# Advancing categories with functors of functors

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**Abstract**

In this document, we review the basics of category theory and functors calculus, and we implement some novel concepts. In particular, we review the concepts of category, functors, and natural transformations. Furthermore we introduce the novel concept of func- tors of functors. We illustrate these concepts with diagrams to ameliorate the expression of these ideas. We conclude that these concepts open new avenues and are advancing category.

Prepared for submission

# Advancing categories with functors of functors

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**Abstract.** In this document, we review the basics of category theory and functors calculus, and we implement some novel concepts. In particular, we review the concepts of category, functors, and natural transformations. Furthermore we introduce the novel concept of functors of functors. We illustrate these concepts with diagrams to ameliorate the expression of these ideas. We conclude that these concepts open new avenues and are advancing category theory.

**Keywords:** category theory, functors

## Contents



*Paper dedicated to all the dreamers and philosophers!*

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### 1.7 Data availability statement

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# 3 Introduction

Category theory is the analogue of theory which takes a astronaut's eye view of mathematics. From the outerspace, details become imperceptible, but we can distinguish patterns that were impossible to detect from Earth surface's level. How does the the direct sum of two vector spaces match the lowest common multiple of two numbers ? What do free groups, fields of fractions, and discrete topological spaces, have in common? We will explore answers to these and many akin questions, seeing patterns in mathematics and physics that you may never have seen before. This documents is based on Leinster [1], and Wikipedia contributors [2].

Note that category theory, has many contemporarily and recent applications, such as in physics, in Feynman diagrams [3], cosmology and functors of actions [4], statistics and computer science [5], and even epidemiology [6]. Therefore, it is highly importance to develop and establish further the concepts of category theory.

# 4 Preliminaries

We start by the concepts introduced in Leinster [1], and Wikipedia contributors [2], and then we expand and develop them further. Category theory is a branch of mathematics which explores ideas to generalise all subjects of mathematics in terms of categories, independent of what their objects represent. Aesthetically, every branch of modern mathematics can be stated in terms of categories. These statements often reveal deep insights and similarities between apparently different areas of mathematics. By definition, category theory accommodates an alternative foundation for mathematics to proposed axiomatic foundations such as set theory. Generically, the objects and their relations could be abstract entities of any kind, while the notion of category accommodates an abstract and fundamental way to state mathematical entities and their relations.

In mathematics, a category is a collection of objects that are linked by relations with a direction. We can call these relations or links as directed links, directed relations or arrows, these are formally named as morphisms. A category has two basic properties: the ability to compose the relations associatively and the existence of an identity relation for each object. A simple example is the category of sets, whose objects are sets and whose relations are functions. Usually, an abstract category is simply stated as category and this notion should be distinguished from a concrete category. The notion of concrete category is beyond the scope of this document.

The most important concept in this chapter is that of *universal property*. The further one deepens its knowledge in mathematics, especially pure mathematics, the more universal properties one meets. We will study different manifestations of this concept.

A first example of a universal property can be simply put as follows. In this context, the following meanings of the words 'map', 'mapping', and 'function' are identical.

**Example 1.** *Let's denote with 1 a set with one element. The nomenclature of this element can be ignored. Then 1 has the following property:*

$$\begin{aligned} &\textit{For all sets, } S\textit{, there exists a unique map from } S \textit{ to } 1. \\ &\forall\, S,\, \exists\, m_u : S \to 1 \end{aligned} \tag{4.1}$$

*Proof.* In other words lets consider the following. Let $S$ be a set, there exist a map $S \to 1$, because we can define a map $m : S \to 1$, by taking $m(s)$ to be the single element of 1 for each $s \in S$. This is a *unique map* $S \to 1$, because there is no other choice for this subject: Any function $S \to 1$ must send each element of $S$ to the single element 1. $\square$

Note that phrases formulated as "there exists a unique such-and-such fulfilling some condition" are common in category theory. The phrase meaning is that there is only one such-and-such fulfilling the condition. The existence part is proven by showing that there is at least one, while the uniqueness part is proven by showing that there is at most one; i.e. any two such-and-suches fullfilling the condition are equal. The property starts with the words 'For all sets $S$', which informs something about the relation between 1 and every set $S$: namely, there is a unique function from $S$ to 1.

Such properties are called "universal" since they state how the object stated (in this case the set 1) relates to the entire universe in which it lives (in this case the universe of sets).

**Example 2.** *Another example of categories is the rings, with the multiplicative identity* 1. *Same-wise, homomorphism of rings are considered to preserve mutliplicative identities. The ring* $\mathbb{Z}$ *has the property that for all rings* $R$*, there exists a unique homomorphism,* $\mathbb{Z} \to R$*, i.e. we write:*

$$\forall\ R,\ \exists\ \textit{unique}\ hm : \mathbb{Z} \to R\ . \tag{4.2}$$

*Proof of existence. Let* $R$ *be a ring. Define a function* $\phi : \mathbb{Z} \to R$ *by*

$$\phi(n) = \begin{cases} 1 + \cdots + (n-2)\ \textit{times} + \cdots + 1\ , & \textit{if}\, n > 0 \\ 0 & ,\ \textit{if}\, n = 0 \\ -\phi(-n) & ,\ \textit{if}\, n < 0 \end{cases} \tag{4.3}$$

*where* $n \in \mathbb{Z}$. *A series of elementary checks confirms that* $\phi$ *is homomorphism. For example by checking that for* $\phi(1+1) = 1+1 = \phi(1) + \phi(1) = 1+1 = 2$.
*Proof of uniqueness. Let* $R$ *be a ring and let the existence of the homomorphism be* $\psi : \mathbb{Z} \to R$. *It suffices to show that the aforementioned* $\phi$ *is equal to another homomorphism, named as* $\psi$. *We know that these homomorphism preserve multiplicative identities, such as*

$$\phi(1) = 1 = \psi(1) \tag{4.4}$$

*they preserve addition, i.e.*

$$\phi(n) = 1 + \ldots (n-2) \textit{times} \cdots + 1 = \psi(1) + \ldots (n-2) \textit{times} \cdots + \psi(1) = \psi(n)\ ,\ \forall\ n > 0\ . \tag{4.5}$$

*They also preserve additive identity, i.e. zero, which means*

$$\phi(0) = 0 = \psi(0)\ . \tag{4.6}$$

*and they preserve negatives*

$$\phi(n) = -\phi(-n) = -\psi(-n) = \psi(n)\ ,\ \forall\ n < 0\ . \tag{4.7}$$

Note that essentially there is only one object which can satisfy a given universal property. The word essentially brings emphasis of the fact that two objects satisfying the same universal property need not literally be equal, but they are always isomoprhic. For example

**Definition 1.** *Lemma Let* $X$ *be a ring with the property that for all rings,* $R$*, there exists a unique homomorphism from* $X$ *to* $R$*, and then* $X \simeq \mathbb{Z}$. *we write:*

$$\forall\ R,\ X \to R\ :\ X \simeq \mathbb{Z} \tag{4.8}$$

# 5 Category theory

Following from Leinster [1], in this section we give the basic definitions of category theory, and we try to generalise them, and remove some degeneracies introduced by previous authors. Firstly we start defining the categories, then the functors, including the novel concept of functors of functors, and then natural formalism.

## 5.1 Category definitions

Here, we describe the definitions of a category.

### 5.1.1 Category: Definition 0

Generically in mathematics, informally, a category (sometimes called an abstract category to distinguish it from a concrete catergory) is a collection of objects that are linked with some directed links, e.g. arrows. Any category have two basic properties: the ability to compose the directed links associatively and the existence of an identity directed link for each object to the same object.

**Example 3.** *A simple example is the category of sets, whose objects are sets and whose directed links are functions. There exist, the set* $A, B, C$*, with their identities,* $id_A \in A$*,* $id_C \in C$*,* $id_C \in C$ *and the functions between them* $f : A \to B$*,* $g : B \to C$*, and their composition,* $f \circ g : A \to C$*. This category usually is denoted with a bold number,* ***3****.*

There are many equivalent definitions of a category. One commonly used definition is as follows.

**Definition 2.** *A category* $C$ *consist of*

- *a class of objects denoted with* $ob(C)$
- *a class of morphism, or links, or arrows, or maps between the objects, denoted wtih* $hom(C)$*.*
- *a domain, or source object class function, denoted with* $dom : hom(C) \to ob(C)$*,*
- *a codomain or target object class function, denoted with* $cod : hom(C) \to ob(C)$*,*
- *for every three objects* $a$*,* $b$*, and* $c$*a binary operation*

$$hom(a, b) \circ_b inhom(b, c) \to hom(a, c) \tag{5.1}$$

  *exists called composition of morphism; the composiiton of* $f : a \to b$ *and* $g : b \to c$ *is written as* $f \circ_{comp} g$ *or simply* $f \circ g$ *or* $fg$*.*

*such that the following axioms hold:*

- *Assosiativity. if* $f : a \to b$*,* $g : b \to c$ *and* $h : c \to d$*, then* $(f \circ g) \circ h = f \circ (g \circ h)$
- *Identity. For every object* $x$*, there exists a morphism* $id_x : x \to x$*, called the identity morphism for* $x$*, such that every morphism* $f : a \to x$ *satisfies* $id_x \circ f \to f$ *and every morphism* $g : x \to b$ *satisfies* $g \circ id_x = g$*.*

*Note that here hom(a,b) denotes the subclass of morphisms f in hom(C) such that dom(f) = a and cod(f) = b. Such morphism is often written as* $f : a \to b$.

