a < b < c$, there exist δ_0 and $\alpha_0 > 0$ such that

$$|\delta| \geq \delta_0 \Rightarrow \mu(a + \delta, b + \delta) \geq \alpha_0\mu(\delta, c + \delta).$$

$(\mathbf{H}_4)$ For all $\tau \in \mathbb{R}$ there exist $\beta > 0$ and a bounded interval I such that

$$\mu\Big(\{a+\tau : a\in A\}\Big) \le \beta\mu(A) \text{ when } A\in\mathcal{N} \text{ and satisfies } A\cap I=\varnothing.$$

Proposition 3.6. *Assume that $(\boldsymbol{H}_2)$ holds. Then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$ is a Banach space with the norm $\|.\|_\infty$*

Proof. We can see that $\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$ is a vector subspace of $BC(\mathbb{R}, L^p(\Omega,H))$. To complete the proof is enough to prove that $(\mathbf{H}_2)$ holds. Then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$ is closed in $BC(\mathbb{R}, L^p(\Omega,H))$. Let $(f_n)_n$ be a sequence in $\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$ such that $\lim\limits_{n\to+\infty} f_n=f$ uniformly in $BC(\mathbb{R}, L^p(\Omega,H))$. From $\nu(\mathbb{R})=+\infty$, it follows that $\nu([-\tau,\tau])>0$ for τ sufficiently large. Let $n_0\in\mathbb{N}$ such taht for all $n\ge n_0$, $\|f_n-f\|_\infty<\varepsilon$. Let $n\ge n_0$, then we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(t)\|^p\Big)d\mu(t) &\le \frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)-f(t)\|^p\Big)d\mu(t)\\
&\quad+\frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)\|^p\Big)d\mu(t)\\
&\le \frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{t\in\mathbb{R}}\mathbb{E}\|f_n(t)-f(t)\|^p\Big)d\mu(t)\\
&\quad+\frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)\|^p\Big)d\mu(t)\\
&\le 2^{p-1}\|f_n-f\|_\infty^p\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])}+\frac{2^{p-1}}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_n(t)\|^p\Big)d\mu(t).
\end{aligned}$$

We deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{+\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(t)\|^p\Big)d\mu(t)\le 2^{p-1}\delta\varepsilon \quad\text{for any}\quad \varepsilon>0.$$

■

Next result is a characterisation of p-th mean (μ,ν)-ergodic processes of class r.

Theorem 3.7. *Asume that $(\boldsymbol{H}_2)$ holds and let $\mu,\nu\in\mathcal{M}$ and I be a bounded interval (eventually $I=\varnothing$). Assume that $f\in BC(\mathbb{R}, L^p(\Omega,H))$. The following assertions are equivalent*

i) $f\in\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$

ii) $\displaystyle\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)=0.$

iii) For any $\varepsilon>0$, $\displaystyle\lim_{\tau\to+\infty}\frac{\mu\Big(\{t\in[-\tau,\tau]\setminus I:\mathbb{E}\|f(\theta)\|^p>\varepsilon\}\Big)}{\nu([-\tau,\tau]\setminus I)}=0$

Proof. The proof uses the same arguments of the proof of Theorem 2.22 in [28].
i) ⇔ *ii)*. Denote By $A=\mu(I)$ and $B=\displaystyle\int_I\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)$.
Since the interval I is bounded and the process f is stochastically bounded continuous. Then A, B and C are finite.
For $\tau>0$, such that $I\subset[-\tau,\tau]$ and $\nu([-\tau,\tau]\setminus I)>0$, we have

$$\begin{aligned}\frac{1}{\nu([-\tau,\tau])\setminus I}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta\|^p\Big)d\mu(t) &= \frac{1}{\nu[-\tau,\tau]-A}\Big[\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)-B\Big]\\ &= \frac{\nu([-\tau,\tau])}{\nu[-\tau,\tau]-A}\Big[\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}|f(\theta)\|^p\Big)d\mu(t)\\ &\quad -\frac{B}{\nu([-\tau,\tau])}\Big]\end{aligned}$$

From above equalities and the fact $\nu(\mathbb{R})=+\infty$, we deduce $ii)$ is equivalent to

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)=0,$$

that $i)$. $iii)\Rightarrow ii)$ Denote by A^ε_τ and B^ε_τ the following sets

$$A^\varepsilon_\tau=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p>\varepsilon\Big\}\text{ and }B^\varepsilon_\tau=\Big\{t\in[-\tau,\tau]\setminus I:\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\le\varepsilon\Big\}.$$

Assume that $ii)$ holds, that is

$$\lim_{\tau\to+\infty}\frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau])\setminus I)}=0. \tag{3.2}$$

From the equality

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)=\int_{A^\varepsilon_\tau}\Big(\sup_{\theta\in[t-r,t]}\|f(\theta)\|^p\Big)d\mu(t)+\int_{B^\varepsilon_\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)$$

we deduce that for τ sufficient large

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ \le\ \|f\|_\infty\times\frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)}+\varepsilon\frac{\mu(B^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)}$$

Since $\mu(\mathbb{R})=\nu(\mathbb{R})=\infty$ and by using $(\mathbf{H}_2)$ then for all $\varepsilon>0$ we have

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\le\delta\varepsilon$$

Consequently $ii)$ holds.
$ii)\Rightarrow iii)$

$$\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ge\int_{A^\varepsilon_\tau}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)$$

$$\frac{1}{\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ge\varepsilon\frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)}$$

$$\frac{1}{\varepsilon\nu([-\tau,\tau]\setminus I)}\int_{[-\tau,\tau]\setminus I}\Big(\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta)\|^p\Big)d\mu(t)\ge\frac{\mu(A^\varepsilon_\tau)}{\nu([-\tau,\tau]\setminus I)},$$

for τ sufficiently large, we obtain equation (3.2), that is $iii)$. ∎

Definition 3.8. *Let $\mu,\nu \in \mathcal{M}$. A function $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be (μ,ν)-ergodic in p-th mean sense in $t \in \mathbb{R}$ uniformly with the respect to $x \in \mathcal{K}$, if $f \in BC(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \mathbb{E}\|f(t,x)\|^p d\mu(t) = 0,$$

where $\mathcal{K} \subset L^p(\Omega, H)$ is compact.

We denote $\mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$ the set of all such functions.

Definition 3.9. *Let $\mu,\nu \in \mathcal{M}$. A function $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be p-th mean (μ,ν)-ergodic of class r in $t \in \mathbb{R}$ uniformly with the respect to $x \in \mathcal{K}$, if $f \in BC(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and satisfies*

$$\lim_{\tau\to+\infty} \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta,x)\|^p d\mu(t) = 0,$$

where $\mathcal{K} \subset L^p(\Omega, H)$ is compact.

We denote $\mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$ the set of all such functions.

Definition 3.10. *Let $\mu_1, \mu_2 \in \mathcal{M}$. We say that μ_1 is equivalent to μ_2 , denoting this as $\mu_1 \sim \mu_2$ if there exist constants α and $\beta > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha\mu_1(A) \le \mu_2(A) \le \beta\mu_1(A)$, when $A \in \mathcal{N}$ satisfies $A \cap I = \varnothing$.*

Remark 3.11. *The relation $\sim$ is an equivalence relation on $\mathcal{M}$.*

Theorem 3.12. *Let $\mu_1, \nu_1, \mu_2, \nu_2 \in \mathcal{M}$. If $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.*

Proof. Since $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, there exists some constants $\alpha_1, \alpha_2, \beta_1, \beta_2 > 0$ and a bounded interval I (eventually $I = \varnothing$) such that $\alpha_1\mu_1(A) \le \mu_2(A) \le \beta_1\mu_2(A)$ and $\alpha_2\nu_1(A) \le \nu_2(A) \le \beta_1\nu_1(A)$ for each $A \in \mathcal{N}$ satisfies $A \cap I = \varnothing$, i.e

$$\frac{1}{\beta_2\nu_1(A)} \le \frac{1}{\nu_2(A)} \le \frac{1}{\alpha_2\nu_1(A)}$$

Since $\mu_1 \sim \mu_2$ and $\mathcal{N}$ is the Lebesgue σ-field for τ sufficiently large,

$$\begin{aligned}
\frac{\alpha_1\mu_1\Big(\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\beta_2\mu_2([-\tau,\tau] \setminus I)} &\le \frac{\mu_2\Big(\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\nu_2([-\tau,\tau] \setminus I)} \\
&\le \frac{\beta_1\mu_1\Big(\{t \in [-\tau,\tau] \setminus I : \sup\limits_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p > \varepsilon\}\Big)}{\alpha_2\nu([-\tau,\tau] \setminus I)}.
\end{aligned}$$

By using Theorem 3.7, we deduce that $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.

