# The Maximum Principle with Terminal State Constraints for Optimal Control of Mean-Field FBSDE Driving by Teugels Martingales

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### Abstract

This paper studies the problem of optimal control with state constraints for mean-field type stochastic systems, which is governed by fully coupled forward-backward stochastic differential equations(FBSDE) with Teugels martingales. In this system, the coefficients contain not only the state processes but also its marginal distribution, and the cost function is of mean-field type as well. We use an equivalent backward formulation to deal with the terminal state constraint, and then we obtain a stochastic maximum principle by Ekeland's variational principle. In addition, we discuss a stochastic LQ control problem with state constraints.

# The Maximum Principle with Terminal State Constraints for Optimal Control of Mean-Field FBSDE Driving by Teugels Martingales *

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**Abstract** This paper studies the problem of optimal control with state constraints for mean-field type stochastic systems, which is governed by a fully coupled forward-backward stochastic differential equation(FBSDE) with Teugels martingales. In this system, the coefficients contain not only the state processes but also its marginal distribution, and the cost function is of mean-field type as well. We use an equivalent backward formulation to deal with the terminal state constraint, and then we obtain a stochastic maximum principle by Ekeland's variational principle. In addition, we discuss a stochastic linear-quadratic (LQ) control problem with state constraints.

**Keywords** mean-field forward-backward stochastic differential equations, Lévy processes, Teugels martingales, adjoint equation, state constraints, stochastic maximum principle

## 1 Introduction

In the classical case, many random phenomena can be described by a mathematic model of stochastic differential equations. However there also exist some cases which should characterize the individuals mutual interactions. Such models may be identified by mean-field stochastic systems, where the mean-field term is used to model the interactions among agents and approach the expected value when the number of individuals tends to infinity. The rigorous investigation of continuous time mean-field stochastic differential equations was initiated by McKean [1] in 1966. Since then, the interest in mean-field theory has increased and many applications have been found in physics, engineering, economics, finance and game theory. Dawson [2] examined the dynamics and fluctuations in the critical situation with a mean-field model exhibiting bistable macroscopic behavior. Yong [3] discussed corresponding mean-field stochastic LQ problems by a variational method and decoupling technique. Andersson and Djehiche [4] solved the Markowitz mean-variance portfolio selection problem based on the stochastic maximum principle of mean-field type. See also Ahuja et al. [5], Wang et al. [6], Ma and Huang [7] for the mean-field games of stochastic systems.

The maximum principle is an important approach to study the modern optimal control theories. It was first studied by Pontryagin et al. [8] and has been developed widely by many authors, including Kushner [9, 10], Bensoussan [11], Peng [12], Xu [13], also see Wu [14] for the fully coupled FBSDEs.

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Moreover, for the control systems related to Teugels martingales can realistically describe the randomness in the world, the models with Teugels martingales are of great importance in applications such as the jump-type behavior in financial markets. Thanks to a very useful predictable representation property obtained in Nualart and Schoutens [15, 16], stochastic differential equations driven by Teugels martingales have been investigated increasingly. For backward stochastic systems associated with Lévy process, existence and uniqueness of solutions were given by Bahlali et al. [17], and the maximum principle has been studied by Meng and Tang [18]. A survey of stochastic linear quadratic problems with Lévy process has been investigated by Tand and Wu [19]. The study regarding forward-backward stochastic control system driven by Teugels martingales were considered, see Baghery et al. [20], and for the maximum principle see Wang and Huang [21]. However, mean-field forward and backward stochastic differential equations (MFFBSDEs) driven by Teugels martingales theory and state constraints are not considered in the above control problems, which inspires our work.

This paper is concerned with the mean-field optimal control problem with terminal state constraint. It has a widely application especially in economics and finance, such as the optimization of recursive utilities under constraints [22] , and the mean-variance portfolio selection with bankruptcy prohibition in a complete market model [23]. The difficulty is that the classical theory is generally incapable of solving this problem as our terminal state constraint is a sample-wise constraint [24]. So we would like to adopt some recently developed methods called dual method and terminal perturbation method (refer to [25, 26]). Firstly, we transform the forward and backward stochastic system into a purely backward stochastic system. Then we derive an equivalent control system on which the terminal state $x_T$ is regarded as the control variable. Meanwhile, the initial condition of the forward equation turns to be an additional constraint. It is fortunately that this constraint can be tackled by Ekeland's variational principle which ensured our transformation feasibly.

Motivated by the above discussion, this paper investigates the stochastic control problem for fully coupled MFFBSDEs, and gets the results of variational inequality and stochastic maximum principle. The paper is organized as follows. Section 2 presents some preliminaries used in this paper and formulate the optimal control problem. In Section 3, we reformulate the control problem as an equivalent backward system under some assumptions, then we obtain the variational inequality and the maximum principle by Ekeland's variational principle. Section 4 is devoted to the LQ stochastic optimal control problem with terminal state constraint.

# 2 Preliminaries and Problem Formulation

## 2.1 Preliminaries and Basic Notions

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space on which we define two mutually independent stochastic processes: a $d$-dimensional standard Brownian motion $\{W_t\}_{t\geq 0}$ and a $m$-valued Lévy process $\{L_t\}_{t\geq 0}$ with a Lévy measure $\nu(d\theta)$ such that $\int_R (1\wedge\theta^2)\nu(d\theta) < \infty$. Let $\mathcal{F}_t^W$ and $\mathcal{F}_t^L$ be the $P$-completed natural filtration generated by $\{W_t\}_{t\geq 0}$ and $\{L_t\}_{t\geq 0}$ respectively. Set $\mathcal{F}_t = \mathcal{F}_t^W \vee \mathcal{F}_t^L \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of the $P$-null set.

The real-valued Lévy process $L = \{L_t, t \geq 0\}$ has stationary and independent increments with $L_0 = 0$. Denote the left limit process by $L_{t-} = \lim\limits_{s\to t, s<t} L_s, t \geq 0$ and the jump size at time $t$ by $\triangle L_t = L_t - L_{t-}$. We suppose that for some $\varepsilon > 0$ and $\lambda > 0$ , $\int_{(-\varepsilon,\varepsilon)^c} exp(\lambda|\theta|)\nu(d\theta) < \infty$. Let's recall a convenient basis for martingale representation provided by the Teugels martingales in Nualart-Schoutens[15]. We denote the power-jump processes by $L_t^{(1)} = L_t$ and $L_t^{(i)} = \sum\limits_{0<s\leq t} (\triangle L_s)^i$ for $i \geq$

2. $Y_t^{(i)} = L_t^{(i)} - E[L_t^{(i)}]$ is the compensated power jump process of order $i$. Then the family of Teugels martingales $\{H_t^{(i)}\}_{i=1}^{\infty}$ associated with the Lévy process $\{L_t\}_{t\geq 0}$ is defined by $H_t^{(i)} = c_{i,i}Y_t^{(i)} + c_{i,i-1}Y_t^{(i-1)} + c_{i,i-2}Y_t^{(i-2)} + ... + c_{i,1}Y_t^{(1)}$, where the coefficients $c_{i,k}$ correspond to orthonormalization of the polynomials $1, x, x^2, ...$ with respect to the measure $\mu(dx) = x^2\nu(dx) + \sigma^2\delta_0(dx)$. Moreover, $\{H_t^{(i)}\}_{i=1}^{\infty}$ are pairwise strongly orthogonal and their predictable quadratic variation processes satisfy $\langle H^{(i)}, H^{(j)}\rangle_t = \delta_{ij}t$.

For any given Hilbert space $H$, we denote by $|\cdot|$ the norm of $H$ and by $\langle\cdot,\cdot\rangle$ the scalar product of $H$. Then we introduce the following spaces:

$L^2(\Omega, \mathcal{F}_t, H) := \big\{\xi : \Omega \to H \mid \xi$ is $\mathcal{F}_t$-measurable, and $E|\xi|^2 < \infty\big\}$.

$S^2_{\mathcal{F}}(0,T;H) := \big\{x : [0,T]\times\Omega \to H \mid (x_t)_{0\leq t\leq T}$ is $\mathcal{F}_t$-adapted and *càdlàg* processes such that $E(\sup\limits_{0\leq t\leq T} |x_t|^2) < \infty\big\}$.

$M^2_{\mathcal{F}}(0,T;H) := \big\{\varphi : [0,T]\times\Omega \to H \mid (\varphi_t)_{0\leq t\leq T}$ is $\mathcal{F}_t$-progressively measurable process such that $E\int_0^T |\varphi_t|^2 dt < \infty\big\}$.