*We write* $f : a \to b$*, and we say "f is a morphism from a to b". We write hom(a, b) (or* $hom_C(a, b)$ *to denote also the category of the morphism) to denote the hom-class of all morphisms from a to b. From these axioms, we can prove that there is exactly one identity morphism for every object.*

Generically, in mathematics, morphism, is defined as follows.

**Definition 3.** ***Morphism*** *Informally, a morphism is a structure preserving map from one mathematical structure to another of the same type. They are sometime called directed links, directed relations or arrows.*

**Example 4.** *In set theory, morphisms are functions; in linear algebra, linear transformations; in group theory, group homomorphisms; in topology, continuous functions; and so on.*

In category theory, morphism is a broadly similar idea: the mathematical objects involved need not be sets, and the relations between them may be something other than maps, although the morphism between the objects of a given category have to behave similarly to maps in that they have to admit an associative operation similar to function composition. Therefore, in category theory a morphism is an abstraction of the homomorphism from group theory.

**Definition 4.** ***Morphism****. A category C consists of two classes, one of objects and the other of morphisms. There are two objects that are associated to every morphism, the source and the target. A morphism f with a source X and target Y is written as* $f : X \to Y$*. Diagrammatically represented by an arrow named f from X to Y.*

Note that morphisms are basically links with a direction, or else directed links.

Note that for common categoties, objects are usually sets (often with some additional structure) and morphism are functions from an object to another object. Therefore the source and the target of a morphism are often called domain and codomain.

### 5.1.2 Category: Definition 1

From Leinster [1], we have the following definition for the category.

**Definition 5.** *A category* $\mathcal{C}$ *consists of*

- *a collection of objects, denoted with* $\mathrm{ob}(\mathcal{C})$*;*
- *for two such objects,* $A, B \in \mathrm{ob}(\mathcal{C})$*, a collection of maps or arrows or morphisms or directed links exists from A to B, denoted with* $\mathcal{C}(A, B)$*;*
- *for three such objects* $A, B, C \in \mathrm{ob}(\mathcal{C})$*, a map*

$$\mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \tag{5.2}$$
$$(f, g) \mapsto f \circ g \tag{5.3}$$

  *exists and it is called composition*

- *for each* $A \in \mathrm{ob}(\mathcal{C})$ *an element* $\mathrm{id}_A \in \mathcal{C}(A, A)$ *exists, and it is called identity on A.*

*satisfying the axioms:*

- *associativity,* $\mathbb{a}$

$$\forall,\ f \in \mathcal{C}(A,B),\ g \in \mathcal{C}(B,C),\ \text{and } h \in \mathcal{C}(C,D) \tag{5.4}$$

  *we have*

$$(f \circ g) \circ h = f \circ g(\circ h)\ . \tag{5.5}$$

- *identity law*

$$\forall\ f \in \mathcal{C}(A,B) \Rightarrow \mathrm{id}_A \circ f = f \circ \mathrm{id}_B = f\ . \tag{5.6}$$

### 5.1.3 Category from Definition 1 to definition 2: Proof

To define the category, we can have an adaptive definition from Leinster [1], in which we can remove any degeneracy that exists in Leinster [1] formalism, as follows.

**Definition 6.** *A category $\mathcal{C}$ consists of the following five items:*

- *a collection of objects, denoted with $\mathcal{O} = \mathcal{O}(\mathcal{C}) = \mathrm{ob}(\mathcal{C})$;*
- *for any two such objects, $A, B \in \mathrm{ob}(\mathcal{C})$, a collection of maps or arrows or morphisms or directed links exists from $A$ to $B$, denoted with*

$$\mathbb{m} = \mathcal{C}(A,B) \tag{5.7}$$
$$\mathbb{m} : A \to B \tag{5.8}$$

- *for three such objects $A, B, C \in \mathrm{ob}(\mathcal{C})$ (which means, $A, B, C \in \mathcal{O}$) a map*

$$\mathbb{c} : \mathcal{C}(A,B) \times \mathcal{C}(B,C) \to \mathcal{C}(A,C) \tag{5.9}$$
$$\mathbb{c} : (f,g) \mapsto f \circ g \tag{5.10}$$

  *exists and it is called composition, denoted with $\mathbb{c}$,*

- *for each $A \in \mathrm{ob}(\mathcal{C})$ (which means $A \in \mathcal{O}$) an element $\mathrm{id}_A \in \mathcal{C}(A,A)$ exists, and it is called identity on $A$. Since this is true $\forall\ A \in \mathcal{O}$, we can all this as the existence of $\mathrm{id}_{\mathcal{O}}$. and*
- *associativity, $\mathbb{a}$ is defined as*

$$\mathbb{a} : (f \circ g) \circ h = f \circ g \circ (\circ h)\ , \tag{5.11}$$

$$\forall,\ f \in \mathcal{C}(A,B),\ g \in \mathcal{C}(B,C),\ \text{and } h \in \mathcal{C}(C,D)\ . \tag{5.12}$$

*This means that collectively we can define a category through a signature. Therefore, generically, a category, $\mathcal{C}$, is considered as a pentuple, or 5-tuple of the set of objects, $\mathcal{O}$, the set of mormphisms, $\mathbb{m}$, the property of composition, $\mathbb{c}$, identity of any object, $\mathrm{id}_{\mathcal{O}}$, and assiosiativity, $\mathbb{a}$ given by*

$$\mathcal{C} = \{\mathcal{O}, \mathbb{m}; \mathbb{c}, \mathrm{id}_{\mathcal{O}}, \mathbb{a}\}\ . \tag{5.13}$$

Note that here we have used a mathematical singature, to describe a category, which is a new way to view categories.

Note that we can unexpectedly use the mnemonic rule to remember this definition. The initials of the signature of the category, form the word Comcia sounds like "Komcia", which sounds like the greek word "komsos" (plural "Komsa", or "Komsia"), meaning elegant.

### 5.1.4 Category: Definition 2

To define the category, we can have an adaptive definition from Leinster [1], in which we can remove any degeneracy that exists in Leinster [1], as follows.

**Definition 7.** *A category $\mathcal{C}$ consists of the following five items:*

- *a collection or class of objects, denoted with*

$$\mathcal{O} = \mathcal{O}(\mathcal{C}) \tag{5.14}$$

- *for any two such objects, $A, B \in \mathcal{O}(\mathcal{C})$, a collection of maps or arrows or morphisms or directed links exists from $A$ to $B$, denoted with*

$$\mathcal{C}(A, B) : A \to B \tag{5.15}$$

- *for each $A \in \mathcal{O}(\mathcal{C})$ an element $\mathrm{id}_A \in \mathcal{C}(A, A)$ exists, and it is called identity on $A$. Since this is true $\forall\ A \in \mathcal{O}$, we can all this as the existence of $\mathrm{id}_{\mathcal{O}}$, and we write:*

$$\forall A \in \mathcal{O}(\mathcal{C})\ :\ \exists\ \mathrm{id}_A \in \mathcal{C}(A, A) \tag{5.16}$$

- *for three such objects $A, B, C \in \mathcal{O}$ and $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, a map*

$$\mathbb{c} : \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \tag{5.17}$$

$$\mathbb{c} : (f, g) \mapsto f \circ g \tag{5.18}$$

*exists and it is called $\mathbb{c}$omposition, denoted with $\mathbb{c}$,*

*and*

- *$\mathbb{a}$ssociativity,*

$$\mathbb{a} : (f \circ g) \circ h = f \circ g(\circ h)\ , \tag{5.19}$$

$$\forall,\ f \in \mathcal{C}(A, B),\ g \in \mathcal{C}(B, C),\ \text{and } h \in \mathcal{C}(C, D)\ . \tag{5.20}$$

*Therefore in general a category, $\mathcal{C}$, is considered as a pentuple, or 5-tuple of the set of objects, $\mathcal{O}$, the set of mormphisms, $\mathcal{C}$, the property of composition, $\mathbb{c}$, identity of any object, $\mathrm{id}_{\mathcal{O}}$, and assiosiativity, $\mathbb{a}$ given by*

$$\mathcal{C} = \{\mathcal{O}, \mathcal{C}; \mathbb{c}, \mathrm{id}_{\mathcal{O}}, \mathbb{a}\} \tag{5.21}$$

[1]

[1] Note that we can use the mnemonic rule to remember this signature. The initials of the singature, form the word, *Coccia*, which sounds like the greek word, *Kotsia*, meaning guts.