Let $\mu,\nu \in \mathcal{M}$, we denote by

$$cl(\mu,\nu) = \Big\{\overline{\omega}_1, \overline{\omega}_2 \in \mathcal{M} : \mu_1 \sim \mu_2, \nu_1 \sim \nu_2\Big\}$$

Lemma 3.13. [14] *Let $\mu \in \mathcal{M}$ satisfy $(\boldsymbol{H}_4)$. Then the measures μ and μ_τ are equivalent for all $\tau \in \mathbb{R}$.*

Lemma 3.14. [14] ($\boldsymbol{H}_4$) *implies*

$$\text{for all } \sigma > 0, \quad \limsup_{\tau\to+\infty} \frac{\mu([-\tau-\sigma,\tau+\sigma])}{\nu([-\tau,\tau])} < \infty.$$

Theorem 3.15. *Assume that* ($\boldsymbol{H}_4$) *holds. Then* $\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$ *is invariant by translation.*

Proof. The proof is inspired by Theorem 3.5 in [13].
Let $f \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$ and $a \in \mathbb{R}$. Since $\nu(\mathbb{R}) = +\infty$, there exists $a_0 > 0$ such that $\nu([-\tau-|a|,\tau+|a|]) > 0$ for $|a| > a_0$. Denote

$$M_a(\tau) = \frac{1}{\nu_a([\tau,\tau])}\int_{-\tau}^{\tau}\big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p\big)d\mu_a(t) \ \ \forall \tau > 0 \text{ and } a \in \mathbb{R},$$

where ν_a is the positive measure define by equation (3.1) By using Lemma (3.13), it follows that ν and ν_a are equivalent, μ and μ_a are equivalent and by Theorem (3.12), we have $\mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu_a,\nu_a,r) = \mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu,\nu,r)$, therefore $f \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega,H),\mu_a,\nu_a,r)$ that is $\lim\limits_{t\to+\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.
For all $A \in \mathcal{N}$, we denote χ_A the characteristic function of A. By using definition of the μ_a, we obtain that

$$\int_{[-\tau,\tau]} \chi_A(t)d\mu_a(t) = \int_{[-\tau,\tau]} \chi_A(t)d\mu_a(t+a) = \int_{[-\tau+a,\tau+a]} \chi_A(t)d\mu_a(t).$$

Since $t \mapsto \sup\limits_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p$ is the pointwise limit of an increasing sequence of function see([19, Theorem 1.17, p.15]), we deduce that

$$\int_{[-\tau,\tau]} \sup_{\theta\in[t-r,t]} \mathbb{E}\|f(\theta)\|^p d\mu_a(t) = \int_{[-\tau+a,\tau+a]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t).$$

We denote by $a^+ = \max(a,0)$ and $a^- = \max(-a,0)$. Then we have $|a|+a = 2a^+$, $|a|-a = 2a^-$ and $[-\tau+a-|a|,\tau+a+|a|] = [-\tau-2a^-,\tau+2a^+]$. Therefore we obtain

$$M_a(\tau+|a|) = \frac{1}{\nu([-\tau-2a^-,\tau+2a^+])}\int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \tag{3.3}$$

From (3.3) and the following inequality

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \leq \frac{1}{\nu([-\tau,\tau])}\int_{[-\tau-2a^-,\tau+2a^+]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)|^p d\mu(t),$$

we obtain

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \leq \frac{\nu([-\tau-2a^-,\tau+2a^+])}{\nu([-\tau,\tau])} \times M_a(\tau+|a|).$$

This implies

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) \leq \frac{\nu([-\tau-2|a|,\tau+2|a|])}{\nu([-\tau,\tau])} \times M_a(\tau+|a|). \tag{3.4}$$

From equation (3.3) and equation (3.4) and using Lemma3.14, we deduce that

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]} \sup_{\theta\in[t-a-r,t-a]} \mathbb{E}\|f(\theta)\|^p d\mu(t) = 0,$$

which equivalent to

$$\frac{1}{\nu([-\tau,\tau])}\int_{[-\tau,\tau]}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f(\theta-a)\|^p d\mu(t)=0,$$

that is $f_a \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. We have proved that $f \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then $f_{-a} \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for all $a \in \mathbb{R}$, that is $\mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ invariant by translation. ■

Proposition 3.16. *The space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ is invariant by translation, that is for all $a \in \mathbb{R}$ and $f \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, $f_a \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.*

4. p-th mean (μ, ν)-pseudo almost automorphic processes

In this section, we define p-th mean (μ, ν) -pseudo almost automorphic and their properties.

Definition 4.1. [4] *A continuous function stochastic process $f : \mathbb{R} \to L^p(\Omega, H)$ is said to be almost automorphic process in the p-th mean sense if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g : \mathbb{R} \to L^p(\Omega, H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n)-g(t)\|^p=0$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n)-f(t)\|^p=0$$

for each $t \in \mathbb{R}$.

We denote the space of all such stochastic processes by $AA(\mathbb{R}, L^p(\Omega, H))$

Lemma 4.2. [4] *The space $AA(\mathbb{R}, L^p(\Omega, H))$ of p-th mean almost automorphic stochastic processes equipped with the norm $\|.\|_\infty$ is a Banach space.*

Definition 4.3. [4] *A continuous function stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$, $(t, x) \mapsto f(t, x)$ is said to be almost automorphic process in the p-th mean sense in $t \in \mathbb{R}$ uniformly with respect to $x \in K$, if for every sequence of real numbers $(s_m)_{m\in\mathbb{N}}$, there exists a subsequence $(s_n)_{n\in\mathbb{N}}$ and a stochastic process $g : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ such that*

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n,x)-g(t,x)\|^p=0$$

is well defined for each $t \in \mathbb{R}$ and

$$\lim_{n\to\infty}\mathbb{E}\|g(t-s_n,x)-f(t,x)\|^p=0$$

for each $t \in \mathbb{R}$, where $K \subset L^p(\Omega, H)$ is compact.

We denote the space of all such stochastic processes by $AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$.

Lemma 4.4. [4] *If x and y are two automorphic processes in p-th mean sense, then*

(1) $x + y$ is almost automorphic in p-th mean sense ;

(2) for every scalar λ, λx is almost automorphic in p-th mean sense ;

(3) there exists a constant $M > 0$ such that

$$\sup_{t\in\mathbb{R}}\mathbb{E}\|x(t)\|^p\leq M,$$

that is, x is bounded in $L^p(\Omega, H)$.

We now introduce some new spaces used in the sequel.

Definition 4.5. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu)$.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Definition 4.6. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Proposition 4.7. [28] *Assume that ($\boldsymbol{H}_3$) holds. Then the decomposition of (μ,ν)-pseudo almost automorphic function in the form $f = g + \phi$, where $g \in AA(\mathbb{R}, X)$ and $\phi \in \mathscr{E}(\mathbb{R}, X, \mu, \nu)$ is unique.*

Remark 4.8. *Let $X = L^p(\Omega, H)$. Then the Proposition 4.7 always holds.*

Proposition 4.9. [11] *Assume that ($\boldsymbol{H}_3$) holds. Then the decomposition of (μ,ν)-pseudo almost automorphic function of class r in the form $\phi = \phi_1 + \phi_2$, where $\phi_1 \in AA(\mathbb{R}, X)$ and $\phi_2 \in \mathscr{E}(\mathbb{R}, X, \mu, \nu, r)$ is unique.*

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu)$.

Definition 4.10. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic of class r in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

We denote by $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ the space of all such stochastic processes.

Proposition 4.11. *Assume that ($\boldsymbol{H}_2$) holds. Let $\mu, \nu \in \mathcal{M}$. The space $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ endowed with the uniform topology norm is a Banach space.*

Proof. This Proposition is the consequence of Lemma 4.2 and Proposition 3.6 ■

Definition 4.12. *Let $\mu,\ \nu \in \mathcal{M}$. A bounded continuous stochastic process $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$ is said to be (μ,ν)-pseudo almost automorphic of class r in p-th mean sense, if it can decomposed as follows*

$$f = g + \phi,$$

where $g \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$ and $\phi \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

We denote the space of all such stochastic processes by $PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.

Proposition 4.13. *Let μ_1, μ_2, ν_1 and $\nu_2 \in \mathcal{M}$ if $\mu_1 \sim \mu_2$ and $\nu_1 \sim \nu_2$, then $PAA(\mathbb{R}, L^p(\Omega, H), \mu_1, \nu_1, r) = PAA(\mathbb{R}, L^p(\Omega, H), \mu_2, \nu_2, r)$.*

This Proposition is just a consequence of Theorem 3.12.

Theorem 4.14. *Assume that ($\boldsymbol{H}_3$) holds. Let $\mu, \nu \in \mathcal{M}$ and $\phi \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then the function $t \to \phi_t$ belongs to $PAA((C[-r, 0], L^p(\Omega, H), \mu, \nu, r)$.*

Proof. Assume that $\phi = g + h$, where $g \in AA(\mathbb{R}, L^p(\Omega, H))$ and $h \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Then we ca see that $\phi_t = g_t + h_t$ and g_t is p-th mean almost automorphic process. Let us denote

$$M_a = \frac{1}{\nu_a([-\tau, \tau])} \int_{-\tau}^{\tau} \sup_{\theta \in [t-r,t]} \mathbb{E}\|h(\theta)\|^p d\mu_a(t),$$

where μ_a and ν_a are the positive measures defined by equation (3.1). By using Lemma 3.13 it follows that μ and μ_a are equivalent, ν and ν_a are equivalent by using theorem 3.12 $\mathscr{E}_p(\mathbb{R}L^p(\Omega, H), \mu, \nu, r) = \mathscr{E}_p(\mathbb{R}L^p(\Omega, H), \mu_a, \nu_a, r)$ therefore $f \in \mathscr{E}_p(\mathbb{R}L^p(\Omega, H), \mu_a, \nu_a, r)$ that is $\lim_{\tau\to\infty} M_a(\tau) = 0$ for all $a \in \mathbb{R}$.

