The H[?] optimal Control Problem of CSVIU Systems

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The H_∞ optimal Control Problem of CSVIU Systems

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Abstract

The paper devises a H_∞-norm theory for the CSVIU (control and state variations increase uncertainty) class of stochastic systems. This system model appeals to stochastic control problems to express the state evolution of a possibly nonlinear dynamic system restraint to poor modeling. Contrary to other H_∞ stochastic formulations that mimic deterministic models dealing with finite energy disturbances, the focus is on the H_∞ control with infinity energy disturbance signals. Thus, the approach portrays the persistent perturbations due to the environment more naturally. In this regard, it requires a refined connection between a suitable notion of stability and the systems' energy or power finiteness. It delves into the control solution employing the relations between H_∞ optimization and differential games, connecting the worst-case stability analysis of CSVIU systems with a perturbed Lyapunov type of equation. The norm characterization relies on the optimal cost induced by the *min-max* control strategy. The rise of a pure saddle point is linked to the solvability of a modified Riccati-type equation in a form known as a *generalized game-type Riccati equation*, which yields the solution of the CSVIU dynamic game. The emerging optimal disturbance compensator produces inaction regions in the sense that, for sufficiently minor deviations from the model, the optimal action is constant or null in the face of the uncertainty involved. A numerical example illustrates the synthesis.

KEYWORDS:
H_∞-norm control, stochastic stability, stochastic detectability, uncertainties stochastic models, stochastic dynamic games, min-max optimal control

1 | INTRODUCTION

To design a control system, one must count on a mathematical model of the process in view; thus, the model's trustworthiness concerning the plant is fundamental. Nevertheless, as a rule of thumb, uncertainties and inaccuracies are always present, and developing a control agent that can act satisfactorily, even while facing adverse or unknown dynamics, can be difficult. Stochastic systems offer well-known powerful tools to represent uncertainties and disturbances of the current system.

The principle of CSVIU, Control and State Variations Increase Uncertainty [1,2,3], leads to a class of models that endure a not sound reliable instance using stochastic tools to deal with it. The reason is either that there are fewer data to validate a model fully or it is too complex. A control model could only be a rough caricature purposefully taken from the existing system. The CSVIU class of stochastic models applies to control systems and filtering problems [4] to convey the state evolution of a dynamic system with just a rough model of the existing plant. This paradigm looms many significant engineering and physics problems as well as other essential fields such as macroeconomic and finance, health (tumor growth [4]), and biology (maritime fishing [5]) problems.

Stochastic control theory aims at the stability analysis, optimal control, and performance of a related system problem subject to randomness. The special focus here is developing a H_∞-type norm theory for CSVIU systems. Let us start by examining the time-invariant system below, defined in a filtered probability space $(\Omega, \mathcal{F}, P, \{\mathcal{F}_k\}_{k\geq 0})$,

$$\hat{\Theta}_{\text{ctr}} := \begin{cases} x(k+1) = Ax(k) + (\sigma_x + \overline{\sigma}_x \operatorname{diag}(|x(k)|))\varepsilon^x(k) \\ \qquad\qquad + Bu(k) + (\sigma_u + \overline{\sigma}_u \operatorname{diag}(|u(k)|))\varepsilon^u(k) + \sigma\omega(k), \\ z(k) = Cx(k) + Du(k) + F\omega(k), \quad x(0) = x, \end{cases} \tag{1}$$

where $x = \{x(k)\}_{k\geq 0}, x_k \in \mathbb{R}^n$ and $u = \{u(k)\}_{k\geq 0}, u_k \in \mathbb{R}^m$ are the *state and control processes*; $z = \{z(k)\}_{k\geq 0}$ is the *output process*. Matrices $A, \sigma_x, \overline{\sigma}_x \in \mathbb{R}^{n\times n}$, $B, \sigma_u, \overline{\sigma}_u \in \mathbb{R}^{n\times m}$, $\sigma \in \mathbb{R}^{n\times r}$, $C \in \mathbb{R}^{p\times n}$ and $F \in \mathbb{R}^{p\times r}$ are the CSVIU model *system and output matrices*. Sequence $\omega = \{\omega(k)\}_{k\geq 0}$, $\omega(k) \in \mathbb{R}^r$ is named *disturbance trajectory*. It is a bounded deterministic function representing an undesirable persistent disturbance input. The $n+m$-dimensional process formed by $\{\varepsilon^x(k), \varepsilon^u(k)\}_{k\geq 0}$ is an i.i.d. noise sequence that accounts for the imperfections of the system due to poor modeling. It is a zero mean process with the joint covariance in the identity matrix form. The filtration $\{\mathcal{F}_k\}_{k\geq 0}$ contains the sub-σ-algebras $\mathcal{F}_k \subset \mathcal{F}$ generated by the random variables $\varepsilon^x(0), \varepsilon^u(0), \dots, \varepsilon^x(k), \varepsilon^u(k)$. For $x \in \mathbb{R}^n$, $\operatorname{diag}(|x|)$ is the n-dimensional diagonal matrix formed by taking $|x| = [\,|x_1|\ |x_2|\ \cdots\ |x_n|\,]^{\intercal}$ as its diagonal, where $|\cdot|$ is the absolute value.

The control process u belongs to the set of $\{\mathcal{F}_k\}$-adapted m-dimensional processes in feedback form, $\mathcal{U} := \{u : u(k) = u(k, x(k)) \in \mathbb{R}^m, 0 \leq k \leq \kappa\}$ with κ finite or infinite. $\mathcal{U}$ forms the admissible class of controls, and $k \to (x_k, u_k)$ or $k \to (z_k, x_k, u_k)$ refer to an admissible pair or triple of system $\hat{\Theta}_{\text{ctr}}$.

The CSVIU model $\hat{\Theta}_{\text{ctr}}$ can deal with poorly known stochastic systems. To illustrate, for $f : \mathbb{R}^n \to \mathbb{R}^n$ and σ as above, say that the actual system obeys the difference equation,

$$\zeta(k+1) = f(\zeta(k)) + \sigma\omega(k), \tag{2}$$

for some initial condition, near a point $\bar{\zeta}$. Assuming that the derivatives of f exist near such a point, let matrix A represent the Jacobian of f evaluate at $\bar{\zeta}$. In this regard, set $x(k) := \zeta(k) - \bar{\zeta}$ to write from (2) that

$$x(k+1) = Ax(k) + \Big(f(\bar{\zeta}) - \bar{\zeta}\Big) + \Big(\frac{\partial f}{\partial \zeta}\Big|_{\bar{\zeta}} - A\Big)x(k) + o(|\zeta(k) - \bar{\zeta}|^2) + \sigma\omega(k) \tag{3}$$

If $\bar{\zeta}$ is an equilibrium point for the zero noise version of (2), the first difference on the rhs of (3) is null, whereas the second difference also is provided that f is precisely known. When system (2) is not well known, these terms are hardly null since the assumed equilibrium point, and the dynamic matrix A may be rough representations of the actual values of such a vector and matrix. In addition, since (3) without the larger order terms turns into a linear approximation for (2), the simple linear representation.

$$x(k+1) = Ax(k) + \sigma\omega(k)$$

could be a deceiving model and falls short of a convincing portrait of the actual system. In addition to such a natural deficiency, precision is lost as the state drifts away from the better-known point $\bar{\zeta}$. The declining trust in the model due to the residuals pointed out above is accounted into the model $\hat{\Theta}_{\text{ctr}}$ through the additional noise terms associated with the sequence $\{\varepsilon^x(k)\}_{k\geq 0}$. They are model motivated and differ from the "natural" or original perturbation $\{\omega(k)\}_{k\geq 0}$.

The term $\sigma_x\varepsilon^x(k)$ accounts for the error due to the linear approximation residue, the offset $f(\bar{\zeta}) - \bar{\zeta}$. On the other hand, $\bar{\sigma}_x \operatorname{diag}(|x_t|)\varepsilon^x(k)$ considers the impact due to the adoption of an inaccurate representation of the dynamic system matrix A vis-a-vis the existing system. It also attempts to account for second and higher-order terms, as each state vector component displaces from the reference point $\bar{\zeta}$ (or $x = 0$). The deviations depending on the componentwise distances $|\zeta_i(k) - \bar{\zeta}| = |x_i(x)|, i = 1, \dots, n$ appears in system (1) in the term $\operatorname{diag}(|x|)$. These terms induce increasing variance in proportion to the square of such displacements to arouse uncertainty. Mutatis mutandis, the same reasoning applies to the system's control dependent terms $\hat{\Theta}_{\text{ctr}}$.

The H_∞-norm criterion evaluates the robustness of the system's stability. It measures the system's capacity to behave well while facing disturbances. Various current publications have clarified the growing interest in extending the H_∞-norm theory to stochastic systems and filtering. The representation of uncertainties of poorly known systems through stochastic perturbations is not uncommon in the control literature. For example, the stochastic multiplicative disturbance theory (SMD)[6,7] portrays the deviation of the linearized model from the system's nominal model via state and control-dependent noise terms.

Nevertheless, when it comes to the H_∞ problem, previous works assume ℓ_2 finite-energy exogenous disturbances[8,9,10,11,12,13,14,15], regardless of deterministic or stochastic approaches. They all presuppose a disturbance dependent-noise or a state-multiplicative disturbance that vanishes when the system reaches equilibrium[10,16,13,17,18], or deal with the finite horizon case[19,20]. Compared with other literature on system models, a noticeable feature of the CSVIU method is that infinite-energy ℓ_2

disturbance signals are accounted for in an infinite horizon approach. More naturally, it portrays the persistent perturbations a plant may face due to the environment. Although this representation of the disturbance appears less artificial, it requires an elaborate connection between the appropriate notion of stability and the finiteness of systems energy. Stability analysis appears in Section 2.

Energy and Cost measures

The systems' performance demands a quantitative treatment; still, due to the persistent nature of the disturbance, the usual notions of norms for signal and systems do not apply to system $\hat{\Theta}_{\rm ctr}$. We introduce some ℓ_2 energy measurements appropriate for stochastic signals and present the intended induced cost measure.

Consider the n-dimensional stochastic process $k \to y(k)$, $k \geq 0$ adapted to the previously introduced filtration $\{\mathcal{F}_k\}_{k\geq 0}$. For some $Q \succeq 0$ and $\kappa > 0$, define the $\ell_2(\Omega, \mathcal{F}, P)$ mean energy measurement,

$$\mathcal{E}^{\kappa}_{2,Q}(y) := E_x\Big[\sum_{k=0}^{\kappa} \|y(k)\|^2_Q \Big| x(0) = 0\Big], \quad y(0) = 0, \tag{4a}$$

where the expectation $E\big[\,\cdot\,|x(0)\big]$ stands for $E\big[\,\cdot\,|\mathcal{F}_0\big]$, and $\|y\|^2_Q = \langle y, Qy\rangle$ is a Q-weighted square norm of y for some $Q \succ 0$ ($\cdot \succ 0$ indicates positive definiteness). Define also the *Q-mean power measure* of $y(\cdot)$, given by

$$\hat{\mathcal{E}}_{2,Q}(y) = \limsup_{\kappa\to\infty} \frac{1}{\kappa}\mathcal{E}^{\kappa}_{2,Q}(y), \quad y(0) = 0, \tag{4b}$$

The measurements $\lim_{\kappa\to\infty} \mathcal{E}^{\kappa}_{2,Q}(y)$ and $\hat{\mathcal{E}}_{2,Q}(y)$ stand as possible stochastic H_2-norms[3,6].

An average performance mean measure is appropriate to deal with stochastic systems under persistent disturbance. In this paper, the disturbance ω comes as deterministic signal that has finite Q-mean power, namely, $\hat{\mathcal{E}}_{2,Q}(\omega) < \infty$ for each $Q \succ 0$. For the pertinent H_∞-norm problem, given $\gamma > 0$, a stochastic H_∞-norm problem is induced from quadratic functionals. Set the finite time horizon H_∞-measure as,

$$\mathrm{J}^{\kappa}_{\infty}(y) := \mathcal{E}^{\kappa}_{2,Q}(y) - \gamma^2\mathcal{E}^{\kappa}_{2,Q}(\omega), \quad \gamma = \left(\frac{\mathcal{E}^{\kappa}_{2,Q}(y)}{\mathcal{E}^{\kappa}_{2,Q}(\omega)}\right)^{1/2}, \quad \text{and} \quad \gamma^* = \sup_{0<\mathcal{E}^{\kappa}_{2,Q}(\omega)<\infty} \gamma, \tag{5}$$

for some $\kappa > 0$, in which γ is the energy gain and γ^* is the norm of the system. Hence, the norm γ^* gives the maximum factor by which the system magnifies the Q-mean power of any input. Define also a corresponding infinite horizon cost,

$$\mathfrak{P}_\infty(y) := \limsup_{\kappa\to\infty} \frac{1}{\kappa}\mathrm{J}^{\kappa}_{\infty}(y), \quad y(0) = 0, \tag{6}$$

whenever finite. The cost functional in (6) is a Cèsaro's summation form employed as a stochastic norm in[6,3]. In particular, if we set $y = z$, the measure in (6) accounts for the effect of input ω on the output z with emphasis on the asymptotic behavior of state and output processes.

The norm γ^* is also called the "Min-Max attenuation bound". In truth, this refers to the well-known Dynamic Game Theory, which introduces the prospect of differential games into the context of optimal controls. In general, it is of interest to obtain the value $\gamma^* \geq 0$ under which the upper value of the associated game with cost function $\mathfrak{P}_\infty$ is bounded above by zero, and the corresponding control law that achieves such an upper value. This chore motivates this paper, and given some gain $\gamma \geq \gamma^*$, we seek a stabilizing controller and sufficient conditions for it to exist.

As stated in[21], the hallmark of the theory of dynamic games resides in three chief features: the multiple agents involved in the game, the search for an optimal behavior of the agents, and the long-standing consequences of decisions. In addition, the formulation of H_∞ optimal control problems from a game-theoretical approach allows us to include in this list "the resilience concerning variability in the environment".

The framing of differential games applies here to the CSVIU model, and the structure of the quadratic functional (5) enables the evaluation of an optimal control policy or, equivalently, an optimal disturbance compensation control from the solution of a saddle point problem. A first characterization for the H_∞ norm of CSVIU systems arises from this construct:

Definition 1. The H_∞ norm is the gain $\gamma > 0$ that nullifies the Min-Max optimal cost $\min_u \max_\omega \mathfrak{P}_\infty(z)$.

It is undeniable that a stability sense is essential in dealing with dynamic systems, and the following definition accommodates the Q-mean power measure in (4b) to a stability notion that fits stochastic systems subject to persistent disturbance excitations. Let us consider $\hat{\Theta}$ the uncontrolled version of system $\hat{\Theta}_{\rm ctr}$, namely, for $\hat{\Theta}$, the matrices $B, \sigma^u, \bar{\sigma}^u$ and D are null. With that, the notion of stochastic stability and stabilizability reads as follows.

Definition 2. i) System $\hat{\Theta}$ is *stochastic stable* if the measurement $\hat{\mathcal{E}}_{2,Q}(x) \le \bar{c} < \infty$, $\forall x(0) = x_0 \in \mathbb{R}^n$ and $Q \succ 0$.
ii) System $\hat{\Theta}_{\mathrm{ctr}}$ is *stochastic stabilizable* if there exists $u \in \mathcal{U}$ that turns $\hat{\Theta}_{\mathrm{ctr}}$ into a stochastic stable system.

Since the stability notion stands for the worst-case scenario induced by disturbance ω, one can say of Definition 2 that $\hat{\Theta}$ is stable (or $\hat{\Theta}_{\mathrm{ctr}}$ is stabilizable) with respect to ω.

Section 2 discusses the relation between the power measurement in (6) and system stability in Definition 2 (i). A class of perturbed Lyapunov equations connects to the worst-case stability analysis, and the section sets up a link between the limit in (6) and the notion of stochastic stability.

As mentioned, the paper exploits the relation between H_∞ optimization and differential games to solve the H_∞ state feedback control problem. With this aim, Section 3 presents the linear-quadratic CSVIU dynamic game and solves the problem of optimal disturbance attenuation and stabilization for the class of uncertain dynamic models in (1). Exciting advances in the optimal control global law structure for CSVIU systems appear in[3]. The paper uses these findings to furnish the global disturbance compensator solution to the H_∞-optimal control problem from the general optimal saddle point of a CSVIU dynamic game. It also characterizes the so-called "inaction regions" of the CSVIU H_∞-optimal compensator.

Section 3.2 focus on the class of generalized game-type Riccati equations and their relation with the system's energy gain γ. It provides a more handleable definition for the system's norm relying on the Riccati equation's solvability and defines the *Suboptimal Problem*. The suboptimal norm problem has a practical appeal since it prevents the system from working close to stability boundaries. The suboptimal setting supplies a solution $\gamma > \gamma^*$ with $\gamma^* > 0$ being the norm of the system in Definition 1. Section 4 presents a stochastic algorithm for computing the expected value of a random term present in the optimal control law. It applies it to a numerical example to illustrate the use of the algorithm and the solution of a suboptimal norm problem. At the end, Section 5 offers some conclusions.

