On Hurwitz stability for families of polynomials

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April 20, 2023

Abstract

The robustness of a linear system in the view of parametric variations requires a stability analysis of a family of polynomials. If the parameters vary in a compact set A, then obtaining necessary and sufficient conditions to determine stability on the family F A is one of the most important tasks in the field of robust control. Two interesting classes of families arise when A is a *diamond* or a *box* of dimension $n+1$. These families will be denoted by F D n and F B n , respectively. In this paper a study is presented to contribute to the understanding of Hurwitz stability of families of polynomials F A . As a result of this study and the use of classical results found in the literature, it is shown the existence of an extremal polynomial f (α * , x) whose stability determines the stability of the entire family F A . In this case f (α * , x) comes from minimizing determinants and sometimes f (α * , x) coincides with a Kharitonov's polynomial. Thus another extremal property of Kharitonov's polynomials has been found. To illustrate the versatility/generality of our approach, this is addressed to families such as F D n and F B n , when n[?]5. Furthermore, the study is also used to obtain the maximum robustness of the parameters of a polynomial. To exemplify the proposed results, first, a family F D n is taken from the literature to compare and corroborate the effectiveness and the advantage of our perspective. Followed by two examples where the maximum robustness of the parameters of polynomials of degree 3 and 4 are obtained. Lastly, a family F B 5 is proposed whose extreme polynomial is not necessarily a Kharitonov's polynomial.

Received: Added at production | Revised: Added at production | Accepted: Added at production

DOI: xxx/xxxx

ARTICLE TYPE

On Hurwitz stability for families of polynomials

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Summary

The robustness of a linear system in the view of parametric variations requires a stability analysis of a family of polynomials. If the parameters vary in a compact set A, then obtaining necessary and sufficient conditions to determine stability on the family $\mathfrak{F}_A$ is one of the most important tasks in the field of robust control. Two interesting classes of families arise when A is a *diamond* or a *box* of dimension $n+1$. These families will be denoted by $\mathfrak{F}_{D_n}$ and $\mathfrak{F}_{B_n}$, respectively. In this paper a study is presented to contribute to the understanding of Hurwitz stability of families of polynomials $\mathfrak{F}_A$. As a result of this study and the use of classical results found in the literature, it is shown the existence of an extremal polynomial $f(\alpha^*, x)$ whose stability determines the stability of the entire family $\mathfrak{F}_A$. In this case $f(\alpha^*, x)$ comes from minimizing determinants and sometimes $f(\alpha^*, x)$ coincides with a Kharitonov's polynomial. Thus another extremal property of Kharitonov's polynomials has been found. To illustrate the versatility/generality of our approach, this is addressed to families such as $\mathfrak{F}_{D_n}$ and $\mathfrak{F}_{B_n}$, when $n \leq 5$. Furthermore, the study is also used to obtain the maximum robustness of the parameters of a polynomial. To exemplify the proposed results, first, a family $\mathfrak{F}_{D_n}$ is taken from the literature to compare and corroborate the effectiveness and the advantage of our perspective. Followed by two examples where the maximum robustness of the parameters of polynomials of degree 3 and 4 are obtained. Lastly, a family $\mathfrak{F}_{B_5}$ is proposed whose extreme polynomial is not necessarily a Kharitonov's polynomial.

KEYWORDS:

Polynomial family; Hurwitz stability; interval polynomials; diamond polynomials, robust stability; extremal property.

1 | INTRODUCTION

In the control theory framework, the design and tuning of a controller that ensures the efficient handling of a system is undoubtedly of great relevance. In this sense, three important aspects must be considered[1]: its *fragility* to the variation of its own parameters, its *performance* to a load disturbance or set-point change, and its *robustness* to the changes in the controlled process characteristics.

One of the most recurrent problems in physical (real-world) systems is that the proposed design and tuning of controllers have poor performance or below expectations. This may be due to the fact that these systems present parametric variations. In the framework of control theory, this problem is known as robustness of a controller under parametric uncertainty[2]. Among the main achievements proposed in this field is the obtaining of conditions to determine if a linear time invariant control system remains stable as the parameters vary over a set.

In this context, one of the most relevant problems is to determine the stability of a whole family of polynomials through the stability of a subfamily of it[3]. Currently, this continues to be a field of opportunity given the great variety of families of polynomials that can be proposed and its possible subfamilies. Above all, if the conditions required to test stability are more relaxed and less complicated to apply. In the literature there are many results about stability of families of polynomials, among which are those that can be described as interval polynomials[4], polytope of polynomials[5], rays and cones of polynomials[6,7], segments of polynomials[8] and diamond of polynomials[9], to mention a few. All of them use fundamental results such as the Edge Theorem[5,10] and its generalization[11], the Zero Exclusion Principle[12,13], the concept of convex directions[14], and the boundary crossing theorem[15], among others. In[4], the stability analysis of systems with parametric uncertainty is through a study of their corresponding family of interval polynomials. The result surprised the scientific community when it was shown that to prove the stability of a family of interval polynomials it is only necessary to test the stability of four polynomials. Since then, much research has been focused on this type of families with the intention of extending and improving this result. In[16], given a Hurwitz stable polynomial of the form $P(s) = s^n + \sum_{k=1}^{n} t_k s^{n-k}$, the Kharitonov's Theorem is used to obtain the maximum interval centered at $t = [t_1, t_2, \dots, t_n]$. In other words, the maximum $\epsilon_{max} > 0$ is determined such that the polynomial $P(s)$ remains stable on the interval $(t_i - \underline{w}_i\epsilon_{max}, t_i + \overline{w}_i\epsilon_{max})$. Here, $P(s)$ is stable for $t_i \in (\underline{w}_i, \overline{w}_i)$ by Kharitonov's Theorem. While, in[17], a method is proposed to obtain the largest hypersphere with center at $t = [t_1, t_2, \dots, t_n]$ for which a polynomial preserves its stability, depending on the case. However this method is purely geometric. In[18], an elementary proof of Kharitonov's Theorem using simple complex plane geometry is given. In[19], some results are obtained to determine the stability of a family of interval polynomials of degree n. Here, the stability of the family for $n = 3, 4$ and 5 is guaranteed by the stability of $1, 2$ and 3 of the Kharitonov's polynomials, respectively. In[20], a generalization of Kharitonov's Theorem is presented. This generalization provides necessary and sufficient conditions for the stability of a family of polynomials of the form $P(s) = \sum_{i=1}^{m} Q_i(s)P_i(s)$, where $Q_i(s)$ are fixed polynomials and $P_i(s)$ are interval polynomials. Here the problem is focused on solving the robust stability of the corresponding transfer function of the closed-loop system of a plant with a controller, whose coefficients (some or all) are subject to perturbations within prescribed ranges. The family of transfer functions corresponding to this box in the parameter space is referred to as an interval plant. Subsequently, in[21] the design of controllers type P, PI and PID for these interval plants is proposed. While, in[22] Kharitonov's Theorem is used to tune a PID controller for uncertain plants. Unlike previous papers, using Kharitonov's Theorem, the controller coefficients are selected within a non-conservative stability region, called the Kharitonov region, to stabilize uncertain plants and fulfill system specifications in terms of gain margins and phase margins. Finally, in[23] using Newton and Marclaurin inequalities some general necessary conditions for the stability of $\mathfrak{F}_{B_n}$ are founded.