$l^2(H) := \big\{ \{f^{(i)}\}_{i\geq 1} \mid \{f^{(i)}\}_{i\geq 1}$ is $H$-valued sequences and satisfies $\sum\limits_{i=1}^{\infty} \|f^{(i)}\|_H^2 < \infty\big\}$.

$l^2_{\mathcal{F}}(0,T;H) := \big\{\{f^{(i)}\}_{i\geq 1} : [0,T]\times\Omega \to l^2(H) \mid \{f_t^{(i)}\}_{0\leq t\leq T}$ is $\mathcal{F}_t$-predictable processes for each $i \geq 1$ , and $E\int_0^T \sum\limits_{i=1}^{\infty} \|f_t^{(i)}\|_H^2 dt < \infty\ \big\}$.

Recall the more general Itô's formula about semimartingales. Let $X = \{X_t : t \in [0,T]\}$ be a *càdlàg* semimartingale, and $[X] = \{[X]_t : t \in [0,T]\}$ is the quadratic variation, $F$ is a $C^2$ real valued function, then $F(X)$ is also a semimartingale and the following Itô's formula holds

$$F(X_t) = F(X_0) + \int_0^t F'(X_{s-})dX_s + \frac{1}{2}\int_0^t F''(X_{s-})d[X]_s^c + \sum_{0<s\leq t}\big\{F(X_s) - F(X_{s-}) - F'(X_{s-})\Delta X_s\big\},$$

where $[X]^c$ is the continuous part of the quadratic variation $[X]$.

Let us consider the following fully coupled MFFBSDE with Teugels martingales:

$$\begin{cases} dx_t = \bar{b}(t, x_t, Ex_t, y_t, Ey_t, z_t, Ez_t, r_t)dt + \bar{\sigma}(t, x_t, Ex_t, y_t, Ey_t, z_t, Ez_t, r_t)dW_t \\ \qquad\quad + \sum\limits_{j=1}^{\infty} \bar{g}^{(j)}(t, x_{t-}, Ex_{t-}, y_{t-}, Ey_{t-}, z_t, Ez_t, r_t)dH_t^{(j)}, \\ dy_t = -\bar{f}(t, x_t, Ex_t, y_t, Ey_t, z_t, Ez_t, r_t)dt + z_t dW_t + \sum\limits_{j=1}^{\infty} r_t^{(j)} dH_t^{(j)}, \\ x_0 = a, \quad y_T = h(x_T, Ex_T). \end{cases} \tag{1}$$

where $W_t$ is a $R^d$ valued Brownian motion and $\{H_t^{(j)}\}_{j=1}^{\infty}$ is a family of Teugels martingales independent of $W_t$; $a$ is a $\mathcal{F}_0$ measurable random variable, and

$$\begin{aligned} &\bar{b} : \Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m) \to R^n, \\ &\bar{\sigma} : \Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m) \to R^{n\times d}, \\ &\bar{g} : \Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m) \to l^2(R^n), \\ &\bar{f} : \Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m) \to R^m, \\ &h : \Omega\times R^n\times R^n \to R^m \end{aligned}$$

are $\mathcal{F}_t$ progressively measurable processes.

Given a $m\times n$ full-rank matrix G. Let

$$\lambda=\begin{pmatrix} x\\ y\\ z\end{pmatrix}, E\lambda=\begin{pmatrix} Ex\\ Ey\\ Ez\end{pmatrix}, A(t,\lambda,E\lambda,r)=\begin{pmatrix} -G^T\bar f\\ G\bar b\\ G\bar\sigma\end{pmatrix}(t,\lambda,E\lambda,r).$$

Throughout this paper, we assume the following:

(H1) (i) $\bar b$, $\bar\sigma$, $\bar g$, $\bar f$ are uniformly Lipschitz with respect to $(\lambda, E\lambda, r)$ ,

(ii) for each $\lambda\in R^n\times R^m\times R^{m\times d}$ and $r\in l^2(R^m)$, $A(\cdot,\lambda,E\lambda,r)$ is in $M^2(0,T)$,

(iii) $h(x,Ex)$ is uniformly Lipschitz with respect to $x\in R^n$, and $h(x,Ex)$ is in $L^2(\Omega,\mathcal{F}_T,P)$.

Here, we denote $M^2(0,T)=S^2_{\mathcal{F}}(0,T;R^n)\times S^2_{\mathcal{F}}(0,T;R^m)\times M^2_{\mathcal{F}}(0,T;R^{m\times d})\times l^2_{\mathcal{F}}(0,T;l^2(R^m))$ as the natural space for solutions of equation (1). Set $\Delta\lambda=\lambda-\lambda'=(\Delta x,\Delta y,\Delta z)=(x-x',y-y',z-z')$ and $\Delta r=r-r'$, let's assume the following monotonicity conditions

(H2)$\langle A(t,\lambda,E\lambda,r)-A(t,\lambda',E\lambda',r'),\Delta\lambda\rangle+\sum_{j=1}^{\infty}\langle G\bar g^{(j)}(t,\lambda,E\lambda,r)-G\bar g^{(j)}(t,\lambda',E\lambda',r'),\Delta r^{(j)}\rangle$

$$\le -\beta_1(|G\Delta x|^2+|GE\Delta x|^2)-\beta_2(|G^T\Delta y|^2+|G^TE\Delta y|^2+|G^T\Delta z|^2+|G^TE\Delta z|^2+\sum_{j=1}^{\infty}\|G^T\Delta r^{(j)}\|^2),$$

$\langle h(x,Ex)-h(x',Ex'),G\Delta x\rangle\ge\mu_1(|G\Delta x|^2+|GE\Delta x|^2)$,

where $\beta_1$ and $\beta_2$ are given nonnegative constants with $\beta_1+\beta_2>0$, $\beta_2+\mu_1>0$. Moreover we have $\beta_1>0$ (resp., $\beta_2>0$) when $m>n$ (resp., $n>m$).

Then it can be very useful to indicate the following result for the well-posedness of the state equation.

**Lemma 2.1** (Existence and uniqueness of MFFBSDE driven by Teugels martingales)

Assume that (H1) and (H2) hold, the fully coupled MFFBSDE driven by Teugels martingales (1) admits a unique adapted solution $(x_t,y_t,z_t,r_t)$.

The above lemma can be proved by the technique similar to that of [27]. Besides using Itô's formula and constructing a contraction mapping, which is the main idea in the derivation, Jensen's inequality is also helpful to deal with the mean-field variables in the process. More details about the derivation can be seen in the appendix.

## 2.2 Formulation of the Control Problem

The control system is described by the following MFFBSDE:

$$\begin{cases} dx_t=\bar b(t,x_t,Ex_t,y_t,Ey_t,z_t,Ez_t,r_t,u_t)dt+\bar\sigma(t,x_t,Ex_t,y_t,Ey_t,z_t,Ez_t,r_t,u_t)dW_t\\ \qquad +\sum_{j=1}^{\infty}\bar g^{(j)}(t,x_{t-},Ex_{t-},y_{t-},Ey_{t-},z_t,Ez_t,r_t,u_t)dH_t^{(j)},\\ dy_t=-\bar f(t,x_t,Ex_t,y_t,Ey_t,z_t,Ez_t,r_t,u_t)dt+z_tdW_t+\sum_{j=1}^{\infty}r_t^{(j)}dH_t^{(j)},\\ x_0=a,\quad y_T=h(x_T,Ex_T),\end{cases}\tag{2}$$

where

$$\begin{aligned}&\bar b:\Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m)\times R^k\to R^n,\\ &\bar\sigma:\Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m)\times R^k\to R^{n\times d},\\ &\bar g:\Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m)\times R^k\to l^2(R^n),\\ &\bar f:\Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m)\times R^k\to R^m.\end{aligned}$$

We define the admissible control set by $\mathcal{U}_{ad}=\{u(\cdot)\mid u(\cdot)\in M^2(0,T;R^k)\}$. Under the assumptions (H1) and (H2), we know (2) has a unique solution for any admissible controls $u(\cdot)\in\mathcal{U}_{ad}$. Our control problem consists in minimizing the cost function as follows:

$$\bar{J}(u)=E\Big\{\int_0^T \bar{L}(t,x_t,Ex_t,y_t,Ey_t,z_t,Ez_t,r_t,u_t)dt+\phi(x_T,Ex_T)+\psi(y_0)\Big\}. \tag{3}$$

where $\bar{L}:\Omega\times[0,T]\times R^n\times R^n\times R^m\times R^m\times R^{m\times d}\times R^{m\times d}\times l^2(R^m)\times R^k\to R$, $\phi:R^n\times R^n\to R$, $\psi:R^m\to R$. We also assume

(H3)(i) $\bar{b},\bar{\sigma},\bar{g},\bar{f},h$ and $\bar{L},\phi,\psi$ are continuous and continuously differentiable in their arguments,

(ii) The derivatives of $\bar{b},\bar{\sigma},\bar{g},\bar{f}$ and $h$ with respect to their variables are bounded,

(iii) The derivatives of $\bar{L}$ are bounded by $C(1+|x|+|Ex|+|y|+|Ey|+|z|+|Ez|+|r|+|u|)$, and the derivatives of $\phi$ and $\psi$ are bounded by $C(1+|x|+|Ex|)$ and $C(1+|y|)$ respectively. Here $C>0$ is a constant, which can be different from line to line.