2 | ENERGY MEASUREMENTS AND STABILITY

The section deals with the worst-case system's analysis and states sufficient conditions for the stochastic system $\hat{\Theta}$ to be stable with a power gain γ. Here, one seeks to clarify the intended connection between the measure (6) and the notion of stochastic stability in Definition 2. It further weakens the stability requirement test by introducing a stochastic detectability notion.

Notation

For a square matrix $Y \in \mathbb{R}^{n\times n}$, $Y_d \in \mathbb{R}^n$ indicates the main diagonal of Y, and the diagonal matrix $\mathrm{Diag}(Y) \in \mathbb{R}^{n\times n}$ is made up by the main diagonal of Y, Y_d, and zero elsewhere. Let $\mathbb{S}^n$ stand for the real vector space of n-dimensional symmetric matrices endowed with the inner product $\langle X, Y\rangle = \mathrm{tr}(XY)$, where $\mathrm{tr}(\cdot)$ is the trace operator. $\mathbb{S}^{n+}$ denotes the cone of positive semidefinite matrix therein. For any $Y \in \mathbb{S}^{n+}$, $Y \succ 0$ ($Y \succeq 0$) designates Y as a positive (semi-) definite matrix and, if $Y, Z \in \mathbb{S}^{n+}$, $Z \succeq Y \Leftrightarrow Z - Y \succeq 0$. Analogously, $Y \prec 0$ ($Y \preceq 0$) denotes a negative (semi-) definite matrix. Besides, $\|Y\|$ stands for any matrix norm, and when Y is square, $r_\sigma(Y)$ is the spectral radius, and $\lambda^+(Y)$ and $\lambda^-(Y)$ denote its largest and smallest eigenvalues, respectively.

For a vector $v \in \mathbb{R}^n$, define $|v| := [\, |v_1| \; |v_2| \; \cdots \; |v_n| \,]^\intercal$ and set $\mathrm{diag}(v)$ as the diagonal matrix made up by v in the main diagonal and zero elsewhere. For $w, v \in \mathbb{R}^n$, $\langle w, v\rangle$ denotes the usual inner product, $w \cdot v$ denotes the Hadamard product, and the square (semi-)norms $\|v\|_Y^2$ stands for $\langle v, Yv\rangle$, in which $Y \in \mathbb{S}^{n+}$.

Associated with the data of a CSVIU system and a positive real number γ, define the operators $\Upsilon_\gamma(Y) : \mathbb{S}^{n+} \to \mathbb{S}^r$, $\Psi(Y) : \mathbb{S}^{n+} \to \mathbb{R}^{r\times n}$, $\Gamma(Y) : \mathbb{S}^{n+} \to \mathbb{R}^{r\times m}$, $\mathcal{Z}_x : \mathbb{S}^{n+} \to \mathbb{S}^{n+}$, $\mathcal{W}_x : \mathbb{S}^{n+} \to \mathbb{S}^n$ and $\varphi : \mathbb{S}^{n+} \to \mathbb{R}$, given by:

$$\Upsilon_\gamma(Y) = \sigma^\intercal Y \sigma + F^\intercal F - \gamma^2 I_r, \tag{7a}$$

$$\Psi(Y) = \sigma^\intercal Y A + F^\intercal C, \tag{7b}$$

$$\Gamma(Y) = \sigma^\intercal Y B + F^\intercal D, \tag{7c}$$

$$\mathcal{Z}_x(Y) = \mathrm{Diag}(\overline{\sigma}_x^\intercal Y \overline{\sigma}_x), \tag{7d}$$

$$\mathcal{W}_x(Y) = \mathrm{Diag}(\overline{\sigma}_x^\intercal Y \sigma_x + \sigma_x^\intercal Y \overline{\sigma}_x), \tag{7e}$$

$$\varphi(Y) = \mathrm{tr}\{\sigma_x^\intercal Y \sigma_x\}. \tag{7f}$$

In addition, when $\Upsilon_\gamma(Y)$ is invertible, set $M_\gamma(Y) : \mathbb{S}^{n+} \to \mathbb{S}^n$ and $\mathcal{L}_\gamma : \mathbb{S}^{n+} \to \mathbb{S}^{n+}$ as

$$M_\gamma(Y) = -\Psi(Y)^\intercal \Upsilon_\gamma(Y)^{-1}\Psi(Y), \tag{7g}$$

$$\mathcal{L}_\gamma(Y) = A^\intercal Y A + \mathcal{Z}_x(Y) + M_\gamma(Y). \tag{7h}$$

Finally, the adjoint of $\mathcal{L}_\gamma$ in $\mathbb{S}^n$ is the operator written as

$$\mathcal{L}^\star_\gamma(Y) = AYA^\intercal + \mathrm{Diag}(\overline{\sigma}_x Y \overline{\sigma}_x^\intercal) + M_\gamma(Y). \tag{7i}$$

Note that if for some $Y \in \mathbb{S}^{n+}$ and $\gamma > 0$, $\Upsilon_\gamma(Y) \prec 0$, then $-\Upsilon_\gamma$ and M_γ are linear-positive operators, i.e., $Y \succeq 0$ implies that $-\Upsilon_\gamma(Y), M_\gamma(Y) \succeq 0$. Indeed, $\mathcal{Z}_x, \varphi$ also are and $\mathcal{L}_\gamma$ is linear-positive with the preceding assumption on M_γ. A linear-positive operator Π is also monotone, namely, if $Y \succeq X$ for $Y, X \in \mathbb{S}^{n+}$ then $\Pi(Y) \succeq \Pi(X)$. We also consider the notation

$$A_{\mathrm{cl}}(Y) := A - \sigma\Upsilon_\gamma(Y)^{-1}\Psi(Y) \tag{7j}$$

A useful tool in the study of the CSVIU model is the *signal vector function*, namely, the vector function $\mathcal{S} : \mathbb{R}^n \to \{-1, 0, +1\}^n$ defined for $x \in \mathbb{R}^n$ as,

$$\mathcal{S}(x) = \left[\mathrm{sign}(x_1) \ \cdots \ \mathrm{sign}(x_n)\right]^\intercal, \tag{8}$$

with the convention $\mathrm{sign}(0) = 0$. Note that for any $w, v \in \mathbb{R}^n$, and $Y \in \mathbb{S}^n$ the following identities hold true,

$$\langle w, |v|\rangle = \langle \mathcal{S}(v) \cdot w, v\rangle = \langle \mathcal{S}(v), w \cdot v\rangle, \tag{9a}$$

$$\mathrm{tr}\{Y \,\mathrm{diag}(|v|)\} = \langle \mathcal{S}(v), \mathrm{Diag}(Y)v\rangle. \tag{9b}$$

A control problem considers a single exogenous variable, the control variable, and a criterion to be optimized. On the other hand, differential or difference game theory generalizes the control problem to two variables/players, presenting a non-cooperative behavior and acting to achieve conflicting goals in a zero-sum game. Player 1, represented by the control input, tries to stabilize the system and minimize operating costs simultaneously. Player 2, in turn, inflicts the worst disturbance to deviate the system from its reference. If the game renders a saddle point, a finite optimal solution arises, and the game solution is coined indifferently as $\min_u \max_\omega$ or $\max_\omega \min_u$ in either order. We choose to deal with the former ordering and adopt an uncontrolled and compact representation to start for simplicity.

Let us denote $\sigma(x) := \left[\sigma \ \ \sigma_x + \bar{\sigma}_x \,\mathrm{diag}(|x|)\right]$ and $\zeta_k := [\omega_k \ \ \varepsilon^x_k]^\intercal$. The dynamic equation of the uncontrolled system $\hat{\Theta}$ in a compact notation reads as,

$$x_{k+1} = Ax_k + \sigma(x_k)\zeta_k, \quad x_0 = x \in \mathbb{R}^n. \tag{10}$$

An extra advantage of approaching system $\hat{\Theta}$ first is that the stochastic stability analysis in the sequel proves useful for the controlled system $\hat{\Theta}_{\mathrm{ctr}}$. The following lemma takes this standing.

Lemma 1. Set $\gamma > 0$, and consider sequences $\{\mathbf{Z}_k\}, \mathbf{Z}_k \in \mathbb{S}^{n+}, \{v_k\}, v_k \in \mathbb{R}^n$ and $\{g_k\}, g_k \in \mathbb{R}$, $k = 0, 1, \dots, \kappa$ satisfying the difference equations with final conditions,

$$\mathbf{Z}_k = \mathcal{L}_\gamma(\mathbf{Z}_{k+1}) + C^\intercal C, \qquad \mathbf{Z}_\kappa = \Phi \in \mathbb{S}^{n+}, \tag{11a}$$

$$v_k = A_{\mathrm{cl}}(\mathbf{Z}_{k+1})^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{Z}_{k+1})\mathcal{S}(x_k), \qquad v_\kappa = \theta \in \mathbb{R}^n, \tag{11b}$$

$$g_k = g_{k+1} + \varphi(\mathbf{Z}_{k+1}) - \frac{1}{4}\|\sigma^\intercal v_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{Z}_{k+1})^{-1}}, \qquad g_\kappa = \tau \geq 0. \tag{11c}$$

Provided that $\Upsilon_\gamma(\mathbf{Z}_k) \prec 0$ for each $k = 0, \dots, \kappa$, then

$$\max_\omega \quad \mathrm{J}^{\kappa-1}_\infty(z) = \|x\|^2_{\mathbf{Z}_0} + E\left[\langle v_0, x_0\rangle + g_0 | x_0 = x\right] - E\left[\|x_\kappa\|^2_\Phi + \langle \theta, |x_\kappa|\rangle | x_0 = x\right] - \tau. \tag{12}$$

holds, for $x_0 = x$, where ω stands for the finite disturbance sequence $\{\omega(k)\}_{k=0,\dots,\kappa-1}$.

Proof. Consider the auxiliary function $V : \mathbb{N} \times \mathbb{R}^n \to \mathbb{R}$ depending on the sequences $\mathbf{Z}, r$ and g of compatible dimension as,

$$V(k, x) := x^\intercal \mathbf{Z}_k x + \langle r_k, |x|\rangle + g_k, \quad x \in \mathbb{R}^n, \omega \in \mathbb{R}^r.$$

Provided that $\Upsilon_\gamma(\mathbf{Z}_\kappa) = \Upsilon_\gamma(\Phi) \prec 0$, then from (7g) $M_\gamma(\mathbf{Z}_\kappa) \succeq 0$, which in turn, implies from (7h) that $\mathcal{L}_\gamma(\mathbf{Z}_\kappa) \succeq 0$. Now, from (11a) one has that $\mathbf{Z}_{\kappa-1} \succeq 0$. Under the assumptions, an induction argument shows that $\mathbf{Z}_k \in \mathbb{S}^{n+}$, $\forall 0 \leq k \leq \kappa$. Taking into account the identities in (9), the dynamic of the system $\hat{\Theta}$, and the representation $\langle r, |x|\rangle = \langle \mathcal{S}(x), r \cdot x\rangle$, one can evaluate the variation of V along a path $k \to x_k$. Using the compact notation in (10), the difference of successive time steps is calculated as

$$\begin{aligned} V(k+1, x_{k+1}) - V(k, x_k) &= \|x_{k+1}\|^2_{\mathbf{Z}_{k+1}} + \langle s_{k+1}, r_{k+1} \cdot x_{k+1}\rangle + g_{k+1} - (\|x_k\|^2_{\mathbf{Z}_k} + \langle s_k, r_k \cdot x_k\rangle + g_k) \\ &= \|Ax_k\|^2_{\mathbf{Z}_{k+1}} + 2(Ax_k)^\intercal \mathbf{Z}_{k+1}\sigma(x_k)\zeta_k + \|\sigma(x_k)\zeta_k\|^2_{\mathbf{Z}_{k+1}} - \|x_k\|^2_{\mathbf{Z}_k} + \\ &\qquad \langle s_{k+1}, r_{k+1} \cdot \left(Ax_k + \sigma(x_k)\zeta_k\right)\rangle - \langle s_k, r_k \cdot x_k\rangle + g_{k+1} - g_k \end{aligned} \tag{13}$$

where we set $s_k = \mathcal{S}(x_k)$, $s_{k+1} = \mathcal{S}(x_{k+1})$. One can check that,

$$\begin{aligned} E\left[\|\sigma(x_k)\zeta_k\|^2_Y |x_k = x\right] &= \omega_k^\intercal \sigma^\intercal Y \sigma \omega_k + \mathrm{tr}\left\{[\sigma_x + \overline{\sigma}_x \operatorname{diag}(|x|)]^\intercal Y [\sigma_x + \overline{\sigma}_x \operatorname{diag}(|x|)]\right\} \\ &= \|\sigma\omega_k\|^2_Y + \|x\|^2_{\mathcal{Z}_x(Y)} + \mathrm{tr}\{\mathcal{W}_x(Y) \operatorname{diag}(|x|)\} + \varphi(Y) \\ &= \|\sigma\omega_k\|^2_Y + \|x\|^2_{\mathcal{Z}_x(Y)} + \langle \mathcal{S}(x), \mathcal{W}_x(Y)x\rangle + \varphi(Y) \end{aligned}$$

and note that $\mathcal{W}_x(Y)x = \mathcal{W}_{x_d}(Y) \cdot x$. Returning to (13) one can write,

$$\begin{aligned} &V(k+1, x_{k+1}) - V(k, x_k,) \\ &\quad = \|Ax_k\|^2_{\mathbf{Z}_{k+1}} + \|x_k\|^2_{\mathcal{Z}_x(\mathbf{Z}_{k+1})} - \|x_k\|^2_{\mathbf{Z}_k} + \langle s_{k+1}, r_{k+1} \cdot Ax_k\rangle + \langle s_k, \left(\mathcal{W}_{x_d}(\mathbf{Z}_{k+1}) - r_k\right) \cdot x_k\rangle + \\ &\qquad \|\sigma\omega_k\|^2_{\mathbf{Z}_{k+1}} + 2\langle \sigma^\intercal \mathbf{Z}_{k+1} A x_k, \omega_k\rangle + \langle s_{k+1}, r_{k+1} \cdot \sigma\omega_k\rangle + g_{k+1} + \varphi(\mathbf{Z}_{k+1}) - g_k + m_k \end{aligned} \tag{14}$$

where the process $k \to m_k$ is

$$m_k := \langle 2\mathbf{Z}_{k+1}(Ax_k + \sigma\omega_k) + r_{k+1} \cdot s_{k+1}, (\sigma_x + \bar{\sigma}_x \operatorname{diag}(|x_k|))\varepsilon_k^x\rangle,$$

comprising each term of (14) such that $E[m_k|x_k] = 0$, namely, $k \to m_k$ is a zero $\{\mathcal{F}_k\}$-martingale. Note that the difference in (14) depends explicitly on x_k and ω_k only, and we denote $\Delta V(x_k, \omega_k) := V(k+1, x_{k+1}) - V(k, x_k)$ for short. By adding and subtracting the terms $\|z_k\|^2 - \gamma^2\|\omega_k\|^2$, one gets that

$$\begin{aligned} \Delta V(x_k, \omega_k) &= \langle x_k, \left(A^\intercal \mathbf{Z}_{k+1} A + \mathcal{Z}_x(\mathbf{Z}_{k+1}) + C^\intercal C - \mathbf{Z}_k\right) x_k\rangle + \langle A^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{Z}_{k+1}) s_k - v_k, x_k\rangle + \\ &\quad \langle s_{k+1}, r_{k+1} \cdot Ax_k\rangle + \langle s_k, \left(\mathcal{W}_{x_d}(\mathbf{Z}_{k+1}) - r_k\right) \cdot x_k\rangle + \|\omega_k\|^2_{(\sigma^\intercal \mathbf{Z}_{k+1}\sigma + F^\intercal F - \gamma^2 I_r)} + \\ &\quad 2\langle (F^\intercal C + \sigma^\intercal \mathbf{Z}_{k+1} A) x_k, \omega_k\rangle + \langle s_{k+1}, r_{k+1} \cdot \sigma\omega_k\rangle + g_{k+1} + \varphi(\mathbf{Z}_{k+1}) - g_k + m_k - (\|z_k\|^2 - \gamma^2\|\omega_k\|^2) \end{aligned} \tag{15}$$