### 5.1.5 Category: Definition 3

To define the category, we can have an adaptive definition from Leinster [1], in which we can remove any degeneracy that exists in Leinster [1], as follows.

**Definition 8.** *In general, a category, $\mathcal{C}$, is considered as a pentuple, or 5-tuple of the set of objects, $\mathcal{O}$, the set of mormphisms, $\mathbb{m}$, the property of composition, $\mathbb{c}$, set of identities of any object, $\mathrm{id}_{\mathcal{O}}$, and assiosiativity, $\mathbb{a}$ given by*

$$\mathcal{C} = \{\mathcal{O}, \mathbb{m}; \mathbb{c}, \mathrm{id}_{\mathcal{O}}, \mathbb{a}\} \tag{5.22}$$

*where*

- *a class of objects, denoted with*

$$\mathcal{O} = \mathcal{O}(\mathcal{C}) \tag{5.23}$$

- *for any four such objects, i.e.*

$$\forall\ A, B, C, D \in \mathcal{O}(\mathcal{C})\ , \tag{5.24}$$

  - *a collection of maps or arrows or morphisms or directed links exists from $A$ to $B$, denoted with*

$$\exists\ \mathbb{m} = \mathcal{C}(A, B) : A \to B \tag{5.25}$$

  - *for each $A \in \mathcal{O}(\mathcal{C})$ an identity element $\mathrm{id}_A \in \mathcal{C}(A, A)$ exists, and it is called identity on $A$. Since this is true $\forall\ A \in \mathcal{O}$, we can describe this as the existence of a set of identity elements $\mathrm{id}_{\mathcal{O}}$, and we write:*

$$\exists\ \mathrm{id}_A \in \mathcal{C}(A, A), \mathrm{id}_{\mathcal{O}} \tag{5.26}$$

  - *for three such functions, i.e.*

$$\forall\ f \in \mathcal{C}(A, B)\ ,\ \forall\ g \in \mathcal{C}(B, C)\ ,\ \forall\ h \in \mathcal{C}(C, D)\ , \tag{5.27}$$

  *there exists $\mathbb{c}$omposition morphism denoted with*

$$\exists\ \mathbb{c}\ :\ \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \tag{5.28}$$

  *or*

$$\exists\ \mathbb{c}\ :\ (f, g) \to f \circ g \tag{5.29}$$

  *and*

  - *$\mathbb{a}$ssosiativity property, denoted with*

$$\exists\ \mathbb{a}\ :\ [\mathcal{C}(A, B) \times \mathcal{C}(B, C)] \times \mathcal{C}(C, D) = \mathcal{C}(A, B) \times [\mathcal{C}(B, C) \times \mathcal{C}(C, D)] \tag{5.30}$$

  *or*

$$\mathbb{a}\ :\ (f \circ g) \circ h = f \circ (g \circ h) \tag{5.31}$$

*or symbolically*
*a category is*

$$\mathcal{C} = \{\mathcal{O}, \mathbb{m}; \mathbb{c}, \mathrm{id}_{\mathcal{O}}, \mathbb{a}\} \tag{5.32}$$

*where*

$$\forall\ A, B, C \in \mathcal{O}(\mathcal{C}) = \mathcal{O} \tag{5.33}$$

$$\exists\ \mathbb{m} = \mathcal{C}(A, B)\ :\ A \to B \tag{5.34}$$

$$\exists\ \mathrm{id}_A \in \mathcal{C}(A, A), \mathrm{id}_{\mathcal{O}} \tag{5.35}$$

*and*

$$\forall\ f \in \mathcal{C}(A, B)\ ,\ \forall\ g \in \mathcal{C}(B, C)\ ,\ \forall\ h \in \mathcal{C}(C, D)\ , \tag{5.36}$$

$$\exists\ \mathbb{c}\ :\ (f, g) \to f \circ g \tag{5.37}$$

$$\exists\ \mathbb{a}\ :\ (f \circ g) \circ h = f \circ (g \circ h)\ . \tag{5.38}$$

*or symbolically, in a more concise way, we write:*
*Category is*

$$\mathcal{C} = \{\mathcal{O}, \mathbb{m}; \mathbb{c}, \mathrm{id}_{\mathcal{O}}, \mathbb{a}\} \tag{5.39}$$

*where*

$$\forall\ A, B, C \in \mathcal{O}(\mathcal{C}) = \mathcal{O} \tag{5.40}$$

$$\exists\ \mathbb{m} = \mathcal{C}(A, B)\ :\ A \to B \tag{5.41}$$

$$\exists\ \mathrm{id}_A \in \mathcal{C}(A, A), \mathrm{id}_{\mathcal{O}} \tag{5.42}$$

*and*

$$\forall\ \mathcal{C}(A, B)\ ,\ \forall\ \mathcal{C}(B, C)\ ,\ \forall\ \mathcal{C}(C, D)\ , \tag{5.43}$$

$$\exists\ \mathbb{c}\ :\ \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C) \tag{5.44}$$

$$\exists\ \mathbb{a}\ :\ [\mathcal{C}(A, B) \times \mathcal{C}(B, C)] \times \mathcal{C}(C, D) = \mathcal{C}(A, B) \times [\mathcal{C}(B, C) \times \mathcal{C}(C, D)] \tag{5.45}$$

We illustrate a simple abstract definition of category theory in Fig. 1. We represent the category $\mathcal{C}$ as a black line containing all its entities, such as the object, $\mathcal{O}$ and its elements, $C, C' \in \mathcal{O}$, as well as the morphism between $\mathcal{C}(C, C')$. The object collection, $\mathcal{O}$, including its elements, $C, C' \in \mathcal{O}$ is illustrated with the same black line line, which contains the elements $C, C'$. The symbol of morphism, $\mathcal{C}(C, C')$, is illustrated with a directed link between the object $C$ to the object $C'$.

We represent the full definition of category via a simple diagram, as shown in Fig. 2. This diagram shows the relation between, objects, morphisms, identities, $\mathbb{c}$omposition, and $\mathbb{a}$ssociativity of a category. Any category $\mathcal{C}$ can be represented by a universal set with a black

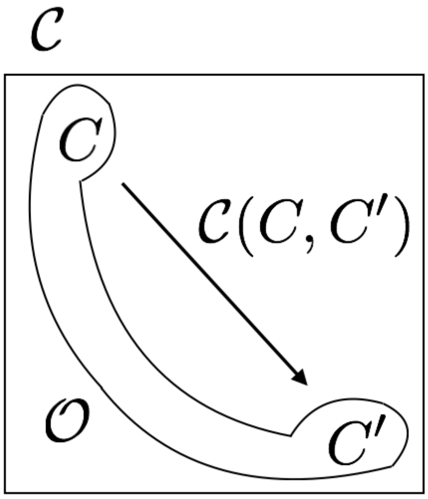


**Figure 1**. Here we illustrate a simple abstract definition of category theory. We illustrate the category $\mathcal{C}$ as a black line containing all its entities. The Object collection, $\mathcal{O}$, including its elements, $C, C' \in \mathcal{O}$ is illustrated with the same black line line, which contains the elements $C, C'$. The symbol of morphism, $\mathcal{C}(C, C')$, is illustrated with a directed link between the object $C$ to the object $C'$. [See section 5]

line. Any object collection, $\mathcal{O}$, can be represented with a thin line containing all its elements, i.e. $A, B, C, D \in \mathcal{O}$. Its element of the identity collection, $\mathrm{id}_{\mathcal{O}}$ can be represented with a curved almost circle arrow around its element, formulated as $\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, \mathrm{id}_D \in \mathrm{id}_{\mathcal{O}}$, while any identity element collection, $\mathrm{id}_{\mathcal{O}}$ is represented with dotted line containing all its elements. We also represent the morphisms simply as $\mathcal{C}(A, B), \mathcal{C}(B, C), \mathcal{C}(C, D), \mathcal{C}(A, C), \mathcal{C}(A, D) \in \mathcal{C}$.

In particular, we can see a morphism from an element A to element B symbolised by an arrow, named as $\mathcal{C}(A, B)$. The same holds for all morphism stated in the former paragraph. Then, we can see the identity property which is basically a curved almost circular morphism which takes an element A and maps it to itself. The same holds for all the identities stated in the former paragraph. Furthermore, we can observe the composition property since we can see the the morphism $\mathcal{C}(A, B) \times \mathcal{C}(B, C)$ is equivalent to $\mathcal{C}(A, C)$. We can also observe the assosiativity property, which is basically shown by the morphism $\mathcal{C}(A, D)$ which is equivalent to $[\mathcal{C}(A, B) \times \mathcal{C}(B, C)] \times \mathcal{C}(C, D)$ and equivalent to $\mathcal{C}(A, B) \times [\mathcal{C}(B, C) \times \mathcal{C}(C, D)]$. We can also show the assosiativity property algebraically as

$$
\begin{aligned}
\mathcal{C}(A, D) &= \mathcal{C}(A, B) \times \mathcal{C}(B, C) \times \mathcal{C}(C, D) && (5.46)\\
&= [\mathcal{C}(A, B) \times \mathcal{C}(B, C)] \times \mathcal{C}(C, D) && (5.47)\\
&= \mathcal{C}(A, C) \times \mathcal{C}(C, D) && (5.48)\\
&= \mathcal{C}(A, B) \times [\mathcal{C}(B, C) \times \mathcal{C}(C, D)] && (5.49)\\
&= \mathcal{C}(A, B) \times \mathcal{C}(B, D)\ . && (5.50)
\end{aligned}
$$

## 5.2 Functors definitions

Here, we describe the definitions of a functor, including functor of functors.