On the other hand for $\tau > 0$, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\sup_{\theta\in[-r,0]} \mathbb{E}\|h(\theta+\xi)\|^p a \Big) d\mu(t) \leq \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-2r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-2r,t-r]} \mathbb{E}\|h(\theta)\|^p + \sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \Big(\sup_{\theta\in[t-2r,t-r]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) + \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau-r}^{\tau-r} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) + \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{1}{\nu([-\tau,\tau])} \int_{-\tau-r}^{\tau+} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t+r) + \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \|h(\theta)\|^p \Big) d\mu(t) \\
\leq\ & \frac{\mu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])} \Big(\frac{1}{\mu([-\tau-r,\tau+r])} \int_{-\tau-r}^{\tau} \Big(\sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t+r) \Big) + \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t).
\end{aligned}$$

Consequently

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \Big(\sup_{\theta\in[-r,0]} \mathbb{E}\|h(\theta+\xi)\|^p \Big) d\mu(t) \quad \leq \quad & \frac{\mu([-\tau-r,\tau+r])}{\nu([-\tau,\tau])} \times M_\delta(\tau+r) \\
& + \frac{1}{\nu([-\tau,\tau])} \int_{-\tau}^{\tau} \sup_{\theta\in[t-r,t]} \mathbb{E}\|h(\theta)\|^p \Big) d\mu(t),
\end{aligned}$$

which shows usind Lemma 3.13 and Lemma 3.14 that ϕ_t belongs to $PAA(C[-r,0], \mu, \nu, r)$. Thus we obtain the desired result. ■

Next, we study the composition of (μ, ν)-pseudo almost automorphic process in p-th mean sense.

Theorem 4.15. [5] *Let $f : \mathbb{R} \times L^p(\Omega, H) \to L^p(\Omega, H)$, $(t, x) \mapsto f(t, x)$ be almost automorphic in p-th sense in $t \in \mathbb{R}$, for each $x \in L^p(\Omega, H)$ and assume that f satisfies the lipschitz condition in the following sense*

$$\mathbb{E}\|f(t,x) - f(t,y)\|^p \leq L\|x - y\|^p \ \forall x, y \in L^p(\Omega, H),$$

where L is positive number. Then $t \mapsto f(t, x(t)) \in AA(\mathbb{R}, L^p(\Omega, H))$ for any $x \in AA(\mathbb{R}, L^p(\Omega, H))$.

Theorem 4.16. *Let $(\boldsymbol{H}_2)$ holds and $\mu, \nu \in \mathcal{M}$ satisfy $(\boldsymbol{H}_4)$. Suppose that $f \in PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$ satisfies the Lipschitz condition in the second variable that is, there exists a positive number L such that for any $x, y \in L^p(\Omega, H)$,*

$$\mathbb{E}\|f(t,x) - f(t,y)\|^p \leq L\|x - y\|^p, \ t \in \mathbb{R}.$$

Then $t \mapsto f(t, x(t)) \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ for any $x \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$.

Proof. Since $x \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, then we can decompose $x = x_1 + x_2$, where $x_1 \in AA(\mathbb{R}, L^p(\Omega, H))$ and $x_2 \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Otherwise, since $f \in PAA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$

then $f = f_1 + f_2$, where $f_1 \in AA(\mathbb{R} \times L^p(\Omega, H))$ and $f_2 \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. Then the function f can be decomposed as follows

$$\begin{aligned} f(t, x(t)) &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1)] + [f(t, x_1(t)) - f_1(t, x_1(t))] \\ &= f_1(t, x_1(t)) + [f(t, x(t)) - f(t, x_1(t))] + f_2(t, x_1(t)). \end{aligned}$$

Using Theorem4.15, we have $t \mapsto f_1(t, x_1) \in AA(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H))$. It remains to show that the both functions $t \mapsto [f(t, x_1(t)) - f_1(t, x_1(t))]$ and $t \mapsto +f_2(t, x_1(t))$ belong to $\mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$.
We have

$$\mathbb{E}\|f(t, x(t)) - f(t, x_1(t))\|^p \leq L\|x(t) - x_1(t)\|^p$$

$$\sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p \leq L \sup_{\theta \in [t-r,t]} \|x(\theta) - x_1(\theta)\|^p.$$

It follows that

$$\begin{aligned} \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p d\mu(t) &\leq \frac{L}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|x(\theta) - x_1(\theta)\|^p d\mu(t) \\ &\leq \frac{L}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|x_2(\theta)\|^p d\mu(t) \end{aligned}$$

Since $x_2 \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ then

$$\lim_{\tau \to +\infty} \frac{L}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|x_2(\theta)\|^p d\mu(t) = 0.$$

We deduce that

$$\limsup_{\tau \to +\infty} \frac{1}{\nu([-\tau, \tau])} \int_{[-\tau,\tau]} \sup_{\theta \in [t-r,t]} \mathbb{E}\|f(\theta, x(\theta)) - f(\theta, x_1(\theta))\|^p d\mu(t) = 0,$$

therefore $[f(t, x(t)) - f(t, x_1(t))] \in \mathscr{E}_p(\mathbb{R} \times L^p(\Omega, H), L^p(\Omega, H), \mu, \nu, r)$. Now to complete the proof it is enough to prove that $t \mapsto f_2(t, x_1(t)) \in \mathscr{E}_p(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$

In fact for each $t \in \mathbb{R}$, we have

$$\begin{aligned} \|f_2(t, x) - f_2(t, y)\|^p &= \|f(t, x) - f_1(t, x) - f_1(t, y) + f(t, y)\|^p \\ &\leq 2^{p-1}\|f(t, x) - f(t, y)\|^p + 2^{p-1}\|f_1(t, x) - f_1(t, y)\|^p. \end{aligned}$$

By using the Lipschitz condition, we have

$$\begin{aligned} \mathbb{E}\|f_2(t, x) - f_2(t, y)\|^p &\leq 2^{p-1}\mathbb{E}\|f(t, x) - f(t, y)\|^p + 2^{p-1}\mathbb{E}\|f_1(t, x) - f_1(t, y)\|^p \\ &\leq 2^p\|x - y\|^p \end{aligned}$$

Since $K = \overline{\{x_1(t) : t \in \mathbb{R}\}}$ is compact. Then for $\varepsilon > 0$, there exists a finite number $x_1, \ldots, x_m$ such that

$$K \subset \bigcup_{i=1}^{m} B\Big(x_i, \frac{\varepsilon}{2^{2p-1}L}\Big),$$

where $B\Big(x_i, \frac{\varepsilon}{2^{2p-1}L}\Big) = \big\{x \in K, \|x_i - x\|^p \leq \frac{\varepsilon}{2^{2p-1}L}\big\}$. Its implies that

$$K \subset \bigcup_{i=1}^{m} \Big\{x \in K, \forall\, t \in \mathbb{R}, \|f_2(t, x) - f_2(t, x_i)\|^p \leq \frac{\varepsilon}{2^{p-1}}\Big\}$$

Let $t \in \mathbb{R}$ and $x \in K$, there exists $i_0 \in \{1, \dots, m\}$ such that

$$\mathbb{E}\|f_2(t,x) - f_2(t,x_{i_0})\|^p \leq \frac{\varepsilon}{2^{p-1}},$$

therefore

$$\begin{aligned}
\mathbb{E}\|f_2(t,x_1(t))\|^p &\leq 2^{p-1}\|f_2(t,x_1(t)) - f_2(t,x_{i_o}(t))\|^p + 2^{p-1}\mathbb{E}\|f_2(t,x_{i_o}(t))\|^p \\
&\leq \varepsilon + 2^{p-1}\mathbb{E}\|f_2(t,x_{i_o}(t))\|^p \\
&\leq \varepsilon + 2^{p-1}\sum_{i=1}^{m}\mathbb{E}\|f_2(t,x_{i_o}(t))\|^p.
\end{aligned}$$

It follows that

$$\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_1(\theta))\|^p d\mu(t) \leq \Big(\frac{\varepsilon\mu([-\tau,\tau])}{\nu([-\tau,\tau])} + \sum_{i=1}^{m}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_i(\theta))\|^p d\mu(t)\Big).$$

By the fact $\forall i \in \{1,\dots,m\}$, $\lim_{\tau\to+\infty}\sum_{i=1}^{m}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_i(\theta))\|^p d\mu(t) = 0$, we deduce that

$$\limsup_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\mathbb{E}\|f_2(\theta,x_1(\theta))\|^p d\mu(t) \leq \varepsilon\delta.$$

Therefore $t \mapsto f_2(t,x_1(t)) \in \mathscr{E}_p(\mathbb{R}\times L^p(\Omega,H), L^p(\Omega,H),\mu,\nu,r)$. ■

Proposition 4.17. *Assume that ($\boldsymbol{H}_4$) holds. Then the space $PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ is invariant by translation, that is $f \in PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ implies $f_\alpha \in PAA(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$ for all $\alpha\in\mathbb{R}$.*

Lemma 4.18. [6] *Let $G : [0,T]\times\Omega \to \mathscr{L}(L^p(\Omega,H))$ be an $\mathcal{F}_t$-adapted measurable stochastic process satisfying*

$$\int_0^T \mathbb{E}\|G(t)\|^2 < \infty$$

almost surely, where $\mathscr{L}(L^p(\Omega,H))$ denote the space of all linear operators from $L^p(\Omega,H)$ to itself. Then for any $p \geq 1$, there exists a constant $C_p > 0$ such that

$$\mathbb{E}\sup_{0\leq s\leq T}\Big\|\int_0^T G(s)dW(s)\Big\|^p \leq C_p\mathbb{E}\Big(\int_0^T \|G(s)\|^2 ds\Big)^{p/2}, T > 0.$$

We make the following assumption
($\mathbf{H}_5$) g is a stochastically bounded process in p-th mean sense.