The optimal control problem can be formulated as follows.

Problem A.

$$\begin{aligned}&Minimize\ \ \bar{J}(u)\\ &s.t.\ \ u\in\mathcal{U}_{ad},\ \ x_T\in M,\ \ M\subseteq R^n\ is\ convex.\end{aligned}$$

An essential feature is that the above optimal control has a terminal state constraint. Since it is difficult to deal with the stochastic control with sample-wise state constraints, we tackle it smoothly by the backward stochastic differential equation(BSDE) theory and terminal perturbation method.

## 2.3 The equivalent problem in backward formulation

Let us transform the mean-field forward-backward control system into an equivalent backward form, and get an equivalent control problem. Moreover, the stochastic maximum principle will be derived. For this, we need the following additional assumption:

(H4) The mapping $u_t\to\begin{pmatrix}\bar{\sigma}(t,\lambda,E\lambda,r,u) & 0\\ 0 & \bar{g}(t,\lambda,E\lambda,r,u)\end{pmatrix}$ is a bijection for any $(t,\lambda,E\lambda,r)$.

Therefore, if we set $p_t=\bar{\sigma}(t,\lambda,E\lambda,r,u)$, $q_t=\bar{g}(t,\lambda,E\lambda,r,u)$ under (H4), then there exists the inverse function $\bar{\sigma}^{-1}$, $\bar{g}^{-1}$, such that $u_t=\begin{pmatrix}\bar{\sigma}^{-1}(t,\lambda,E\lambda,r,u) & 0\\ 0 & \bar{g}^{-1}(t,\lambda,E\lambda,r,u)\end{pmatrix}$. In this way, the system (2) can be rewritten as

$$\begin{cases} dx_t=-b(t,\lambda_t,E\lambda_t,r_t,p_t,q_t)dt+p_tdW_t+\sum\limits_{j=1}^{\infty}q_t^{(j)}dH_t^{(j)},\\ dy_t=-f(t,\lambda_t,E\lambda_t,r_t,p_t,q_t)dt+z_tdW_t+\sum\limits_{j=1}^{\infty}r_t^{(j)}dH_t^{(j)},\\ x_0=a,\ \ y_T=h(x_T,Ex_T),\end{cases} \tag{4}$$

where $b(t,\lambda_t,E\lambda_t,r_t,p_t,q_t)=-\bar{b}(t,\lambda_t,E\lambda_t,r_t,u_t)$, $f(t,\lambda_t,E\lambda_t,r_t,p_t,q_t)=\bar{f}(t,\lambda_t,E\lambda_t,r_t,u_t)$.

Without loss of generality, $\begin{pmatrix}p_t & 0\\ 0 & q_t\end{pmatrix}$ could be regarded as the control variable. By the existence and uniqueness theorem of mean-field BSDE, selecting $\begin{pmatrix}p_t & 0\\ 0 & q_t\end{pmatrix}$ is equivalent to selecting the terminal state $x_T$. Therefore we obtain an equivalent backward control system as follows:

$$\begin{cases} dx_t=-b(t,\lambda_t,E\lambda_t,r_t,p_t,q_t)dt+p_tdW_t+\sum\limits_{j=1}^{\infty}q_t^{(j)}dH_t^{(j)},\\ dy_t=-f(t,\lambda_t,E\lambda_t,r_t,p_t,q_t)dt+z_tdW_t+\sum\limits_{j=1}^{\infty}r_t^{(j)}dH_t^{(j)},\\ x_T=\xi,\ \ y_T=h(\xi,E\xi),\end{cases} \tag{5}$$

where the control variable is the random variable $\xi \in U$ , $U = \{\xi \mid E|\xi|^2 < \infty, \xi \in M, a.s.\}$.

The equivalent cost function is

$$J(\xi) = E\Big\{\int_0^T L(t, x_t, Ex_t, y_t, Ey_t, z_t, Ez_t, r_t, p_t, q_t)dt + \phi(x_T, Ex_T) + \psi(y_0)\Big\}, \tag{6}$$

where $L(t, x_t, Ex_t, y_t, Ey_t, z_t, Ez_t, r_t, p_t, q_t) = \bar{L}(t, x_t, Ex_t, y_t, Ey_t, z_t, Ez_t, r_t, u_t)$.

Consequently, the optimal control problem is given rise to the following equivalent optimization problem.

Problem B.

$$Minimize \;\; J(\xi)$$

$$s.t. \;\; \xi \in U, \;\; x_0^\xi = a,$$

here $x_0^\xi$ is the solution of Equation (5) at time 0 under $\xi$.

In this way, the terminal state turns into a control variable $\xi$, and the initial condition $x_0^\xi = a$ is considered as a constraint. That is, it is more feasible since a control constraint is much easier to be dealt with than a state constraint. Moreover, $b$, $f$, $L$ also satisfy the similar conditions in $(H_3)$ according to their definitions. From now on, we focus on Problem B to describe the maximum principle of the optimal control.

# 3 Maximum Principle

## 3.1 Variational equation

Let $\xi^* \in U$ be an optimal control of Problem B and $(x_t^*, y_t^*, z_t^*, r_t^*, p_t^*, q_t^*)$ be the state process of (5) with $\xi^*$. As $U$ is convex, for each $0 \le \varepsilon \le 1$, $\xi \in U$, we know $\xi^\varepsilon = \xi^* + \varepsilon(\xi - \xi^*) \in U$. The corresponding trajectory of (5) with $\xi^\varepsilon$ is denoted by $(x_t^\varepsilon, y_t^\varepsilon, z_t^\varepsilon, r_t^\varepsilon, p_t^\varepsilon, q_t^\varepsilon)$.

Consider the following mean-field BSDEs called the variational equations:

$$\begin{cases} dX_t = -\big[b_x^* X_t + b_{\tilde{x}}^* EX_t + b_y^* Y_t + b_{\tilde{y}}^* EY_t + b_z^* Z_t + b_{\tilde{z}}^* EZ_t + b_r^* R_t + b_p^* P_t + b_q^* Q_t\big]dt \\ \qquad\quad +P_t dW_t + \sum\limits_{j=1}^{\infty} Q_t^{(j)} dH_t^{(j)}, \\ dY_t = -\big[f_x^* X_t + f_{\tilde{x}}^* EX_t + f_y^* Y_t + f_{\tilde{y}}^* EY_t + f_z^* Z_t + f_{\tilde{z}}^* EZ_t + f_r^* R_t + f_p^* P_t + f_q^* Q_t\big]dt \\ \qquad\quad +Z_t dW_t + \sum\limits_{j=1}^{\infty} R_t^{(j)} dH_t^{(j)}, \\ X_T = \xi - \xi^*, \;\; Y_T = h_x(\xi^*, E\xi^*) \cdot (\xi - \xi^*) + h_{\tilde{x}}(\xi^*, E\xi^*) \cdot E(\xi - \xi^*), \end{cases} \tag{7}$$

here $b_a^* = b_a(t, x_t^*, Ex_t^*, y_t^*, Ey_t^*, z_t^*, Ez_t^*, r_t^*, p_t^*, q_t^*)$, $f_a^* = f_a(t, x_t^*, Ex_t^*, y_t^*, Ey_t^*, z_t^*, Ez_t^*, r_t^*, p_t^*, q_t^*)$, which is a first order partial derivatives to $a$ ($a = x, \tilde{x}, y, \tilde{y}, z, \tilde{z}, r, p, q$ respectively), $\tilde{x}$, $\tilde{y}$, $\tilde{z}$ are written for $Ex$, $Ey$, $Ez$ respectively to ease notations.

The above equations (7) are obviously composed of two linear mean-field BSDEs with Teugels martingales. Under the assumptions, it is easy to check that (7) have the unique adapted solution $(X_t, Y_t, Z_t, R_t, P_t, Q_t)$.