When coming across a novel types of objects, we usually ask ourself, if there is a map between such objects. In the previous case we have discussed the object of category. The map between categories is called *functors*.

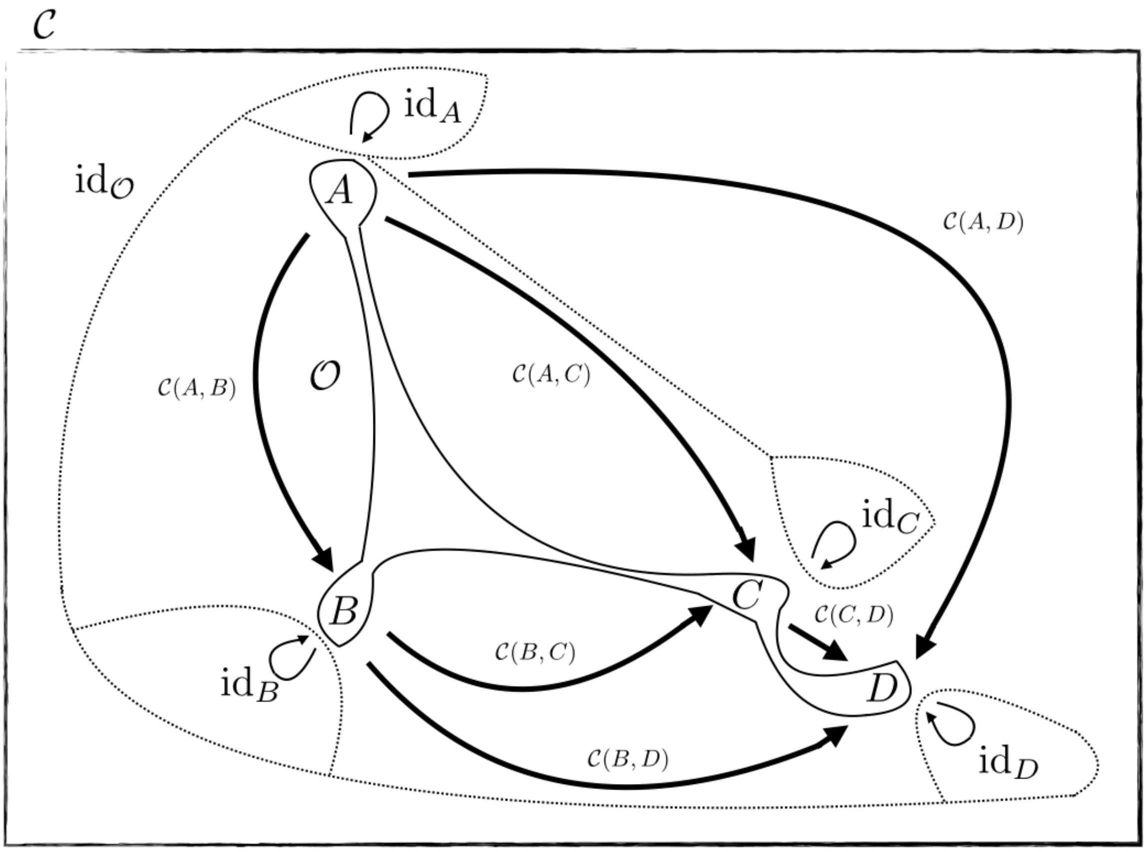


**Figure 2**. Here we illustrate the definition of category theory. The Category $\mathcal{C}$ is represented by a universal set with a black line. The Object collection, $\mathcal{O}$, including its elements, $A, B, C, D \in \mathcal{O}$ is represented with a thin line, the identity element collection, $\mathrm{id}_{\mathcal{O}}$ is represented with dotted line, while its element of the identity collection is represented with a curved almost circle arrow around its element, formulated as $\mathrm{id}_A, \mathrm{id}_B, \mathrm{id}_C, \mathrm{id}_D \in \mathrm{id}_{\mathcal{O}}$. We also represent the relation between the morphism simply as $\mathcal{C}(A, B), \mathcal{C}(B, C), \mathcal{C}(C, D), \mathcal{C}(A, C), \mathcal{C}(A, D) \in \mathcal{C}$. This diagram shows the relation between, objects, morphisms, identities, composition, and associativity of a category. [See section 5]

To define the category, we can have an adaptive definition from Leinster [1], in which we can remove any degeneracy that exists in Leinster [1], as follows.

First we consider the definition from of functors from Leinster [1].

#### 5.2.1 Functor definition 0

**Definition 9.** ***Functor***. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, while elements of the objects are $C \in \mathcal{O}(\mathcal{C}) = \mathrm{ob}(\mathcal{C})$ and $D \in \mathcal{O}(\mathcal{D})$. A functor*

$$F : \mathcal{C} \to \mathcal{D} \tag{5.51}$$

*consists of*

- *a function of the objects of those catevories*

$$\mathrm{ob}(\mathcal{C}) \to \mathrm{ob}(\mathcal{D}) \tag{5.52}$$

$$\mathcal{O}(\mathcal{C}) \to \mathcal{O}(\mathcal{D}) \tag{5.53}$$

*written also as*

$$C \mapsto F(C) \tag{5.54}$$

- *for each $C, C' \in \mathcal{C}$, a function*

$$\mathcal{C}(C, C') \to \mathcal{D}\left[F(C), F(C')\right], \tag{5.55}$$

*written as*

$$f \mapsto F(f) \tag{5.56}$$

*where $f \in \mathcal{C}(C, C')$ and $F(f) \in \mathcal{D}\left[F(C), F(C')\right]$,*

*satisfying the following axiomatic properties:*
*composition heritage written as*

$$F(f \circ f') = F(f) \circ F(f'),\ \forall\, C \xrightarrow{f} C' \xrightarrow{f'} C'' \in \mathcal{C} \tag{5.57}$$

*identity heritage written as*

$$F(1_C) = 1_{F(C)},\ \forall\, C \in \mathcal{C} \tag{5.58}$$

Notice that we have enriched the notation from Leinster [1] to define, so that we can get rid of some degeneracies.Therefore we can have a simpler notation for the definition of functor which is the following.

### 5.2.2 Functor definition 1

**Definition 10.** ***Functor***. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories, while elements of the objects are $C, C' \in \mathcal{O}(\mathcal{C})$ and $D \in \mathcal{O}(\mathcal{D})$, and $f \in \mathcal{C}(C, C')$ and $F(f) \in \mathcal{D}\left[F(C), F(C')\right]$. A functor*

$$F : \mathcal{C} \to \mathcal{D} \tag{5.59}$$

*consists of*

- *a function of the objects of those catevories*

$$\mathcal{O}(\mathcal{C}) \to \mathcal{O}(\mathcal{D}) \tag{5.60}$$

*written also as*

$$C \mapsto F(C) \tag{5.61}$$

- *a function*

$$\mathcal{C}(C, C') \to \mathcal{D}\left[F(C), F(C')\right], \tag{5.62}$$

*written as*

$$f \mapsto F(f) \tag{5.63}$$

*satisfying the following axiomatic properties:*

*composition heritage written as*

$$F(f \circ f') = F(f) \circ F(f'), \; \forall \, C \xrightarrow{f} C' \xrightarrow{f'} C'' \in \mathcal{C} \tag{5.64}$$

*identity heritage written as*

$$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}, \; \forall \, C \in \mathcal{C} \tag{5.65}$$

We can understand the definition of functors with the following sketch, as shown in Fig. 3. This functor $F$ is a map between two categories, $\mathcal{C}$ and $\mathcal{D}$ and it is illustrated directed line, from the category $\mathcal{C}$, to the category $\mathcal{D}$. The category $\mathcal{C}$ is illustrated with a black lined box, with two elements, $C$ and $C'$, and a map between them, $\mathcal{C}(C, C')$. The category $\mathcal{D}$ is illustrated with a black lined box, with two elements, $D$ and $D'$, and a map between them, $\mathcal{D}(D, D')$.