Proposition 4.19. *Assume that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$) and ($\boldsymbol{H}_5$) hold and the semigroup $(U(t))_{t\geq 0}$ is hyperbolic. If f is bounded on $\mathbb{R}$, then there exists a unique bounded solution u of equation (1.1) on $\mathbb{R}$, given by*

$$\begin{aligned}
u_t &= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&+ \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)
\end{aligned}$$

where $\widetilde{B}_\lambda = \lambda(\lambda I - \mathcal{A}_{\mathcal{U}})^{-1}$ for$\lambda > \widetilde{\omega}$, Π^s and Π^u are projections of C_0 onto the stable and unstable subspaces respectively.

Proof. Let

$$u_t = v(t) + \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s),$$

where

$$v(t) = \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds$$

Let us first prove that u_t exists. The existence of $v(t)$ have proved by [23]. Now we show that the limit $\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds$ exists.

For each $t\in\mathbb{R}$ and by Lemma 4.18, we have

$$\begin{aligned}
\mathbb{E}\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\le C_p\mathbb{E}\Big(\int_{-\infty}^{t}\|\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))\|^2 ds\Big)^{p/2}\\
&\le C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\mathbb{E}\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}\|g(s)\|^2ds\Big)^{p/2}\\
&\le C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\mathbb{E}\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}\|g(s)\|^2ds\Big)^{p/2}\\
&\le C_p(\overline{M}\widetilde{M})|\pi^s|)^p\sum_{n=1}^{+\infty}\mathbb{E}\Big(\int_{t-n}^{t-n+1}e^{-2(\frac{p-2}{p})\omega(t-s)}\\
&\quad\times e^{-\frac{4}{p}\omega(t-s)}\|g(s)\|^2ds\Big)^{p/2}.
\end{aligned}$$

By using Hölder inequality, we obtain

$$\begin{aligned}
\mathbb{E}\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\le C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big[\Big(\int_{t-n}^{t-n+1}\big(e^{-2(\frac{p-2}{p})\omega(t-s)}\big)^{\frac{p}{p-2}}ds\Big)^{\frac{p}{p-2}}\Big]^{p/2}\\
&\quad\times\mathbb{E}\Big[\Big(\int_{t-n}^{t-n+1}\big(e^{-\frac{4}{p}\omega(t-s)}\|g(s)\|^2\big)^{\frac{p}{2}}ds\Big)^{\frac{2}{p}}\Big]^{p/2}\\
&\le C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\\
&\quad\times\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}\mathbb{E}\|g(s)\|^pds\Big).
\end{aligned}$$

Since g stochastic bounded process in p-th mean sense, then there exists, $M>0$ such that $\mathbb{E}\|g(s)\|^P\le M$.

It follows that

$$\begin{aligned}\mathbb{E}\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\leq C_pM(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big(\int_{t-n}^{t-n+1}e^{-2\omega(t-s)}ds\Big)^{p/2}\\ &\leq C_pM(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\frac{1}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2}\times e^{-\omega pn}\\ &\leq C_pM(\overline{M}\widetilde{M})|\Pi^s|)^p\frac{1}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2}\times\sum_{n=1}^{+\infty}e^{-\omega pn}.\end{aligned}$$

Since the serie $\sum_{n=1}^{+\infty}e^{-\omega pn}=1-\frac{1}{1-e^{-\omega p}}=\frac{e^{-\omega p}}{1-e^{-\omega p}}<\infty.$
It follows that

$$\mathbb{E}\|\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p\leq\gamma, \tag{4.1}$$

where

$$\gamma=\frac{C_pM(\overline{M}\widetilde{M})|\Pi^s|)^p}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2}\times\frac{e^{-\omega p}}{1-e^{-\omega p}}.$$

Set

$$F(n,s,t)=\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s)) \text{ for } n\in\mathbb{N} \text{ for } s\leq t.$$

For n is sufficiently large and $\sigma\leq t$, we have

$$\begin{aligned}\mathbb{E}\|\int_{-\infty}^{\sigma}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p &\leq C_p(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big(\int_{\sigma-n}^{\sigma-n+1}e^{-2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\\ &\quad\times\Big(\int_{\sigma-n}^{\sigma-n+1}e^{-2\omega(t-s)}\mathbb{E}\|g(s)\|^pds\Big)\\ &\leq C_pM(\overline{M}\widetilde{M})|\Pi^s|)^p\sum_{n=1}^{+\infty}\Big(\int_{\sigma-n}^{\sigma-n+1}e^{-2\omega(t-s)}ds\Big)^{p/2}\\ &\leq C_pM(\overline{M}\widetilde{M})|\Pi^s|)^p\frac{1}{(2\omega)^{p/2}}(e^{2\omega}-1)^{p/2}e^{-\omega p(t-\sigma)}\times\sum_{n=1}^{+\infty}e^{-\omega pn}\\ &\leq \gamma e^{-\omega p(t-\sigma)}.\end{aligned}$$

It follow that for n and m sufficiently large and $\sigma \leq t$, we have

$$\begin{aligned}
\mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^p &\leq \mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(n,s,t)dW(s) + \int_{\sigma}^{t} F(n,s,t)dW(s) \\
&\quad - \int_{-\infty}^{\sigma} F(m,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^p \\
&\leq 3^{p-1}\mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(n,s,t)dW(s)\Big\|^p + 3^{p-1}\mathbb{E}\Big\| \int_{-\infty}^{\sigma} F(m,s,t)dW(s)\Big\|^p \\
&\quad +3^{p-1}\mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^p \\
&\leq 2\times 3^{p-1}\gamma e^{-\omega p(t-\sigma)} \\
&\quad +3^{p-1}\mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s) - \int_{\sigma}^{t} F(m,s,t)dW(s)\Big\|^p
\end{aligned}$$

Since $\lim_{n\to+\infty} \mathbb{E}\Big\| \int_{\sigma}^{t} F(n,s,t)dW(s)\Big\|^p$ exists, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^p \leq 2\times 3^{p-1}\gamma e^{-\omega p(t-\sigma)}.$$

If $\sigma \to -\infty$, then

$$\limsup_{n,m\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s) - \int_{-\infty}^{t} F(m,s,t)dW(s)\Big\|^p = 0.$$

We deduce that

$$\lim_{n\to\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} F(n,s,t)dW(s)\|^p = \lim_{n\to\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)\Big\|^p$$

exists.

Therefore the limit $\lim_{n\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)$ exists. In addition, one can see from the equation (4.1) that the function

$$\eta_1 : t \to \lim_{n\to+\infty} \mathbb{E}\Big\| \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_n X_0 g(s))dW(s)\|^p$$

is bounded on $\mathbb{R}$. Similarly, we can show that the function

$$\eta_2 : t \to \lim_{n\to+\infty} \mathbb{E}\Big\| \int_{t}^{+\infty} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_n X_0 g(s))dW(s)\Big\|^p$$

is well defined and bounded on $\mathbb{R}$. ■

Proposition 4.20. *Assume that ($\boldsymbol{H}_5$) holds. Let $f, g \in AA(\mathbb{R}, X)$ and Γ be the mapping defined for $t \in \mathbb{R}$ by*

$$\begin{aligned}
\Gamma(f,g)(t) &= \Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\
&\quad + \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) + \lim_{\lambda\to+\infty} \int_{+\infty}^{t} \mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big](0).
\end{aligned}$$

Then $\Gamma(f,g)\in AA(\mathbb{R},L^p(\Omega,H))$.

Proof. The proof of this Proposition will be in two steps.