Set

$$\begin{aligned} \hat{x}_t^\varepsilon &= \varepsilon^{-1}(x_t^\varepsilon - x_t^*) - X_t, & \hat{y}_t^\varepsilon &= \varepsilon^{-1}(y_t^\varepsilon - y_t^*) - Y_t, & \hat{z}_t^\varepsilon &= \varepsilon^{-1}(z_t^\varepsilon - z_t^*) - Z_t, \\ \hat{r}_t^\varepsilon &= \varepsilon^{-1}(r_t^\varepsilon - r_t^*) - R_t, & \hat{p}_t^\varepsilon &= \varepsilon^{-1}(p_t^\varepsilon - p_t^*) - P_t, & \hat{q}_t^\varepsilon &= \varepsilon^{-1}(q_t^\varepsilon - q_t^*) - Q_t. \end{aligned}$$

We have the following convergence results:

**Lemma 3.1** Let condition (H1)-(H4) hold, we have

$$\begin{aligned} &\lim_{\varepsilon\to 0}\sup_{0\le t\le T} E|\hat{x}_t^\varepsilon|^2=0, \quad \lim_{\varepsilon\to 0} E\int_0^T |\hat{z}_t^\varepsilon|^2dt=0, \quad \lim_{\varepsilon\to 0} E\int_0^T \|\hat{r}_t^\varepsilon\|^2dt=0, \\ &\lim_{\varepsilon\to 0}\sup_{0\le t\le T} E|\hat{y}_t^\varepsilon|^2=0, \quad \lim_{\varepsilon\to 0} E\int_0^T |\hat{p}_t^\varepsilon|^2dt=0, \quad \lim_{\varepsilon\to 0} E\int_0^T \|\hat{q}_t^\varepsilon\|^2dt=0. \end{aligned} \tag{8}$$

*Proof.* It is easy to check that

$$\left\{\begin{aligned} d\hat{x}_t^\varepsilon =& -\Big\{\varepsilon^{-1}[b(t,\lambda_t^\varepsilon,E\lambda_t^\varepsilon,r_t^\varepsilon,p_t^\varepsilon,q_t^\varepsilon)-b(t,\lambda_t^*,E\lambda_t^*,r_t^*,p_t^*,q_t^*)]-b_x^*X_t-b_{\tilde{x}}^*EX_t-b_y^*Y_t \\ &-b_{\tilde{y}}^*EY_t-b_z^*Z_t-b_{\tilde{z}}^*EZ_t-b_r^*R_t-b_p^*P_t-b_q^*Q_t\Big\}dt+\hat{p}_t^\varepsilon dW_t+\sum_{j=1}^\infty \hat{q}_t^{(j)\varepsilon}dH_t^{(j)}, \\ =& -\Big\{b_x(t)\hat{x}_t^\varepsilon+b_{\tilde{x}}(t)E\hat{x}_t^\varepsilon+b_y(t)\hat{y}_t^\varepsilon+b_{\tilde{y}}(t)E\hat{y}_t^\varepsilon+b_z(t)\hat{z}_t^\varepsilon+b_{\tilde{z}}(t)E\hat{z}_t^\varepsilon+b_r(t)\hat{r}_t^\varepsilon \\ &+b_p(t)\hat{p}_t^\varepsilon+b_q(t)\hat{q}_t^\varepsilon+A_t^\varepsilon\Big\}dt+\hat{p}_t^\varepsilon dW_t+\sum_{j=1}^\infty \hat{q}_t^{(j)\varepsilon}dH_t^{(j)}, \\ d\hat{y}_t^\varepsilon =& -\Big\{\varepsilon^{-1}[f(t,\lambda_t^\varepsilon,E\lambda_t^\varepsilon,r_t^\varepsilon,p_t^\varepsilon,q_t^\varepsilon)-f(t,\lambda_t^*,E\lambda_t^*,r_t^*,p_t^*,q_t^*)]-f_x^*X_t-f_{\tilde{x}}^*EX_t-f_y^*Y_t \\ &-f_{\tilde{y}}^*EY_t-f_z^*Z_t-f_{\tilde{z}}^*EZ_t-f_r^*R_t-f_p^*P_t-f_q^*Q_t\Big\}dt+\hat{z}_t^\varepsilon dW_t+\sum_{j=1}^\infty \hat{r}_t^{(j)\varepsilon}dH_t^{(j)}, \\ =& -\Big\{f_x(t)\hat{x}_t^\varepsilon+f_{\tilde{x}}(t)E\hat{x}_t^\varepsilon+f_y(t)\hat{y}_t^\varepsilon+f_{\tilde{y}}(t)E\hat{y}_t^\varepsilon+f_z(t)\hat{z}_t^\varepsilon+f_{\tilde{z}}(t)E\hat{z}_t^\varepsilon+f_r(t)\hat{r}_t^\varepsilon \\ &+f_p(t)\hat{p}_t^\varepsilon+f_q(t)\hat{q}_t^\varepsilon+B_t^\varepsilon\Big\}dt+\hat{z}_t^\varepsilon dW_t+\sum_{j=1}^\infty \hat{r}_t^{(j)\varepsilon}dH_t^{(j)}, \\ \hat{x}_T^\varepsilon =& 0, \\ \hat{y}_T^\varepsilon =& \varepsilon^{-1}[h(\xi^\varepsilon,E\xi^\varepsilon)-h(\xi^*,E\xi^\star)]-h_x(\xi^*,E\xi^*)\cdot(\xi-\xi^*)-h_{\tilde{x}}(\xi^*,E\xi^*)\cdot E(\xi-\xi^*), \end{aligned}\right. \tag{9}$$

where

$$b_a(t)=\int_0^1 b_a\Big(t,A(\alpha,t),EA(\alpha,t),B(\alpha,t),EB(\alpha,t),C(\alpha,t),EC(\alpha,t),D(\alpha,t),F(\alpha,t),G(\alpha,t)\Big)d\alpha$$

$$f_a(t)=\int_0^1 f_a\Big(t,A(\alpha,t),EA(\alpha,t),B(\alpha,t),EB(\alpha,t),C(\alpha,t),EC(\alpha,t),D(\alpha,t),F(\alpha,t),G(\alpha,t)\Big)d\alpha$$

$a=x,\tilde{x},y,\tilde{y},z,\tilde{z},r,p,q$, respectively,

$$A(\alpha,t)=x_t^*+\alpha\varepsilon(X_t+\hat{x}_t^\varepsilon), \quad B(\alpha,t)=y_t^*+\alpha\varepsilon(Y_t+\hat{y}_t^\varepsilon), \quad C(\alpha,t)=z_t^*+\alpha\varepsilon(Z_t+\hat{z}_t^\varepsilon),$$

$$D(\alpha,t)=r_t^*+\alpha\varepsilon(R_t+\hat{r}_t^\varepsilon), \quad F(\alpha,t)=p_t^*+\alpha\varepsilon(P_t+\hat{p}_t^\varepsilon), \quad G(\alpha,t)=q_t^*+\alpha\varepsilon(Q_t+\hat{q}_t^\varepsilon),$$

and

$$\begin{aligned} A_t^\varepsilon =& [b_x(t)-b_x^*]X_t+[b_{\tilde{x}}(t)-b_{\tilde{x}}^*]EX_t+[b_y(t)-b_y^*]Y_t+[b_{\tilde{y}}(t)-b_{\tilde{y}}^*]EY_t+[b_z(t)-b_z^*]Z_t \\ &+[b_{\tilde{z}}(t)-b_{\tilde{z}}^*]EZ_t+[b_r(t)-b_r^*]R_t+[b_p(t)-b_p^*]P_t+[b_q(t)-b_q^*]Q_t, \end{aligned}$$

$$\begin{aligned} B_t^\varepsilon =& [f_x(t)-f_x^*]X_t+[f_{\tilde{x}}(t)-f_{\tilde{x}}^*]EX_t+[f_y(t)-f_y^*]Y_t+[f_{\tilde{y}}(t)-f_{\tilde{y}}^*]EY_t+[f_z(t)-f_z^*]Z_t \\ &+[f_{\tilde{z}}(t)-f_{\tilde{z}}^*]EZ_t+[f_r(t)-f_r^*]R_t+[f_p(t)-f_p^*]P_t+[f_q(t)-f_q^*]Q_t. \end{aligned}$$