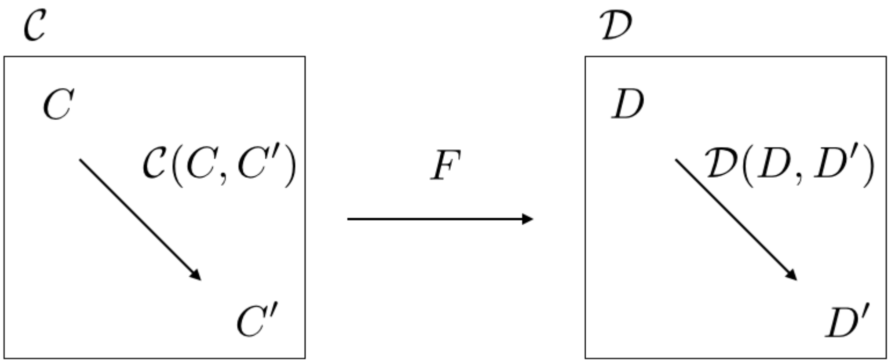


**Figure 3**. Here we illustrate the definition of functor. The functor $F$ is a map between two categories, $\mathcal{C}$ and $\mathcal{D}$ and it is illustrated directed line, from the category $\mathcal{C}$, to the category $\mathcal{D}$. The category $\mathcal{C}$ is illustrated with a black lined box, with two elements, $C$ and $C'$, and a map between them, $\mathcal{C}(C, C')$. The category $\mathcal{D}$ is illustrated with a black lined box, with two elements, $D$ and $D'$, and a map between them, $\mathcal{D}(D, D')$. [See section 5.2.2]

**Definition 11.** ***Remarks***. *The definition of functor is sketched so that from each chain*

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} C_{n-1} \xrightarrow{f_n} C_n \tag{5.66}$$

*of maps in $\mathcal{C}$, ($\forall n \geq 0$), it is possible to built exactly one map*

$$F(C_0) \to F(C_n) \in \mathcal{D} \, . \tag{5.67}$$

*For example given the maps*

$$C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} C_2 \xrightarrow{f_3} C_3 \xrightarrow{f_4} C_4 \in \mathcal{C} \, , \tag{5.68}$$

*we can build maps of the form*

$$F(C_0) \xrightarrow{F(f_1)F(f_2)F(f_3)F(f_4)} F(C_4) \tag{5.69}$$

*which is equivalent to*

$$F(C_0) \overset{F(f_1 f_2)F(f_3 f_4)}{\longrightarrow} F(C_4) \tag{5.70}$$

*which is equivalent to*

$$F(C_0) \overset{F(f_1)F(f_2 f_3)F(f_4)}{\longrightarrow} F(C_4) \tag{5.71}$$

*which is equivalent to*

$$F(C_0) \overset{F(f_1)F(f_2)F(f_3 f_4)}{\longrightarrow} F(C_4) \tag{5.72}$$

### 5.2.3 Functors of functors

Functors is applied to categories. Could a functor applied to two different functors, that have different domain and codomain ? This is only possible if a functor can be considered as a category as well. In this case, a functor of functors can be denote with $\mathbb{F}$. Before reaching this term, we are going to define the definition for the function of functors, denoted with $\mathcal{F}$.

**Definition 12.** ***Functor of functors***. *Let $\mathcal{C}, \mathcal{D}, \mathcal{E}, \mathcal{H}$ be four categories, while elements of the objects are $C, C' \in \mathcal{O}(\mathcal{C}), D \in \mathcal{O}(\mathcal{D}), E, E' \in \mathcal{O}(\mathcal{E}), H \in \mathcal{O}(\mathcal{H})$, while there exists the functions, while there exists the functors:*

$$F : \mathcal{C} \to \mathcal{D} \tag{5.73}$$
$$G : \mathcal{E} \to \mathcal{H} \tag{5.74}$$

*which both belong to the functor category, denoted with $\tilde{\mathcal{F}}$, i.e.*

$$F, G \in \tilde{\mathcal{F}} \tag{5.75}$$

*while there are the functions of objects*

$$f : \mathcal{O}(\mathcal{C}) \to \mathcal{O}(\mathcal{D}) \tag{5.76}$$
$$g : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{H}) \tag{5.77}$$
$$\tag{5.78}$$

*written also as*

$$C \mapsto F(C) \tag{5.79}$$
$$E \mapsto G(E) \tag{5.80}$$

*or*

$$F : C \to D \tag{5.81}$$
$$G : E \to H \tag{5.82}$$

*while there is also the functions:*

$$\mathcal{C}(C, C') \to \mathcal{D}\left[F(C), F(C')\right] \tag{5.83}$$
$$\mathcal{E}(E, E') \to \mathcal{H}\left[G(E), G(E')\right] \ . \tag{5.84}$$

*Note that also that*

$$f \in \mathcal{C}(C, C') \tag{5.85}$$
$$F(f) \in \mathcal{D}\left[F(C), F(C')\right] \tag{5.86}$$
$$g \in \mathcal{E}(E, E') \tag{5.87}$$
$$G(g) \in \mathcal{H}\left[G(E), G(E')\right] \ , \tag{5.88}$$

*while*

$$f \mapsto F(f) \tag{5.89}$$
$$g \mapsto G(g) \tag{5.90}$$

*and*

$$f \in \mathcal{C}(C, C') \tag{5.91}$$
$$F(f) \in \mathcal{D}\left[F(C), F(C')\right] \tag{5.92}$$
$$g \in \mathcal{E}(E, E') \tag{5.93}$$
$$G(g) \in \mathcal{H}[G(E), G(E')] \ . \tag{5.94}$$

262 *Note that the functor $F$ satisfies the axiomatic properties:*

- *composition of functions heritage*

$$F(f \circ f') = F(f) \circ F(f'),\ \forall\ C \xrightarrow{f} C' \xrightarrow{f'} C'' \in \mathcal{C} \tag{5.95}$$

- *identity heritage*

$$F(\mathrm{id}_C) = \mathrm{id}_{F(C)}\ \forall\ C \in \mathcal{C} \tag{5.96}$$

263 *while the functor $G$ satisfy the axiomatic properties*

- *composition of functions heritage*

$$G(g \circ g') = G(g) \circ G(g'),\ \forall\ E \xrightarrow{g} E' \xrightarrow{g'} E'' \in \mathcal{E} \tag{5.97}$$

- *identity heritage*

$$G(\mathrm{id}_E) = \mathrm{id}_{G(E)}\ \forall\ E \in \mathcal{E} \tag{5.98}$$

*Given the aforementioned information, a functional of functors exists*

$$\mathcal{F} : \mathcal{D} \to \mathcal{E} \ . \tag{5.99}$$

*which is written analytically as*

$$\mathcal{F} : \mathcal{D}[F(C), F(C')] \to \mathcal{E}\left\{\mathcal{F}[F(C)], \mathcal{F}[F(C')]\right\} \tag{5.100}$$
$$\tag{5.101}$$

*when*

$$\mathcal{C}(C,C') \to \mathcal{D}\left[F(C),F(C')\right] \equiv \mathcal{D}\left(D,D'\right) \tag{5.102}$$

$$\mathcal{D}\left(D,D'\right) \to \mathcal{E}\left\{\mathcal{F}\left(D,D'\right)\right\} \equiv \mathcal{E}\left\{\mathcal{F}\left[F(C),F(C')\right]\right\} \equiv \mathcal{E}\left(E,E'\right)\ . \tag{5.103}$$

*Given the aforementioned information, a functor of functors exists as*

$$\mathbb{F} : \tilde{\mathcal{F}} \to \tilde{\mathcal{F}} \tag{5.104}$$

264 *and it consists of*

- *a functor of the functions of the objects of those functor categories*

$$\mathbb{F} : F \to G \tag{5.105}$$

$$\mathbb{F} : F\left[\mathcal{C}(C,C)\right] \to G\left[\mathcal{E}(E,E)\right] \tag{5.106}$$

- *a functor of functors*

$$\mathbb{F} : \begin{array}{l} \mathcal{C}(C,C') \to \mathcal{D}\left[F(C),F(C')\right] \\ \qquad\qquad\qquad \downarrow \\ \mathcal{E}(E,E') \to \mathcal{H}\left[G(E),G(E')\right] \end{array} \tag{5.107}$$

*written also as*

$$\mathbb{F} : F \to G \tag{5.108}$$

$$\mathbb{F} : \mathcal{C} \to \mathcal{D} \to \mathcal{E} \to \mathcal{H} \tag{5.109}$$

$$\mathbb{F} : \mathcal{C} \to \mathcal{H} \tag{5.110}$$

$$\mathbb{F} : \left\{C,C'\right\} \to \left\{H,H'\right\} \tag{5.111}$$

*written also more analytically as*

$$\mathbb{F} : \left\{\begin{array}{c} D = F(C) \\ D' = F(C') \end{array}\right\} \to \left\{\begin{array}{c} H = G(E) \\ H' = G(E') \end{array}\right\} \tag{5.112}$$

*or*

$$\left\{\begin{array}{c} D = F(C) \\ D' = F(C') \end{array}\right\} \xrightarrow{\mathbb{F}} \left\{\begin{array}{c} H = G(E) \\ H' = G(E') \end{array}\right\} \tag{5.113}$$