In this paper a study is presented to contribute to the understanding of Hurwitz stability of families of polynomials $\mathfrak{F}_A$ when A is a compact set. It is shown that the stability of a family $\mathfrak{F}_A$ of polynomials of degree n whose coefficients vary in a compact parametric set A is determined by the stability of an extremal polynomial. This extremal polynomial comes from minimizing certain determinants and in some cases the extremal polynomial coincides with a Kharitonov's polynomial. Thus, another extremal property of Kharitonov's polynomials is obtained. The previous results are oriented towards a *family of diamond polynomials* $\mathfrak{F}_{D_n}$ and a *family of interval polynomials* $\mathfrak{F}_{B_n}$. This study suggests that a fewer number of conditions is required to test its stability compared with results found in the literature.

The rest of the paper is organized as follows. In Section 2, statement of the problem and specific contributions are given. While in Section 3, some criteria and concepts necessary to obtain the proposed results are presented. In Section 4, the results concerning the Hurwitz stability of families of polynomials are stated. Continuing with Section 5, where the proposed results are applied to some polynomials. The document ends with some conclusions in Section 6.

2 | PROBLEM STATEMENT AND CONTRIBUTION

We now formulate the problem to be studied, followed by a brief description of the contribution.

Problem statement

A function of the form

$$f(\alpha, x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_n x^n, \tag{1}$$

where $\alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1}$ and $\alpha_n \neq 0$, is called a *polynomial* of *degree n* with real *coefficients*. The polynomial (1) is said to be *stable* (*or Hurwitz stable*) if and only if all its roots lie in the open left-half of the complex plane.

For a set $A \subseteq \mathbb{R}^{n+1}$,

$$\mathfrak{F}_A = \{ f(\alpha, x) \, : \, \alpha \in A \}$$

is the family of all polynomials of the form (1) whose coefficients belong to A. As mentioned above, the topology of A determines the type of *family of polynomials* that is studied. In particular:

- If $A = D_n$ is an $(n+1)$-dimensional ball (*diamond*) with center $c = (c_0, c_1, \dots, c_n) \in \mathbb{R}^{n+1}$, and radius $r > 0$ with respect to the 1-norm[24] of the form

$$D_n = \{ \alpha \in \mathbb{R}^{n+1} \, : \, |\alpha_0 - c_0| + |\alpha_1 - c_1| + \cdots + |\alpha_n - c_n| \leq r \}, \tag{2}$$

  the collection $\mathfrak{F}_{D_n}$ is known as a *family of diamond polynomials*. This diamond has $2(n+1)$ vertices of the form $(c_0, \dots, c_{j-1}, c_j \pm r, c_{j+1}, \dots, c_n)$ for $j = 0, 1, \dots, n$.

- If $A = B_n$ is an $(n+1)$-dimensional *box* of the form

$$\begin{aligned} B_n &= [l_0, u_0] \times [l_1, u_1] \times \cdots \times [l_n, u_n] \\ &= \{ \alpha = (\alpha_0, \alpha_1, \dots, \alpha_n) \in \mathbb{R}^{n+1} \, : \, l_j \leq \alpha_j \leq u_j \text{ for } j = 0, 1, \dots, n \}, \end{aligned} \tag{3}$$

  the collection $\mathfrak{F}_{B_n}$ is known as a *family of interval polynomials*. The sets $[l_j, u_j]$ represent the interval uncertainty in the coefficients of polynomial (1). This box has 2^{n+1} vertices of the form $(\alpha_0, \alpha_1, \dots, \alpha_n)$, where $\alpha_j \in \{l_j, u_j\}$ for $j = 0, 1, \dots, n$.

Definition 1. [15] A family $\mathfrak{F}_A$ is said to be *stable* (or *Hurwitz stable*) if and only if each of its elements is a stable polynomial.

A criterion to determine the stability of a diamond polynomials is the following.

Theorem 1. (Barmish[9]) Assume that $|c_j| > r$ for $j = 0, 1, \dots, n$. The family $\mathfrak{F}_{D_n}$ is stable if and only if the eight polynomials

$$\begin{aligned} f(q^{1,2}, x) &= f(c, x) \pm r \\ f(q^{3,4}, x) &= f(c, x) \pm rx \\ f(q^{5,6}, x) &= f(c, x) \pm rx^{n-1} \\ f(q^{7,8}, x) &= f(c, x) \pm rx^n \end{aligned} \tag{4}$$

are stable. Here q^j, $j = 1, \dots, 8$, are vertices of the diamond D_n.

The best-known criterion to determine the stability of a family of interval polynomials is the following.

Theorem 2. (Kharitonov[4]) The family $\mathfrak{F}_{B_n}$ is stable if and only if the four polynomials

$$\begin{aligned} f(k^1, x) &= l_0 + l_1 x + u_2 x^2 + u_3 x^3 + l_4 x^4 + l_5 x^5 + \cdots \\ f(k^2, x) &= u_0 + u_1 x + l_2 x^2 + l_3 x^3 + u_4 x^4 + u_5 x^5 + \cdots \\ f(k^3, x) &= l_0 + u_1 x + u_2 x^2 + l_3 x^3 + l_4 x^4 + u_5 x^5 + \cdots \\ f(k^4, x) &= u_0 + l_1 x + l_2 x^2 + u_3 x^3 + u_4 x^4 + l_5 x^5 + \cdots \end{aligned} \tag{5}$$

are stable. Here k^j, $j = 1, \dots, 4$, are vertices of the box B_n.