In addition, setting $v_k := s_k \cdot r_k$ and $v_{k+1} := s_{k+1} \cdot r_{k+1}$, it comes as

$$\begin{aligned} \Delta V(x_k, \omega_k) + (\|z_k\|^2 - \gamma^2\|\omega_k\|^2) &= \|x_k\|^2_{(A^\intercal \mathbf{Z}_{k+1} A + \mathcal{Z}_x(\mathbf{Z}_{k+1}) + C^\intercal C - \mathbf{Z}_k)} + \langle A^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{Z}_{k+1}) s_k - v_k, x_k\rangle + \|\omega_k\|^2_{\Upsilon_\gamma(\mathbf{Z}_{k+1})} + \\ &\quad \langle 2\Psi(\mathbf{Z}_{k+1}) x_k + \sigma^\intercal v_{k+1}, \omega_k\rangle + g_{k+1} + \varphi(\mathbf{Z}_{k+1}) - g_k + m_k \\ &= \|x_k\|^2_{(A^\intercal \mathbf{Z}_{k+1} A + \mathcal{Z}_x(\mathbf{Z}_{k+1}) + C^\intercal C - \mathbf{Z}_k)} + \langle A^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{Z}_{k+1}) s_k - v_k, x_k\rangle + \\ &\quad \|\omega_k - \omega_k^0\|^2_{\Upsilon_\gamma(\mathbf{Z}_{k+1})} - \|\omega_k^0\|^2_{\Upsilon_\gamma(\mathbf{Z}_{k+1})} + g_{k+1} + \varphi(\mathbf{Z}_{k+1}) - g_k + m_k \end{aligned} \tag{16}$$

where $\omega_k^0 := -\Upsilon_\gamma(\mathbf{Z}_{k+1})^{-1}\left(\Psi(\mathbf{Z}_{k+1}) x_k + \frac{1}{2}\sigma^\intercal v_{k+1}\right)$. Now, we evaluate an upper bound for the one step variation $\Delta V(x_k, \omega_k)$ concerning the disturbance value ω_k. Since, from the assumptions, $\Upsilon_\gamma(\mathbf{Z}_{k+1}) \prec 0$, we get from eqs. (11) that

$$\begin{aligned} &\Delta V(x_k, \omega_k) + (\|z_k\|^2 - \gamma^2\|\omega_k\|^2) \le \\ &\quad \langle x_k, \left(A^\intercal \mathbf{Z}_{k+1} A + \mathcal{Z}_x(\mathbf{Z}_{k+1}) + C^\intercal C - \mathbf{Z}_k\right) x_k\rangle + \langle A^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{Z}_{k+1}) s_k - v_k, x_k\rangle + \\ &\quad g_{k+1} - \|\omega_k^0\|^2_{\Upsilon_\gamma(\mathbf{Z}_{k+1})} + \varphi(\mathbf{Z}_{k+1}) - g_k + m_k = \\ &\quad \langle x_k, \left(\mathcal{L}_\gamma(\mathbf{Z}_{k+1}) + C^\intercal C - \mathbf{Z}_k\right) x_k\rangle + \langle A_{\mathrm{cl}}(\mathbf{Z}_{k+1})^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{Z}_{k+1}) s_k - v_k, x_k\rangle + \\ &\quad g_{k+1} + \varphi(\mathbf{Z}_{k+1}) - \frac{1}{4}\|\sigma^\intercal v_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{Z}_{k+1})^{-1}} - g_k + m_k \end{aligned} \tag{17}$$

Thus, if the sequences $\{\mathbf{Z}_k, v_k, g_k\}_{k=0,\dots,\kappa}$ satisfy (11), one readily gets

$$\Delta V(x_k, \omega_k) \le -(\|z_k\|^2 - \gamma^2\|\omega_k\|^2) + m_k$$

and the equality is attained whenever $\omega_k = \omega_k^0$. Hence,

$$E[\Delta V(x_k, \omega_k^0)|x_k] = -E[\|z_k\|^2 - \gamma^2\|\omega_k^0\|^2|x_k],$$

and writing a telescoping sum with such a general term, taking into account that $\hat{\Theta}$ forms a Markovian process, one gets that

$$\mathrm{J}_{\infty}^{\kappa-1}(z) = \mathcal{E}_{2,I}^{\kappa-1}(z) - \gamma^2 \mathcal{E}_{2,I}^{\kappa-1}(\omega) \leq \sum_{k=0}^{\kappa-1} E[\|z_k\|^2 - \gamma^2 \|\omega_k^0\|^2 | x_0] = E[V(0, x_0, \omega_0^0) - V(\kappa, x_\kappa, \omega_\kappa^0) | x_0]$$

$$= E\left[\|x_0\|_{\mathbf{Z}_0}^2 + \langle v_0, x_0 \rangle + g_0 | x_0\right] - E\left[\|x_\kappa\|_{\Phi}^2 + \langle \theta, |x_\kappa| \rangle + \tau | x_0\right] \quad (18)$$

Equality is attained above by the proper sequence $\{\omega_k^0\}_{k=0,\ldots,\kappa-1}$. Thus, (12) holds true. □

2.1 | Stability of Worst-case Disturbance

The maximization in Lemma 1 sets up the worst disturbance input under the assumption of the negative definiteness of the sequences $\{\Upsilon_\gamma(\mathbf{Z}_k)\}_{k=0,\ldots,\kappa}$. A disturbance input is prone to create instability, and the present approach allows us to develop worst-case stability conditions for the class of systems in (1).

This section deals with the stability notion in Definition 2 and provides sufficient conditions based on the worst possible disturbance scenario. Stability should precede system performance (norm evaluations), but a stochastic detectability notion can lightly but tightly connect the former to the latter, which is pursued here. On this footing, consider the family of matrix equations of type,

$$(I - \mathcal{L}_\gamma)(Y) = Q \quad (19)$$

for some $Q \succeq 0$. These equations play a vital role in the worst-case analysis, essential in establishing conditions for stochastic recurrence in terms of some Lyapunov perturbed equations and are studied in connection with positive operators in ordered Banach spaces, e.g.,[22,23,6,7].

Proposition 1 (prop. 3.1[3])**.** The following statements are equivalent

i) $\mathcal{L}_\gamma$ is an inverse-positive operator,

ii) $\mathcal{L}_\gamma$ is d-stable,

iii) There exists $\mathbf{Z} \succ 0$ such that $(I - \mathcal{L}_\gamma)(\mathbf{Z}) \succ 0$,

iv) A is d-stable relative to $\mathcal{Z}_x + M_\gamma$,

v) All eigenvalues of A lay in the open unit disk and $r_\sigma\big((I - \mathbb{A})^{-1}(\mathcal{Z}_x + M_\gamma)\big) < 1$, where $\mathbb{A}(Y) := A^\intercal Y A$ for $Y \in \mathbb{R}^{n \times n}$.

If all eigenvalues of $A_{\mathrm{cl}}(Y)$ lay in the open unit disk, then any of conditions (i)–(v) is equivalent to stochastic stability of $\hat{\Theta}$.

Proof. The proof follows from Proposition 3.1[3], by setting $\alpha = 1$ and replacing the operator $\mathcal{L}^{\alpha=1} \equiv \mathbb{A} + \mathcal{Z}_x$ by $\mathcal{L}_\gamma$. □

Remark 1. Note that the requirement on the spectral radius of $A_{\mathrm{cl}}(Y)$ is neither necessary nor sufficient for (19) to hold, or bears no connection with the conditions *(i)–(v)* in Proposition 1. It is associated with the ℓ_1 part of the CSVIU dynamics and ensuing costs but not related to the quadratic (or ℓ_2) cost part that attaches with the more usual type of Lyapunov condition (19).

The following lemma provides sufficient conditions for stability of the system $\hat{\Theta}$ according to Definition 2, taking into account the disturbance input ω.

Lemma 2. If for $Q = C^\intercal C \succ 0$ there is $\mathbf{Z} \succ 0$, the solution of (19), with $r_\sigma(A_{\mathrm{cl}}(\mathbf{Z})) < 1$ and $\Upsilon_\gamma(\mathbf{Z}) \prec 0$. Then the system in (10) is stochastically stable.

Proof. Note that $\mathbf{Z} \succ 0$, the solution of (19), is also the unique nonnegative stationary solution of the matrix difference in (11a). In connection, set in Lemma 1 $\mathbf{Z}_\kappa = \mathbf{Z}$, $v_\kappa = 0$ and $g_\kappa = 0$, and it yields, for any $x_0 = x$ that

$$\|x\|_{\mathbf{Z}}^2 + E_x\left[\langle v_0^{(\kappa)}, x \rangle + g_0^{(\kappa)}\right] = \max_{\omega} \mathrm{J}_{\infty}^{\kappa-1}(z) + E_x\left[\|x_\kappa\|_{\mathbf{Z}}^2\right] \quad (20)$$

where we add the horizon κ as superscript, and $v_0^{(\kappa)}, g_0^{(\kappa)}$ are respectively,

$$v_k^{(\kappa)} = \sum_{\ell=k}^{\kappa-1} \left(A_{\mathrm{cl}}(\mathbf{Z})^\intercal\right)^{\ell} \mathcal{W}_x(\mathbf{Z}) \mathcal{S}(x_\ell), \quad g_k^{(\kappa)} = \sum_{\ell=k+1}^{\kappa} \left(\varphi(\mathbf{Z}) - \frac{1}{4} \|\sigma^\intercal v_\ell^{(\kappa)}\|_{\Upsilon_\gamma(\mathbf{Z})^{-1}}^2\right) \quad (21)$$

evaluated at $k = 0$. Note that $v_k^{(\kappa)}, k = 0, \dots, \kappa$ are well-defined random vectors for each x_0 and κ; moreover, $\lim_{\kappa\to\infty} |v_k^{(\kappa)}| \le \bar{v}$, where, from the assumption,

$$\bar{v} := r_\sigma\Big(\big(I - A_{\mathrm{cl}}(\mathbf{Z})^\intercal\big)^{-1}\Big)\left|\mathcal{W}_{x_d}(\mathbf{Z})\right|. \tag{22}$$

Now, denote $\bar{\mathrm{J}}_\infty^\kappa(z) := \max_\omega \mathrm{J}_\infty^\kappa(z)$, and from (20) and (21),

$$\bar{\mathrm{J}}_\infty^\kappa(z) + E_x[\|x_{\kappa+1}\|_{\mathbf{Z}}] = \|x\|_{\mathbf{Z}}^2 + \left\langle E_x[v_0^{(\kappa+1)}], x\right\rangle + \sum_{k=1}^{\kappa+1}\left(\varphi(\mathbf{Z}) - \frac{1}{4}E_x\left[\|\sigma^\intercal v_k^{(\kappa)}\|^2_{\Upsilon_\gamma(\mathbf{Z})^{-1}}\right]\right),$$

and it readily follows that

$$\limsup_{\kappa\to\infty}\frac{1}{\kappa}\bar{\mathrm{J}}_\infty^\kappa(z) \le \limsup_{\kappa\to\infty}\frac{1}{\kappa}\left(\|x + \frac{1}{2}\mathbf{Z}^{-1}\bar{v}\|_{\mathbf{Z}}^2 + \kappa\left(\varphi(\mathbf{Z}) + \frac{\|\bar{v}\|^2}{4}\|\sigma^\intercal\|^2_{\Upsilon_\gamma(\mathbf{Z})^{-1}}\right)\right) \le \varphi(\mathbf{Z}) + \frac{\|\bar{v}\|^2}{4}\|\sigma^\intercal\|^2_{\Upsilon_\gamma(\mathbf{Z})^{-1}}$$

and since for any number $\epsilon > 0$,

$$\bar{\mathrm{J}}_\infty^\kappa(z) \ge (1-\epsilon)\bar{\mathrm{J}}_\infty^\kappa(Cx) + (1-\epsilon^{-1})\bar{\mathrm{J}}_\infty^\kappa(F\omega) \ge \lambda^-(1-\epsilon)\bar{\mathrm{J}}_\infty^\kappa(x) + (1-\epsilon^{-1})\bar{\mathrm{J}}_\infty^\kappa(F\omega)$$

where λ^- stands for the smallest eigenvalue of $C^\intercal C$, one can write that

$$\limsup_{\kappa\to\infty}\frac{1}{\kappa}\bar{\mathrm{J}}_\infty^\kappa(x) \le \frac{1}{\lambda^-(1-\epsilon)}\left[\varphi(\mathbf{Z}) + \frac{\|\bar{v}\|^2}{4}\|\sigma^\intercal\|^2_{\Upsilon_\gamma(\mathbf{Z})^{-1}} + \limsup_{\kappa\to\infty}\frac{1}{\kappa}(\epsilon^{-1}-1)\bar{\mathrm{J}}_\infty^\kappa(F\omega)\right]$$

and the last term in the rhs is finite by assumption. Identify $\bar{c} < \infty$ in Definition 2 with the rhs above, which shows that $\hat{\Theta}$ is stochastically stable despite the worst disturbance input ω. □

2.1.1 | Stochastic Detectability

Stochastic stability conditions stated in Corollary 2 relies on a strictly positive solution from the perturbed Lyapunov equation in (19). Detectability notions weaken the positive definite scenario to a positive semidefinite solution ensuring stochastic stability. They allow a broader framework for the Lyapunov analysis.

Definition 3. System $\hat{\Theta}$ is $(C, A, \bar{\sigma}_x, M_\gamma)$ detectable if $XC^\intercal \neq 0$ holds true for every nonzero eigenvector $X \ge 0$ of $\mathcal{L}_\gamma^\star$ corresponding to the eigenvalue $|\lambda| \ge 1$.

This notion is called $(C, \mathcal{L}_\gamma)$-detectable for short.

The detectability notion in Definition 3 is analogous to the concept of exact detectability for SMD systems, in which a Hautus test is crucial for the detectability of these stochastic systems, see[24,25,26]. The proof of the following proposition is a straightforward conclusion from the Lemma 3.7 of[22] and Proposition 1.

Proposition 2. Suppose that,

$$(I - \mathcal{L}_\gamma)(Y) = C^\intercal C$$

has a solution $\mathbf{Z} \succeq 0$. If $\hat{\Theta}$ is $(C, \mathcal{L}_\gamma)$-detectable and if, all eigenvalues of $A_{\mathrm{cl}}(\mathbf{Z})$ lies in the open unit disk, then $\hat{\Theta}$ is stochastically stable.

Under the assumption of detectability, the following corollary connects the finiteness of cost $\mathfrak{P}_\infty(z)$ with the corresponding stochastic stability notion.

Corollary 1 (Detectability and Stochastic Stability)**.** Suppose that system $\hat{\Theta}$ is $(C, \mathcal{L}_\gamma)$-detectable for some $\gamma > 0$. Then,

i) System $\hat{\Theta}$ is stochastically stable if (19) with $Q = C^\intercal C$ has a solution $\mathbf{Z} \succeq 0$, and $r_\sigma(A_{\mathrm{cl}}(\mathbf{Z})) < 1$;

ii) If $\hat{\Theta}$ is stochastically stable then $\hat{\mathcal{E}}_{2,Q}(x(\cdot)) < \infty$ for $Q = C^\intercal C$ and for any $x(0) = x$, $\mathfrak{P}_\infty(z) < \infty$, given by

$$\mathfrak{P}_\infty(z) = \varphi(\mathbf{Z}) - \lim\inf_{\kappa\to\infty}\frac{1}{4\kappa}\sum_{k=1}^{\kappa+1}E_x\left[\|\sigma^\intercal v_k^{(\infty)}\|^2_{\Upsilon_\gamma(\mathbf{Z})^{-1}}\right]$$

in regarding to equations (21).

Remark 2. Lemma 2 states sufficient conditions for stochastic stability by requiring stability of $\hat{\Theta}$ with respect to the disturbance ω. Nevertheless, if system (10) is detectable, Corollary 1 (i) weakens the requirements for stochastic stability of $\hat{\Theta}$, and the finiteness of the cost $\mathfrak{P}_\infty(z)$ is tied to the stability of the system by Corollary 1 (ii).

3 | CSVIU LINEAR-QUADRATIC DYNAMIC GAME

To design a control system means to project a control mechanism capable of keeping z small despite disturbance ω. In this sense, we seek a stabilizing control u that aims to compensate for the unpredictable behavior consequences of the disturbance ω.

This section considers a two-person stochastic zero-sum game. The minimizing player is the control input $u \in \mathcal{U}$, and the maximizing player is the disturbance $\omega \in \mathcal{V}$. Both $\mathcal{U}$ and $\mathcal{V}$ are regarded to be compact sets. To ensure a pure strategy saddle point solution, let $\mathcal{U}$ and $\mathcal{V}$ be convex sets. A Markovian strategy suffices for both players to attain the respective optimal performances. The payoff functional $J(u,\omega) := J_\infty^k$ in (5) is continuous in the pair $(u,\omega) \in \mathcal{U} \times \mathcal{V}$ and we take advantage of its strictly convexity-concavity property to obtain the original optimal saddle point solution for the CSVIU dynamic game.