*or*

$$\mathbb{F} : \left\{\begin{array}{c} F(C) \\ F(C') \end{array}\right\} \to \left\{\begin{array}{c} G(E) \\ G(E') \end{array}\right\} \tag{5.114}$$

*or in a cleaner form*

$$\left\{\begin{array}{c} F(C) \\ F(C') \end{array}\right\} \xrightarrow{\mathbb{F}} \left\{\begin{array}{c} G(E) \\ G(E') \end{array}\right\} \tag{5.115}$$

265 *satisfying the following axiomatic properties:*

- *composition heritage heritage, written as*

$$\mathbb{F}\left[F \bigcirc G\right] = \mathbb{F}\left[F\right] \bigcirc \mathbb{F}\left[G\right] \ \forall\, F, G \in \tilde{\mathcal{F}} \tag{5.116}$$

- *the identity heritage heritage, written as*

$$\mathbb{F}\left[\mathrm{id}_F\right] = \mathrm{id}_{\mathbb{F}[F]}\ ,\ \forall\, F \in \tilde{\mathcal{F}} \tag{5.117}$$

*We can also define the functor of functors via the following sketch, shown in Fig. 4. To build this functor of functors, we need two different functors. The first functor, $F$ is a map between two categories, $\mathcal{C}$ and $\mathcal{D}$ and it is illustrated with a directed line, from the category $\mathcal{C}$, to the category $\mathcal{D}$. The category $\mathcal{C}$ is illustrated with a black lined box, with two elements, $C$ and $C'$, and a map between them, denoted with $\mathcal{C}(C, C')$. The category $\mathcal{D}$ is illustrated with a black lined box, with two elements, $D$ and $D'$, and a map between them, denoted with $\mathcal{D}(D, D')$. The second functor, $G$ is a map between two categories, $\mathcal{E}$ and $\mathcal{H}$ and it is illustrated with a directed line, from the category $\mathcal{E}$, to the category $\mathcal{H}$. The category $\mathcal{E}$ is illustrated with a black lined box, with two elements, $E$ and $E'$, and a map between them, $\mathcal{H}(H, H')$. The category $\mathcal{H}$ is illustrated with a black lined box, with two elements, $H$ and $H'$, and a map between them, $\mathcal{H}(H, H')$. Then the functor of functors, $\mathbb{F}$, is illustrated with a bold black directed line, from the functor $F$ to the functor $G$.*

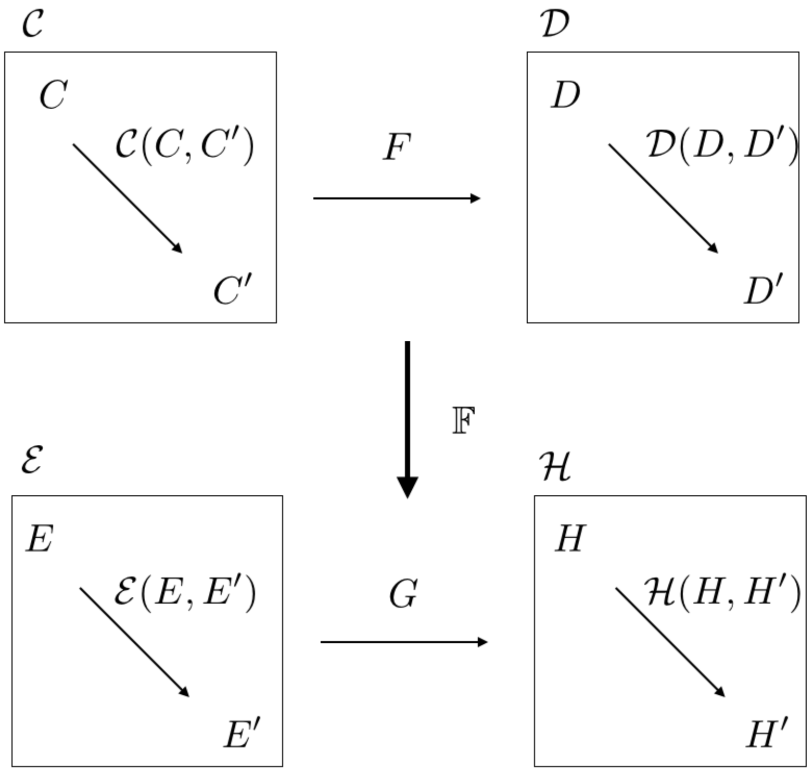


**Figure 4**. Here we illustrate the simple definition of functors of functors, denoted with $\mathbb{F}$. To build this functor of functors, we need two different functors. The first functor, $F$ is a map between two categories, $\mathcal{C}$ and $\mathcal{D}$ and it is illustrated with a directed line, from the category $\mathcal{C}$, to the category $\mathcal{D}$. The category $\mathcal{C}$ is illustrated with a black lined box, with two elements, $C$ and $C'$, and a map between them, $\mathcal{C}(C, C')$. The category $\mathcal{D}$ is illustrated with a black lined box, with two elements, $D$ and $D'$, and a map between them, $\mathcal{D}(D, D')$. The second functor, $G$ is a map between two categories, $\mathcal{E}$ and $\mathcal{H}$ and it is illustrated with a directed line, from the category $\mathcal{E}$, to the category $\mathcal{H}$. The category $\mathcal{E}$ is illustrated with a black lined box, with two elements, $E$ and $E'$, and a map between them, $\mathcal{H}(H, H')$. The category $\mathcal{H}$ is illustrated with a black lined box, with two elements, $H$ and $H'$, and a map between them, $\mathcal{H}(H, H')$. Then the functor of functors, $\mathbb{F}$, from the functor $F$ to the functor $G$, is illustrated with a bold black directed line. The functor of functor is denoted with $\mathbb{F} = \mathbb{F}(F, G)$. [See section 5.2.3]

**Definition 13.** *We can then define the full definition of functors of functors, $\mathcal{F}$. To construct that, we need also the category of functors, denoted with $\mathcal{C}_{\mathcal{F}}$.*

*We can show that the definition of functors of functors, $\mathcal{F}$, is give by the following illustration, Fig. 5. In particular, we illustrate the full definition of functors of functors, denoted with $\mathbb{F}$. To build this functor of functors, we need four different functors and eight different categories. The eight different categories are listed as $\{\mathcal{A},'\mathcal{A}, \mathcal{B},'\mathcal{B}, \mathcal{C},'\mathcal{C}, \mathcal{D},'\mathcal{D}\}$ illustrated with square boxes. Each category has its own objects, and morphism. In particular, the objects $\{A, A'\} \in \mathcal{A}$, $\{B, D'\} \in \mathcal{B}$, $\{C, D'\} \in \mathcal{C}$, $\{D, D'\} \in \mathcal{D}$, while $\{'A, A'\} \in' \mathcal{A}$, $\{'B, D'\} \in' \mathcal{B}$, $\{'C, D'\} \in' \mathcal{C}$, and $\{'D, D'\} \in' \mathcal{D}$. Then we have their corresponding functors for each pair. In particular we have the functor $I$ for the pair of categories, $\mathcal{A}$ and $'\mathcal{A}$, which is defined as $I_{\mathcal{A}} := I(\mathcal{A},'\mathcal{A})$. Analogically we have the functors, $G_{\mathcal{B}} := I(\mathcal{B},'\mathcal{B})$, $F_{\mathcal{C}} := F(\mathcal{C},'\mathcal{C})$, and $J_{\mathcal{D}} := F(\mathcal{D},'\mathcal{D})$. The category of functors or functor category, denoted with $\mathcal{C}_{\mathbb{F}}$, consists of* *the objects $I, G, F, J$, which are functors and their relations defined by the functor of functors $\mathbb{F}(I, G)$, $\mathbb{F}(G, F)$, and $\mathbb{F}(F, J)$.*