From Theorems1 and 2 it follows that:

- There exist eight points (vertices) of the diamond D_n such that determine the stability of the family $\mathfrak{F}_{D_n}$

$$\begin{aligned} q^{1,2} &= (c_0 \pm r, c_1, c_2, \dots, c_{n-2}, c_{n-1}, c_n) \\ q^{3,4} &= (c_0, c_1 \pm r, c_2, \dots, c_{n-2}, c_{n-1}, c_n) \\ q^{5,6} &= (c_0, c_1, c_2, \dots, c_{n-2}, c_{n-1} \pm r, c_n) \\ q^{7,8} &= (c_0, c_1, c_2, \dots, c_{n-2}, c_{n-1}, c_n \pm r). \end{aligned} \tag{6}$$

- There exists four points (vertices) of the the box B_n such that determine the stability of the family $\mathfrak{F}_{B_n}$

$$\begin{aligned} k^1 &= (l_0, l_1, u_2, u_3, l_4, l_5, \dots) \\ k^2 &= (u_0, u_1, l_2, l_3, u_4, u_5, \dots) \\ k^3 &= (l_0, u_1, u_2, l_3, l_4, u_5, \dots) \\ k^4 &= (u_0, l_1, l_2, u_3, u_4, l_5, \dots). \end{aligned} \tag{7}$$

Undoubtedly, the following questions arise:

- For a set A, is it possible to determine necessary and sufficient conditions to ensure stability on the entire family $\mathfrak{F}_A$?
- Is it possible to reduce the number of polynomials (or points) needed to determine the stability of $\mathfrak{F}_{D_n}$ or $\mathfrak{F}_{B_n}$?
- For $\mathfrak{F}_{D_n}$ or $\mathfrak{F}_{B_n}$, are necessarily vertices of a diamond or a box, respectively, the points (or polynomials) that determine the stability of these families?

Remark 1. As far as we know, one of the best attempts to reduce the number of Kharitonov's polynomials needed to test the stability of a family $\mathfrak{F}_{B_n}$ is given in Anderson[19]. There, for $n = 3, 4$, and 5, the number of Kharitonov's polynomials required to check stability of $\mathfrak{F}_{B_n}$ is one, two, and three, respectively, instead of four. That is, the stability of $\mathfrak{F}_{B_{3,4,5}}$, is determined by one, two, and three, points of the form (7), respectively. Furthermore, the authors assure that the four Kharitonov's polynomials (or the four points (7)) are required for $n \geq 6$.

Contributions of this manuscript

- The evidence from this study suggests the existence of an *extreme point* $\alpha^* \in A \subseteq \mathbb{R}^{n+1}$, with A compact set, such that the stability of the polynomial $f(\alpha^*, x)$, which we will call *extremal polynomial*, determines the stability of the entire family $\mathfrak{F}_A$ of degree n.
- A result is proposed to show that the *extreme point* $\alpha^* \in A \subseteq \mathbb{R}^{n+1}$ is on the boundary of A.
- Based on the previous item, the findings of this study points towards the idea that the stability of families such as $\mathfrak{F}_{D_n}$ and $\mathfrak{F}_{B_n}$ can be guaranteed by the existence of at least one *extremal polynomial* $f(\alpha^*, x)$ and/or failing that, by fulfilling simple inequalities.
- In this case $f(\alpha^*, x)$ comes from minimizing determinants and sometimes $f(\alpha^*, x)$ coincides with a Kharitonov's polynomial. Then, we get another extremal property of Kharitonov's polynomials.
- To illustrate the results obtained, necessary and sufficient conditions to determine stability on $\mathfrak{F}_{D_n}$ and $\mathfrak{F}_{B_n}$ are given for $n \leq 5$. This results suggest that our approach can contribute for the understanding and relaxation of conditions to determine stability in families of polynomials.

3 | PRELIMINARY RESULTS

In this section we state the background theorems needed to prove the main results.

Lemma 1. (Stodola[25]) If $f(\alpha, x)$ of the form (1) is a stable polynomial, then all its coefficients have the same sign.

Remark 2. In what follows, we should assume that $\alpha_j > 0$ for $j = 0, 1, 2, \dots, n$.

The Hurwitz *principal determinants* $\Delta_1, \Delta_2, \dots, \Delta_n$ of the of the polynomial (1) are defined by

$$\Delta_k(\alpha) = \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & \cdots & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \cdots & 0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \alpha_{2k-3} & \alpha_{2k-4} & \alpha_{2k-5} & \cdots & \alpha_{k-2} \\ \alpha_{2k-1} & \alpha_{2k-2} & \alpha_{2k-3} & \cdots & \alpha_k \end{vmatrix}, \tag{8}$$

where $\alpha_{2i-j} = 0$ for $2i - j < 0$ or $2i - j > n$, $i, j = 1, 2, \ldots, k$, and $k = 1, 2, \ldots, n$. Using Remark 2, one of the classical criteria on the stability of polynomials can be formulated as follows.

Theorem 3. (Liénard-Chipart criterion[26]) The polynomial (1) is stable if and only if the determinants

$$\Delta_1(\alpha),\ \Delta_3(\alpha),\ \Delta_5(\alpha),\ \ldots$$

or

$$\Delta_2(\alpha),\ \Delta_4(\alpha),\ \Delta_6(\alpha),\ \ldots$$

are positive.

Next we will use the previous criterion to obtain a point $\alpha^* \in A$ such that its corresponding extremal polynomial $f(\alpha^*, x)$ determines the stability of the entire family $\mathfrak{F}_A$.

4 | MAIN RESULTS

In this section the above is used to obtain necessary and sufficient conditions of stability for a family $\mathfrak{F}_A$, when A is a compact set. This result is oriented to a *family of diamond polynomials* $\mathfrak{F}_{D_n}$ and a *family of interval polynomials* $\mathfrak{F}_{B_n}$.

4.1 | On the family $\mathfrak{F}_A$

We next say what is meant to be an extremal polynomial and its existence for a family of polynomials $\mathfrak{F}_A$, with A compact set.

Definition 2. A polynomial $f(\alpha^*, x)$ of the form (1) is said to be a *extremal polynomial* for a family $\mathfrak{F}_A$ if the stability of $f(\alpha^*, x)$ determines the stability of the entire family $\mathfrak{F}_A$.

Note that obtaining an extremal polynomial $f(\alpha^*, x)$ in $\mathfrak{F}_A$ is equivalent to obtaining an *extreme point* α^* in A. To obtain extremal polynomials or extreme points a criterion of stability is required. Next we use the Liénard-Chipart criterion.

Suppose that the coefficients of polynomial (1) belong to a compact set A whose elements have positive components. Since the determinants (8) are continuous functions of α, there exist points $a^1, a^2, \ldots, a^n \in A$ such that

$$\begin{aligned}
\Delta_1(a^1) &= \min\{\Delta_1(\alpha) : \alpha \in A\},\\
\Delta_2(a^2) &= \min\{\Delta_2(\alpha) : \alpha \in A\},\\
&\vdots\\
\Delta_n(a^n) &= \min\{\Delta_n(\alpha) : \alpha \in A\}.
\end{aligned} \tag{9}$$

Now, it is shown that the points $a^1, a^2, \ldots, a^n$ cannot be interior points of the set A. Then this points are on the boundary of A. A more precise but less general result can be found in Bialas[27]: for a stable family $\mathfrak{F}_{B_n}$ it is proved that a^n corresponds to a vertex of the box B_n .

Theorem 4. Assume determinants (9) are positive. Then these minimum values cannot be attained at interior points of the set A.