Step 1 : We will show that $\Gamma(f,g)$ is continuous. For $t_0\in\mathbb{R}$, we have

$$\begin{aligned}
\mathbb{E}\|\Gamma(f,g)(t)-\Gamma(f,g)(t_0)\|^p &= \mathbb{E}\Big\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\
&\quad+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\\
&\quad-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds-\lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\
&\quad-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big\|^p\\
&\leq 4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p\\
&\quad+4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds-\lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\|^p\\
&\quad+4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p\\
&\quad+4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p\\
&= 4^{p-1}(I_1+I_2+I_3+I_4).
\end{aligned}$$

We have

$I_1=\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p.$

Let $\sigma=s-t+t_0$ and by Hölder inequality, we have

$$\begin{aligned}
I_1 &= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-\sigma)\Pi^s(\widetilde{B}_\lambda X_0 f(\sigma+t-t_0))d\sigma-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p\\
&\leq \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s+t-t_0)-f(s)])ds\|^p\\
&\leq \mathbb{E}\Big(\overline{M}\widetilde{M}\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}|\Pi^s|\|f(s+t-t_0)-f(s)\|ds\Big)^p\\
&\leq \mathbb{E}\Big(\overline{M}\widetilde{M}|\Pi^s|\int_{-\infty}^{t_0}e^{-\frac{\omega(p-1)(t_0-s)}{p}}\times e^{-\frac{\omega(t-s)}{p}}\|f(s+t-t_0)-f(s)\|ds\Big)^p\\
&\leq (\overline{M}\widetilde{M}|\Pi^s|)^p\mathbb{E}\Bigg[\Big(\int_{-\infty}^{t}\Big(e^{-\frac{\omega(p-1)(t_0-s)}{p}}\Big)^{\frac{p}{p-1}}ds\Big)^{\frac{p-1}{p}}\times\Big(\int_{-\infty}^{t}\Big(e^{-\frac{\omega(t-s)}{p}}\|f(s+t-t_0)-f(s)\|\Big)^p ds\Big)^{\frac{1}{p}}\Bigg]^p\\
&\leq (\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}ds\Big)^{p-1}\times\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds\\
&\leq \frac{(\overline{M}\widetilde{M}|\Pi^s|)^p}{\omega^{p-1}}\times\int_{-\infty}^{t_0}e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds.
\end{aligned}$$

For an arbitrary sequence of real $\{t_n\}$ with $t_n \to t$ as $n \to +\infty$. By Lemma 4.4 and the defiition of $AA(\mathbb{R}, L^p(\Omega, H))$ we deduce that $f \in BC(\mathbb{R}, L^p(\Omega, H))$. So

$$e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p \to 0 \text{ as } , \to +\infty.$$

Hence

$$e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t_n-t_0)-f(s)\|^p \leq 2^p e^{-\omega(t_0-s)}\|f\|_\infty^p,$$

for every n sufficiently large. Note that

$$\int_{-\infty}^{t_0} 2^p e^{-\omega(t_0-s)}\|f\|_\infty^p ds < \infty.$$

Then according to Lebesgue dominated convergence theorem, we have

$$\lim_{n\to+\infty}\int_{-\infty}^{t_0} e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds = 0.$$

Since the arbitrariness of $\{t_n\}$, we deduce that

$$\lim_{t\to t_0}\int_{-\infty}^{t_0} e^{-\omega(t_0-s)}\mathbb{E}\|f(s+t-t_0)-f(s)\|^p ds = 0,$$

which implies that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds\|^p = 0 \tag{4.2}$$

Similarly, we can se that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds - \lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\|^p = 0. \tag{4.3}$$

Let $\widetilde{W}(\tau) = W(\tau+t-t_0) - W(t-t_0)$. One can see that $\widetilde{W}$ is a Winner process and has the same distribution as W. Let $\tau = s-t+t_0$. Then by Lemma 4.18 and Hölder inequality, we have

$$\begin{aligned}
I_3 &= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \\
&= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(\tau+t-t_0))dW(\tau+t-t_0) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p \\
&= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(\tau+t-t_0))d\widetilde{W}(\tau) - \lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))d\widetilde{W}(s)\|^p \\
&= \mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s+t-t_0)-g(s)])d\widetilde{W}(s) \\
&\leq C_p(\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t_0} e^{-2\omega(t_0-s)}ds\Big)^{\frac{P-2}{2}} \times \int_{-\infty}^{t_0} e^{-2\omega(t_0-s)}\mathbb{E}\|g(s+t-t_0)-g(s)\|^p ds.
\end{aligned}$$

By the similar arguments as above, we obtain

$$\lim_{t\to t_0}\int_{-\infty}^{t_0} e^{-2\omega(t_0-s)}\mathbb{E}\|g(s+t-t_0)-g(s)\|^p ds = 0,$$

which implies that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{-\infty}^{t_0}\mathcal{U}^s(t_0-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p=0. \tag{4.4}$$

Similarly, we can see that

$$\lim_{t\to t_0}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)-\lim_{\lambda\to+\infty}\int_{+\infty}^{t_0}\mathcal{U}^u(t_0-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\|^p=0. \tag{4.5}$$

From equations (4.2), (4.3), (4.4) and (4.5), we deduce that

$$\lim_{t\to t_0}\mathbb{E}\|\Gamma(f,g)(t)-\Gamma(f,g)(t_0)\|^p=0$$

and yield the continuity of $\Gamma(f,g)$.

Step 2 : Since $f,g\in AA(\mathbb{R},L^p(\Omega,H))$. Thus, for every sequence of real numbers $(s_m)m\in\mathbb{N}$, there exists a subsequence $(s_n)n\in\mathbb{N}$ and stochastic processes $\widetilde{f},\widetilde{g}:\mathbb{R}\to L^p(\Omega,H)$ which each $t\in\mathbb{R}$ such that

$$\lim_{n\to\infty}\mathbb{E}\|f(t+s_n)-\widetilde{f}(t)\|^p=0,\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{f}(t-s_n)-f(t)\|^p=0$$

and

$$\lim_{n\to\infty}\mathbb{E}\|g(t+s_n)-\widetilde{g}(t)\|^p=0,\quad \lim_{n\to\infty}\mathbb{E}\|\widetilde{g}(t-s_n)-g(t)\|^p=0.$$

Let

$$\begin{aligned} w(t+s_n) &= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s+s_s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s+s_n))ds \\ &\quad +\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s+s_n) \\ &\quad +\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s+s_n))dW(s+s_n)\Big] \end{aligned}$$

and

$$\begin{aligned} \widetilde{w}(t) &= \Big[\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\ &\quad +\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big]. \end{aligned}$$

Then we have

$$\begin{aligned}
\mathbb{E}\|w(t+s_n)-\widetilde{w}(t)\|^p &\leq 4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s+s_n)-f(s)]ds\|^p\\
&+4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0[f(s+s_n)-f(s)]ds\|^p\\
&4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s+s_n)-g(s)]d\widetilde{W}(s)\|^p\\
&+4^{p-1}\mathbb{E}\|\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0[g(s+s_n)-g(s)]d\widetilde{W}(s)\|^p,
\end{aligned}$$

where $\widetilde{W}(s)=W(s+s_n)-W(s)$. Note that W and $\widetilde{W}$ are two Wiener processes and have the same distribution. Then we have

$$\begin{aligned}
\mathbb{E}\|w(t+s_n)-\widetilde{w}(t)\|^p &\leq 4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t}e^{-\omega(t-s)}ds\Big)^{p-1}\times\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds\\
&+4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p\Big(\int_{+\infty}^{t}e^{\omega(t-s)}ds\Big)^{p-1}\times\int_{+\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds\\
&+4^{p-1}C_p(\overline{M}\widetilde{M}|\Pi^s|)^p\Big(\int_{-\infty}^{t}e^{-2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds\\
&+4^{p-1}C_p(\overline{M}\widetilde{M}|\Pi^u|)^p\Big(\int_{-\infty}^{t}e^{2\omega(t-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{+\infty}^{t}e^{2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds.
\end{aligned}$$

By similarly arguments as above, we have
$\lim_{n\to+\infty}\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds=0$, $\lim_{n\to+\infty}\int_{+\infty}^{t}e^{\omega(t-s)}\mathbb{E}\|f(s+s_n)-f(s)\|^p ds=0$,
$\lim_{n\to+\infty}\int_{-\infty}^{t}e^{-2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds=0$ and $\lim_{n\to+\infty}\int_{+\infty}^{t}e^{2\omega(t-s)}\mathbb{E}\|g(s+s_n)-g(s)\|^p ds=0$.
Thus

$$\lim_{n\to+\infty}\mathbb{E}\|w(t+s_n)-\widetilde{w}(t)\|^p=0.$$

Similarly, we have

$$\lim_{n\to+\infty}\mathbb{E}\|\widetilde{w}(t-s_n)-w(t)\|^p=0.$$

Therefore by **Steps 1** and **2**, we proved that $\Gamma(f,g)\in AA(\mathbb{R},L^p(\Omega,H))$.

Theorem 4.21. *Assume that* ($\boldsymbol{H}_3$) *and* ($\boldsymbol{H}_5$) *hold. Let* $f,g\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$*, then* $\Gamma(f,g)\in\mathscr{E}_p(\mathbb{R},L^p(\Omega,H),\mu,\nu,r)$.