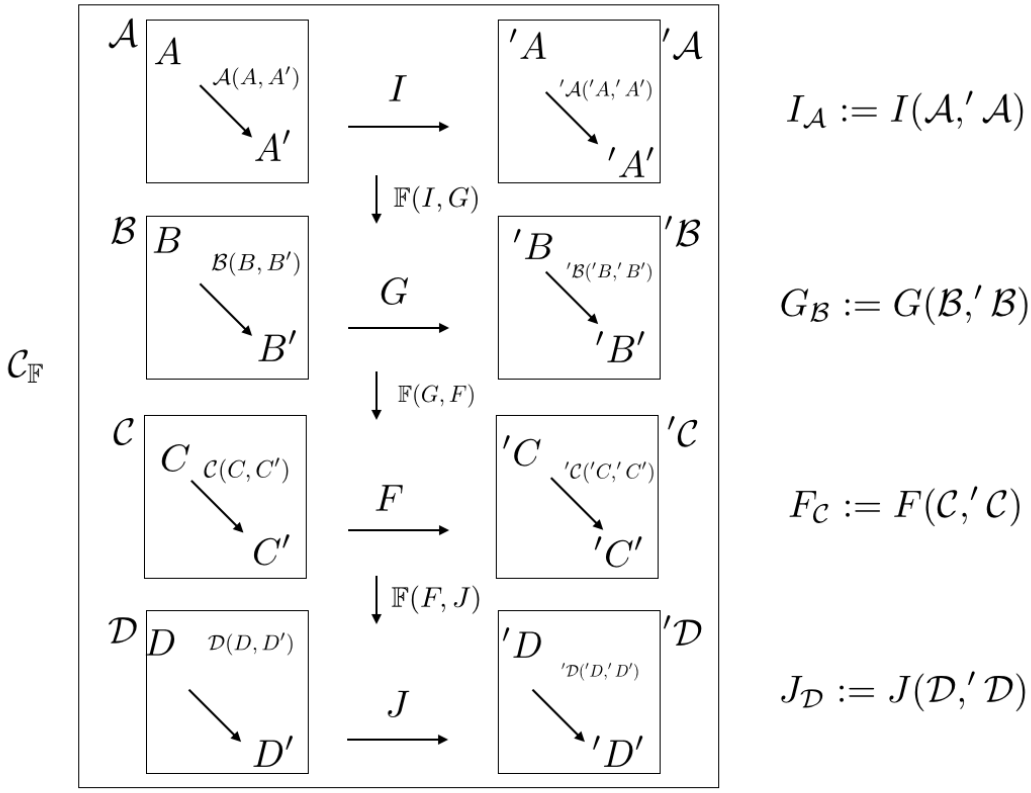


**Figure 5**. Full definition of functor of functors. Here we illustrate the full definition of functors of functors, denoted with $\mathbb{F}$. To build this functor of functors, we need four different functors and eight different categories. The eight different categories are listed as $\{\mathcal{A},' \mathcal{A}, \mathcal{B},' \mathcal{B}, \mathcal{C},' \mathcal{C}, \mathcal{D},' \mathcal{D}\}$ illustrated with square boxes. Each category has its own objects, and morphism. In particular, the objects $\{A, A'\} \in \mathcal{A}$, $\{B, D'\} \in \mathcal{B}$, $\{C, D'\} \in \mathcal{C}$, $\{D, D'\} \in \mathcal{D}$, while $\{'A, A'\} \in' \mathcal{A}$, $\{'B, D'\} \in' \mathcal{B}$, $\{'C, D'\} \in' \mathcal{C}$, and $\{'D, D'\} \in' \mathcal{D}$. Then we have their corresponding functors for each pair. In particular we have the functor I for the pair of categories, $\mathcal{A}$ and $'\mathcal{A}$, which is defined as $I_{\mathcal{A}} := I(\mathcal{A},' \mathcal{A})$. Analogically we have the functors, $G_{\mathcal{B}} := I(\mathcal{B},' \mathcal{B})$, $F_{\mathcal{C}} := F(\mathcal{C},' \mathcal{C})$, and $J_{\mathcal{D}} := F(\mathcal{D},' \mathcal{D})$. The category of functors or functor category, denoted with $\mathcal{C}_{\mathbb{F}}$, consists of the objects $I, G, F, J$, which are functors and their relations defined by the functor of functors $\mathbb{F}(I, G)$, $\mathbb{F}(G, F)$, and $\mathbb{F}(F, J)$. [See section 5.2.3]

We can show that the definition of category of functors, $\mathcal{C}_{\mathcal{F}}$, is give by the following illustration, Fig. 6. In particular we illustrate the full definition of category of functors of functor category, $\mathcal{C}_{\mathbb{F}}$, which includes the functors of functors, denoted with $\mathbb{F}$. To build this functor of functors, we need four different functors and four different categories, implying the need of another four. The four different categories are listed as $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$, and illustrated as indices of the functors, $\{I, G, F, J\}$. In particular, each source category, denoted with $\mathcal{C}$ has a corresponding target category, denoted with $'\mathcal{C}$, which is created by each corresponding functor, $\mathcal{F}$, denoted with $F_{\mathcal{C}} := F(\mathcal{C},' \mathcal{C})$. The same applies for the other functors, i.e. $I_{\mathcal{A}}, G_{\mathcal{B}}, J_{\mathcal{D}}$. Each category has its own objects, and morphism, but we omit this information here. The category of functors or functor category, denoted with $\mathcal{C}_{\mathbb{F}}$, consists of the objects $\{I_{\mathcal{A}}, G_{\mathcal{B}}, F_{\mathcal{C}}, J_{\mathcal{D}}\} \in \mathcal{O} \equiv \mathcal{O}\,(\mathcal{C}_{\mathbb{F}})$, which are functors. Then, to illustrate the identity of functor of functors property, we have the collection of identity of functors of functors, denoted with $\mathrm{id}_{\mathcal{O}}$ and the identities of functors of functors denoted with $\{\mathrm{id}_{I_{\mathcal{A}}}, \mathrm{id}_{G_{\mathcal{B}}}, \mathrm{id}_{F_{\mathcal{C}}}, \mathrm{id}_{J_{\mathcal{D}}}, \} \in \mathrm{id}_{\mathcal{O}}$, which they map its functor to itself. The functors of functors are $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) := \mathbb{F}(I_{\mathcal{A}}, G_{\mathcal{B}})$, $\mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}}) := \mathbb{F}(G_{\mathcal{B}}, F_{\mathcal{C}})$, and $\mathcal{C}_{\mathbb{F}}(F_{\mathcal{A}}, J_{\mathcal{D}}) := \mathbb{F}(F_{\mathcal{C}}, J_{\mathcal{D}})$. Furthermore, we also illustrate the relations $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, F_{\mathcal{C}})$, $\mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, J_{\mathcal{D}})$, and $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, J_{\mathcal{D}})$, to complete the associativity relation.