Proof. For $k = 1, 2, \ldots, n$, suppose that $a^k = (a_0^k, a_1^k, \ldots, a_n^k)$ is an interior point of A. Then there exists an $\varepsilon > 0$ in the interval $(0, 2\|a^k\|)$, such that $B_\varepsilon(a^k) \subset A$, where $B_\varepsilon(a^k)$ denotes the ball with center a^k and radius ε with respect to the Euclidean norm.

Let

$$\alpha = \begin{cases} (a_0^k, \lambda a_1^k, a_2^k, \lambda a_3^k, \ldots, a_{2m-2}^k, \lambda a_{2m-1}^k), & n = 2m-1\\ (a_0^k, \lambda a_1^k, a_2^k, \lambda a_3^k, \ldots, \lambda a_{2m-1}^k, a_{2m}^k), & n = 2m \end{cases}$$

where $m \geq 1$, and $\lambda = 1 - \frac{\varepsilon}{2\|a^k\|}$. By this choice, $0 < \lambda < 1$, and

$$\|\alpha - a^k\| = (1-\lambda)\left[(a_1^k)^2 + (a_3^k)^2 + \cdots + (a_{2m-1}^k)^2\right]^{1/2} \leq (1-\lambda)\|a^k\| = \frac{\varepsilon}{2} < \varepsilon.$$

Thus $\alpha \in B_\varepsilon(a^k)$, and by the multi-linearity of the determinant Δ_k, we have that

$$\Delta_k(\alpha) = \lambda^l \Delta_k(a^k) \quad \text{for} \quad k = 2l-1 \ \text{ or } \ k = 2l,$$

where $l = 1, 2, \ldots, m$. Since $\lambda < 1$, we have that $\Delta_k(\alpha) < \Delta_k(a^k)$ for $k = 1, 2, \ldots, n$. This contradiction proves the result. □

On the other hand, using the Theorem 1 we define

$$\Delta_o(\alpha) := \min\{\Delta_1(\alpha), \Delta_3(\alpha), \Delta_5(\alpha), \dots\} \tag{10}$$

and

$$\Delta_e(\alpha) := \min\{\Delta_2(\alpha), \Delta_4(\alpha), \Delta_6(\alpha), \dots\}. \tag{11}$$

Since Δ_o and Δ_e are continuous functions, there exist points $\alpha_o^*, \alpha_e^* \in A$ such that

$$\Delta_o(\alpha_o^*) = \min\{\Delta_o(\alpha) : \alpha \in A\}$$

and

$$\Delta_e(\alpha_e^*) = \min\{\Delta_e(\alpha) : \alpha \in A\}.$$

Theorem 5. Let A be a compact set. The family $\mathfrak{F}_A$ is stable if and only if there exists a stable extremal polynomial $f(\alpha^*, x)$, where α^* is α_e^* or α_o^*.

Proof. Note that

$$\Delta_o(\alpha_o^*) = \min\{\Delta_1(a^1), \Delta_3(a^3), \Delta_5(a^5), \dots\},$$

and

$$\Delta_e(\alpha_e^*) = \min\{\Delta_2(a^2), \Delta_4(a^4), \Delta_6(a^6), \dots\}.$$

Then the family $\mathfrak{F}_A$ is stable if and only if $\Delta_o(\alpha_o^*) > 0$ or $\Delta_o(\alpha_e^*) > 0$. The result follows from Definition 2. □

Remark 3. Since an extremal polynomial can be obtained from the inequalities $\Delta_o(\alpha_o^*) > 0$ or $\Delta_e(\alpha_e^*) > 0$, or an equivalent form, in what follows we will use the most convenient of these cases.

4.2 | On the family $\mathfrak{F}_{D_n}$

In this section we consider a compact set A which is a diamond or a ball in the 1-norm of the form (2).

Define the finite set

$$Q = \{q^1, q^2, q^3, q^4, q^5, q^6, q^7, q^8\} \subset D_n,$$

where q^i, $i = 1, \dots, 8$, are defined in (6). These are eight vertices of the *diamond* D_n. By Remark 3, we choose the odd determinants. Then there exist points $\alpha_o \in Q$, and $\alpha_o^* \in D_n$ such that

$$\Delta_o(\alpha_o) = \min\{\Delta_o(\alpha) : \alpha \in Q\}$$

and

$$\Delta_o(\alpha_o^*) = \min\{\Delta_o(\alpha) : \alpha \in D_n\}.$$

Note that the points α_o and α_o^* are not necessarily equal, whence the extremal polynomials $f(\alpha_o, x)$ and $f(\alpha_o^*, x)$ are not necessarily the same, but either one determines the stability of the family $\mathfrak{F}_{D_n}$.

Theorem 6. The family $\mathfrak{F}_{D_n}$ is stable if only if the extremal polynomial $f(q^*, x)$ is stable, where q^* is α_o^* or α_o.

Proof. We have that

$$\Delta_o(\alpha_o^*) \le \Delta_o(\alpha_o),$$

and by Theorems 1 and 5 the family $\mathfrak{F}_{D_n}$ is stable if and only if $\Delta_o(\alpha_o) > 0$ or $\Delta_o(\alpha_o^*) > 0$. Thus there always exists an extremal polynomial that determines the stability of the entire family $\mathfrak{F}_{D_n}$. □

4.3 | On the family $\mathfrak{F}_{B_n}$

Since most of the scientific community has directed its attention on families of the form (3), below we will emphasize our contribution with several results on this class of families.

Define the finite set

$$K = \{k^1, k^2, k^3, k^4\} \subset B_n,$$

where k^j for $j = 1, \dots, 4$ is defined in (7). By Remark 3, we choose the odd determinants. Then there exist points $\alpha_o \in K$, and $\alpha_o^* \in B_n$ such that

$$\Delta_o(\alpha_o) = \min\{\Delta_o(\alpha) : \alpha \in K\}$$

and

$$\Delta_o(\alpha_o^*) = \min\{\Delta_o(\alpha) : \alpha \in B_n\}.$$

Analogous to the previous section, the stability of the family $\mathfrak{F}_{B_n}$ is determined by any of the two extremal polynomials, which are not necessarily equal.

Theorem 7. The family $\mathfrak{F}_{B_n}$ is stable if only if the extremal polynomial $f(k^*, x)$ is stable, where k^* is α_o^* or α_o.

Proof. We have that

$$\Delta_o(\alpha_o^*) \leq \Delta_o(\alpha_o),$$

and the family $\mathfrak{F}_{B_n}$ is stable if and only if $\Delta_o(\alpha_o) > 0$ or $\Delta_o(\alpha_o^*) > 0$. Thus there always exists an extremal Kharitonov's polynomial that determines the stability of the entire family $\mathfrak{F}_{B_n}$. This complements the result obtained by Anderson[19]. □

Remark 4. In some cases, $k^* = \alpha_o^*$ corresponds to coefficients of a Kharitonov's polynomial, which is always true for $k^* = \alpha_o$. Thus these polynomials have an other extremal property than the one obtained in[28].