With the fact that $[H^{(i)},H^{(j)}]_t-\langle H^{(i)},H^{(j)}\rangle_t$ is a $\mathcal{F}_t$-martingale, we apply Itô's formula to $|\hat{x}_t^\varepsilon|^2+|\hat{y}_t^\varepsilon|^2$, then

$$\begin{aligned} E|\hat{x}_t^\varepsilon|^2+E|\hat{y}_t^\varepsilon|^2&+E\int_t^T\Big(|\hat{p}_s^\varepsilon|^2+|\hat{z}_s^\varepsilon|^2\Big)ds+E\int_t^T\Big(\|\hat{r}_s^\varepsilon\|^2+\|\hat{q}_s^\varepsilon\|^2\Big)ds \\ &\le E|\hat{y}_T^\varepsilon|^2+cE\int_t^T\Big(|\hat{x}_s^\varepsilon|^2+|\hat{y}_s^\varepsilon|^2\Big)ds+c_1E\int_t^T\Big(|\hat{p}_s^\varepsilon|^2+|\hat{z}_s^\varepsilon|^2\Big)ds \\ &\quad +c_1E\int_t^T\Big(\|\hat{r}_s^\varepsilon\|^2+\|\hat{q}_s^\varepsilon\|^2\Big)ds+c_2E\int_t^T\Big(|A_s^\varepsilon|^2+|B_s^\varepsilon|^2\Big)ds, \end{aligned}$$

where $c_1 < 1$ and $\varepsilon > 0$ is sufficient small. In the light of Gronwall's inequality, it follows that

$$\begin{aligned}\sup_{0\le t\le T} E|\hat{x}_t^\varepsilon|^2 + \sup_{0\le t\le T} E|\hat{y}_t^\varepsilon|^2 + E\int_0^T \big(|\hat{p}_t^\varepsilon|^2 + |\hat{z}_t^\varepsilon|^2\big)dt + E\int_0^T \big(\|\hat{r}_t^\varepsilon\|^2 + \|\hat{q}_t^\varepsilon\|^2\big)dt \\ \le c' E|\hat{y}_T^\varepsilon|^2 + c' E\int_0^T \big(|A_t^\varepsilon|^2 + |B_t^\varepsilon|^2\big)dt.\end{aligned} \tag{10}$$

It is easy to check that $\lim_{\varepsilon\to 0} E \mid \hat{y}_T^\varepsilon \mid^2 = 0$. Moreover, we can get $\lim_{\varepsilon\to 0} E\int_0^T \big(|A_t^\varepsilon|^2 + |B_t^\varepsilon|^2\big)dt = 0$ from the Lebesgue's dominated convergence theorem. Then, let $\varepsilon \to 0$ in (10), the desired results (8) are obtained.

## 3.2 Variational inequality

Now let's recall the following Ekeland's variational principle to deal with the initial state constraint $x_0^\xi = a$.

**Lemma 3.2** (Ekeland's variational principle) Let $(V, d(\cdot,\cdot))$ be a complete metric space and $F(\cdot) : V \to R$ be a proper lower semi-continuous function bounded from below. Suppose that for every $\varepsilon > 0$, there exists $u \in V$ such that $F(u) \le \inf_{v\in V} F(v) + \varepsilon$, then there exists $u_\varepsilon \in V$ such that

(i) $F(u_\varepsilon) \le F(u)$,

(ii) $d(u, u_\varepsilon) \le \sqrt{\varepsilon}$,

(iii) $F(v) + \sqrt{\varepsilon} d(v, u_\varepsilon) \ge F(u_\varepsilon), \forall v \in V$.

The metric in $U$ is defined by $d(\xi_1, \xi_2) = (E|\xi_1 - \xi_2|^2)^{\frac{1}{2}}$ for $\xi_1, \xi_2 \in U$. Obviously, $(U, d(\cdot,\cdot))$ is a complete metric space.

We firstly consider the case where $L(t, \lambda, E\lambda, r, p, q) = 0$ in the cost function (6). For the given optimal control $\xi^* \in U$, we give a penalty function $F(\cdot) : U \to R$ by

$$F(\xi) = \Big\{|x_0^\xi - a|^2 + max^2\Big(0, \phi(\xi, E\xi) - \phi(\xi^*, E\xi^*) + \psi(y_0^\xi) - \psi(y_0^*) + \delta\Big)\Big\}^{\frac{1}{2}}, \tag{11}$$

where $\delta > 0$ is an arbitrary constant. It is easy to check that $F(\cdot)$ is a continuous function of $\xi$ defined on U. From the Ekeland's variational principle, we have the following variational inequality.

**Theorem 3.1** Under the assumptions (H1)-(H4), let $\xi^*$ be an optimal control to Problem B. Then for any $\xi \in U$, there exists $h_0 \ge 0$ and $h_1 \in R^n$ with $h_0 + |h_1| \ne 0$, the following variational inequality holds

$$\langle h_1, X_0\rangle + h_0\langle \phi_x(\xi^*, E\xi^*) + \phi_{\tilde{x}}(\xi^*, E\xi^*), \xi - \xi^*\rangle + h_0\langle \psi_y(y_0^*), Y_0\rangle \ge 0. \tag{12}$$

*Proof.* From the penalty function $F(\cdot)$, we have the following properties:

$$F(\xi^*) = \delta; \qquad F(\xi) > 0, \quad \xi \in U; \qquad F(\xi^*) \le \inf_{\xi\in U} F(\xi) + \delta.$$

According to Lemma 3.2, there exists $\xi^\delta \in U$ satisfying

(i) $F(\xi^\delta) \le F(\xi^*)$,

(ii) $d(\xi^\delta, \xi^*) \le \sqrt{\delta}$,

(iii) $F(\xi) + \sqrt{\delta} d(\xi, \xi^\delta) \ge F(\xi^\delta), \forall \xi \in U$.

Since $U$ is convex, for any $\xi \in U$ and $0 \le \rho \le 1$, it is obviously that $\xi^\rho = \xi^\delta + \rho(\xi - \xi^\delta) \in U$. Denote $(x_t^\rho, y_t^\rho, z_t^\rho, r_t^\rho, p_t^\rho, q_t^\rho)$ and $(x_t^\delta, y_t^\delta, z_t^\delta, r_t^\delta, p_t^\delta, q_t^\delta)$ to be the solution of (5) with $\xi = \xi^\rho$, $\xi^\delta$ respectively.

In the same manner, let $(X_t^\delta, Y_t^\delta, Z_t^\delta, R_t^\delta, P_t^\delta, Q_t^\delta)$ be the solution of variation equation (7) when $\xi^*$ is substituted by $\xi^\delta$. Therefore, we have

$$F(\xi^\rho) - F(\xi^\delta) \geq -\sqrt{\delta} d(\xi^\rho, \xi^\delta). \tag{13}$$

By using Lemma 3.1, it follows that

$$\begin{aligned} &\lim_{\rho\to 0} E \sup_{0\leq t\leq T} |\rho^{-1}(x_t^\rho - x_t^\delta) - X_t^\delta|^2 = 0, \qquad \lim_{\rho\to 0} E \sup_{0\leq t\leq T} |\rho^{-1}(y_t^\rho - y_t^\delta) - Y_t^\delta|^2 = 0, \\ &\lim_{\rho\to 0} E\int_0^T |\rho^{-1}(z_t^\rho - z_t^\delta) - Z_t^\delta|^2 dt = 0, \qquad \lim_{\rho\to 0} E\int_0^T |\rho^{-1}(p_t^\rho - p_t^\delta) - P_t^\delta|^2 dt = 0, \\ &\lim_{\rho\to 0} E\int_0^T \|\rho^{-1}(r_t^\rho - r_t^\delta) - R_t^\delta\|^2 dt = 0, \qquad \lim_{\rho\to 0} E\int_0^T \parallel \rho^{-1}(q_t^\rho - q_t^\delta) - Q_t^\delta \parallel^2 dt = 0. \end{aligned} \tag{14}$$

Thus

$$\begin{aligned} &x_t^\rho - x_t^\delta = \rho X_t^\delta + o(\rho), \qquad y_t^\rho - y_t^\delta = \rho Y_t^\delta + o(\rho), \qquad z_t^\rho - z_t^\delta = \rho Z_t^\delta + o(\rho), \\ &p_t^\rho - p_t^\delta = \rho P_t^\delta + o(\rho), \qquad r_t^\rho - r_t^\delta = \rho R_t^\delta + o(\rho), \qquad q_t^\rho - q_t^\delta = \rho Q_t^\delta + o(\rho), \end{aligned}$$

Consequently, let us define the following expansions:

$$\begin{aligned} &|x_0^\rho - a|^2 - |x_t^\delta - a|^2 = 2\rho\langle x_0^\delta - a, X_0^\delta\rangle + o(\rho), \\ &|\phi(\xi^\rho, E\xi^\rho) - \phi(\xi^*, E\xi^*) + \psi(y_0^\rho) - \psi(y_0^*) + \delta|^2 - |\phi(\xi^\delta, E\xi^\delta) - \phi(\xi^*, E\xi^*) + \psi(y_0^\delta) - \psi(y_0^*) + \delta|^2 \\ &= 2\rho\big[\phi(\xi^\delta, E\xi^\delta) - \phi(\xi^*, E\xi^*) + \psi(y_0^\delta) - \psi(y_0^*) + \delta\big] \\ &\quad \cdot\big[\langle \phi_x(\xi^\delta, E\xi^\delta) + \phi_{\bar{x}}(\xi^\delta, E\xi^\delta), \xi - \xi^\delta\rangle + \langle \psi_y(y_0^\delta), Y_0^\delta\rangle\big] + o(\rho). \end{aligned} \tag{15}$$