In addition to operators in (7), let us define $\mathcal{Z}_u : \mathbb{S}^{n+} \to \mathbb{S}^{n+}$, $\mathcal{W}_u : \mathbb{S}^{n+} \to \mathbb{S}^n$, and $\varphi_1 : \mathbb{S}^{n+} \to \mathbb{R}$ for any $Y \in \mathbb{S}^{n+}$, such that,

$$\mathcal{Z}_u(Y) = \mathrm{Diag}(\overline{\sigma}_u^\intercal Y \overline{\sigma}_u), \tag{23a}$$

$$\mathcal{W}_u(Y) = \mathrm{Diag}(\overline{\sigma}_u^\intercal Y \sigma_u + \sigma_u^\intercal Y \overline{\sigma}_u), \tag{23b}$$

$$\varphi_1(Y) = \mathrm{tr}\{Y(\sigma_x\sigma_x^\intercal + \sigma_u\sigma_u^\intercal)\}, \tag{23c}$$

Except for $\mathcal{W}_u$, the operators in (23) are linear-positive operators. Define also $\Sigma : \mathbb{R}^{n\times n} \to \mathbb{R}^{m\times n}$, $\Lambda : \mathbb{R}^{n\times n} \to \mathbb{R}^{m\times m}$, and when Υ_γ is invertible, $\Delta(Y) : \mathbb{R}^{n\times n} \to \mathbb{R}^{m\times m}$, with

$$\Sigma(Y) = B^\intercal Y A + D^\intercal C, \tag{24a}$$

$$\Lambda(Y) = B^\intercal Y B + \mathcal{Z}_u(Y) + D^\intercal D, \tag{24b}$$

$$\Delta(Y) = \Lambda(Y) - \Gamma(Y)^\intercal \Upsilon_\gamma(Y)^{-1}\Gamma(Y). \tag{24c}$$

When Υ_γ and Δ are invertible, consider additionally Ric $: \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ such that

$$\mathrm{Ric}(Y) = \mathcal{L}_\gamma(Y) - \Sigma_{\mathrm{cl}}(Y)^\intercal \Delta(Y)^{-1}\Sigma_{\mathrm{cl}}(Y) + C^\intercal C, \tag{25}$$

where Σ_{cl} refers to the matrix

$$\Sigma_{\mathrm{cl}}(Y) := \Sigma(Y) - \Gamma(Y)^\intercal \Upsilon_\gamma(Y)^{-1}\Psi(Y). \tag{26}$$

In addition to the notation $A_{\mathrm{cl}}(Y) = A - \sigma\Upsilon_\gamma(Y)^{-1}\Psi(Y)$, consider also $B_{\mathrm{cl}}(Y) := B - \sigma\Upsilon_\gamma(Y)^{-1}\Gamma(Y)$.

3.1 | Optimal saddle point solution

To alleviate expressions, we adopt the compact notation,

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k + \sigma_u(x_k,u_k)\zeta_{0,k} \\ z_k &= Cx_k + Du_k + F\omega(k), \quad k \geq 0, \end{aligned} \tag{27}$$

for the controlled CSVIU system $\hat{\Theta}_{\mathrm{ctr}}$ in (1). Here, the standard noise vector is $\zeta_{0,k} = \begin{bmatrix}\omega(k) & \varepsilon^x(k) & \varepsilon^u(k)\end{bmatrix}^\intercal$ with standard deviation matrix $\sigma_u(x,u) = \begin{bmatrix}\sigma & \sigma_x + \bar{\sigma}_x \,\mathrm{diag}(|x|) & \sigma_u + \bar{\sigma}_u \,\mathrm{diag}(|u|)\end{bmatrix}$. Besides, let us take into account some sequences $\{\mathbf{X}_k\}, \mathbf{X}_k \in \mathbb{S}^{n+}, \{\mathfrak{r}_k\}, \mathfrak{r}_k \in \mathbb{R}^n$ and $\{\mathfrak{g}_k\}, \mathfrak{g}_k \geq 0$, $k = 0,1,\dots$ to introduce the function $W : \mathbb{N}\times\mathbb{R}^n \to \mathbb{R}$,

$$W(k,x) := x^\intercal \mathbf{X}_k x + \langle \mathfrak{r}_k, |x|\rangle + \mathfrak{g}_k, \quad x \in \mathbb{R}^n. \tag{28}$$

Lemma 3. Set some $\gamma > 0$ and consider sequences $\{\mathbf{X}_k\}, \mathbf{X}_k \in \mathbb{S}^{n+}, \{\mathfrak{r}_k\}, \mathfrak{r}_k \in \mathbb{R}^n$ and $\{\mathfrak{g}_k\}, \mathfrak{g}_k \in \mathbb{R}$, $k = 0,1,\dots,\kappa$. Provided that $\Upsilon_\gamma(\mathbf{X}_k) \prec 0$ for each $k = 0,\dots,\kappa$, then

$$\begin{aligned} &W(k+1,x_{k+1}) - W(k,x_k) + [\|z_k\|^2 - \gamma^2\|\omega_k\|^2] \leq \\ &\quad \|Ax_k\|^2_{\mathbf{X}_{k+1}} + \|x_k\|^2_{\mathcal{Z}_x(\mathbf{X}_{k+1})} + \|x_k\|^2_{C^\intercal C} - \|x_k\|^2_{\mathbf{X}_{k+1}} + \langle A^\intercal\eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle + \max_\omega \left[\|\omega - \omega_k^0\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})}\right] + \\ &\quad \|u_k\|^2_{\Lambda(\mathbf{X}_{k+1})} + \langle 2\Sigma(\mathbf{X}_{k+1})x_k + B^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s_k^u, u_k\rangle - \|\omega_k^0\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})} + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) + m_k \end{aligned} \tag{29}$$

and

$$
\begin{aligned}
&W(k+1,x_{k+1}) - W(k,x_k) + [\|z_k\|^2 - \gamma^2\|\omega_k^0\|^2] \geq \\
&\quad x_k^\intercal\big(\operatorname{Ric}(\mathbf{X}_{k+1}) - \mathbf{X}_k\big)x_k + \langle (A_{\mathrm{cl}}(\mathbf{X}_{k+1}) - B_{\mathrm{cl}}(\mathbf{X}_{k+1})\Delta(\mathbf{X}_{k+1})^{-1}\Sigma_{\mathrm{cl}}(\mathbf{X}_{k+1}))^\intercal \eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle \\
&\quad + \min_u \Big[\|u - u_k^0\|^2_{\Delta(\mathbf{X}_{k+1})} - \big\langle \Sigma_{\mathrm{cl}}(\mathbf{X}_{k+1})^\intercal \Delta(\mathbf{X}_{k+1})^{-1}\mathcal{W}_u(\mathbf{X}_{k+1})s_k^u, x_k\big\rangle - \frac{1}{4}\|B_{\mathrm{cl}}(\mathbf{X}_{k+1})^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s_k^u\|^2_{\Delta(\mathbf{X}_{k+1})^{-1}}\Big] \\
&\qquad + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) - \frac{1}{4}\|\sigma^\intercal\eta_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}} + m_k \qquad (30)
\end{aligned}
$$

where,

$$\omega_k^0 := -\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}\Big(\Psi(\mathbf{X}_{k+1})x_k + \Gamma(\mathbf{X}_{k+1})u_k + \frac{1}{2}\sigma^\intercal\eta_{k+1}\Big), \tag{31a}$$

$$u_k^0 := -\Delta(\mathbf{X}_{k+1})^{-1}\Big[\Sigma(\mathbf{X}_{k+1})x_k + \frac{1}{2}\left(B^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s_k^u\right) - \Gamma(\mathbf{X}_{k+1})^\intercal\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}\Big(\Psi(\mathbf{X}_{k+1})x_k + \frac{1}{2}\sigma^\intercal\eta_{k+1}\Big)\Big], \tag{31b}$$

s_k and s_k^u denotes $\mathcal{S}(x_k)$ and $\mathcal{S}(u_k)$, respectively, $\eta_k := s_k \cdot \mathfrak{r}_k$ and $\eta_{k+1} := s_{k+1}\cdot\mathfrak{r}_{k+1}$, and $k \to m_k$ is a zero $\{\mathcal{F}_k\}$-martingale. Equality in (29) and (30) are attained whenever $\omega_k = \omega_k^0$ and u_k is set to be equal to the minimizer in (30).

The expressions in Lemma 3 are verified in Appendix A.

Remark 3. To keep in view an important feature of the optimal solution, set in (29), $\omega = \omega_k^0$, to attain the equality. After some algebraic manipulations, the absolute values involved in the original form can be retrieved, and one can write the one-stage difference $J_u : \mathbb{R}^m \to \mathbb{R}$, solely as a function of the control action $u_k = u$,

$$
\begin{aligned}
J_u &:= W(k+1,x_{k+1}) - W(k,x_k) + [\|z_k\|^2 - \gamma^2\|\omega_k\|^2] = \\
&\quad \|Ax_k\|^2_{\mathbf{X}_{k+1}} + \|x_k\|^2_{\mathcal{Z}_x(\mathbf{X}_{k+1})} + \|x_k\|^2_{C^\intercal C} - \|x_k\|^2_{\mathbf{X}_{k+1}} + \|x_k\|^2_{\Psi(\mathbf{X}_{k+1})^\intercal\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}\Psi(\mathbf{X}_{k+1})} + \\
&\quad \langle A_{\mathrm{cl}}(\mathbf{X}_{k+1})^\intercal\eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) - \frac{1}{4}\|\sigma^\intercal\eta_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}} + m_k + \\
&\quad \|u\|^2_{\Delta(\mathbf{X}_{k+1})} + \langle 2\Sigma_{\mathrm{cl}}(\mathbf{X}_{k+1})x_k + B_{\mathrm{cl}}(\mathbf{X}_{k+1})^\intercal\eta_{k+1}, u\rangle + \langle \mathcal{W}_{u_d}(\mathbf{X}_{k+1}), |u|\rangle \\
&= \|u\|^2_{\Delta(\mathbf{X}_{k+1})} + \langle 2\Sigma_{\mathrm{cl}}(\mathbf{X}_{k+1})x_k + B_{\mathrm{cl}}(\mathbf{X}_{k+1})^\intercal\eta_{k+1}, u\rangle + \langle \mathcal{W}_{u_d}(\mathbf{X}_{k+1}), |u|\rangle + f_k, \qquad (32)
\end{aligned}
$$

in which, f_k comprises the remaining terms in (29) that does not depend on control u at time k. To ensure strict convexity of J_u assume that $\Upsilon_\gamma(\mathbf{X}_k) \prec 0, \forall k$, which implies that $\Delta(\mathbf{X}_{k+1}) \succ 0, \forall k$ and recall that $\mathcal{W}_u$ a linear-positive operator.

Stabilizing Controllers

For any $G \in \mathbb{R}^{m\times n}$, let us denote $\mathcal{A} := A + BG$ and $\mathcal{C} := C + DG$ and consider the operators $\mathcal{M}_\gamma : \mathbb{S}^{n+} \to \mathbb{S}^n$ and $\mathscr{L}_\gamma : \mathbb{S}^{n+} \to \mathbb{S}^{n+}$, similar to (7g) and (7h), respectively, as,

$$\mathcal{M}_\gamma(Y) = \left[\sigma^\intercal Y\mathcal{A} + F^\intercal\mathcal{C}\right]^\intercal\Upsilon_\gamma(Y)^{-1}\left[\sigma^\intercal Y\mathcal{A} + F^\intercal\mathcal{C}\right] \tag{33a}$$

$$\mathscr{L}_\gamma(Y) = \mathcal{A}^\intercal Y\mathcal{A} + \mathcal{Z}_x(Y) + \mathcal{M}_\gamma(Y) \tag{33b}$$

In the following, we establish conditions under which linear feedback made up of G, $k \to u_k = Gx_k$, to be a *stabilizing controller* for system $\hat{\Theta}_{\mathrm{ctr}}$ in (1), despite the worst possible disturbance.

Recall the stochastic stability analysis in Section 2 for system $\hat{\Theta}$. The following lemma parallels the stability results for a $\hat{\Theta}_{\mathrm{ctr}}$ system, which is stochastically stabilizable by a linear feedback control G. For the proof, see Appendix B.

Lemma 4. For any $G \in \mathbb{R}^{m\times n}$, define the sequences $\{\mathbf{X}_k\}$, $\{\nu_k\}$, $\{\mathfrak{g}_k\}$, $k = 0,\ldots,\kappa-1$, in which,

$$\mathbf{X}_k = \mathscr{L}_\gamma(\mathbf{X}_{k+1}) + G^\intercal\mathcal{Z}_u(\mathbf{X}_{k+1})G + \mathcal{C}^\intercal\mathcal{C}, \tag{34a}$$

$$\nu_k = \left(A_{\mathrm{cl}}(\mathbf{X}_{k+1}) + B_{\mathrm{cl}}(\mathbf{X}_{k+1})G\right)^\intercal\nu_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})\mathcal{S}(x_k) + G^\intercal\mathcal{W}_u(\mathbf{X}_{k+1})\mathcal{S}(u_k), \tag{34b}$$

$$\mathfrak{g}_k = \mathfrak{g}_{k+1} + \varphi_1(\mathbf{X}_{k+1}) - \frac{1}{4}\|\sigma^\intercal\nu_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}}, \tag{34c}$$

with final conditions, $\mathbf{X}_\kappa = 0$, $\nu_\kappa = 0$ and $\mathfrak{g}_\kappa = 0$. Then, if $\Upsilon_\gamma(\mathbf{X}_k) \prec 0$, $k = 0,\ldots\kappa-1$, for $k \to u_k = Gx_k$,

$$\sup_{\{\omega_k\}_{k=0,\ldots,\kappa-1}} E\Big[\sum_{k=0}^{\kappa-1}\|z_k\|^2 - \gamma^2\|\omega_k\|^2\Big|x_0\Big] = \|x_0\|^2_{\mathbf{X}_0} + E[\langle\nu_0,|x_0|\rangle + \mathfrak{g}_0|x_0] \tag{35}$$

Moreover, if the $\hat{\Theta}_{\mathrm{ctr}}$ system is $(C, \mathcal{L}_\gamma)$-detectable, and

$$(I - \mathscr{L}_\gamma - G^\intercal \mathcal{Z}_u G)(Y) = C^\intercal C, \tag{36}$$

has a solution $\mathbf{Y} \geq 0$ with $\Upsilon_\gamma(\mathbf{Y}) \prec 0$ and $r_\sigma\left(A_{\mathrm{cl}}(\mathbf{Y}) + B_{\mathrm{cl}}(\mathbf{Y})G\right) < 1$. Then, $\hat{\Theta}_{\mathrm{ctr}}$ is stochastically stabilizable and $k \to u_k = Gx_k$ stabilizes $\hat{\Theta}_{\mathrm{ctr}}$ in the sense of Definition 2 (ii).

3.2 | Optimal Stabilizing Controllers

The next theorem provides conditions for the existence of an optimal stabilizing controller.

Theorem 1. Let $\gamma > 0$, suppose that $\hat{\Theta}_{\mathrm{ctr}}$ is $(C, \mathcal{L}_\gamma)$-detectable and there is a solution $\mathbf{P} \geq 0$ to $(I - \mathrm{Ric})(Y) = 0$ with $\Upsilon_\gamma(\mathbf{P}) \prec 0$. In addition, assume that $r_\sigma(A_{\mathrm{cl}}(\mathbf{P}) + B_{\mathrm{cl}}(\mathbf{P})G) < 1$, where $G := -\Delta(\mathbf{P})^{-1}\Sigma_{\mathrm{cl}}(\mathbf{P})$. For any admissible pair $k \to (x_k, u_k)$, let us consider the difference equations for $k \to v_k$ and $k \to g_k$, $k \geq 0$ given by

$$v_k = \left(A_{\mathrm{cl}}(\mathbf{P}) + B_{\mathrm{cl}}(\mathbf{P})G\right)^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{P})\mathcal{S}(x_k), \tag{37a}$$

$$g_k = g_{k+1} + \varphi_1(\mathbf{P}) - \frac{1}{4}\|\sigma^\intercal v_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{P})^{-1}} + \rho_k, \tag{37b}$$

with the values $\lim_{k\to\infty} v_k, g_k$ finite and arbitrary. In (37b),

$$\rho_k := \min_{u\in\mathbb{R}^m} \rho_k(u) \tag{37c}$$

$$\text{with } \rho_k(u) := \|u - u_k^0\|^2_{\Delta(\mathbf{P})} - \frac{1}{4}\left\|B_{\mathrm{cl}}(\mathbf{P})^\intercal \eta_{k+1} - \mathcal{W}_u(\mathbf{P})\mathcal{S}(u)\right\|^2_{\Delta(\mathbf{P})^{-1}} + \left\langle \mathcal{W}_u(\mathbf{P})G_k x_k, \mathcal{S}(u)\right\rangle, \tag{37d}$$

$$\text{and } u_k^0 := -\Delta(\mathbf{P})^{-1}\left[\Sigma(\mathbf{P})x_k + \frac{1}{2}\left(B^\intercal \eta_{k+1} + \mathcal{W}_u(\mathbf{P})\mathcal{S}(u)\right) - \Gamma(\mathbf{P})^\intercal \Upsilon_\gamma(\mathbf{P})^{-1}\left(\Psi(\mathbf{P})x_k + \frac{1}{2}\sigma^\intercal \eta_{k+1}\right)\right],$$

where $\eta_{k+1} = E[v_{k+1}|x_k]$. Set $\bar{u}_k = \arg\min_{u\in\mathbb{R}^m} \rho_k(u)$, $k \geq 0$. Then, $k \to \bar{u}_k$ attains the minimum power average cost $\mathfrak{P}^*_\infty := \min_u \max_\omega \mathfrak{P}_\infty$, given by

$$\mathfrak{P}^*_\infty = \varphi_1(\mathbf{P}) + \limsup_{\kappa\to\infty} \frac{1}{\kappa} E\left[\sum_{k=0}^{\kappa-1} \rho_k - \frac{1}{4}\|\sigma^\intercal \eta_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{P})^{-1}} \Big| x_0 = 0\right] \tag{38}$$

Moreover, it stabilizes system $\hat{\Theta}_{\mathrm{ctr}}$ in the sense of Definition 2 (ii).