Next we obtain explicit necessary and sufficient conditions for the family $\mathfrak{F}_{B_n}$ to be stable when $n \leq 5$.

Proposition 1. The families $\mathfrak{F}_{B_1}$, $\mathfrak{F}_{B_2}$, and $\mathfrak{F}_{B_3}$ are stable if and only if the extremal polynomials $f(\alpha^*, x) = \alpha_0 + l_1 x$, $f(\alpha^*, x) = \alpha_0 + l_1 x + l_2 x^2$, and $f(\alpha^*, x) = u_0 + l_1 x + l_2 x^2 + u_3 x^3$ are stable, respectively.

Proof. If $n = 1$, then $B_1 = [l_0, u_0] \times [l_1, u_1]$, $\Delta_o = \Delta_1$, and

$$\Delta_o(\alpha_o^*) = l_1.$$

In this case $\alpha^* = \alpha_o^* = (\alpha_0, l_1)$, and the extremal polynomial is $f(\alpha^*, x) = \alpha_0 + l_1 x$.

If $n = 2$, then $B_2 = [l_0, u_0] \times [l_1, u_1] \times [l_2, u_2]$, $\Delta_o = \Delta_1$, and $\Delta_e = \Delta_2$. By Remark 3 we choose Δ_e, thus

$$\Delta_e(\alpha_e^*) = l_1 l_2.$$

In this case, $\alpha^* = \alpha_e^* = (\alpha_0, l_1, l_2)$, and the extremal polynomial is $f(\alpha^*, x) = \alpha_0 + l_1 x + l_2 x^2$.

If $n = 3$, then $B_3 = [l_0, u_0] \times [l_1, u_1] \times [l_2, u_2] \times [l_3, u_3]$, $\Delta_o = \min\{\Delta_1, \Delta_3\}$, and $\Delta_e = \Delta_2$. By Remark 3 we choose Δ_e, thus

$$\Delta_e(\alpha_e^*) = l_1 l_2 - u_0 u_3.$$

In this case, $\alpha^* = \alpha_e^* = (u_0, l_1, l_2, u_3)$, and the extremal polynomial is $f(\alpha^*, x) = u_0 + l_1 x + l_2 x^2 + u_3 x^3$. □

Remark 5. As an interesting fact, note that the components of the point α^* corresponds to the coefficients of the Kharitonov's polynomial $f(k^4, x) = u_0 + l_1 x + l_2 x^2 + u_3 x^3$. An equivalent conclusion is presented in[19], see Remark 1.

Proposition 2. For $n \geq 3$, a necessary condition for the family $\mathfrak{F}_{B_n}$ to be stable is

$$l_1 l_2 - u_0 u_3 > 0. \tag{12}$$

Proof. For $n \geq 3$, the determinants

$$\Delta_2(\alpha) = \begin{vmatrix} \alpha_1 & \alpha_0 \\ \alpha_3 & \alpha_2 \end{vmatrix} = \alpha_1\alpha_2 - \alpha_0\alpha_3$$

do not change. Since

$$\min_{\alpha \in B_n} \Delta_2(\alpha) = l_1 l_2 - u_0 u_3, \tag{13}$$

the family $\mathfrak{F}_{B_n}$ is stable if (13) is positive. $\square$

If $l_3 = u_3 = 1$, the conditions stated in Proposition 1 can be seen geometrically in Figure 1. Four of the eight vertices of the box B_3 (one red dot and three blue dots) correspond to the coefficients of Kharitonov's polynomials. The red vertex is $\alpha^* = (u_0, 1_1, l_2, u_3)$. It is observed that as long as the box B_3 keeps between the surfaces of the hyperbolic paraboloid $\alpha_0 = \alpha_1\alpha_2$ and the plane $\alpha_0 = 0$, the family $\mathfrak{F}_{B_3}$ is stable.

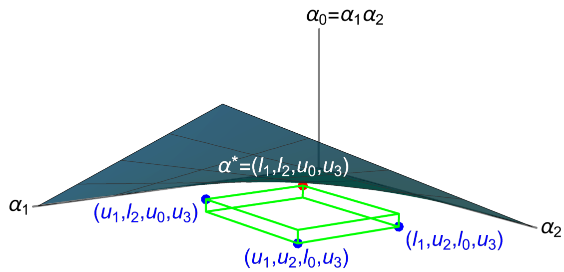

Figure 1 Stability of the family $\mathfrak{F}_{B_3}$.

Theorem 8. Let $d = l_2^2 - 4u_0u_4 > 0$. The family of interval polynomials $\mathfrak{F}_{B_4}$ is stable if and only if the extremal polynomial $f(\alpha^*, x)$, where $\alpha^* = (u_0, u_1, l_2, l_3, u_4), (u_0, l_1, l_2, u_3, u_4)$ or $(u_0, l_1, l_2, l_3, u_4)$, is stable or equivalently one of the inequalites

$$\frac{l_2 - \sqrt{d}}{2u_0} u_1 < l_3 \leq u_3 < \frac{l_2 + \sqrt{d}}{2u_0} l_1 \tag{14}$$

or

$$\min\left\{\Delta_3(u_0, u_1, l_2, l_3, u_4), \Delta_3(u_0, l_1, l_2, u_3, u_4), \Delta_3(u_0, l_1, l_2, l_3, u_4)\right\} > 0. \tag{15}$$

is satisfied.

Proof. From (10) we have that

$$\Delta_o(\alpha) = \min\left\{\Delta_1(\alpha), \Delta_3(\alpha)\right\}.$$

As mentioned in Remark 3, to obtain an extremal polynomial, we discard Δ_1 since it is always positive, and minimize a function related to Δ_3. By Theorem 4, the family $\mathfrak{F}_{B_4}$ is stable if and only if

$$\Delta_3(\alpha) = \alpha_1\alpha_2\alpha_3 - \alpha_0\alpha_3^2 - \alpha_1^2\alpha_4 > 0 \ \text{ for all } \ \alpha \in B_4.$$

Let $F_3(\alpha) = \Delta_3(u_0, \alpha_1, l_2, \alpha_3, u_4) = l_2\alpha_1\alpha_3 - u_0\alpha_3^2 - u_4\alpha_1^2$. Since

$$\min_{\alpha \in B_4} \Delta_3(\alpha) = \min_{\alpha \in B_4} F_3(\alpha),$$

we can minimize F_3 instead of Δ_3.