For the given $\delta > 0$, we need to consider the penalty function in the following two cases.
**Case 1.** There exists $\rho_0 > 0$, such that for all $0 \leq \rho \leq \rho_0$, it holds that

$$\phi(\xi^\rho, E\xi^\rho) - \phi(\xi^*, E\xi^*) + \psi(y_0^\rho) - \psi(y_0^*) + \delta \geq 0.$$

Therefore,

$$\begin{aligned} \lim_{\rho\to 0} \frac{F(\xi^\rho) - F(\xi^\delta)}{\rho} &= \lim_{\rho\to 0} \frac{1}{F(\xi^\rho) + F(\xi^\delta)} \cdot \frac{F^2(\xi^\rho) - F^2(\xi^\delta)}{\rho} \\ &= \frac{1}{F(\xi^\delta)}\Big\{\langle x_0^\delta - a, X_0^\delta\rangle + \big[\phi(\xi^\delta, E\xi^\delta) - \phi(\xi^*, E\xi^*) + \psi(y_0^\delta) - \psi(y_0^*) + \delta\big]\cdot \\ &\qquad \big[\langle \phi_x(\xi^\delta, E\xi^\delta) + \phi_{\bar{x}}(\xi^\delta, E\xi^\delta), \xi - \xi^\delta\rangle + \langle \psi_y(y_0^\delta), Y_0^\delta\rangle\big]\Big\}. \end{aligned}$$

Set

$$\begin{aligned} h_0^\delta &= \frac{1}{F(\xi^\delta)}\big[\phi(\xi^\delta, E\xi^\delta) - \phi(\xi^*, E\xi^*) + \psi(y_0^\delta) - \psi(y_0^*) + \delta\big], \\ h_1^\delta &= \frac{1}{F(\xi^\delta)}\big[x_0^\delta - a\big]. \end{aligned}$$

From (13), we obtain

$$\langle h_1^\delta, X_0^\delta\rangle + h_0^\delta \cdot \big[\langle \phi_x(\xi^\delta, E\xi^\delta) + \phi_{\bar{x}}(\xi^\delta, E\xi^\delta), \xi - \xi^\delta\rangle + \langle \psi_y(y_0^\delta), Y_0^\delta\rangle\big] \geq -\sqrt{\delta}\big[E|\xi - \xi^\delta|^2\big]^{\frac{1}{2}}. \tag{16}$$

**Case 2.** There exists a positive sequence $\{\rho_n\}$ satisfying $\rho_n \to 0$ such that

$$\phi(\xi^{\rho_n}, E\xi^{\rho_n}) - \phi(\xi^*, E\xi^*) + \psi(y_0^{\rho_n}) - \psi(y_0^*) + \delta \leq 0.$$

By the definition of the penalty function $F(\cdot)$, we know that $F(\xi^{\rho_n}) = \{|x_0^{\rho_n} - a|^2\}^{\frac{1}{2}}$. As $F(\cdot)$ is continuous, we have $F(\xi^\delta) = \{|x_0^\delta - a|^2\}^{\frac{1}{2}}$, where $x_0^{\rho_n} \to x_0^\delta$ $(n \to \infty)$. Then

$$\lim_{\rho\to 0} \frac{F(\xi^{\rho_n}) - F(\xi^\delta)}{\rho_n} = \lim_{\rho\to 0} \frac{1}{F(\xi^{\rho_n}) + F(\xi^\delta)} \frac{F^2(\xi^{\rho_n}) - F^2(\xi^\delta)}{\rho_n} = \frac{\langle x_0^\delta - a, X_0^\delta\rangle}{F(\xi^\delta)}.$$

Set $h_0^\delta = 0$, $h_1^\delta = \dfrac{1}{F(\xi^\delta)}\big[x_0^\delta - a\big]$, it follows from (13) that

$$\langle h_1^\delta, X_0^\delta \rangle \geq -\sqrt{\delta}\big[E|\xi - \xi^\delta|^2\big]^{\frac{1}{2}}. \tag{17}$$

Above all, we have $h_0^\delta \geq 0$, $|h_0^\delta|^2 + |h_1^\delta|^2 = 1$ and the inequality (16) holds.

From the above, there exists a subsequence of $(h_0^\delta, h_1^\delta)$ which will converge to a limit denoted by $(h_0, h_1)$. Since $d(\xi^\delta, \xi^*) \leq \sqrt{\delta}$, then we have $\xi^\delta \to \xi^*$ in U as $\delta \to 0$. Therefore, from the regularity of the solutions of mean-field BSDEs, we have $X^\delta \to X^*$, $Y^\delta \to Y^*$ as $\delta \to 0$. Let $\delta \to 0$ in (16), we get the variational inequality (12).

Next, we consider the case where $L(t, \lambda, E\lambda, r, p, q) \neq 0$. By using the similar analysis, the variational inequality can be derived as follows:

**Theorem 3.2** Under the assumptions (H1)-(H4), let $\xi^*$ be an optimal control to Problem B. Then for any $\xi \in U$, there exists $h_0 \geq 0$ and $h_1 \in R^n$ with $h_0 + |h_1| \neq 0$, the following variational inequality holds

$$\begin{aligned}
&\langle h_1, X_0\rangle + h_0\langle \phi_x(\xi^*, E\xi^*) + \phi_{\tilde{x}}(\xi^*, E\xi^*), \xi - \xi^*\rangle + h_0\langle \psi_y(y_0^*), Y_0\rangle \\
&+h_0 E\int_0^T \big[\langle L_x^*, X_t\rangle + \langle L_{\tilde{x}}^*, EX_t\rangle + \langle L_y^*, Y_t\rangle + \langle L_{\tilde{y}}^*, EY_t\rangle \\
&+\langle L_z^*, Z_t\rangle + \langle L_{\tilde{z}}^*, EZ_t\rangle + \langle L_r^*, R_t\rangle + \langle L_p^*, P_t\rangle + \langle L_q^*, Q_t\rangle\big]dt \geq 0,
\end{aligned} \tag{18}$$

here $L_a^* = L_a(t, x_t^*, Ex_t^*, y_t^*, Ey_t^*, z_t^*, Ez_t^*, r_t^*, p_t^*, q_t^*)$, $a = x, \tilde{x}, y, \tilde{y}, z, \tilde{z}, r, p, q$, respectively.

## 3.3 Maximum principle

To derive the maximum principle, we first introduce the Hamiltonian function $\mathcal{H}(\cdot)$ associated with the mean-field stochastic control system as follows:

$$\mathcal{H}(t, \lambda, E\lambda, r, p, q, m, n) = \langle m, b(t, \lambda, E\lambda, r, p, q)\rangle + \langle n, f(t, \lambda, E\lambda, r, p, q)\rangle + h_0 L(t, \lambda, E\lambda, r, p, q),$$

here $(m_t, n_t)$ is the solution of the following adjoint equations:

$$\begin{cases}
dm_t = \big[b_x^* m_t + E(b_{\tilde{x}}^* m_t) + f_x^* n_t + E(f_{\tilde{x}}^* n_t) + h_0 L_x^* + h_0 L_{\tilde{x}}^*\big]dt \\
\qquad +\big[b_p^* m_t + f_p^* n_t + h_0 L_p^*\big]dW_t + \sum\limits_{j=1}^{\infty}\big[b_{q^{(j)}}^* m_t + f_{q^{(j)}}^* n_t + h_0 L_{q^{(j)}}^*\big]dH_t^{(j)}, \\
dn_t = \big[b_y^* m_t + E(b_{\tilde{y}}^* m_t) + f_y^* n_t + E(f_{\tilde{y}}^* n_t) + h_0 L_y^* + h_0 L_{\tilde{y}}^*\big]dt \\
\qquad +\big[b_z^* m_t + E(b_{\tilde{z}}^* m_t) + f_z^* n_t + E(f_{\tilde{z}}^* n_t) + h_0 L_z^* + h_0 L_{\tilde{z}}^*\big]dW_t \\
\qquad +\sum\limits_{j=1}^{\infty}\big[b_{r^{(j)}}^* m_t + f_{r^{(j)}}^* n_t + h_0 L_{r^{(j)}}^*\big]dH_t^{(j)}, \\
m_0 = h_1, \quad n_0 = h_0\psi_y(y_0^*).
\end{cases} \tag{19}$$