Remark 4. For any pair $k \to (u_k, x_k)$, the worst case disturbance is given by $\bar{\omega}_k = -\Upsilon_\gamma(\mathbf{P})^{-1}\left(\Psi(\mathbf{P})x_k + \Gamma(\mathbf{P})u_k + \frac{1}{2}\sigma^\intercal \eta_{k+1}\right)$ The maximum in $\mathfrak{P}^*_\infty$ is attained by $k \to \bar{\omega}_k$ when $(u_k, x_k) = (\bar{u}_k, \bar{x}_k), \forall k \geq 0$. The pair $k \to (\bar{u}_k, \bar{\omega}_k)$ is said to be the *optimal saddle point solution* of the stochastic game.

Proof. The first part of the proof comprises a finite-time evaluation of the cost inducing the power norms and provides bounds for processes $k \to \eta_k$ and $k \to \rho_k$. Let us consider the sequences produced by the following set of difference equations,

$$\mathbf{X}_{k-1} = \mathrm{Ric}(\mathbf{X}_k), \tag{39a}$$

$$\upsilon_{k-1} = (A_{\mathrm{cl}}(\mathbf{X}_k) + B_{\mathrm{cl}}(\mathbf{X}_k)G_k)^\intercal \upsilon_k + \mathcal{W}_x(\mathbf{X}_k)\mathcal{S}(x_{k-1}), \tag{39b}$$

$$\mathfrak{g}_{k-1} = \mathfrak{g}_k + \varphi_1(\mathbf{X}_k) - \frac{1}{4}\|\sigma^\intercal n_k\|^2_{\Upsilon_\gamma(\mathbf{X}_k)^{-1}} + \varrho_k^*, \quad k = 1, \ldots, \kappa, \tag{39c}$$

for $\mathbf{X}_\kappa = 0, \eta_\kappa = 0$ and $\mathfrak{g}_\kappa = 0$, and where $n_k = E[\upsilon_k|x_{k-1}]$, $G_k = -\Delta(\mathbf{X}_k)^{-1}\Sigma_{\mathrm{cl}}(\mathbf{X}_k)$ and $\varrho_k^* = \min_{u\in\mathbb{R}^m} \varrho_k(u)$, with

$$\varrho_k(u) := \|u - u^0_{k-1}\|^2_{\Delta(\mathbf{X}_k)} - \frac{1}{4}\left\|B_{\mathrm{cl}}(\mathbf{X}_k)^\intercal n_k + \mathcal{W}_u(\mathbf{X}_k)\mathcal{S}(u)\right\|^2_{\Delta(\mathbf{X}_k)^{-1}} + \left\langle \mathcal{W}_u(\mathbf{X}_k)G_k x_{k-1}, \mathcal{S}(u)\right\rangle \tag{39d}$$

Here, u^0_{k-1} is as in (31b). Let us denote $u^*_{k-1} = \arg\min_{u\in\mathbb{R}^m} \varrho_k(u)$ and $\omega^*_k = \omega^0_k$ appearing in (31a). Then, provided that $\Upsilon_\gamma(\mathbf{X}_k) \prec 0$ for each k, from Lemma 3 one gets that (30) holds as equality for the pair (u^*_k, ω^*_k).

Consider the feedback law and the disturbances $k \to (u^*_k, \omega^*_k)$, $k = 0, \ldots, \kappa - 1$ for some $\kappa > 0$. By setting $\mathbf{X}_\kappa = 0, \eta_\kappa = 0$ and $\mathfrak{g}_\kappa = 0$, one has that $W(x_\kappa, u^*_\kappa, \omega^*_\kappa) = 0$. Since the process $\hat{\Theta}_{\mathrm{ctr}}$ is Markovian, taking into account (30), (39), and by creating a telescoping sum, the inequality,

$$\begin{aligned}
\mathrm{J}^{\kappa}_{\infty}(z^*) &= W(x_0,u_0^*,\omega_0^*) = E[W(x_0,u_0^*,\omega_0^*) - W(x_\kappa,u_\kappa^*,\omega_\kappa^*)|x_0] \\
&= E\Big[\sum_{k=0}^{\kappa-1} E\left(W(x_k,u_k^*,\omega_k^*) - W(x_{k+1},u_{k+1}^*,\omega_{k+1}^*)|x_k\right)\Big|x_0\Big] = E\Big[\sum_{k=0}^{\kappa-1}\|z_k^*\|^2 - \gamma^2\|\omega_k^*\|^2 + \varrho_k^* - \varrho_k(u_k^*)|x_0\Big] \\
&\le E\Big[\sum_{k=0}^{\kappa-1}\|z_k\|^2 - \gamma^2\|\omega_k^*\|^2 + \varrho_k(u_k^*) - \varrho_k(u_k)|x_0\Big] \le E\Big[\sum_{k=0}^{\kappa-1}\|z_k\|^2 - \gamma^2\|\omega_k^*\|^2|x_0\Big] = \mathrm{J}^{\kappa}_{\infty}(z),
\end{aligned}\tag{40}$$

holds, no matter the choice of the feedback law and the corresponding output $k \to (u_k, z_k)$, $k = 0,\dots,\kappa-1$. In other words,

$$W(x_0,u_0^*,\omega_0^*) \le \mathrm{J}^{\kappa}_{\infty}(z),\tag{41}$$

and equality is attained, provided that $k \to u_k^*$, $k = 0,\dots,\kappa-1$ is applied in the rhs evaluation.

Now, we explicitly indicate the horizon κ as $\mathbf{X}_k^{(\kappa)}$ or $\upsilon_k^{(\kappa)}, \forall k \le \kappa$ for the solutions of (39a)–(39b) with $\mathbf{X}_\kappa^{(\kappa)} = 0$ and $\upsilon_\kappa^{(\kappa)} = 0$, respectively. From the assumptions we get that, $0 \le \mathbf{X}_k^{(\kappa)} \uparrow \mathbf{P}$ for each k in the semipositive definite sense, as $\kappa \to \infty$, where $\mathbf{P}$ is the unique solution of $(I - \mathrm{Ric})(Y) = 0$. This is exactly equivalent to say that $\mathbf{P} \ge 0$ satisfies $(I - \mathscr{L}_\gamma - G^\intercal \mathcal{Z}_u)(Y) = C^\intercal C$ in Lemma 4 with $G = -\Delta(\mathbf{P})^{-1}\Sigma_{\mathrm{cl}}(\mathbf{P})$.

From the assumptions in the theorem, $\hat{\Theta}_{\mathrm{ctr}}$ is $(C,\mathcal{L}_\gamma)$-detectable and $r_\sigma(A_{cl} + B_{cl}G) < 1$. In view of Lemma 4 we conclude that $\hat{\Theta}_{\mathrm{ctr}}$ is stochastically stabilizable and $k \to u_k = Gx_k$ stabilizes $\hat{\Theta}_{\mathrm{ctr}}$ in the sense of Definition 2 (ii).

To show that the optimal control $k \to \bar{u}_k$ also stabilizes $\hat{\Theta}_{\mathrm{ctr}}$, consider that

$$\upsilon_k^{(\infty)} = \lim_{\kappa\to\infty}\upsilon_k^{(\kappa)} = \lim_{\kappa\to\infty}\sum_{\ell=0}^{\kappa}(\mathcal{A}^\intercal)^n\mathcal{W}_x(\mathbf{X}_{k+\ell}^{(\kappa)})\mathcal{S}(x_{k+\ell}) = \sum_{\ell=0}^{\infty}(\mathcal{A}^\intercal)^n\mathcal{W}_x(\mathbf{P})\mathcal{S}(x_{k+\ell}) = \nu_k, \quad \forall k \ge 0\tag{42}$$

where $\mathcal{A} = A + BG$; therefore, $|\upsilon_k^{(\infty)}| \le \upsilon_{\mathrm{M}}$ where,

$$\upsilon_{\mathrm{M}} = r_\sigma\left((I - \mathcal{A}^\intercal)^{-1}\right)\left|\mathcal{W}_{x_d}(\mathbf{P})\right|\tag{43}$$

hence $k \to \upsilon_k^{(\infty)}$ is a bounded n-valued processes. Monotone convergence of each $\mathbf{X}_k^{(\kappa)}$ to $\mathbf{P}$ leads to convergence of $\upsilon_k^{(\infty)}$ to ν_k, and the bound $|\nu_k| \le \upsilon_{\mathrm{M}}$ applies uniformly. Hence, $\varrho_k(u)$ in (39d) converges to $\rho_k(u)$ in (37d), and with that, some bounds on $k \to \rho_k$ can be found. For this, let us set here $u_k = Gx_k = -\Delta(\mathbf{P})^{-1}\Sigma_{\mathrm{cl}}(\mathbf{P})x_k$. Then, one can evaluate,

$$\begin{aligned}
\rho_k(Gx_k) &= \|Gx_k - u_k^0\|^2_{\Delta(\mathbf{P})} - \frac{1}{4}\left\|B_{\mathrm{cl}}(\mathbf{P})^\intercal\eta_{k+1} - \mathcal{W}_u(\mathbf{P})\mathcal{S}(Gx_k)\right\|^2_{\Delta(\mathbf{P})^{-1}} + \left\langle\mathcal{W}_u(\mathbf{P})Gx_k, \mathcal{S}(Gx_k)\right\rangle \\
&= \left\langle\mathcal{W}_u(\mathbf{P})Gx_k, \mathcal{S}(Gx_k)\right\rangle = \langle\mathcal{W}_{u_d}(\mathbf{P}), |Gx_k|\rangle \ge 0
\end{aligned}$$

and $\rho_k(Gx_k) \ge \left\langle\mathcal{W}_u(\mathbf{P})Gx_k, \mathcal{S}(u)\right\rangle$, $\forall u \in \mathbb{R}^m$ holds true. Now, since also $|\eta_k| \le \upsilon_{\mathrm{M}}$,

$$\begin{aligned}
\rho_k = \rho_k(\bar{u}_k) &\ge -\frac{1}{4}\left\|B_{\mathrm{cl}}(\mathbf{P})^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{P})\mathcal{S}(\bar{u}_k)\right\|^2_{\Delta(\mathbf{P})^{-1}} + \left\langle\mathcal{W}_u(\mathbf{P})G_kx_k, \mathcal{S}(\bar{u}_k)\right\rangle \\
&\ge -\frac{1}{4}\left(\|B_{\mathrm{cl}}(\mathbf{P})^\intercal\|^2_{\Delta(\mathbf{P})^{-1}}\|\upsilon_{\mathrm{M}}\|^2_{\Delta(\mathbf{P})^{-1}} + \|\mathcal{W}_{u_d}(\mathbf{P})\|^2_{\Delta(\mathbf{P})^{-1}}\right) - \rho_k(Gx_k)
\end{aligned}\tag{44}$$

To get an upper bound for $k \to \rho_k$, let us set again $u_k = Gx_k$. Then, from optimality, we evaluate,

$$\rho_k \le \rho_k(Gx_k) = \langle\mathcal{W}_u(\mathbf{P}), |G_kx_k|\rangle\tag{45}$$

which, together with (44), shows that

$$|\rho_k| \le a + \langle\mathcal{W}_u(\mathbf{P}), |G_kx_k|\rangle = a + \rho_k(Gx_k)$$

holds for some $a > 0$.

For the control $k \to \bar{u}_k = \arg\min \rho_k(u)$ denote $k \to (\bar{u}_k, \bar{x}_k, \bar{z}_k)$ the control and corresponding state and output. Let also $k \to (\tilde{u}_k, \tilde{x}_k, \tilde{z}_k)$ be the triple produced by the control $\tilde{u}_k = G\tilde{x}_k$, and set $\mathbf{X}_\kappa = \mathbf{P}$ and $\eta_\kappa = \eta_\kappa^{(\infty)}$ as in (42) to get that

$$\limsup\frac{1}{\kappa}\mathrm{J}^{\kappa}_{\infty}(z^*) = \limsup\frac{1}{\kappa}\Big(\min_u\max_\omega E\Big[\sum_{k=0}^{\kappa-1}\|z_k\|^2 - \gamma^2\|\omega_k\|^2|x_0\Big]\Big) = \limsup\frac{1}{\kappa}\Big(E\Big[\sum_{k=0}^{\kappa-1}\|\bar{z}_k\|^2 - \gamma^2\|\bar{\omega}_k\|^2|x_0\Big]\Big)$$

and similarly to (40), by replacing $\mathbf{X}_{k-1}, \mathbf{X}_k \to \mathbf{P}$, $\varrho_k \to \rho_k$, $\eta_k \to \eta_k^{(\infty)}$,

$$\limsup \frac{1}{\kappa}\Big(\min_u \max_\omega E\big[\sum_{k=0}^{\kappa-1} \|z_k\|^2 - \gamma^2\|\omega_k\|^2|x_0\big]\Big) \le \limsup \frac{1}{\kappa}\Big(\max_\omega E\big[\sum_{k=0}^{\kappa-1} \|\tilde{z}_k\|^2 - \gamma^2\|\omega_k\|^2 + \rho_k(\bar{u}_k) - \rho_k(G\tilde{x}_k)|x_0\big]\Big)$$

$$\le \limsup \frac{1}{\kappa} E\big[\sum_{k=0}^{\kappa-1} \|\bar{z}_k\|^2 - \gamma^2\|\bar{\omega}_k\|^2|x_0\big] + \limsup \frac{1}{\kappa} E\big[\sum_{k=0}^{\kappa-1} \rho_k - \rho_k(G\tilde{x}_k)|x_0\big]$$

$$\le \limsup \frac{1}{\kappa} J_\infty^\kappa(\tilde{z}) + \limsup \frac{1}{\kappa} E[\sum_{k=0}^{\kappa-1} 2\langle \mathcal{W}_u(\mathbf{P}), |G_k\tilde{x}_k|\rangle + a] \tag{46}$$

the first term in the rhs above is precisely the power norm $\mathfrak{P}_\infty(\tilde{z})\Big|_{\tilde{u}=G\tilde{x}} < \infty$ for the stabilizing control $\tilde{u}_k, k \ge 0$, whereas, the second term can be bounded as

$$\limsup \frac{1}{\kappa} E[\sum_{k=0}^{\kappa-1} 2\langle \mathcal{W}_u(\mathbf{P}), |G_k\tilde{x}_k|\rangle + a] \le \limsup \frac{1}{\kappa} E[\sum_{k=0}^{\kappa-1} \|G\tilde{x}_k\|^2 + a + b] = \hat{\mathcal{E}}_{2,Q}(\tilde{x}) + a + b < \infty$$

where $Q = G^\intercal G$, and we use Corollary 1(ii). Thus, the stabilizing control $\tilde{u}_k, k \ge 0$ provides the ultimate evaluation, showing that the optimal power H_∞-norm, $\mathfrak{P}_\infty^*$ is finite and that $k \to \bar{u}_k$ is an optimal stabilizing control.