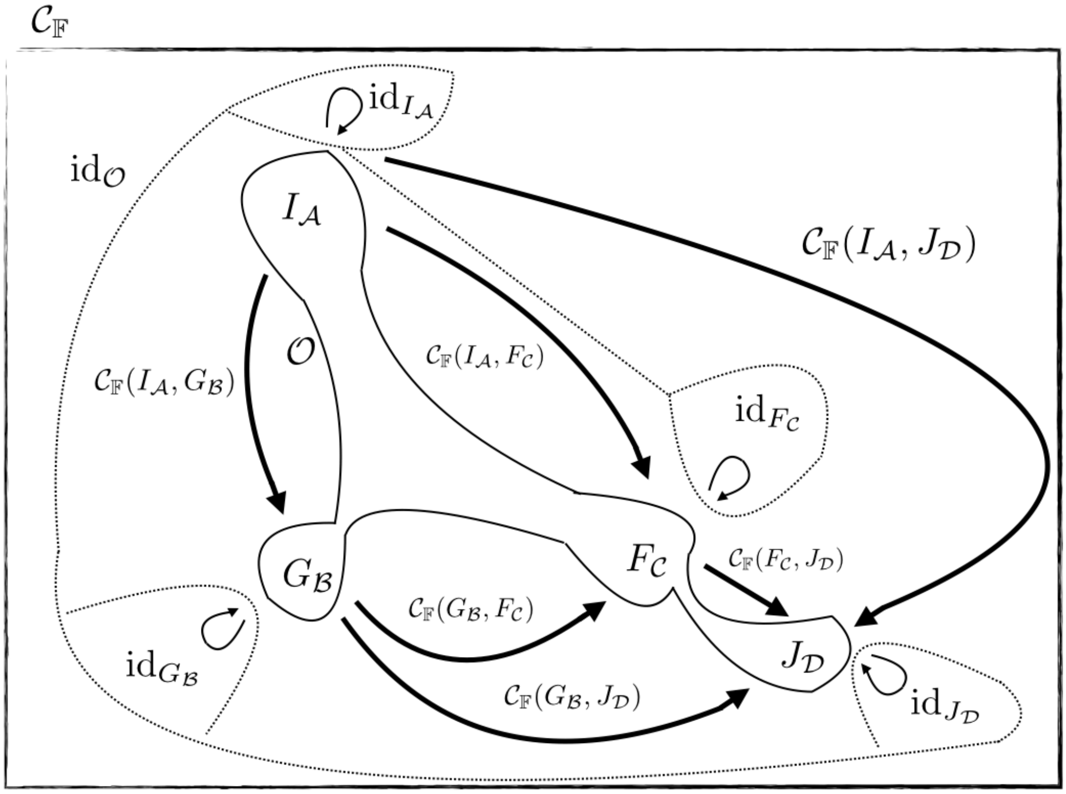


**Figure 6**. Full definition of category of functors or functors category. Here we illustrate the full definition of category of functors of functor category, $\mathcal{C}_{\mathbb{F}}$, which includes the functors of functors, denoted with $\mathbb{F}$. To build this functor of functors, we need four different functors and four different categories, implying the need of another four. The four different categories are listed as $\{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}\}$, and illustrated as indices of the functors, $\{I, G, F, J\}$. In particular, each source category, denoted with $\mathcal{C}$ has a corresponding target category, denoted with $'\mathcal{C}$, which is created by each corresponding functor, $\mathcal{F}$, denoted with $F_{\mathcal{C}} := F(\mathcal{C}, '\mathcal{C})$. The same applies for the other functors, i.e. $I_{\mathcal{A}}, G_{\mathcal{B}}, J_{\mathcal{D}}$. Each category has its own objects, and morphism, but we omit this information here. The category of functors or functor category, denoted with $\mathcal{C}_{\mathbb{F}}$, consists of the objects $\{I_{\mathcal{A}}, G_{\mathcal{B}}, F_{\mathcal{C}}, J_{\mathcal{D}}\} \in \mathcal{O} \equiv \mathcal{O}(\mathcal{C}_{\mathbb{F}})$, which are functors. Then, to illustrate the identity of functor of functors property, we have the collection of identity of functors of functors, denoted with $\mathrm{id}_{\mathcal{O}}$ and the identities of functors of functors denoted with $\{\mathrm{id}_{I_{\mathcal{A}}}, \mathrm{id}_{G_{\mathcal{B}}}, \mathrm{id}_{F_{\mathcal{C}}}, \mathrm{id}_{J_{\mathcal{D}}}, \} \in \mathrm{id}_{\mathcal{O}}$, which they map its functor to itself. The functors of functors are $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) := \mathbb{F}(I_{\mathcal{A}}, G_{\mathcal{B}})$, $\mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}}) := \mathbb{F}(G_{\mathcal{B}}, F_{\mathcal{C}})$, and $\mathcal{C}_{\mathbb{F}}(F_{\mathcal{A}}, J_{\mathcal{D}}) := \mathbb{F}(F_{\mathcal{C}}, J_{\mathcal{D}})$. Furthermore, we also illustrate the relations $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, F_{\mathcal{C}})$, $\mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, J_{\mathcal{D}})$, and $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, J_{\mathcal{D}})$, to complete the associativity relation. [See section 5.2.3]

In particular, we can see a morphism from an element $I_{\mathcal{A}}$ to element $G_{\mathcal{B}}$ symbolised by a directed link, named as $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}})$. The same holds for all morphism stated in the former paragraph. Then, we can see the identity property for each functor of functors, which is basically a curved almost circular morphism, denoted with, $\mathrm{id}_{I_{\mathcal{A}}}$, which takes an element $I_{\mathcal{A}}$ and maps it to itself. The same holds for all the identities stated in the former paragraph. Furthermore, we can observe the composition property since we can see the the morphism $\mathcal{C}_{\mathbb{F}}(|_{\mathcal{A}}, G_{\mathcal{B}}) \times \mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}})$ is equivalent to $\mathcal{C}(I_{\mathcal{A}}, F_{\mathcal{C}})$. We can also observe the assosiativity property, which is basically shown by the morphism $\mathcal{C}_{\mathcal{F}}(I_{\mathcal{A}}, J_{\mathcal{D}})$ which is equivalent to $[\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) \times \mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}})] \times \mathcal{C}_{\mathbb{F}}(F_{\mathcal{C}}, J_{\mathcal{D}})$ and equivalent to $\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) \times [\mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}}) \times \mathcal{C}_{\mathbb{F}}(F_{\mathcal{C}}, J_{\mathcal{D}})]$. We can also show the assosiativity property algebraically as

$$\begin{aligned}
\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, J_{\mathcal{D}}) &= \mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) \times \mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}}) \times \mathcal{C}_{\mathbb{F}}(F_{\mathcal{C}}, J_{\mathcal{D}}) && (5.118)\\
&= [\mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) \times \mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}})] \times \mathcal{C}_{\mathbb{F}}(F_{\mathcal{C}}, J_{\mathcal{D}}) && (5.119)\\
&= \mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, F_{\mathcal{C}}) \times \mathcal{C}_{\mathbb{F}}(F_{\mathcal{C}}, J_{\mathcal{D}}) && (5.120)\\
&= \mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) \times [\mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, F_{\mathcal{C}}) \times \mathcal{C}_{\mathbb{F}}(F_{\mathcal{C}}, J_{\mathcal{D}})] && (5.121)\\
&= \mathcal{C}_{\mathbb{F}}(I_{\mathcal{A}}, G_{\mathcal{B}}) \times \mathcal{C}_{\mathbb{F}}(G_{\mathcal{B}}, J_{\mathcal{D}})\ . && (5.122)
\end{aligned}$$

## 5.3 Natural transformation

From Leinster [1], we introduce the definitions of a natural transformation. We introduce those so that we can compare them with the functors of functors concept.

### 5.3.1 Natural transformation definition

The questions that natural transformation answers is the existence of a map or morphism, between, a functor to another ? This answer is positive, and this is applied to functors that have the same domain and co-domain.

**Definition 14.** *A **natural transformation** is a family of maps between the two functors, such that a particular configuration of those maps commutes.*

In mathematical language we write the following.

**Definition 15.** ***Natural transformation** . Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $F$ and $G$ two functors that map from $\mathcal{C}$ to $\mathcal{D}$, and we write*

$$\mathcal{C} \underset{F}{\overset{G}{\rightrightarrows}} \mathcal{D}\ . \tag{5.123}$$

*A natural transformation*

$$\tau : F \to G \tag{5.124}$$

*is a family of maps*

$$[F(C) \to G(C)]_{C \in \mathcal{C}} \in \mathcal{D} \tag{5.125}$$

*such that*

$$\forall C \xrightarrow{f} C' \in \mathcal{C} \tag{5.126}$$

*the square configuration*

$$\begin{tikzcd} F(C) \arrow[r, "F(f)"] \arrow[d, "\tau_C"'] & F(C') \arrow[d, "\tau_{C'}"] \\ G(C) \arrow[r, "G(f)"'] & G(C') \end{tikzcd} \tag{5.127}$$

*commutes. The maps* $\tau_C$ *and* $\tau_{C'}$ *are called the components of* $\tau$*.*

**Definition 16.** *Remark.*

- *The definition of the natural transformation is such that from each map* $C \xrightarrow{f} C' \in \mathcal{C}$*, a unique map of* $F(C) \xrightarrow{\tau_C \circ G(f)} G(C') \in \mathcal{D}$ *can be constructed. When* $f = \mathrm{id}_C$ *this kind of map is* $\tau_C$*. For a generic* $f$*, this map is the diagonal of the square configuration, given by Eq. 5.127, and "unique map" implies that the square configuration commutes.*
- *The natural transformation is also written symbolically as*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau"] & \mathcal{D} \end{tikzcd} \tag{5.128}$$

*which means that* $\tau$ *is a natural transformation from* $F$ *to* $G$*.*

**Example 5.** *Give some examples here.*

**Definition 17.** ***Construction***. *Natural transformations are kinds of maps, so we can compose them. Given natural transformations of the form of*

$$\begin{tikzcd}[column sep=large] \mathcal{C} \arrow[r, bend left=60, "F"] \arrow[r, "G" description] \arrow[r, bend right=60, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\tau"] \arrow[r, phantom, shift right=3, "\Downarrow\sigma"] & \mathcal{D} \end{tikzcd} \tag{5.129}$$

*there is the composite natural transformation given by*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\tau\circ\sigma"] & \mathcal{D} \end{tikzcd} \tag{5.130}$$

*defined by* $\tau \circ \sigma$ *or more specifically,* $\forall\, C \in \mathcal{C}$*,* $\exists(\tau \circ \sigma)_C = \sigma_C \circ \tau_C$*. There is also the identity natural transformation denoted by* $\mathrm{id}_F$*, which fullfils the condition*

$$\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow\mathrm{id}_F"] & \mathcal{D} \end{tikzcd} \tag{5.131}$$

*on any functor* $F$*, defined by*

$$(\mathrm{id}_F)_C = \mathrm{id}_{F(C)} \tag{5.132}$$

### 5.4 Comparison between functors of functors and natural transformation

The main difference between functors of functors and natural transformation is that the functor of functors concept is used to describe the relation between two functors which are used to related either the same two categories, or two and two different categories, while the concept of natural transformation is ristricted used to describe the relation between two functors which relate the same two categories.

In particular if we have a functor $F$ and $G$ between category $C$ and $D$, then there is a natura transformation between $F$ and $G$ denoted with $\tau$. Note that we can also build a functor, $\mathbb{F}$, which related the functors $F$ and $G$. Furthermore we can also build a functor $\mathbb{F}$ which can relate a functor $F$ and a functor $H$. In this case the functor F relates the categories $C$ and $D$, while the functor $H$ relates the categories $A$ and $B$.

## 6 Conclusion

In this work we review the basics of category theory and functors calculus, and we implement some novel concepts. In particular, we review the concepts of category, functors, and natural transformations. We describe the concept of category using the concept of a mathematical signature, which is a novel way to view categories. Furthermore, we introduce the novel concept of functors of functors. We illustrate these concepts with diagrams to ameliorate the expression of these ideas. We also compare the concept of functors of functors with the concept of natural transformation to show the difference and novelty of functors of functors concept. We conclude that these concepts open new avenues for category theory.

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