It is easily verified that

$$F_3(\alpha) = \left(\frac{d}{4u_0}\right)\alpha_1^2 - u_0\left(\alpha_3 - \frac{l_2}{2u_0}\alpha_1\right)^2. \tag{16}$$

Thus

$$F_3(\alpha) > 0 \quad \Longleftrightarrow \quad \frac{l_2 - \sqrt{d}}{2u_0}\alpha_1 < \alpha_3 < \frac{l_2 + \sqrt{d}}{2u_0}\alpha_1 \quad \text{for all} \quad \alpha \in B_4,$$

and we have that

$$\Delta_3(\alpha) > 0 \quad \text{for all} \quad \alpha \in B_4 \qquad \Longleftrightarrow \qquad \frac{l_2 - \sqrt{d}}{2u_0} u_1 < l_3 \le u_3 < \frac{l_2 + \sqrt{d}}{2u_0} l_1,$$

which is the stability condition (14).

Now since

$$\frac{\partial F_3}{\partial \alpha_1}(\alpha) = l_2\alpha_3 - 2u_4\alpha_1 < (>) \ 0 \quad \Longleftrightarrow \quad \alpha_3 < (>) \ \frac{2u_4}{l_2}\alpha_1,$$

$$\frac{\partial F_3}{\partial \alpha_3}(\alpha) = l_2\alpha_1 - 2u_0\alpha_3 < (>) \ 0 \quad \Longleftrightarrow \quad \alpha_3 > (<) \ \frac{l_2}{2u_0}\alpha_1,$$

and

$$\frac{l_2 - \sqrt{d}}{2u_0} < \frac{2u_4}{l_2} < \frac{l_2}{2u_0} < \frac{l_2 + \sqrt{d}}{2u_0},$$

we obtain equation (15). □

Remark 6. The coefficients $(u_0, u_1, l_2, l_3, u_4)$ and $(u_0, l_1, l_2, u_3, u_4)$ correspond to the Kharitonov's polynomials $f(k^2, x)$ and $f(k^4, x)$, respectively, while the coefficients $(u_0, l_1, l_2, l_3, u_4)$ do not.

The polynomial that renders the stability of the family $\mathfrak{F}_{B_4}$ is obtained from equation (15) evaluating the determinant Δ_3 at three points. However, this result can be improved by reducing the number of evaluations to two. To accomplish this, the following elementary lemma is needed.

Lemma 2. Let $\varphi, \psi : B \subset \mathbb{R}^n \to \mathbb{R}$ be continuous functions, where B is a compact set. The assertion

$$\varphi(\alpha) > 0 \quad \Longleftrightarrow \quad \psi(\alpha) > 0 \ \text{ for all } \ \alpha \in B,$$

is equivalent to

$$\min_{\alpha \in B} \varphi(\alpha) > 0 \quad \Longleftrightarrow \quad \min_{\alpha \in B} \psi(\alpha) > 0.$$

Proof. It follows from the definition of minimum and the Weierstrass theorem. □

We now use Lemma 2 to reduce the number of evaluations in Δ_3.

Theorem 9. Let $d = l_2^2 - 4u_0u_4 > 0$. The family of interval polynomials $\mathfrak{F}_{B_4}$ is stable if and only if the extremal polynomial $f(\alpha^*, x)$, where $\alpha^* = (u_0, l_1, l_2, u_3)$ or (u_0, u_1, l_2, l_3), is stable or equivalently

$$\min\{G_3(u_0, l_1, l_2, u_3, u_4), G_3(u_0, u_1, l_2, l_3, u_4)\} > 0, \tag{17}$$

where

$$G_3(\alpha) = \sqrt{d}\alpha_1 - 2u_0 \left| \alpha_3 - \frac{l_2}{2u_0}\alpha_1 \right|.$$

Proof. From (16), we obtain the equivalence

$$\Delta_3(\alpha) > 0 \quad \Longleftrightarrow \quad G_3(\alpha) > 0 \quad \text{for all} \quad \alpha \in B_4.$$

Since

$$\frac{\partial G_3}{\partial \alpha_1}(\alpha) = \begin{cases} \sqrt{d} + l_2, \ \alpha_3 > \frac{l_2}{2u_0}\alpha_1 \\ \sqrt{d} - l_2, \ \alpha_3 < \frac{l_2}{2u_0}\alpha_1 \end{cases}$$

and

$$\frac{\partial G_3}{\partial \alpha_3}(\alpha) = \begin{cases} -2u_0, & \alpha_3 > \frac{l_2}{2u_0}\alpha_1 \\ 2u_0, & \alpha_3 < \frac{l_2}{2u_0}\alpha_1 \end{cases}$$

we see that

$$\min_{\alpha \in B_4} G_3(\alpha) = \min\{G_3(u_0, u_1, l_2, l_3, u_4), G_3(u_0, l_1, l_2, u_3, u_4)\}.$$

By Lemma 2, we obtain equation (17). □

Remark 7. We note that:

- The coefficients $(u_0, u_1, l_2, l_3, u_4)$ and $(u_0, l_1, l_2, u_3, u_4)$ correspond to the Kharitonov's polynomials $f(k^2, x)$ and $f(k^4, x)$, respectively.
- As mentioned in Remark 1, in [19] the stability of $\mathfrak{F}_{B_4}$ is determined by two Kharitonov's polynomials. However, according to the Theorem 8, the stability of $\mathfrak{F}_{B_4}$ is determined by inequality (14).

Corollary 1. The family of interval polynomials $\mathfrak{F}_{B_5}$ is stable if and only if the extremal polynomial $f(\alpha_e^*, x)$ is stable, where $\alpha_e^* = a^2$ or a^4.

Proof. We observe that $\Delta_e(\alpha_e^*) > 0$ if and only if $\Delta_2(a^2) > 0$ and $\Delta_4(a^4) > 0$. By Theorem 5, we have that $\alpha_e^* = a^2$ or a^4. □

Due to the complexity of the calculations, few researchers have addressed this problem for $n \geq 5$. Next, a necessary condition for the stability of the family $\mathfrak{F}_{B_5}$ is given, while in Example 3, $f(\alpha_e^*, x)$ is obtained for a particular family $\mathfrak{F}_{B_5}$.

Proposition 3. A necessary condition for the family $\mathfrak{F}_{B_5}$ to be stable is

$$l_3^2 - 4u_1u_5 > 0 \quad \text{and} \quad l_2^2 - 4u_0u_4 > 0. \tag{18}$$

Proof. Assume that $\Delta_4(\alpha) > 0$ for all $\alpha \in \boldsymbol{B}_3$. The determinant

$$\Delta_4(\alpha) = \begin{vmatrix} \alpha_1 & \alpha_0 & 0 & 0 \\ \alpha_3 & \alpha_2 & \alpha_1 & \alpha_0 \\ \alpha_5 & \alpha_4 & \alpha_3 & \alpha_2 \\ 0 & 0 & \alpha_5 & \alpha_4 \end{vmatrix} = (\alpha_3\alpha_4 - \alpha_2\alpha_5)(\alpha_1\alpha_2 - \alpha_0\alpha_3) - (\alpha_0\alpha_5 - \alpha_1\alpha_4)^2$$

is an open downwards parabola in each variable α_j for $j = 0, 1, \ldots, 5$. Using a computer algebra system, it can be seen that if the maximun value of these parabolas is positive, then

$$\alpha_3^2 - 4\alpha_1\alpha_5 > 0 \quad \text{and} \quad \alpha_2^2 - 4\alpha_0\alpha_4 > 0 \quad \text{for all} \quad \alpha \in \boldsymbol{B}_3. \tag{19}$$

Thus we obtain the necessary condition (18). □

5 | ILLUSTRATION OF RESULTS

In this section some applications of the results given in the previous section are presented. First, a family $\mathfrak{F}_{D_n}$ is taken from the literature to corroborate the effectiveness and advantage of our approach. Followed by two example which show that these results can also be used to obtain maximum robustness boxes $\boldsymbol{B}_3$ and $\boldsymbol{B}_4$ of polynomials of degree 3 and 4. Finally, a family $\mathfrak{F}_{B_5}$ is proposed whose extreme polynomial is not necessarily a Kharitonov's polynomial.