Finally, to show (38), set again $\mathbf{X}_\kappa = \mathbf{P}$, $\nu_\kappa = \upsilon_\kappa^{(\infty)}$ as in (42) with $\mathfrak{g}_\kappa = 0$. Then,

$$\mathfrak{P}_\infty^* = \limsup_{\kappa\to\infty} \frac{1}{\kappa}\Big(\min_u \max_\omega E\big[\sum_{k=0}^{\kappa-1} \|z_k\|^2 - \gamma^2\|\omega_k\|^2|x_0 = 0\big]\Big) = \limsup_{\kappa\to\infty} \frac{1}{\kappa}\Big(E[\|x_0\|_{\mathbf{P}} + \langle \nu_0, x_0\rangle + \mathfrak{g}_0^{(\kappa)}]|x_0 = 0]\Big) =$$

$$\limsup_{\kappa\to\infty} \frac{1}{\kappa} E\left[\mathfrak{g}_0^{(\kappa)}\Big|x_0 = 0\right] = \varphi_1(\mathbf{P}) + \limsup_{\kappa\to\infty} \frac{1}{\kappa} E\left[\sum_{k=0}^{\kappa-1} \rho_k - \frac{1}{4}\|\sigma^\intercal\eta_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{P})^{-1}}\Big|x_0 = 0\right] \tag{47}$$

which yields the norm expression in (38). □

The Inaction Region

We first resource to Remark 3 to better understand the optimal solution. The optimal control solution at the k-th stage is precisely the problem of minimizing the difference in (32), appropriately denoted by J_u, wrt the choice of $u_k = u$,

$$J_u = \|u\|^2_{\Delta(\mathbf{P})} + \langle 2\Sigma_{\rm cl}(\mathbf{P})x_k + B_{\rm cl}(\mathbf{P})^\intercal\eta_{k+1}, u\rangle + \langle \mathcal{W}_{u_d}(\mathbf{P}), |u|\rangle + f_k, \tag{48}$$

written here again as a function of $\mathbf{P}$, the solution to $(I - \mathrm{Ric})(Y) = 0$. As pointed out in the remark, with the running assumption in force, $u \to J_{\mathbf{u}}$ is a convex function in which f_k denotes other terms not depending on u.

Eq. (48) exposes a distinct form than that pursued in Theorem 1, and an adequate framework of ideas and interpretations can be brought about. One important alternative derives from the notions of generalized gradients or subgradients of convex function to express the optimality condition simply as $0 \in \partial J_u$. Applied to (48), one gets that the optimal control at time k is expressed by

$$\bar{u}_k = \Delta(\mathbf{P})^{-1}\big(\Sigma_{\rm cl}(\mathbf{P})x_k + \frac{1}{2}(B_{\rm cl}(\mathbf{P})^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{P})\bar{\xi}_k)\big) \tag{49a}$$

where

$$\bar{\xi}_k = \begin{bmatrix} \bar{\xi}_{1,k} \\ \vdots \\ \bar{\xi}_{m,k} \end{bmatrix} \in \mathbb{R}^m \text{ is such that } \bar{\xi}_k^i = \begin{cases} +1, \text{ if } \bar{u}_{i,k} > 0, \\ -1, \text{ if } \bar{u}_{i,k} < 0, \\ \in (-1, +1) \text{ if } \bar{u}_{i,k} = 0, \end{cases} \qquad i = 1, 2, \ldots, m. \tag{49b}$$

The control dynamics introduce the absolute values to model a system in which the controller strives to modulate the current uncertainty. The impact is the emergence of a region in the state space, modulated by the nonnegative diagonal matrix $\mathcal{W}_u(\mathbf{P})$, in which the optimal solution is zero action. An interesting interpretation is that in the face of uncertainties, the optimal policy is not to act if, in the optimality scale, there is no considerable deviation from the zero-state (note that the H_∞-problem is a regulation control problem). This is undoubtedly a robustness approach distinct from the usual worst-case scenario.

As a result of the minimization of function J_u, a partition into the state space emerges, dividing it into three distinct regions for each control input entry. Note first that η_k is solely a function of x_k; thus, (49) is a stationary feedback control $\bar{u}(x)$. Each component $\bar{u}_i(x), i = 1, 2, \ldots, m$ defines three partitions (a disjoint covering) of state space $\mathbb{R}^n$:

$$\mathcal{R}_i^0 := \{x \in \mathbb{R}^n | \bar{u}_i(x) = 0\}, \quad \mathcal{R}_i^+ := \{x \in \mathbb{R}^n | \bar{u}_i(x) > 0\}, \quad \mathcal{R}_i^- := \{x \in \mathbb{R}^n | \bar{u}_i(x) < 0\}.$$

Note that solving (37d) would be much more intricate than (49). The optimal control resembles the classic linear quadratic regulator solution but for an affine term of state x_k and a stochastic term depending on x_k through the signals $k \to \mathcal{S}(x_k)$ and $k \to \mathcal{S}(u_k)$ at each future stage $\kappa \geq k$. The form in Theorem 1 however, allows for the study of stability and the role that a modified deterministic Riccati equation plays.

Suppose η_{k+1} is known. In that case, a full solution to the optimization problem in (49) is presented in[3] (see Theorem 5.12), based on a modified version of the SOR method[27]. Here, in Section 4, a Monte Carlo estimation method is proposed and applied to estimate η_{k+1}.

Asymptotic solutions

The advances in[3] regard the optimal solution written generally as (49). Before that, a framework of asymptotic optimal feedback controller formed the basis for the control understanding of discrete-time CSVIU systems[2] and for the Brownian motion-driven continuous-time case[1] solutions. It relies on the partitions mentioned above and gives interesting insights for simple approximations of the optimal solution.

The intersection $\mathcal{R}^0 = \bigcap_{i=1}^m \mathcal{R}_i^0$ receives special attention, and is said to be the *global inaction region*, where, notably, all control components $\bar{u}_i$ are null. If that region is non-empty, it surrounds the neighborhood of the equilibrium point, and the state vector evolves in open loop while inside it.

On the other hand, let us consider regions of state space in $\mathcal{R}_i^+$ or $\mathcal{R}_i^-$ that are sufficiently distant from $\mathcal{R}_i^0$ for each i, and denote the optimal control signal by $s_u := [\, s_{1,u} \cdots s_{m,u} \,]^\intercal$ with $s_{i,u} \in \{+1, -1\}, 1 \leq i \leq m$. For each i, we define the set $\mathcal{R}_i^{s_{u,i}} = \{x \in \mathbb{R}^n : \operatorname{sign}(\bar{u}_i(x)) = s_{u,i}\}$, and the corresponding *homogeneous signals region* is defined in the intersection $\mathcal{R}^{s_u} = \cap_{i=1}^m \mathcal{R}_i^{s_{u,i}}$, for each s_u; clearly, the signal vector s_u of $\bar{u}(x)$ remains constant for each $x \in \mathcal{R}^{s_u}$.

State vector signals also have an impact on the control solution. Let us consider the open orthants sets, $\mathbb{O}_j, j = 1, \ldots 2^n$ of $\mathbb{R}^n$. A point x is said to lie in an *asymptotic region* of the state space if $x \in \mathcal{R}^{s_u} \cap \mathbb{O}_j$, for some j, and x, is "sufficiently far" from any of the signal switchings boundaries of $\mathbb{O}_j$ and $\mathcal{R}^{s_u}$. With that, inside any asymptotic region, state and optimal control vectors are nonzero with constant signals $\mathcal{S}(x) = s_x$ and s_u, respectively.

Following the assumptions in Theorem 1, consider $\mathbf{P} \geq 0$ that satisfies the modified Riccati equation $(I - \mathrm{Ric})(Y) = 0$. Within one of the asymptotic regions, the state of $\hat{\Theta}_{\mathrm{ctr}}$ evolves and each of the signs, say $k \to \mathcal{S}(x_k) \simeq \bar{s}_x$ and $k \to \mathcal{S}(u_k) \simeq \bar{s}_u$, $\forall k$, are constant during a sufficiently long time period. One can approximate (see (42)),

$$\eta_1(x_0) = E\Big[\sum_{k=1}^{\infty} (\mathcal{A}^\intercal)^k \mathcal{W}_x(\mathbf{P}) \mathcal{S}(x_k) | x_0\Big] \simeq \mathfrak{v}(\bar{s}_x, \bar{s}_u) := E\Big[\sum_{k=1}^{\infty} (\mathcal{A}^\intercal)^k \mathcal{W}_x(\mathbf{P}) \bar{s}_x | x_0\Big] = \mathcal{A}(I - \mathcal{A})^{-\intercal} \mathcal{W}_x(\mathbf{P}) \bar{s}_x \tag{50}$$

with $\mathcal{A} = A + BG$ and $G = -\Delta(\mathbf{P})^{-1}\Sigma_{\mathrm{cl}}(\mathbf{P})$. That leads to an approximate saddle point valid within the asymptotic region associated to the signals $\bar{s}_x, \bar{s}_u$, in which,

$$\bar{u}(x) \simeq -\Delta(\mathbf{P})^{-1}\big(\Sigma_{\mathrm{cl}}(\mathbf{P})x + \frac{1}{2}(B_{\mathrm{cl}}(\mathbf{P})^\intercal \mathfrak{v}(\bar{s}_x, \bar{s}_u) + \mathcal{W}_u(\mathbf{P})\bar{s}_u)\big) \tag{51}$$

$$\bar{\omega}(x, u) \simeq -\Upsilon_\gamma(\mathbf{P})^{-1}\big(\Psi(\mathbf{P})x + \Gamma(\mathbf{P})u + \frac{1}{2}\sigma^\intercal \mathfrak{v}(\bar{s}_x, \bar{s}_u)\big) \tag{52}$$

A strategy to deal with the stochastic feature of the optimization in (49) is to find the inaction region and asymptotic regions with asymptotically valid optimal controllers, to act as approximations. The key point is that for each control action u_i of $u = [\, u_1 \cdots u_m \,]^\intercal$ an inaction region, $\mathcal{R}_i^0$, separates regions $\mathcal{R}_i^+$ and $\mathcal{R}_i^-$. Some interpolation method applies to connect the asymptotic control solutions to the inaction region.

Global saddle point solution and the inaction region

In view of the assumptions in Theorem 1 the optimal saddle point mentioned in Remark 4, $k \to (\bar{u}_k, \bar{\omega}_k)$ is finite and global in $\mathbb{R}^m \times \mathbb{R}^r$. Then, any compact sets $\mathcal{U}$ and $\mathcal{V}$ containing $(\bar{u}_k, \bar{\omega}_k)$ can be taken to apply[28, Theor. 2.3] to conclude that the saddle point solution of the game is a pure strategy solution. The following theorem adapts the results in[3, Sec. 5] to the H_∞ problem. The proof is a very similar extension, thus, omitted here.

Theorem 2. (Global saddle point solution) Under the assumptions of Theorem 1 consider

$$\bar{u}_k = -\Delta(\mathbf{P})^{-1}\big(\Sigma_{\mathrm{cl}}(\mathbf{P})x_k + \frac{1}{2}(B_{\mathrm{cl}}(\mathbf{P})^\intercal \eta_{k+1} + \bar{\xi}_k)\big), \tag{53a}$$

$$\bar{\omega}_k = -\Upsilon_\gamma(\mathbf{P})^{-1}\big(\Psi(\mathbf{P})x_k + \Gamma(\mathbf{P})\bar{u}_k + \frac{1}{2}\sigma^\intercal \eta_{k+1}\big) \tag{53b}$$

for $\bar{\xi}_k = [\,\bar{\xi}_{1,k}\ \cdots\ \bar{\xi}_{m,k}\,]^{\intercal}$ such that $|\bar{\xi}_k| \leq \mathcal{W}_{u_d}(\mathbf{P})$, understood componentwisely, and $\eta_{k+1} = E[\nu_{k+1}|x_k]$ is determined by (37a), for each $k \geq 0$. Then, $k \to (\bar{u}_k, \bar{\omega}_k)$ is the optimal saddle point in Remark 4. Besides, whenever $|\bar{\xi}_{i,k}| < (\mathcal{W}_{u_d}(\mathbf{P}))_i$, $\bar{u}_i = 0$ and $x_k \in \mathcal{R}_i^0$ satisfying,

$$\left|2\langle \Sigma_{\rm cl}(\mathbf{P})_i, x_k\rangle + \langle B_{\rm cl}(\mathbf{P})_i, \eta_{k+1}\rangle\right| < (\mathcal{W}_{u_d}(\mathbf{P}))_i \tag{54}$$

Remark 5. The characterization of an inaction region $\mathcal{R}_i^0$ is given by setting $\bar{u}_{i,k} = 0$ in (53a) and noting that, $\bar{\xi}_{i,k} \in (-c_i, +c_i)$ necessarily, with $c_i = (\mathcal{W}_{u_d}(\mathbf{P}))_i$. Hence, the inequality (54) is satisfied for each $x \in \mathcal{R}_i^0$. The region's boundaries $\mathcal{R}_i^0$ seem to be shaped as parallel hyperplanes; however, $\eta_{k+1} = E[\nu_{k+1}|x_k]$, thus depending on the state point $x_k = x$.

Remark 6. The pair $(\bar{u}_k, \bar{\omega}_k)$ at any stage $k \geq 0$, expresses the "global saddle point solution" of the CSVIU dynamic game. It is a pure strategy solution, and the optimality holds globally, namely, irrelevant of the particular orthants sets $\mathbb{O}_j$ or regions $\mathcal{R}_i^-, \mathcal{R}_i^0$ and $\mathcal{R}_i^+$ of $\mathbb{R}^n$, for each $i = 1, \ldots, m$ and $j = 1, \ldots, 2^n$.

4 | A NUMERICAL EXAMPLE

This section illustrates the design of a suboptimal H_∞-control problem. Recall that, due to the presence of the absolute value function in the dynamic equation (1), the optimal feedback depends on the signal of each of the current control entries, see (49). Concomitantly, future signal values of the state vector along the path impact the solution through the expected value η_{k+1}.

Algorithm 1 comprises an estimate for the expected value η_{k+1} given the present state x_k of the controlled CSVIU process at some stage k, along future time stages $k+1, k+2\ldots$. The akin of the optimal control input solves the subdifferential problem framed in (49), with η_{k+1} replaced by $\hat{\eta}_{k+1}$ as in Step 2, or by the estimate in Remark 7; see the Algorithm 1. The control design presented therein applies to any $\hat{\Theta}_{\rm ctr}$ system that satisfies the conditions in Theorem 1.

The optimal control and the maximum disturbance laws both rely on the conditional expectation $\eta_{k+1} = E[\nu_{k+1}|x_k]$, $k \geq 0$ in which x_k is the current state of the system. The vector η_{k+1} is estimated by Algorithm 1 and

The series involved in the calculus of η_{k+1} is truncated to not fewer than a K stages sum, see (55). The algorithm employs sample paths trajectories to calculate a sample $\eta_{k+1}(x_k^{(i)})$ from signal vectors $\mathcal{S}(x_\ell^{(i)})$ of a controlled realization $\ell \to x_\ell^{(i)}$ for $k \leq \ell \leq k+K$. Recall that if η_{k+1} and x_k are known, the optimal control action u_k is the result of the optimization problem in (49), whose solution is provided by[3, Theor. 5.12] for each k.

The experiment in the sequel approximates the norm γ^* of a system applying Algorithm 1. Given a suboptimal gain $\gamma > 0$, we also consider the impact of the noise terms $\sigma_x \varepsilon^x$ and $\sigma_u \varepsilon^u$ on the system's experimental gain $\hat{\gamma}$ obtained from numerical evaluations.

Consider the following time invariant "poorly-know" system for $k \geq 0$, with single control and noise inputs,

$$x(k+1) = \begin{bmatrix} 0.85 & -1 \\ 1.3 & 0.25 \end{bmatrix} x(k) + \begin{bmatrix} 1 \\ 1.2 \end{bmatrix} u(k) + \begin{bmatrix} 0.2 \\ 1 \end{bmatrix} \omega(k), \tag{57}$$

where $\omega(\cdot)$ is a deterministic persistent disturbance function with a bounded mean. Admitting that sharper mathematical modeling of the actual plant is not achievable, we adopt the CSVIU approach for the reference model in (57), by including the noise-modulating terms,

$$\sigma_x = \begin{bmatrix} 0.22 & 0 \\ 0 & 0.18 \end{bmatrix}, \overline{\sigma}_x = \begin{bmatrix} 0.18 & 0 \\ 0 & 0.45 \end{bmatrix}, \sigma_u = \begin{bmatrix} 0.16 \\ 0.11 \end{bmatrix}, \overline{\sigma}_u = \begin{bmatrix} 0.24 \\ 0.16 \end{bmatrix}, \tag{58}$$

to form a dynamical model system $\hat{\Theta}_{\rm ctr}$ as in (1). Besides, set $C = I_2$, $D = B$, and $F = \sigma$; with such matrix C the system is trivially detectable.