Note that the adjoint equations turns out to be two linear mean-field SDEs. From (H3), we know there admits a unique solution $(m_t, n_t)$ of the above equation. Associated with the Hamiltonian function $\mathcal{H}(\cdot)$, the adjoint equations (19) can also be rewritten as the following stochastic Hamiltonian system's type

$$\begin{cases}
dm_t = \big[\mathcal{H}_x^* + E(\mathcal{H}_{\tilde{x}}^*)\big]dt + \mathcal{H}_p^* dW_t + \sum\limits_{j=1}^{\infty}\mathcal{H}_{q^{(j)}}^* dH_t^{(j)}, \\
dn_t = \big[\mathcal{H}_y^* + E(\mathcal{H}_{\tilde{y}}^*)\big]dt + \big[\mathcal{H}_z^* + E(\mathcal{H}_{\tilde{z}}^*)\big]dW_t + \sum\limits_{j=1}^{\infty}\mathcal{H}_{r^{(j)}}^* dH_t^{(j)}, \\
m_0 = h_1, \quad n_0 = h_0\psi_y(y_0^*),
\end{cases} \tag{20}$$

here $\mathcal{H}_a^* = \mathcal{H}_a(t, x_t^*, Ex_t^*, y_t^*, Ey_t^*, z_t^*, Ez_t^*, r_t^*, p_t^*, q_t^*)$, which is a first order partial derivatives to $a$, $a = x, \tilde{x}, y, \tilde{y}, z, \tilde{z}, r, p, q$ respectively.

Next, by using a technique similar to that of refs. [12, 13], we can easily get the following

**Theorem 3.3** Suppose that (H1)-(H4) hold, let $\xi^*$ be an optimal control to Problem B. Then for any $\xi \in U$, there exists $h_0 \geq 0$ and $h_1 \in R^n$ with $h_0 + |h_1| \neq 0$, the following maximum principle holds true,

$$\langle m_T + h_x(\xi^*, E\xi^*)n_T + h_{\tilde{x}}(\xi^*, E\xi^*)n_T + h_0\phi_x(\xi^*, E\xi^*) + h_0\phi_{\tilde{x}}(\xi^*, E\xi^*), \xi - \xi^* \rangle \geq 0, \quad a.s. \tag{21}$$

*Proof.* By applying Itô's formula to $\langle X_t, m_t \rangle + \langle Y_t, n_t \rangle$, we can write

$$\begin{aligned}
&E\langle \xi - \xi^*, m_T \rangle - \langle X_0, h_1 \rangle + E\langle h_x(\xi^*, E\xi^*) \cdot (\xi - \xi^*) + h_{\tilde{x}}(\xi^*, E\xi^*) \cdot E(\xi - \xi^*), n_T \rangle - \langle Y_0, h_0\psi_y(y_0^*) \rangle \\
&= E\int_0^T \Big\{ \langle X_t, \mathcal{H}_x^* + E(\mathcal{H}_{\tilde{x}}^*) \rangle + \langle Y_t, \mathcal{H}_y^* + E(\mathcal{H}_{\tilde{y}}^*) \rangle + \langle Z_t, \mathcal{H}_z^* + E(\mathcal{H}_{\tilde{z}}^*) \rangle + \langle \mathcal{H}_p^*, P_t \rangle + \langle \mathcal{H}_q^*, Q_t \rangle + \langle \mathcal{H}_r^*, R_t \rangle \\
&\qquad - \langle b_x^* X_t + b_{\tilde{x}}^* EX_t + b_y^* Y_t + b_{\tilde{y}}^* EY_t + b_z^* Z_t + b_{\tilde{z}}^* EZ_t + b_r^* R_t + b_p^* P_t + b_q^* Q_t, m_t \rangle \\
&\qquad - \langle f_x^* X_t + f_{\tilde{x}}^* EX_t + f_y^* Y_t + f_{\tilde{y}}^* EY_t + f_z^* Z_t + f_{\tilde{z}}^* EZ_t + f_r^* R_t + f_p^* P_t + f_q^* Q_t, n_t \rangle \Big\} dt.
\end{aligned}$$

Combing with the variational inequality (18), it can be obtained that

$$\begin{aligned}
&E\langle \xi - \xi^*, m_T \rangle + E\langle h_x(\xi^*, E\xi^*) \cdot (\xi - \xi^*) + h_{\tilde{x}}(\xi^*, E\xi^*) \cdot E(\xi - \xi^*), n_T \rangle \\
&+ h_0\langle \phi_x(\xi^*, E\xi^*) + \phi_{\tilde{x}}(\xi^*, E\xi^*), \xi - \xi^* \rangle \\
&= \langle X_0, h_1 \rangle + \langle Y_0, h_0\psi_y(y_0^*) \rangle + h_0\langle \phi_x(\xi^*, E\xi^*) + \phi_{\tilde{x}}(\xi^*, E\xi^*), \xi - \xi^* \rangle \\
&+ h_0 E\int_0^T \big[ \langle L_x^*, X_t \rangle + \langle L_{\tilde{x}}^*, EX_t \rangle + \langle L_y^*, Y_t \rangle + \langle L_{\tilde{y}}^*, EY_t \rangle + \langle L_z^*, Z_t \rangle + \langle L_{\tilde{z}}^*, EZ_t \rangle \\
&+ \langle L_r^*, R_t \rangle + \langle L_p^*, P_t \rangle + \langle L_q^*, Q_t \rangle \big] dt \geq 0.
\end{aligned}$$

The proof is complete.

## 4 Application: Stochastic LQ control problems with terminal state constraints

In this section, we apply the results to investigate the linear quadratic control problem with terminal state constraints. Consider the following mean-field type linear quadratic control system driven by Teugels martingales:

$$\begin{cases}
dx_t = (a_1 x_t + a_2 Ex_t + a_3 y_t + a_4 Ey_t + a_5 u_t)dt + (b_1 x_t + b_2 Ex_t + b_3 y_t + b_4 Ey_t + b_5 u_t)dW_t \\
\qquad + \sum\limits_{j=1}^{\infty} c_j u_t dH_t^{(j)}, \\
dy_t = -(f_1 x_t + f_2 Ex_t + f_3 y_t + f_4 Ey_t + f_5 z_t + f_6 Ez_t + f_7 u_t)dt + z_t dW_t + \sum\limits_{j=1}^{\infty} r_t^{(j)} dH_t^{(j)}, \\
x_0 = a, \quad y_T = g_1 x_T + g_2 Ex_T,
\end{cases} \tag{22}$$

the object of our control problem is to minimize the cost function:

$$\bar{J}(\cdot) = \frac{1}{2}E[k_1 x_T^2 + k_2 y_0^2],$$

subject to $u(\cdot) \in M^2(0, T; R)$. $a_i, b_i, f_i, g_i$ and $c_j$ are finite real numbers with $b_5 \neq 0$. $k_1$ and $k_2$ are positive constants.

Let $b_1x_t + b_2Ex_t + b_3y_t + b_4Ey_t + b_5u_t = p_t$, $c_ju_t = q_t^{(j)}$, then the system is transformed into:

$$
\begin{cases}
dx_t = \Big[(a_1 - \frac{a_5b_1}{b_5})x_t + (a_2 - \frac{a_5b_2}{b_5})Ex_t + (a_3 - \frac{a_5b_3}{b_5})y_t + (a_4 - \frac{a_5b_4}{b_5})Ey_t + \frac{a_5}{b_5}p_t\Big]dt \\
\qquad +p_tdW_t + \sum\limits_{j=1}^{\infty} q_t^{(j)}dH_t^{(j)}, \\
dy_t = -\Big[(f_1 - \frac{f_7b_1}{b_5})x_t + (f_2 - \frac{f_7b_2}{b_5})Ex_t + (f_3 - \frac{f_7b_3}{b_5})y_t + (f_4 - \frac{f_7b_4}{b_5})Ey_t + f_5z_t \\
\qquad +f_6Ez_t + \frac{f_7}{b_5}p_t\Big]dt + z_tdW_t + \sum\limits_{j=1}^{\infty} r_t^{(j)}dH_t^{(j)}, \\
x_T = \xi, \quad y_T = g_1\xi + g_2E\xi.
\end{cases}
\tag{23}
$$