5.1 | $\mathfrak{F}_{D_n}$ family example

Example 1. Let D_4 be the diamond with center $c = (3.49, 7.98, 6.49, 3, 1)$, and radius $r = 0.5$, see[9]. Then, the family $\mathfrak{F}_{D_4}$ is stable if and only if the extremal polynomial $f(q^*, x) = 3.49 + 7.48x + 6.49x^2 + 3x^3 + x^4$ is stable.

In effect, for the polynomial

$$f(c, x) = 3.49 + 7.98x + 6.49x^2 + 3x^3 + x^4 \tag{20}$$

we have that

$$Q = \{q^{1,2} = (3.49 \pm r, 7.98, 6.49, 3, 1), q^{3,4} = (3.49, 7.98 \pm r, 6.49, 3, 1), \\ q^{5,6} = (3.49, 7.98, 6.49, 3 \pm r, 1), q^{7,8} = (3.49, 7.98, 6.49, 3, 1 \pm r)\}.$$

If $r = 0.5$, then by using Theorem 6 and evaluating $\Delta_o(\alpha) = \min\{\Delta_1(\alpha), \Delta_3(\alpha)\}$ or $\Delta_e(\alpha) = \min\{\Delta_2(\alpha), \Delta_4(\alpha)\}$ at the eight points of Q it is verified that the extreme point is $q^* = \alpha_o = \alpha_e = q^4 = (3.49, 7.48, 6.49, 3, 1)$, and the result follows.

On the other hand, if $r = 0.7$, then the extreme points (extremal polynomials) of the family $\mathfrak{F}_{D_4}$ are:

$$q^* = \alpha_o = q^4 = (3.49, 7.28, 6.49, 3, 1) \Rightarrow f(q^*, x) = 3.49 + 7.28x + 6.49x^2 + 3x^3 + x^4, \quad \text{and} \\ q^* = \alpha_e = q^7 = (3.49, 7.98, 6.49, 3, 1.7) \Rightarrow f(q^*, x) = 3.49 + 7.98x + 6.49x^2 + 3x^3 + 1.7x^4,$$

which shows that an extremal polynomial is not either unique nor fixed.

Furthermore, note that once the center c of a diamond is chosen, determinants Δ_o and Δ_e depend only on the radius r. This parameter can be increased until the stability of one of the polynomials $f(\alpha, x)$ with $\alpha \in Q$ is lost. Thus we obtain a robustness measure, which can be call r^*, of the polynomial (20). Using a numerical method, one obtains the approximation $r^* = 0.9466052$. Figure 2 shows the beaviour of the functions $\Delta_o(r)$ and $\Delta_e(r)$ as r increases.

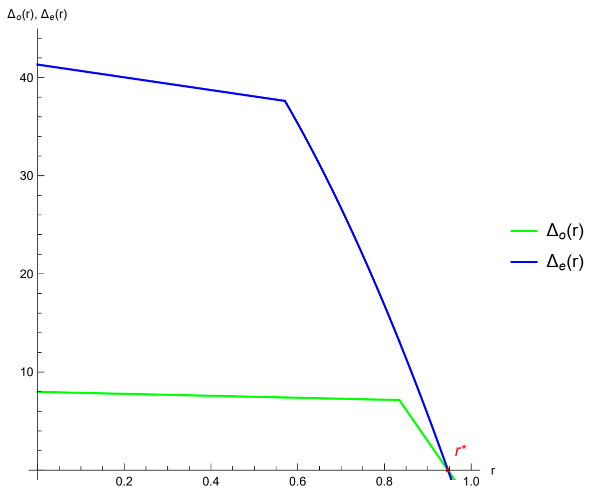

Figure 2 Robustness of polynomial (20)

5.2 | $\mathfrak{F}_{B_n}$ families example

Example 2. Consider a polynomial of the form

$$f(x) = -4r - (4s + 30)x + x^2 + x^3, \tag{21}$$

where r and s are two parameters. Then, the maximum variation of the two previous parameters without losing stability is

$$r \in \left[-\frac{u_0}{4}, -\frac{l_0}{4}\right] \quad \text{and} \quad s \in \left[-\frac{u_1 + 30}{4}, -\frac{l_1 + 30}{4}\right],$$

where $l_1 > u_0$. In effect, applying (12) of Proposition 2, we see that polynomial (21) is stable if and only if

$$0 < l_0 \le -4r \le u_0, \quad 0 < l_1 \le -(4s + 30) \le u_1, \quad \text{and} \quad l_1 > u_0.$$

Note that $\alpha_2 = l_2 = u_2 = 1$, and $\alpha_3 = l_3 = u_3 = 1$. Choosing $l_0 = 4, u_0 = 8, l_1 = 10$ and $u_1 = 50$, we have

$$-2 \leq r \leq -1, \text{ and } -20 \leq s \leq -10.$$

Example 3. Consider a polynomial of the form

$$f(x) = -4r - (4s + 30)x + 5x^2 + tx^3 + x^4, \tag{22}$$

where r, s and t are parameters. Then, the maximum variation of the two previous parameters without losing stability is

$$r \in \left[-\frac{u_0}{4}, -\frac{l_0}{4}\right], \quad s \in \left[-\frac{u_1 + 30}{4}, -\frac{l_1 + 30}{4}\right], \quad \text{and} \quad t \in \left(\frac{l_2 - \sqrt{d}}{2u_0}u_1, \frac{l_2 + \sqrt{d}}{2u_0}l_1\right)$$

where $d = l_2^2 - 4u_0u_4 > 0$. In effect, applying the result (14) of Theorem 8, we see that the polynomial (22) is stable if and only if

$$l_0 \leq -4r \leq u_0, \quad l_1 \leq -(4s + 30) \leq u_1, \quad \text{and} \quad \frac{l_2 - \sqrt{d}}{2u_0}u_1 < l_3 \leq u_3 < \frac{l_2 + \sqrt{d}}{2u_0}l_1, \tag{23}$$

where $d = l_2^2 - 4u_0u_4 > 0$. Note that $\alpha_2 = l_2 = u_2 = 5$, and $\alpha_4 = l_4 = u_4 = 1$. Choosing $l_0 = 1,\ u_0 = 4,\ l_1 = 4$ and $u_1 = 8$, we have

$$-1 \leq r \leq -0.25, \quad -9.5 \leq s \leq -8.5, \quad \text{and} \quad 2 < t \leq 4.$$