Let γ^* be the $\mathcal{H}_\infty$-norm of the system $\hat{\Theta}_{\rm ctr}$ for the optimal $\mathcal{H}_\infty$-problem introduced in (5). A reasonable alternative to the norm γ^* is a "close to optimal" norm, and let δ be a small positive number, the admissible error precision to γ^*. In the first part of the experiment, we set $\delta = 10^{-2}$; starting with some large enough number $\gamma > 0$, we decrease γ of δ steps until finding the value $\gamma_1 = 2.23$. It is the smallest positive scalar within the δ precision that makes simultaneously, $(I - {\rm Ric})(Y) = 0$ solvable and $r_\sigma(\mathcal{A}) < 1$, with $\mathcal{A}$ as in Algorithm 1. The control problem resulting from setting $\gamma_2 = 3$ is also considered for sensitivity analysis. We get the following matrices as the solution of the Riccati equation,

$$\mathbf{P}_{\gamma_1} = \begin{bmatrix} 1.5493 & -0.5468 \\ -0.5468 & 4.0877 \end{bmatrix}, \mathbf{P}_{\gamma_2} = \begin{bmatrix} 1.5101 & -0.5681 \\ -0.5681 & 3.7601 \end{bmatrix}. \tag{59}$$

Algorithm 1 Estimating the vector $\eta_{k+1} = E[v_{k+1}|x_k]$ to solve the optimization problem in (49)

Find $\mathbf{P} \succ 0$ that solves $(I - \mathrm{Ric})(Y) = 0$ for a detectable system $\hat{\Theta}_{\mathrm{ctr}}$. Denote $\mathcal{A} = A + BG$ with $G = -\Delta(\mathbf{P})^{-1}\Sigma_{\mathrm{cl}}(\mathbf{P})$ and verify whether $r_\sigma(\mathcal{A}) < 1$. The present state of system $\hat{\Theta}_{\mathrm{ctr}}$ is set as $x \in \mathbb{R}^n$.

Step 0. Adopt $x_0 = x$ and $\mathcal{S}(x_0^{(i)}) = \mathcal{S}(x_0)$ for each i. For $i = 1$, start a Monte Carlo sequence from a realization of $\hat{\Theta}_{\mathrm{ctr}}$ with the feedback $u_k^{(1)} = Gx_k^{(1)}$ on an interval $0 \le k \le \kappa + K$, providing an initial choice of signal vectors $s_k^{(1)} := \mathcal{S}(x_k^{(1)}), 0 \le k \le \kappa + K$. A set of samples is determined as

$$\eta_{k+1}(x_k^{(i)}) = \sum_{\ell=k+1}^{\kappa+K} (\mathcal{A}^\intercal)^{\ell-k}\mathcal{W}_x(\mathbf{P})\mathcal{S}(x_\ell^{(i)}), \quad 0 \le k \le K \tag{55}$$

with $i = 1$.

Substep 1a. Set $k = 0$ and apply $\eta_{k+1}(x_k^{(i)})$ as it were the true η_{k+1} and solve the optimization problem in (49) according to[3, Theor. 5.12].

Substep 1b. Simulate the one-step time evolution of system $\hat{\Theta}_{\mathrm{ctr}}$ with such control to get a sample $x_{k+1}^{(i)}$, if necessary, replace the signal vector $\mathcal{S}(x_{k+1}^{(i)})$. Solve the optimization problem in (49) for $k + 1$ with $x_{k+1}^{(i)}$ and $\eta_{k+2}(x_{k+1}^{(i)})$ in (55). Set $k = k + 1$ and repeat this substep up to $k = K$.

Substep 1c. Create the signal vector sequence $s_k^{i+1}, 0 \le k \le K + \kappa$ by keeping $s_0^{i+1} = \mathcal{S}(x_0)$, setting $s_k^{i+1} = \mathcal{S}(x_k^{(i)}), 1 \le k \le K$ from the previously substep updated values, and complete the remaining values by adopting $s_k^{(i+1)} = \mathcal{S}^{(i+1)}(x_K), K < k \le \kappa + K$. If $i < I$, set $i = i + 1$ and return to Substep 1a.

Step 2. After a number of repetitions I, discard the first $\imath$ samples and define the estimate

$$\hat{\eta}_1(x) = \sum_{k=1}^{\kappa+K} \left[\left((\mathcal{A}^\intercal)^k \mathcal{W}_x(\mathbf{P}) \right) \frac{1}{I - \imath} \sum_{i=\imath+1}^{I} s_k^{(i)} \right] \tag{56}$$

Solve the optimization problem in (49) for x and $\hat{\eta}_1(x)$ and apply the solution as an approximation to the optimal control at $x_0 = x$.

Step 3. At any successive state, $x_\ell, \ell > 1$, repeat the procedure with x_ℓ in place of x_0.

Remark 7. The number K is a finite horizon approximation for the solution; the maximum length is $\kappa + K$ when $k = 0$. It can be chosen to attain the required precision regarding the numerical relevance of the spectral radius $r_\sigma(\mathcal{A}^K)$.

A variation of the method to speed up possible convergence is to promote partial averages after discarding the first $\imath$ samples. In this case, the expression of $\eta_{k+1}(x_k^{(i)})$ in (55) is substituted in Substeps 1a and 1b by

$$\eta_{k+1}(x_k^{(i)}) = \sum_{\ell=k+1}^{\kappa+K} \left[\left((\mathcal{A}^\intercal)^{\ell-k} \mathcal{W}_x(\mathbf{P}) \right) \frac{1}{i - \imath} \sum_{j=\imath+1}^{i} \mathcal{S}(x_\ell^{(j)}) \right], \quad 0 \le k \le K.$$

and Step 2 would be simply to adopt $\hat{\eta}_1(x) = \eta_1(x_0^{(i)})$ with $i = I$, as above.

Note that $\mathbf{P}_{\gamma_1} \succ \mathbf{P}_{\gamma_2}$ and $\mathrm{eig}(\mathcal{A}_{\gamma_1}) = 0.0502 \pm 0.6062i$, and $\mathrm{eig}(\mathcal{A}_{\gamma_2}) = 0.0425 \pm 0.6045i$ are the corresponding eigenvalues of $\mathcal{A}$.

Choosing the gain γ_1, after successive powers $\left(\mathcal{A}_{\gamma_1}\right)^\kappa$, $\kappa \ge 0$ the sequence approaches the zero matrix for some κ large enough. We set $\kappa = 12$ and $K = 50$, and for an initial condition $x_0 = [10 \quad 10]^\intercal$, Fig. 1 presents an optimal control realization ($*$) and the respective maximum disturbance ($\circ$) for the CSVIU system $\hat{\Theta}_{\mathrm{ctr}}$ assembled in (57)–(58). Fig. 2 shows the corresponding controlled state trajectory.

Now, for the $\mathcal{H}_\infty$-norm estimation, set the initial state $x_0 = 0 \equiv [0\ 0]^\intercal$ and a time horizon of $T = 50$. We estimate the norm as,

$$\hat{\gamma} := \left[\left(\mathcal{E}_{2,I}^T(z) - J_\infty^T(z) \right) / \mathcal{E}_{2,I}^T(\omega) \right]^{1/2}, \tag{60}$$

in which, the two average measures,

$$\mathcal{E}_{2,I}^T(z) = E_x\left[\sum_{k=0}^{T} \|z_k\|^2 \middle| x(0) = 0 \right], \quad \text{and} \quad \mathcal{E}_{2,I}^T(\omega) = E_x\left[\sum_{k=0}^{T} \|\omega_k\|^2 \middle| x(0) = 0 \right],$$

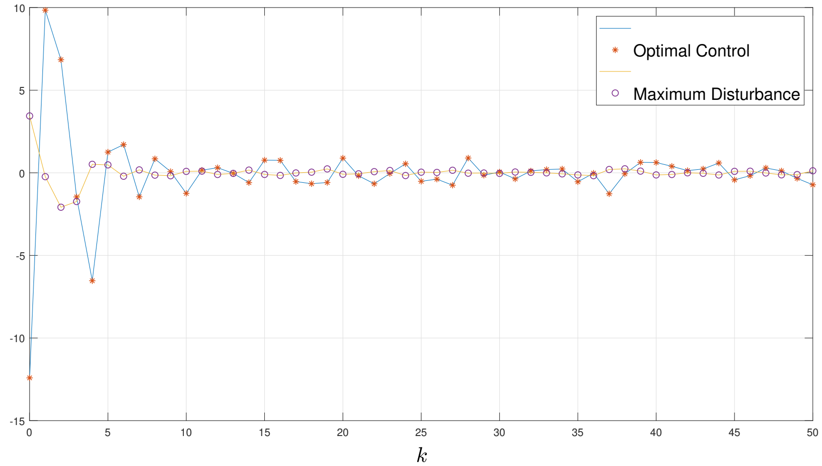

FIGURE 1 Optimal Control and respective Maximum Disturbance

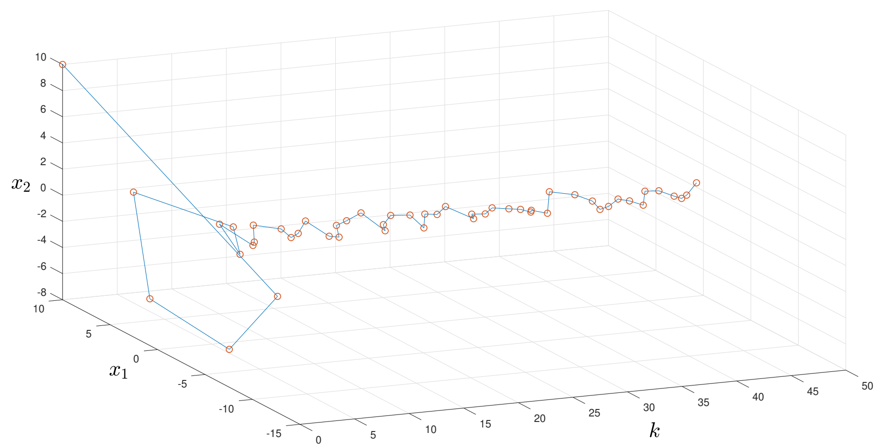

FIGURE 2 System's states trajectory

are estimated by Monte Carlo simulations, and the scalar $\hat{\gamma}$ is the system's performance obtained numerically. We ran eight Monte Carlo experiments, and Tables 1 and 2 present the results. Simulations 1–5 employ decreasing integer values of γ, whereas simulations 6–8 refine the precision of γ. The experiments suggest that decreasing γ also decreases the deviation of the system's experimental gain. Note that, as the γ approaches the norm γ^* of $\hat{\Theta}_{\text{ctr}}$, the estimation error $|\gamma - \hat{\gamma}|$ of the energy gain of the system tends to zero.

The numerical experiments were repeated, this time replacing the matrices σ_x and σ_u in (58) by matrices $10^{-2}\sigma_x$ and $10^{-2}\sigma_u$, respectively. Table 2 furnishes the results, and the estimation error $|\gamma - \hat{\gamma}|$ is null in all the cases due to the minor influence of the noise terms $10^{-2}\sigma_x\varepsilon^x$ and $10^{-2}\sigma_u\varepsilon^u$ on the nominal system.

5 | CONCLUSION

This paper formulates and solves the H_∞-control problem for discrete-time CSVIU systems. Among the significant features of this class of stochastic systems, we stress its ability to account for an infinite energy disturbance signal in an infinite horizon

TABLE 1 Numerical results from eight Monte Carlo experiments.

Experiment	δ	γ	$J_\infty^T(z)$	$\mathcal{E}_{2,I}^T(z)$	$\mathcal{E}_{2,I}^T(\omega)$	$\hat{\gamma}$	$\lvert\gamma-\hat{\gamma}\rvert$
1	-	7	29.6149	29.9791	0.0074	7.0154	0.0154
2	-	6	30.9623	31.4846	0.0145	6.0017	0.0017
3	-	5	27.6991	28.3994	0.0280	5.0011	0.0011
4	-	4	30.4845	31.6952	0.0757	3.9992	0.0008
5	-	3	29.8476	32.1277	0.2533	3.0003	0.0003
6	-	2.5	28.3880	32.1417	0.6006	2.5000	0
7	$1\cdot10^{-2}$	2.23	33.5676	39.2997	1.1527	2.2300	0
8	$1\cdot10^{-4}$	2.2297	35.0355	41.1868	1.2373	2.2297	0

TABLE 2 Monte Carlo experiments, for $10^{-2}\sigma_x$, $10^{-2}\sigma_u$

Experiment	$\bar{\delta}$	γ	$J_\infty^T(z)$	$\mathcal{E}_{2,I}^T(z)$	$\mathcal{E}_{2,I}^T(\omega)$	$\hat{\gamma}$	$\lvert\gamma-\hat{\gamma}\rvert$
1	-	7	0.0026	0.0027	7.0709e-07	7.0000	0
2	-	6	0.0029	0.0030	1.3963e-06	6.0000	0
3	-	5	0.0036	0.0037	3.5527e-06	5.0000	0
4	-	4	0.0029	0.0031	7.7426e-06	4.0000	0
5	-	3	0.0030	0.0032	2.5276e-05	3.0000	0
6	-	2.5	0.0032	0.0036	6.5541e-05	2.5000	0
7	$1\cdot10^{-2}$	2.23	0.0032	0.0038	1.1155e-04	2.2300	0
8	$1\cdot10^{-4}$	2.2297	0.0035	0.0041	1.2328e-04	2.2297	0

approach and underline the inaction region in the state space induced by the optimal solution. In such a behavior resides a novel form of attaining robustness.

Stability conditions under worst-case disturbances are derived. It turns out that stability holds provided that the solution of a perturbed Lyapunov-type equation exists and a detectability notion holds. Differential game machinery is the underpinning technique in the core to sufficient conditions for an optimal stabilizing compensator. The CSVIU dynamic game gives rise to a modified Riccati equation, and the existence of such a controller (as well as the optimal saddle point solution) relies partly on the solvability of this equation. Together with a spectral radius test of an associate matrix, they epitomize the results.

The optimal solution is explored by inspecting the ensuing static minimization and referring to a method with assured convergence. The paper then points out the inaction regions and the idea of asymptotic solutions. To complete the characterization, it frames the solution to the saddle point of the underlying stochastic game, in which the min–max induced cost $\mathfrak{P}_\infty^*$ connects directly to the norm definition and the H_∞ performance of the CSVIU system. The article also provides a critical Monte Carlo method that allows the solution computation at any state value of system $\hat{\Theta}_{\rm ctr}$. An example illustrates the method.

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Conflict of interest

The authors declare no potential conflict of interest.

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$\square$

APPENDIX

A PROOF OF LEMMA 3

Similarly to the proof of Lemma 1, one can evaluate the variation of (28) for some admissible control $k \to u_k$ along successive time steps of the state variable $k \to x_k$ for the dynamical system $\hat{\Theta}_{\mathrm{ctr}}$. Denote the auxiliary function $W : \mathbb{R}\times\mathbb{R}^n \to \mathbb{R}$ depending on the sequences $\mathbf{X}, \mathfrak{r}$ and $\mathfrak{g}$, which applies to the triple $k \to (x_k, u_k, \omega_k)$ as $W(k,x) := x^\intercal \mathbf{X}_k x + \langle \mathfrak{r}_k, |x| \rangle + \mathfrak{g}_k$. We write $\langle \mathfrak{r}, |x| \rangle = \langle \mathcal{S}(x), \mathfrak{r} \cdot x \rangle$ to evaluate,

$$\begin{aligned} W(k+1, x_{k+1}) - W(k, x_k) &= \|x_{k+1}\|^2_{\mathbf{X}_{k+1}} + \langle s_{k+1}, \mathfrak{r}_{k+1} \cdot x_{k+1} \rangle + \mathfrak{g}_{k+1} - (\|x_k\|^2_{\mathbf{X}_k} + \langle s_k, \mathfrak{r}_k \cdot x_k \rangle + \mathfrak{g}_k) \\ &= \|Ax_k + Bu_k\|^2_{\mathbf{X}_{k+1}} + 2(Ax_k + Bu_k)^\intercal \mathbf{X}_{k+1} \sigma_u(x_k, u_k)\zeta_{0,k} + \|\sigma_u(x_k,u_k)\zeta_{0,k}\|^2_{\mathbf{X}_{k+1}} - \|x_k\|^2_{\mathbf{X}_k} \\ &\quad + \langle s_{k+1}, \mathfrak{r}_{k+1} \cdot (Ax_k + Bu_k + \sigma_u(x_k,u_k)\zeta_{0,k}) \rangle - \langle s_k, \mathfrak{r}_k \cdot x_k \rangle + \mathfrak{g}_{k+1} - \mathfrak{g}_k \end{aligned} \tag{A1}$$

where the state signal vector on stages k and $k+1$ are denoted respectively by s_k and s_{k+1}. Also, there holds,

$$E\left[\|\sigma_u(x_k,u_k)\zeta_{0,k}\|^2_Y | x_k = x, u_k = u\right] = \|x\|^2_{\mathcal{Z}_x(Y)} + \langle \mathcal{S}(x), \mathcal{W}_x(Y)x \rangle + \|u\|^2_{\mathcal{Z}_u(Y)} + \langle \mathcal{S}(u), \mathcal{W}_u(Y)u \rangle + \varphi_1(Y).$$