Proof. We have

$$\begin{aligned}
\Gamma(f,g)(t) &= \lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds\\
&+\lim_{\lambda\to+\infty}\int_{-\infty}^{t}\mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s)+\lim_{\lambda\to+\infty}\int_{+\infty}^{t}\mathcal{U}^u(t-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s).
\end{aligned}$$

It follows that

$$\begin{aligned}\mathbb{E}\|\Gamma(f,g)(\theta)\|^p &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\ &\quad + \lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) \\ &\quad + \lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big\|^p.\end{aligned}$$

Then for $\tau>0$, using Lemma 4.18 we have

$$\begin{aligned}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}\mathbb{E}\Big(\overline{M}\widetilde{M}\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}|\Pi^s|\|f(s)\|ds\Big)^p d\mu(t) \\ &\quad + \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}\mathbb{E}\Big(\overline{M}\widetilde{M}\int_{+\infty}^{\theta}e^{\omega(\theta-s)}|\Pi^u|\|f(s)\|ds\Big)^p d\mu(t) \\ &\quad + \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}C_p\mathbb{E}\Big(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}|\Pi^s|^2\|g(s)\|^2ds\Big)^{p/2} d\mu(t) \\ &\quad + \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}4^{p-1}C_p\mathbb{E}\Big(\overline{M}^2\widetilde{M}^2\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}|\Pi^u|^2\|g(s)\|^2ds\Big)^{p/2} d\mu(t)\end{aligned}$$

By using Höder inequality, we obtain

$$\begin{aligned}\mathbb{E}\|\Gamma(f,g)(\theta)\|^p &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s))ds + \lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 f(s))ds \\ &\quad + \lim_{\lambda\to+\infty}\int_{-\infty}^{\theta}\mathcal{U}^s(\theta-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s))dW(s) \\ &\quad + \lim_{\lambda\to+\infty}\int_{+\infty}^{\theta}\mathcal{U}^u(\theta-s)\Pi^u(\widetilde{B}_\lambda X_0 g(s))dW(s)\Big\|^p .vspace*0.25cm\end{aligned}$$

Then for $\tau>0$, using Lemma 4.18 again, we have

$$\begin{aligned}&\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t) \\ &\leq 4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}ds\Big)^{p-1}\times\int_{-\infty}^{\theta}e^{-\omega(\theta-s)}\mathbb{E}\|f(s)\|^pds\Big]d\mu(t) \\ &\quad +4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{+\infty}^{\theta}e^{\omega(\theta-s)}ds\Big)^{p-1}\times\int_{+\infty}^{\theta}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^pds\Big]d\mu(t) \\ &\quad +4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{-\infty}^{\theta}e^{-2\omega(\theta-s)}\mathbb{E}\|g(s)\|^pds\Big]d\mu(t) \\ &\quad +4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big[\Big(\int_{+\infty}^{\theta}e^{2\omega(\theta-s)}ds\Big)^{\frac{p-2}{2}}\times\int_{+\infty}^{\theta}e^{2\omega(\theta-s)}\mathbb{E}\|g(s)\|^pds\Big]d\mu(t).\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t)\\
&\quad\leq \quad \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p}{\omega^{p-1}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\quad\leq \quad \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p}{\omega^{p-1}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\quad\leq \quad \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p}{(2\omega)^{\frac{p-2}{2}}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{2\omega r}\int_{-\infty}^{\theta}e^{-2\omega(t-s)}\mathbb{E}\|g(s)\|^p ds\Big)d\mu(t)\\
&\quad\leq \quad \frac{4^{p-1}(\overline{M}\widetilde{M}|\Pi^u|)^p}{(2\omega)^{\frac{p-2}{2}}}\times\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(t-s)}\mathbb{E}\|g(s)\|^p ds\Big)d\mu(t).
\end{aligned}$$

On the one hand using Fubini's theorem, we have

$$\begin{aligned}
\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{\omega r}\int_{-\infty}^{\theta}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(e^{\omega r}\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\leq e^{\omega r}\int_{-\tau}^{\tau}\int_{-\infty}^{t}e^{-\omega(t-s)}\mathbb{E}\|f(s)\|^p ds d\mu(t)\\
&\leq e^{\omega r}\int_{-\tau}^{\tau}\int_{0}^{\infty}e^{-\omega s}\mathbb{E}\|f(t-s)\|^p ds d\mu(t)\\
&\leq e^{\omega r}\int_{0}^{\infty}e^{-\omega s}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds.
\end{aligned}$$

By Theorem 3.15, we deduce that

$$\lim_{\tau\to+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)\to 0 \text{ for all } s\in\mathbb{R}_+$$

and

$$\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)\leq\frac{e^{-\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|f\|_\infty^p.$$

Similarly, we have

$$\lim_{\tau\to+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)\to 0 \text{ for all } s\in\mathbb{R}_+$$

and

$$\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)\leq\frac{e^{-\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|g\|_\infty^p.$$

Since f and g are two bounded functions, then the functions $s \mapsto \dfrac{e^{-\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|f\|_\infty^p$ and $s \mapsto \dfrac{e^{-2\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|g\|_\infty^p$ belong to $L^1(]0,\infty[)$ in view of the Lebesgue dominated convergence theorem, it follows that

$$e^{\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds = 0$$

and

$$e^{2\omega r}\lim_{\tau\to+\infty}\int_0^{+\infty}\frac{e^{-2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)ds = 0.$$

On the other hand, we have

$$\begin{aligned}
\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t) &\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{t-r}^{+\infty}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\leq \int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{t-r}^{+\infty}e^{\omega(t-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t)\\
&\leq \int_{-\tau}^{\tau}\int_{-\infty}^{r}e^{\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds d\mu(t)\\
&\leq \int_0^{+\infty}e^{\omega s}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds
\end{aligned}$$

By the same arguments, we have

$$\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\int_{\theta}^{+\infty}e^{2\omega(\theta-s)}\mathbb{E}\|f(s)\|^p ds\Big)d\mu(t) \leq \int_0^{+\infty}e^{2\omega s}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)ds$$

Similarly as above, we have the functions $s \mapsto \dfrac{e^{\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|f\|_\infty^p$ and $s \mapsto \dfrac{e^{2\omega s}\nu([(\tau,\tau)])}{\nu([-\tau,\tau])}\|g\|_\infty^p$ belong to $L^1(]0,\infty[)$ in view of the Lebesgue dominated convergence theorem, it follows that

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r}\frac{e^{\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|f(t-s)\|^p d\mu(t)ds = 0$$

and

$$\lim_{\tau\to+\infty}\int_{-\infty}^{r}\frac{e^{2\omega s}}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\|g(t-s)\|^p d\mu(t)ds = 0.$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\sup_{\theta\in[t-r,t]}\Big(\mathbb{E}\|\Gamma(f,g)(\theta)\|^p\Big)d\mu(t) = 0.$$

Thus, we obtain the desired result. ■

Our next objective is to show the existence of p-th ($p \geq 2$) mean (μ,ν)-pseudo almost automorphic solution of class r for the following problem

$$dx(t) = [Ax(t) + L(x_t) + f(t, x_t)]dt + g(t, x_t)dW(t), \text{ for } t \in \mathbb{R}, \tag{4.6}$$

where $f : \mathbb{R} \times \mathcal{C} \to L^p(\Omega, H)$ and $g : \mathbb{R} \times \mathcal{C} \to L^p(\Omega, H)$ are two processes.
For the sequel we make the following assumptions.
($\mathbf{H}_6$) Let $\mu, \nu \in \mathcal{M}$ and $f : \mathbb{R} \times C(([-r, 0], L^p(\Omega, H)) \to L^p(\Omega, H)$ p-th mean $cl(\mu, \nu)$-pseudo almost automorphic of class r such that there exists a positive constant L_f such that

$$\mathbb{E}\|f(t, \phi_1) - f(t, \Phi_2)\|^p \leq L_f \mathbb{E}\|\phi_1 - \phi_2\|^p \text{ for all } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in C(([-r, 0], L^p(\Omega, H)).$$

($\mathbf{H}_7$) Let $\mu, \nu \in \mathcal{M}$ and $g : \mathbb{R} \times C(([-r, 0], L^p(\Omega, H)) \to L^p(\Omega, H)$ p-th mean $cl(\mu, \nu)$ pseudo almost automorphic of class r such that there exists a positive constant L_g such that

$$\mathbb{E}\|g(t, \phi_1) - g(t, \Phi_2)\|^p \leq L_g \mathbb{E}\|\phi_1 - \phi_2\|^p \text{ for all } t \in \mathbb{R} \text{ and } \phi_1, \phi_2 \in C(([-r, 0], L^p(\Omega, H)).$$

($\mathbf{H}_8$) The instable space $U \equiv \{0\}$.