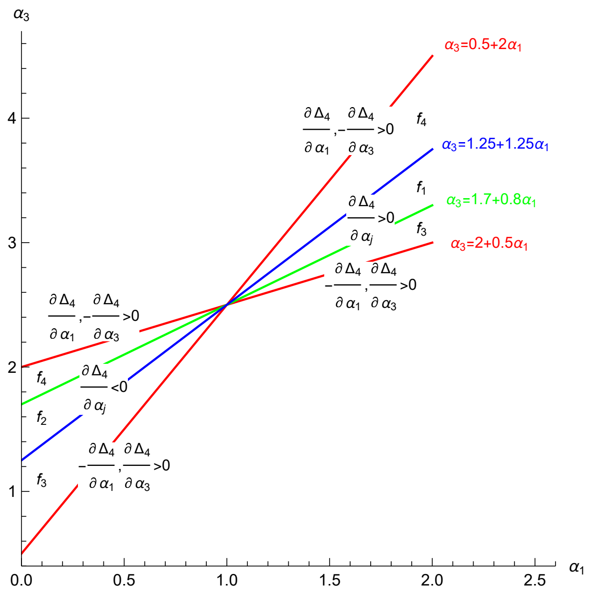

Figure 3 Sign of the partial derivatives $\frac{\partial \Delta_4}{\partial \alpha_1}$ and $\frac{\partial \Delta_4}{\partial \alpha_3}$.

Example 4. In this example, we propose a particular case of $\mathfrak{F}_{B_5}$ where the extremal polynomial is not a Kharitonov's polynomial. Consider a polynomial of the form

$$f(\alpha, x) = \alpha_0 + \alpha_1 x + \alpha_2 x^2 + \alpha_3 x^3 + \alpha_4 x^4 + x^5. \tag{24}$$

It is clear that

$$\min_{\alpha \in B_4} \Delta_2(\alpha) = \Delta_2(a^2),$$

where $a^2 = (u_0, l_1, l_2, u_3, \alpha_4, 1)$, and we define the polynomial

$$f_0(x) = u_0 + l_1 x + l_2 x^2 + u_3 x^3 + \alpha_4 x^4 + x^5. \tag{25}$$

Now, we are going to show that the point a^4 at which the determinant Δ_4 attains its minimum value does not necessarily correspond to the coefficients of a Kharitonov's polynomial.

Writing

$$\Delta_4(\alpha) = -\alpha_4^2\alpha_1^2 + \alpha_2\alpha_4\alpha_1\alpha_3 - \alpha_0\alpha_4\alpha_3^2 + (2\alpha_0\alpha_4 - \alpha_2^2)\alpha_1 + \alpha_0\alpha_2\alpha_3 - \alpha_0^2$$

and using the necessary condition (19), we see that Δ_4 is a hyperbola in the variables α_1, and α_3. Assuming that $\alpha_4 = \alpha_0 = 2$, and $\alpha_2 = 5$, we have that

$$\Delta_4(\alpha) = \frac{1}{2}\left(\alpha_3 + \alpha_1 - \frac{7}{2}\right)^2 - \frac{9}{2}\left(\alpha_3 - \alpha_1 - \frac{3}{2}\right)^2. \tag{26}$$

Figure 3 shows the sign of the partial derivatives $\frac{\partial\Delta_4}{\partial\alpha_1}(\alpha)$, and $\frac{\partial\Delta_4}{\partial\alpha_3}(\alpha)$. The red lines bound the region where $\Delta_4(\alpha) > 0$. If the box $[l_1, u_1]\times[l_3, u_3]$ is inside one of the six triangular regions, the symbols f_1, f_2, f_3 and f_4 indicate that in that region the minimum value of $\Delta_4(\alpha)$ is attained at the coefficients of the polynomials

$$\begin{aligned} f_1(x) &= 2 + l_1x + 5x^2 + l_3x^3 + 2x^4 + x^5 \\ f_2(x) &= 2 + u_1x + 5x^2 + u_3x^3 + 2x^4 + x^5 \\ f_3(x) &= f(k^3, x) = 2 + u_1x + 5x^2 + l_3x^3 + 2x^4 + x^5 \\ f_4(x) &= f(k^4, x) = 2 + l_1x + 5x^2 + u_3x^3 + 2x^4 + x^5, \end{aligned} \tag{27}$$

respectively. Note that f_1 and f_2 are not Kharitonov's polynomials, and since α_4 is fixed, $f_0 = f_4$. Thus the extremal polynomial is one of the polynomials given in (27). For the box $l_1 = 2, u_1 = 2.5, l_3 = 1.7 + 0.8u_1$, and $u_3 = 1.25 + 1.25l_1$ the extremal polynomial is f_1 because $2.5 = \Delta_2(2, l_1, 5, u_3, 2, 1) > \Delta_4(2, l_1, 5, l_3, 2, 1) = 2.24$.

6 | CONCLUSIONS

In this paper a robustness analysis is presented for a class of systems whose characteristic equation is a polynomial of degree n with real coefficients varying in a compact set $A \subset \mathbb{R}^{n+1}$. The evidence from this study indicates that the stability of the entire family $\mathfrak{F}_A$ can be determined by the stability of an extremal polynomial $f(\alpha^*, x)$, whose coefficients correspond to the coordinates of an *extreme point* $\alpha^* = \alpha_o^*$ or α_e^*. Also, it is shown that this point α^* is a point on the boundary of A. In this case $f(\alpha^*, x)$ comes from minimizing determinants and sometimes α^* coincides with the coefficients of a Kharitonov's polynomial, then the results support the idea of having found another extremal property of Kharitonov's polynomials. The versatility/generality of the proposed approach allows it to be oriented and applied to a *family of diamond polynomials* $\mathfrak{F}_{D_n}$ and a *family of interval polynomials* $\mathfrak{F}_{B_n}$ for $n \leq 5$. The results confirm that in some cases the necessary and sufficient conditions to determine stability on families of polynomials found in the literature can be relaxed/reduced, either by obtaining an extreme polynomial or by satisfying simple inequalities. Furthermore, the study can also be used to obtain maximum robustness of a polynomial, as depicted in Examples 1-3. Finally, in contrast to what is expected, for a family $\mathfrak{F}_{B_5}$ is possible to obtain an extremal polynomial which is not a Kharitonov's polynomial.

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