Analogously, denote the control signal vector by $s^u := \mathcal{S}(u)$, and by adding and subtracting the terms $\|z_k\|^2 - \gamma^2\|\omega_k\|^2$, one gets

$$\begin{aligned} W(k+1, x_{k+1}) - W(k, x_k) &= \|Ax_k\|^2_{\mathbf{X}_{k+1}} + \|x_k\|^2_{\mathcal{Z}_x(\mathbf{X}_{k+1})} + \|x_k\|^2_{C^\intercal C} - \|x_k\|^2_{\mathbf{X}_{k+1}} + \langle s_k, \mathcal{W}_x(\mathbf{X}_{k+1})x_k \rangle - \langle s_k, \mathfrak{r}_k \cdot x_k \rangle \\ &+ \|\omega_k\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})} + 2\langle F^\intercal(Cx_k + Du_k), \omega_k \rangle + 2(Ax_k + Bu_k)^\intercal \mathbf{X}_{k+1} \sigma\omega_k + \langle s_{k+1} \cdot \mathfrak{r}_{k+1}, (Ax_k + Bu_k + \sigma\omega_k) \rangle \\ &+ \|u_k\|^2_{\Lambda(\mathbf{X}_{k+1})} + 2\langle B^\intercal \mathbf{X}_{k+1} Ax_k, u_k \rangle + \langle s^u_k, \mathcal{W}_u(\mathbf{X}_{k+1})u_k \rangle + 2\langle D^\intercal Cx_k, u_k \rangle \\ &+ \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) + m_k - [\|z_k\|^2 - \gamma^2\|\omega_k\|^2] \end{aligned} \tag{A2}$$

where the remaining process $k \to m_k$ is given by,

$$m_k = \langle 2\mathbf{X}_{k+1}(Ax_k + Bu_k + \sigma\omega_k) + s_{k+1} \cdot \mathfrak{r}_{k+1}, [\sigma_x + \bar{\sigma}_x \operatorname{diag}(|x|) \;\; \sigma_u + \bar{\sigma}_u \operatorname{diag}(|u|)][\varepsilon^x(k) \;\; \varepsilon^u(k)]^\intercal \rangle$$

is a zero $\{\mathcal{F}_k\}$-martingale. Note that the difference in (A2) depends only on x_k, u_k and ω_k and to that point, let us denote $\Delta W(x_k, u_k, \omega_k) := W(k+1, x_{k+1}) - W(k, x_k)$. Set also $\eta_k := s_k \cdot \mathfrak{r}_k$ and $\eta_{k+1} := s_{k+1} \cdot \mathfrak{r}_{k+1}$. By performing some algebraic manipulations in (A2), having in mind the assumption $\Upsilon_\gamma(\mathbf{X}_k) \prec 0, \forall k$, one gets an upper bound for the lhs below. Set $\omega_k = \omega$,

$$
\begin{aligned}
&\Delta W(x_k, u_k, \omega) + [\|z_k\|^2 - \gamma^2\|\omega\|^2] = \\
&\qquad \|Ax_k\|^2_{\mathbf{X}_{k+1}} + \|x_k\|^2_{\mathcal{Z}_x(\mathbf{X}_{k+1})} + \|x_k\|^2_{C^\intercal C} - \|x_k\|^2_{\mathbf{X}_{k+1}} + \langle A^\intercal \eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle + \\
&\qquad \|\omega\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})} + \langle 2\Psi(\mathbf{X}_{k+1})x_k + 2\Gamma(\mathbf{X}_{k+1})u_k + \sigma^\intercal\eta_{k+1}, \omega\rangle + \|u_k\|^2_{\Lambda(\mathbf{X}_{k+1})} + \\
&\qquad \langle 2\Sigma(\mathbf{X}_{k+1})x_k + B^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s^u_k, u_k\rangle + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) + m_k \\
&\qquad \le \|Ax_k\|^2_{\mathbf{X}_{k+1}} + \|x_k\|^2_{\mathcal{Z}_x(\mathbf{X}_{k+1})} + \|x_k\|^2_{C^\intercal C} - \|x_k\|^2_{\mathbf{X}_{k+1}} + \langle A^\intercal \eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle + \\
&\qquad \max_\omega \|\omega - \omega^0_k\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})} - \|\omega^0_k\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})} + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) + m_k + \\
&\qquad \|u_k\|^2_{\Lambda(\mathbf{X}_{k+1})} + \langle 2\Sigma(\mathbf{X}_{k+1})x_k + B^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s^u_k, u_k\rangle
\end{aligned} \tag{A3}
$$

which shows (29) and the fact that the equality is attained if $\omega_k = \omega^0_k$ in (31a).

Now, suppose that $\omega_k = \omega^0_k$. One gets for some $u_k = u$,

$$
\begin{aligned}
&\Delta W(x_k, u, \omega^0_k) + [\|z_k\|^2 - \gamma^2\|\omega^0_k\|^2] = \\
&\qquad x_k^\intercal\left(A^\intercal \mathbf{X}_{k+1}A + \mathcal{Z}_x(\mathbf{X}_{k+1}) + C^\intercal C - \mathbf{X}_k\right)x_k + \langle A^\intercal\eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle + \\
&\qquad \|u\|^2_{\Delta(\mathbf{X}_{k+1})} + \langle 2\Sigma(\mathbf{X}_{k+1})x_k + B^\intercal\eta_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s^u_k - \Gamma(\mathbf{X}_{k+1})^\intercal\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}\left(2\Psi(\mathbf{X}_{k+1})x_k + \sigma^\intercal\eta_{k+1}\right), u\rangle + \\
&\qquad - \left\|\Psi(\mathbf{X}_{k+1})x_k + \frac{1}{2}\sigma^\intercal\eta_{k+1}\right\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}} + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) + m_k
\end{aligned} \tag{A4}
$$

Now, by one-stage minimization, we get that,

$$
\begin{aligned}
&\Delta W(x_k, u, \omega^0_k) + [\|z_k\|^2 - \gamma^2\|\omega^0_k\|^2] \ge \\
&\qquad x_k^\intercal\left(A^\intercal \mathbf{X}_{k+1}A + \mathcal{Z}_x(\mathbf{X}_{k+1}) + C^\intercal C - \mathbf{X}_k\right)x_k + \langle A^\intercal\eta_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \eta_k, x_k\rangle + \\
&\qquad \min_u\left[\|u - u^0_k\|^2_{\Delta(\mathbf{X}_{k+1})} - \|u^0_k\|^2_{\Delta(\mathbf{X}_{k+1})}\right] - \left\|\Psi(\mathbf{X}_{k+1})x_k + \frac{1}{2}\sigma^\intercal\eta_{k+1}\right\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}} + \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) + m_k
\end{aligned} \tag{A5}
$$

where u^0_k is given by (31b), which also allows us to write (30). The equality is attained if u_k is chosen as the minimizer of the expression within brackets in (A5), and the expressions in the lemma are thus shown.

B PROOF OF LEMMA 4

Let us consider the feedback control $u_k = Gx_k$ and $\omega_k = \omega^0_k$ as in Lemma 3. Denote the auxiliary function $V : \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}$ depending on the sequences $\mathbf{X}$, ν and $\mathfrak{g}$, which applies to the triple $k \to (x_k, u_k = Gx_k, \omega^0_k)$ as,

$$V(x_k, Gx_k, \omega^0_k) := x_k^\intercal \mathbf{X}_k x_k + \langle \nu_k, |x_k|\rangle + \mathfrak{g}_k$$

and its variation $\Delta V(x_k, Gx_k, \omega^0_k) := V(x_{k+1}, Gx_{k+1}, \omega^0_{k+1}) - V(x_k, G_k, \omega^0_k)$. Denote $s_k = \mathcal{S}(x_k)$ and $s^u_k = \mathcal{S}(u_k)$. With the choice of ω^0_k, one gets similar to eq. (A4) in Lemma 3 that

$$
\begin{aligned}
&E[\Delta V(x_k, u, \omega^0_k) + [\|z_k\|^2 - \gamma^2\|\omega^0_k\|^2 | x_k] = \\
&\qquad x_k^\intercal\left(A^\intercal \mathbf{X}_{k+1}A + \mathcal{Z}_x(\mathbf{X}_{k+1}) + C^\intercal C - \mathbf{X}_k\right)x_k + \langle A^\intercal\nu_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k - \nu_k, x_k\rangle + \\
&\qquad - \left\|\Psi(\mathbf{X}_{k+1})x_k + \Gamma(\mathbf{X}_{k+1})u_k + \frac{1}{2}\sigma^\intercal\nu_{k+1}\right\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}} + \|u_k\|^2_{\Lambda(\mathbf{X}_{k+1})} + \langle 2\Sigma(\mathbf{X}_{k+1})x_k + B^\intercal\nu_{k+1} + \mathcal{W}_u(\mathbf{X}_{k+1})s^u_k, u_k\rangle + \\
&\qquad \mathfrak{g}_{k+1} - \mathfrak{g}_k + \varphi_1(\mathbf{X}_{k+1}) = \\
&x_k^\intercal\Big(\mathcal{L}_\gamma(\mathbf{X}_{k+1}) + C^\intercal C - \mathbf{X}_{k+1} + G^\intercal\Delta(\mathbf{X}_{k+1})G + \Sigma(\mathbf{X}_{k+1})^\intercal G + G^\intercal\Sigma(\mathbf{X}_{k+1}) -
\end{aligned}
$$

$$G^\intercal\Gamma(\mathbf{X}_{k+1})^\intercal\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}\Psi(\mathbf{X}_{k+1}) - \Psi(\mathbf{X}_{k+1})^\intercal\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}\Gamma(\mathbf{X}_{k+1})G - \mathbf{X}_{k+1}\Big)x_k +$$
$$\langle\big(A_{\mathrm{cl}}(\mathbf{X}_{k+1}) + B_{\mathrm{cl}}(\mathbf{X}_{k+1})G\big)^\intercal v_{k+1} + \mathcal{W}_x(\mathbf{X}_{k+1})s_k + G^\intercal\mathcal{W}_u(\mathbf{X}_{k+1})s_k^u - v_k, x_k\rangle +$$
$$\mathfrak{g}_{k+1} + \varphi_1(\mathbf{X}_{k+1}) - \frac{1}{4}\|\sigma^\intercal v_{k+1}\|^2_{\Upsilon_\gamma(\mathbf{X}_{k+1})^{-1}} - \mathfrak{g}_k \quad \text{(B6)}$$

Now, from the quadratic term in x_k of(B6) involving $\mathbf{X}_{k+1}$, we get from the operator definitions in (7) and (24) that

$$\mathcal{L}_\gamma(Y) + C^\intercal C + G^\intercal\Delta(Y)G + \Sigma(Y)^\intercal G + G^\intercal\Sigma(Y) - G^\intercal\Gamma(Y)^\intercal\Upsilon_\gamma(Y)^{-1}\Psi(Y) - \Psi(Y)^\intercal\Upsilon_\gamma(Y)^{-1}\Gamma(Y)G =$$
$$A^\intercal Y A + \mathcal{Z}_x(Y) + C^\intercal C + G^\intercal\Lambda(Y)G + \Sigma(Y)^\intercal G + G^\intercal\Sigma(Y) - \big(\Psi(Y) + \Gamma(Y)G\big)^\intercal\Upsilon_\gamma(Y)^{-1}\big(\Psi(Y) + \Gamma(Y)G\big) \quad \text{(B7)}$$

where we denote $Y = \mathbf{X}_{k+1}$ for symplicity. For the last quadratic symmetric term in (B7), we obtain,

$$\big(\bullet\big)^\intercal\Upsilon_\gamma(Y)^{-1}\big(\Psi(Y) + \Gamma(Y)G\big) = \big[\bullet\big]^\intercal\Upsilon_\gamma(Y)^{-1}\big[\sigma^\intercal Y(A + BG) + F^\intercal(C + DG)\big] \quad \text{(B8)}$$

and for the other three terms in (B7),

$$G^\intercal\Lambda(Y)G + \Sigma(Y)^\intercal G + G^\intercal\Sigma(Y) = G^\intercal B^\intercal Y BG + G^\intercal\mathcal{Z}_u(Y)G + G^\intercal D^\intercal DG + A^\intercal Y BG + G^\intercal B^\intercal Y A + C^\intercal DG + G^\intercal D^\intercal C. \quad \text{(B9)}$$

Then, one can write (B7) as

$$(A + BG)^\intercal Y(A + BG) + \mathcal{Z}_x(Y) - \big[\bullet\big]^\intercal\Upsilon_\gamma(Y)^{-1}\big[\sigma^\intercal Y(A + BG) + F^\intercal(C + DG)\big] + G^\intercal\mathcal{Z}_u(Y)G + (C + DG)^\intercal(C + DG)$$
$$= \mathcal{A}^\intercal Y\mathcal{A} + \mathcal{Z}_x(Y) - \big[\bullet\big]^\intercal\Upsilon_\gamma(Y)^{-1}\big[\sigma^\intercal Y\mathcal{A} + F^\intercal\mathcal{C}\big] + G^\intercal\mathcal{Z}_u(Y)G + \mathcal{C}^\intercal\mathcal{C} \quad \text{(B10)}$$

with the notations $\mathcal{A}$ and $\mathcal{C}$ in the lemma. Given the operators defined in the lemma, the quadratic term in x_k of (B6) can be written, from the above as,

$$\mathcal{A}^\intercal\mathbf{X}_{k+1}\mathcal{A} + \mathcal{Z}_x(\mathbf{X}_{k+1}) + \mathcal{M}_\gamma(\mathbf{X}_{k+1}) + G^\intercal\mathcal{Z}_u(\mathbf{X}_{k+1})G + \mathcal{C}^\intercal\mathcal{C} - \mathbf{X}_k = \mathscr{L}_\gamma(\mathbf{X}_{k+1}) + G^\intercal\mathcal{Z}_u(\mathbf{X}_{k+1})G + \mathcal{C}^\intercal\mathcal{C} - \mathbf{X}_k \quad \text{(B11)}$$

where $\mathcal{M}_\gamma(\cdot)$ and $\mathscr{L}_\gamma(\cdot)$ are, mutatis mutandis the same operators in (7g) and (7h), respectively, replacing matrices A and C by $\mathcal{A}$ and $\mathcal{C}$. Considering the set of equations (34), we conclude from (B6) that

$$E[V(x_k, Gx_k, \omega_k^0) - V(x_{k+1}, Gx_{k+1}, \omega_{k+1}^0)|x_k] = E[\|z(k)\|^2 - \gamma^2\|\omega^0(k)\|^2|x_k]$$

Now, set $\mathbf{X}_\kappa = 0$, $v_\kappa = 0$, $\mathfrak{g}_\kappa = 0$ and note that system $\hat{\Theta}_{\mathrm{ctr}}$ with a feedback control is Markovian. Then,

$$E[x_0^\intercal\mathbf{X}_\kappa x_0 + \langle v_0, |x_0|\rangle + \mathfrak{g}_0|x_0] = E\Big[\sum_{k=0}^{\kappa-1}\|z(k)\|^2 - \gamma^2\|\omega^0(k)\|^2\Big|x_0\Big]$$

In addition, to show that $k \to u_k = Gx_k$ stabilizes $\hat{\Theta}_{\mathrm{ctr}}$ in the sense of Definition 2 (ii), denote the sequences $\mathbf{X}_k^{(\kappa)}$, $v_k^{(\kappa)}$ and $\mathfrak{g}_k^{(\kappa)}$, for $k = 0, \ldots, \kappa$, for some horizon $\kappa \in \mathbb{N}$ with null values for $\mathbf{X}_\kappa^{(\kappa)}$, $v_\kappa^{(\kappa)}$ and $\mathfrak{g}_\kappa^{(\kappa)}$. Note that the positive semidefinite ordering $0 \preceq \mathbf{X}_0^{(0)} \preceq \mathbf{X}_0^{(1)} \preceq \cdots$ holds and, from the assumptions, the system $\hat{\Theta}_{\mathrm{ctr}}$ is $(C, \mathcal{L}_\gamma)$-detectable. Using the notation above, this implies that the "uncontrolled" system $\hat{\Theta}$ given by

$$\begin{cases} x_{k+1} = \mathcal{A}x_k + \sigma(x_k)\zeta_k, \\ z_k = \mathcal{C}x_k + F\omega_k, \end{cases}$$

is $(\mathcal{C}, \mathscr{L}_\gamma)$-detectable. If there is a solution $\mathbf{Y} \succeq 0$ to (36) with $\Upsilon_\gamma(\mathbf{Y}) \prec 0$, it is the unique solution since $\mathscr{L}_\gamma(\cdot)$ is a linear-positive operator. Hence, we get that, $0 \preceq \mathbf{X}_k^{(\kappa)} \uparrow \mathbf{Y}$ in the semipositive definite sense, as $\kappa \to \infty$. Moreover, if $r_\sigma(A_{\mathrm{cl}} + B_{\mathrm{cl}}G) < 1$, then Corollary 1 (i) states that the system $\hat{\Theta}$ is stochastically stable, or equivalently that control $u_k = Gx_x$, $k \geq 0$ is stabilizing in the sense of Definition 2 (ii).