Theorem 4.22. *Let $p \geq 2$, assume that ($\boldsymbol{H}_0$), ($\boldsymbol{H}_1$), ($\boldsymbol{H}_4$), ($\boldsymbol{H}_6$) ($\boldsymbol{H}_7$) and ($\boldsymbol{H}_8$) hold. If*

$$(\overline{M}\widetilde{M}|\Pi|^s)^p \Big[\frac{L_f}{\omega^p} + \frac{L_g C_p}{(2\omega)^{\frac{p}{2}}}\Big] < \frac{1}{2^{p-1}},$$

then equation (4.6) *has a unique p-th mean $cl(\mu, \nu)$-pseudo almost automorphic solution of class r.*

Proof. Let x be a function in $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. From Theorem 4.14 the function $t \to x_t$ belongs to $PAA(C([-r, 0]); L^p(\Omega, H), \mu, \nu, r)$. Hence Theorem 4.16 implies that $g(.) = f(., x)$ is in $PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$. Since the unstable space $U \equiv \{0\}$, then $\Pi^u \equiv 0$. Consider the following mapping $H : PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r) \to PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$ defined for $t \in \mathbb{R}$ by

$$(\mathcal{H}x)(t) \;=\; \Big[\lim_{\lambda \to +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 f(s, x_s))ds + \lim_{\lambda \to +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0 g(s, x_s))dW(s)\Big](0).$$

Let $x_1, x_2 \in PAA(\mathbb{R}, L^p(\Omega, H), \mu, \nu, r)$, we have

$$\begin{aligned}
\mathbb{E}\|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|^p \;&=\; \mathbb{E}\Big\| \lim_{\lambda \to +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s, x_{1s}) - f(s, x_{2s})])ds \\
&\quad + \lim_{\lambda \to +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s, x_{1s}) - g(s, x_{2s})])dW(s)\Big\|^p \\
&\leq\; 2^{p-1}\mathbb{E}\Big\| \lim_{\lambda \to +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s, x_{1s}) - f(s, x_{2s})])ds\Big\| \\
&\quad + 2^{p-1}\mathbb{E}\Big\| \lim_{\lambda \to +\infty} \int_{-\infty}^t \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s, x_{1s}) - g(s, x_{2s})])dW(s)\Big\|^p \\
&\leq\; 2^{p-1}(I_1 + I_2).
\end{aligned}$$

By Hölder inequality, it follows that

$$
\begin{aligned}
I_1 &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[f(s,x_{1s})-f(s,x_{2s})])ds \Big\| \\
&\le \mathbb{E}\Big[(\overline{M}\widetilde{M}|\Pi|^s)^p \Big(\int_{-\infty}^{t} e^{-\omega(t-s)}ds \Big)^{p-1} \times \int_{-\infty}^{t} e^{-\omega(t-s)} \|f(s,x_{1s})-f(s,x_{2s})\|^p \Big] \\
&\le (\overline{M}\widetilde{M}|\Pi|^s)^p \Big(\int_{-\infty}^{t} e^{-\omega(t-s)}ds \Big)^{p-1} \times \int_{-\infty}^{t} e^{-\omega(t-s)} \mathbb{E}\|f(s,x_{1s})-f(s,x_{2s})\|^p ds \\
&\le \frac{(\overline{M}\widetilde{M}|\Pi|^s)^p}{\omega^{p-1}} \int_{-\infty}^{t} e^{-\omega(t-s)} L_f \mathbb{E}\|x_{1s}-x_{2s}\|^p ds \\
&\le \frac{(\overline{M}\widetilde{M}|\Pi|^s)^p}{\omega^{p-1}} L_f \sup_{t\in\mathbb{R}} \|x_1(t)-x_2(t)\|^p \Big(\int_{-\infty}^{t} e^{-\omega(t-s)}ds \Big) \\
&\le \frac{(\overline{M}\widetilde{M}|\Pi|^s)^p}{\omega^{p}} L_f \|x_1-x_2\|_\infty^p.
\end{aligned}
$$

By Hölder inequality and by Lemma 4.18, we have

$$
\begin{aligned}
I_2 &= \mathbb{E}\Big\| \lim_{\lambda\to+\infty} \int_{-\infty}^{t} \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s,x_{1s})-g(s,x_{2s})])dW(s) \Big\|^p \\
&\le C_p \mathbb{E}\Big[\lim_{\lambda\to+\infty} \int_{-\infty}^{t} \Big\| \mathcal{U}^s(t-s)\Pi^s(\widetilde{B}_\lambda X_0[g(s,x_{1s})-g(s,x_{2s})]) \|^2 ds \Big]^{p/2} \\
&\le (\overline{M}\widetilde{M}|\Pi|^s)^p C_p \Big(\int_{-\infty}^{t} e^{-\omega(t-s)}ds \Big)^{\frac{p-2}{2}} \times \int_{-\infty}^{t} e^{-2\omega(t-s)} \mathbb{E}\|g(s,x_{1s})-g(s,x_{2s})\|^p ds \\
&\le \frac{C_p(\overline{M}\widetilde{M}|\Pi|^s)^p}{(2\omega)^{\frac{p-2}{2}}} \int_{-\infty}^{t} e^{-2\omega(t-s)} L_g \mathbb{E}\|x_{1s}-x_{2s}\|^p ds \\
&\le \frac{C_p(\overline{M}\widetilde{M}|\Pi|^s)^p}{(2\omega)^{\frac{p-2}{2}}} L_g \sup_{t\in\mathbb{R}} \mathbb{E}\|x_1(t)-x_2(t)\|^p \Big(\int_{-\infty}^{t} e^{-2\omega(t-s)}ds \Big) \\
&\le \frac{C_p(\overline{M}\widetilde{M}|\Pi|^s)^p}{(2\omega)^{\frac{p}{2}}} L_g \|x_1-x_2\|_\infty^p.
\end{aligned}
$$

Thus we have

$$
\mathbb{E}\|(\mathcal{H}x_1)(t)-(\mathcal{H}x_2)(t)\|^p \quad \le \quad 2^{p-1}(\overline{M}\widetilde{M}|\Pi|^s)^p \Big[\frac{L_f}{\omega^p} + \frac{L_g C_p}{(2\omega)^{\frac{p}{2}}} \Big] \|x_1-x_2\|_\infty^p
$$

This means that $\mathcal{H}$ is a strict contraction. Thus by Banach's fixed point theorem, $\mathcal{H}$ has a unique fixed point u in $PAA(\mathbb{R}; L^p(\Omega,H),\mu,\nu,r)$. We conclude that equation (4.6), has one and only one p-th mean $cl(\mu,\nu)$-pseudo almost automorphic solution of class r. ■

Proposition 4.23. *Let $p \ge 2$, assume that $(\boldsymbol{H}_0)$, $(\boldsymbol{H}_1)$ and $(\boldsymbol{H}_4)$ hold, f, g are lipschitz continuous with respect the second argument. If*

$$
Lip(f) = Lip(g) < \frac{1}{2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p \Big[\frac{1}{\omega^p} + \frac{1}{(2\omega)^{\frac{p}{2}}} \Big]},
$$

then(4.6) *has a unique p-th mean* $cl(\mu,\nu)$*-pseudo almost automorphic of class* r*, where* $Lip(f)$ *and* $Lip(g)$ *are respectively the lipschitz constants of* f *and* g*.*

Proof. Let us pose $k = Lip(f) = Lip(g)$, we have

$$\begin{aligned}\mathbb{E}\|(\mathcal{H}x_1)(t) - (\mathcal{H}x_2)(t)\|^p &\leq 2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big[\frac{k}{\omega^p} + \frac{k}{(2\omega)^{\frac{p}{2}}}\Big]\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t) - x_2(t)\|^p \\ &\leq 2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p k\Big[\frac{1}{\omega^p} + \frac{1}{(2\omega)^{\frac{p}{2}}}\Big]\sup_{t\in\mathbb{R}}\mathbb{E}\|x_1(t) - x_2(t)\|^p\end{aligned}$$

Consequently $\mathcal{H}$ is a strict contraction if

$$k < \frac{1}{2^{p-1}(\overline{M}\widetilde{M}|\Pi^s|)^p\Big[\frac{1}{\omega^p} + \frac{1}{(2\omega)^{\frac{p}{2}}}\Big]}$$

■

5. Application

For illustration, we propose to study the existence of solutions for the following model

$$\begin{cases} dz(t,x) = -\frac{\partial^2}{\partial x^2}z(t,x)dt + \Big[\int_{-r}^0 G(\theta)z(t+\theta,x)d\theta + \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big) + \arctan(t) + \int_{-r}^0 h(\theta, z(t+\theta,x)\Big)d\theta\Big]dt \\ \qquad + \Big[\sin\Big(\frac{1}{2+\cos t+\cos\sqrt{3}t}\Big) + \arctan(t) + \int_{-r}^0 h(\theta, z(t+\theta,x)\Big)d\theta\Big]dW(t) \text{ for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \\ z(t,0) = z(t,\pi) = 0 \text{ for } t\in\mathbb{R}, \text{ and } x\in[0,\pi], \end{cases} \tag{5.1}$$

where $G : [-r, 0] \to \mathbb{R}$ is continuous function and $h : [-r, 0] \to \mathbb{R}$ is lipschitz continuous with the respect of the second argument. $W(t)$ is a two-sided standard Brownian motion with values in separable Hilbert space H. To rewrite equation (5.1) in abstract form, we introduce the space $H = L^2((0,\pi))$. Let $A : D(A) \to L^2((0,\pi))$ defined by

$$\begin{cases} D(A) = H^2(0,\pi) \cap H^1(0,\pi) \\ Ay(t) = y''(t) \text{ for } t\in(0,\pi) \text{ and } y\in D(A). \end{cases}$$

Then A generates a C_0-semigroup $(\mathcal{U}(t))_{\geq 0}$ on $L^2((0,\pi))$ given by

$$(\mathcal{U}(t)x)(r) = \sum_{n=1}^{+\infty} e^{-n^2\pi^2 t} < x, e_n >_{L^2} e_n(r),$$

where $e_n(r) = \sqrt{2}\sin(n\pi r)$ for $n = 1, 2\dots$ and $\|\mathcal{U}(t)\| \leq e^{-\pi^2 t}$ for all $t \geq 0$. Thus $\overline{M} = 1$ and $\omega = \pi^2$. Then A satisfies the Hille-Yosida conditions in $L^2(0,\pi)$. Moreover the part A_0 of A in $\overline{D(A)}$ is the generator of compact semigroup. It follows that $(\mathbf{H}_0)$ and $(\mathbf{H}_1)$ are satisfied.
We define $f : \mathbb{R}\times\mathcal{C} \to L^2((0,\pi))$ and $L : \mathcal{C} \to L^2(\Omega, H)$ as follows

$$f(t,\phi)(x) = \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{2}t}\Big) + \arctan(t) + \int_{-r}^0 h(\theta, \phi(\theta)(x))d\theta \text{ for } x\in(0,\pi) \text{ and } t\in\mathbb{R},$$

$$g(t,\phi)(x) = \sin\Big(\frac{1}{2+\cos t+\cos\sqrt{3}t}\Big) + \arctan(t) + \int_{-r}^{0} h(\theta,\phi(\theta)(x))d\theta \text{ for } x\in(0,\pi) \text{ and } t\in\mathbb{R},$$

and

$$L(\phi)(x) = \int_{-r}^{0} G(\theta)(\phi(\theta)(x) \text{ for } -r\leq\theta \text{ and } x\in(0,\phi).$$

Let us pose $v(t) = z(t,x)$. Then equation (5.1) takes the following abstract form

$$dv(t) = [Av(t) + L(v_t) + f(t,v_t)]dt + g(t,v_t)dW(t) \text{ for } t\in\mathbb{R}. \tag{5.2}$$

Consider the measure μ and ν where its Randon-Nikodym derivates are respectively ρ_1 and ρ_2

$$\rho_1(t) = \begin{cases} 1 & \text{for } t>0 \\ e^t & \text{for } t\leq 0 \end{cases}$$

and

$$\rho_2(t) = |t| \text{ for } t\in\mathbb{R}$$

i.e $d\mu(t) = \rho_1(t)dt$ and $d\mu(t) = \rho_2(t)dt$, where dt denotes the Lebesgue measure on $\mathbb{R}$ and

$$\mu(A) = \int_A \rho_1(t)dt \text{ for } \nu(A) = \rho_2(t)dt \text{ for } A\in\mathcal{N}$$

From[14] $\mu,\nu\in\mathcal{M}$ satisfies Hypothesis $(\mathbf{H}_4)$.

$$\lim_{\tau\to+\infty}\frac{\mu([-\tau,\tau])}{\nu([-\tau,\tau])} = \limsup_{\tau\to+\infty}\frac{\displaystyle\int_{-r}^{0} e^t dt + \int_0^{\tau} dt}{\displaystyle 2\int_0^{\tau} tdt} = \limsup_{\tau\to+\infty}\frac{1+e^{-\tau}+\tau}{\tau^2} = 0 < \infty,$$

which implies that $(\mathbf{H}_2)$ is satisfied.

For $t\in\mathbb{R}$, $-\dfrac{\pi}{2}\leq\arctan(\theta)\leq\dfrac{\pi}{2}$, therefore for all $\theta\in[t-r,t]$ $\arctan(t-r)\leq\arctan(t)$. It follows that $|\arctan\theta-\dfrac{\pi}{2}| = \dfrac{\pi}{2}-\arctan\theta \leq |\arctan(t-r)-\dfrac{\pi}{2}| = \dfrac{\pi}{2}-\arctan(t-r)$ which implies that $|\arctan\theta-\dfrac{\pi}{2}|^p \leq |\arctan(t-r)-\dfrac{\pi}{2}|^p$, hence, we have

$$\sup_{\theta\in[t-r,t]}\mathbb{E}|\arctan\theta-\frac{\pi}{2}|^p \leq \mathbb{E}|\arctan(t-r)-\frac{\pi}{2}|^p.$$

On one hand, we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_0^{\tau}\mathbb{E}\sup_{\theta\in[t-r,t]}\mathbb{E}|\arctan\theta-\frac{\pi}{2}|^p dt &\leq \frac{1}{\nu([-\tau,\tau])}\int_0^{\tau}\mathbb{E}|\arctan(t-r)-\frac{\pi}{2}|^p dt \\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_0^{\tau}\mathbb{E}\big(\frac{\pi}{2}-\arctan(t-r)\big)^p dt \\
&\leq \frac{1}{\nu([-\tau,\tau])}\int_0^{\tau}\frac{\pi^p}{2^p}dt \\
&\leq \frac{\pi^p}{2^{p+1}\tau} \to 0, \text{ as } \tau\to+\infty.
\end{aligned}$$

On other hand, we have

$$\begin{aligned}
\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{0}\mathbb{E}\sup_{\theta\in[t-r,t]}\mathbb{E}|\arctan\theta-\frac{\pi}{2}|^{p}dt &\leq \frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{0}\frac{\pi^{p}}{2^{p}}e^{t}dt\\
&\leq \frac{\pi^{p}(1-e^{-\tau})}{2^{p+1}\tau}\to 0,\ \text{as } \tau\to+\infty.
\end{aligned}$$

Consequently

$$\lim_{\tau\to+\infty}\frac{1}{\nu([-\tau,\tau])}\int_{-\tau}^{\tau}\mathbb{E}\sup_{\theta\in[t-r,t]}\mathbb{E}|\arctan\theta-\frac{\pi}{2}|^{p}\,d\mu(t)=0.$$

It follows that $t\mapsto\arctan(t)-\dfrac{\pi}{2}$ is p-th mean (μ,ν)-ergodic of class r, consequently f is uniformly p-th mean (μ,ν)-pseudo almost automorphic of class r. Moreover L is bounded linear operator from $\mathcal{C}$ to $L^{2}(\Omega,H)$.

Let k be the lipschitz constant of h. Then by using Hölder-inequality for every $\varphi_1,\ \varphi_2\in\mathcal{C}$ and $t\geq 0$, we have

$$\begin{aligned}
\mathbb{E}\|f(t,\varphi_1)(x)-f(t,\varphi_2)(x)\|^{p} &= \mathbb{E}\Big\|\int_{-r}^{0}\Big(h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\Big)d\theta\Big\|^{p}\\
&\leq \Big[\int_{-r}^{p}\|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\|d\theta\Big]^{p}\\
&\leq \mathbb{E}\Big[\Big(\int_{-r}^{0}d\theta\Big)^{\frac{p-1}{p}}\times\Big(\int_{-r}^{0}\|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\|^{p}\Big)^{\frac{1}{p}}\Big]^{p}\\
&\leq r^{p-1}\int_{-r}^{0}\mathbb{E}\|h(\theta,\varphi_1(\theta)(x))-h(\theta,\varphi_2(\theta)(x))\|^{p}d\theta\\
&\leq r^{p-1}k\int_{-r}^{0}\mathbb{E}\|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)\|^{p}d\theta\\
&\leq r^{p}k\sup_{-r\leq\theta\leq 0}\mathbb{E}\|\varphi_1(\theta)(x)-\varphi_2(\theta)(x)\|^{p}\\
&\leq r^{p}k\alpha\sup_{-r\leq\theta\leq 0}\mathbb{E}\|\varphi_1(x)-\varphi_2(x)\|^{p}\ \text{for a certain } \alpha\in\mathbb{R}_{+}.
\end{aligned}$$

Consequently, we conclude that f and g are Lipschitz continuous and $cl(\mu,\nu)$-pseudo almost automorphic in p-th mean sense. Moreover, since h is stochastically bounded in p-th mean sense, $i.e$ $\mathbb{E}\|h(t,\phi(t))\|^p \leq \beta$. By Hölder inequality, we have

$$\begin{aligned}
\mathbb{E}\|g(t,\varphi(x))\|^p &= 1+\frac{\pi}{2}+\mathbb{E}\Big\|\int_{-r}^{0} h(\theta,\varphi(\theta)(x))\,d\theta\Big\|^p \\
&\leq \frac{2+\pi}{2}+\Big(\int_{-r}^{0} d\theta\Big)^{p-1}\times\int_{-r}^{0}\mathbb{E}\|h(\theta,\varphi(\theta)(x))\|^p d\theta \\
&\leq \frac{2+\pi}{2}+r^{p-1}\int_{-r}^{0}\beta d\theta \\
&\leq \frac{2+\pi}{2}+r^p\beta \\
&\leq \beta_1,\ \text{with } \beta_1=\frac{2+\pi}{2}+r^p\beta,
\end{aligned}$$

which implies that g satisfies $(\mathbf{H}_5)$.

For hyperbolicity, we suppose that

$(\mathbf{H}_8)$ $\displaystyle\int_{-r}^{0}|G(\theta)|d\theta<1.$

Proposition 5.1. [18] *Assume that $(\boldsymbol{H}_6)$ and $(\boldsymbol{H}_7)$ hold. The the seùigroup $(\mathcal{U}(t))_{t\geq 0}$ is hyperbolic.*

Then by Proposition 4.23, we deduce the following result

Theorem 5.2. *Under above assumptions, if Lip(h) is small enough large, then equation* (5.1) *has unique p-th mean cl(μ,ν)-pseudo almost automorphic solution of class r.*

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