The equivalent cost functional is:

$$
Minimize \;\; J(\cdot) = \frac{1}{2}E[k_1\xi^2 + k_2y_0^2],
$$

$$
s.t. \qquad \xi \in R^+, \;\; x_0^\xi = a.
$$

we can get the following mean-field adjoint equations:

$$
\begin{cases}
dm_t = \Big[(\frac{a_5b_1}{b_5} - a_1)m_t + (\frac{a_5b_2}{b_5} - a_2)Em_t + (f_1 - \frac{f_7b_1}{b_5})n_t + (f_2 - \frac{f_7b_2}{b_5})En_t\Big]dt \\
\qquad +\Big[-\frac{a_5}{b_5}m_t + \frac{f_7}{b_5}n_t\Big]dW_t, \\
dn_t = \Big[(\frac{a_5b_3}{b_5} - a_3)m_t + (\frac{a_5b_4}{b_5} - a_4)Em_t + (f_3 - \frac{f_7b_3}{b_5})n_t + (f_4 - \frac{f_7b_4}{b_5})En_t\Big]dt \\
\qquad +\Big[f_5n_t + f_6En_t\Big]dW_t \\
m_0 = h_1, \quad n_0 = h_0k_2y_0^*,
\end{cases}
\tag{24}
$$

where $y_0^*$ is the solution of (22) associated with $\xi^*$. Then, according to the maximum principle in Theorem 3.3, if $\xi^*$ is optimal control, we conclude that there exist $h_0, h_1 \in R$ with $h_0 \geq 0$ and $h_0 + |h_1| \neq 0$ such that, for any $\xi \geq 0$,

$$
\langle m_T + g_1n_T + g_2En_T + h_0k_1\xi^*, \xi - \xi^*\rangle \geq 0, \quad a.s. \tag{25}
$$

Denote $\Omega_0 := \{\omega \in \Omega \mid \xi^*(\omega) = 0\}$. From the arbitrariness of $\xi$ in (24), we get the adjoint process $(m_\cdot, n_\cdot)$ satisfies

$$
m_T + g_1n_T + g_2En_T \geq 0, \quad a.s. \;\; on \;\; \Omega_0,
$$

and on $\Omega_0^C$, $m_T + g_1n_T + g_2En_T + h_0k_1\xi^* = 0$, $a.s.$. It follows that if $h_0 > 0$ , we have

$$
\xi^* = -h_0^{-1}k_1^{-1}[m_T + g_1n_T + g_2En_T], \quad a.s..
$$

**Appendix. Proof of Lemma 2.1**

We give the existence proof in case of $m \geq n$ as follows.

Consider the following family of mean-field FBSDEs parametrized by $\alpha \in [0, 1]$:

$$
\begin{cases}
dx_t^\alpha = [\alpha\bar{b}(t, \lambda_t^\alpha, E\lambda_t^\alpha, r_t^\alpha) + \phi_t]dt + [\alpha\bar{\sigma}(t, \lambda_t^\alpha, E\lambda_t^\alpha, r_t^\alpha) + \psi_t]dW_t \\
\qquad\quad + \sum\limits_{j=1}^{\infty}[\alpha\bar{g}^{(j)}(t, \lambda_t^\alpha, E\lambda_t^\alpha, r_t^\alpha) + \xi_t^{(j)}]dH_t^{(j)}, \\
-dy_t^\alpha = [(1-\alpha)\beta_1Gx_t^\alpha + \alpha\bar{f}(t, \lambda_t^\alpha, E\lambda_t^\alpha, r_t^\alpha) + \eta_t]dt - z_t^\alpha dW_t - \sum\limits_{j=1}^{\infty} r_t^{(j)}dH_t^{(j)}, \\
x_0^\alpha = a, \quad y_T^\alpha = \alpha h(x_T^\alpha, Ex_T^\alpha) + (1-\alpha)Gx_T^\alpha + \gamma,
\end{cases}
\tag{26}
$$

where $\phi, \psi, \xi$ and $\eta$ are processes in $M^2(0, T)$, and $\gamma \in L^2(\Omega, \mathcal{F}_T, P)$. We can easily check that the above equation has a unique solution when $\alpha = 0$. Moreover, when $\alpha = 1$, the existence of solutions for equation (26) implies that of MFFBSDE (1).

Next, we introduce a continuous dependence lemma to give a priori estimate for the "existence interval" of (26) with respect to $\alpha \in [0,1]$.

**Lemma A** Suppose that (H2.1) and (H2.2) hold and $m \geq n$. If for some $\alpha_0 \in [0,1)$ there exists a solution $(x^{\alpha_0}, y^{\alpha_0}, z^{\alpha_0}, r^{\alpha_0})$ of (26), then there exists a positive constant $\delta_0$ , such that for each $\delta \in [0,\delta_0]$ , there exists a solution $(x^{\alpha_0+\delta}, y^{\alpha_0+\delta}, z^{\alpha_0+\delta}, r^{\alpha_0+\delta})$ of MFFBSDE (26) for $\alpha = \alpha_0 + \delta$.

*Proof.* Since for each $\phi, \psi, \xi, \eta \in M^2(0,T)$, $\alpha \in [0,1)$, there exists a unique solution of (26). Let us define for each quarter $(x_t, y_t, z_t, r_t) \in M^2(0,T; R^{n+m+m\times d}) \times l^2(0,T;R^m)$, the following MFFBSDE:

$$\begin{cases} dX_t = [\alpha_0 \bar{b}(t,\Lambda_t,E\Lambda_t,R_t) + \delta \bar{b}(t,\lambda_t,E\lambda_t,r_t) + \phi_t]dt \\ \qquad\quad +[\alpha_0 \bar{\sigma}(t,\Lambda_t,E\Lambda_t,R_t) + \delta \bar{\sigma}(t,\lambda_t,E\lambda_t,r_t) + \psi_t]dW_t \\ \qquad\quad + \sum\limits_{j=1}^{\infty} [\alpha_0 \bar{g}^{(j)}(t,\Lambda_t,E\Lambda_t,R_t) + \delta \bar{g}^{(j)}(t,\lambda_t,E\lambda_t,r_t) + \xi_t^{(j)}]dH_t^{(j)}, \\ -dY_t = [(1-\alpha_0)\beta_1 GX_t + \alpha_0 \bar{f}(t,\Lambda_t,E\Lambda_t,R_t) + \delta\big(-\beta_1 Gx_t + \bar{f}(t,\lambda_t,E\lambda_t,r_t)\big) \\ \qquad\quad +\eta_t]dt - Z_t dW_t - \sum\limits_{j=1}^{\infty} R_t^{(j)} dH_t^{(j)}, \\ X_0 = a, \quad Y_T = \alpha_0 h(X_T, EX_T) + (1-\alpha_0)GX_T + \delta\big(h(x_T,Ex_T) - Gx_T\big) + \gamma, \end{cases}$$

where $\Lambda_t = (X_t, Y_t, Z_t)$. We are going to prove that the mapping defined by

$$I_{\alpha_0+\delta}(\lambda_t, r_t, x_T) = (\Lambda_t, R_t, X_T)$$

is a contraction.

We set $\Delta\Lambda = (\Delta X, \Delta Y, \Delta Z) = (X - X', Y - Y', Z - Z')$, $\Delta R = R - R'$. Applying Itô's formula to $\langle G\Delta X_t, \Delta Y_t \rangle$, it yields

$$\begin{aligned} &\alpha_0 E\langle \Delta h(X_T,EX_T), G\Delta X_T\rangle + (1-\alpha_0)E|G\Delta X_T|^2 + \delta E\langle \Delta h(x_T,Ex_T) - G\Delta x_T, G\Delta X_T\rangle \\ &= E\int_0^T \langle \alpha_0 \Delta A(t,\Lambda_t,E\Lambda_t,R_t), \Delta\Lambda_t\rangle dt - (1-\alpha_0)\beta_1 E\int_0^T \langle G\Delta X_t, G\Delta X_t\rangle dt \\ &\quad + E\int_0^T \sum_{j=1}^{\infty} \alpha_0 \langle \Delta g^{(j)}(t,\Lambda_t,E\Lambda_t,R_t), G^T \Delta R_t^{(j)}\rangle dt + \delta E\int_0^T \Big\{ \langle G\Delta b_t, \Delta Y_t\rangle \\ &\quad - \langle G\Delta X_t, -\beta_1 G\Delta x_t + \Delta f_t\rangle + \langle \Delta\sigma_t, G^T\Delta Z_t\rangle + \sum_{j=1}^{\infty} \langle \Delta g_t^{(j)}, G^T \Delta R_t^{(j)}\rangle \Big\} dt \\ &\leq E\int_0^T \Big\{ -\beta_1 |G\Delta X_t|^2 - \alpha_0\beta_1 |GE\Delta X_t|^2 - \alpha_0\beta_2 (|G^T\Delta Y_t|^2 + |G^T E\Delta Y_t|^2 + |G^T\Delta Z_t|^2 \\ &\quad + |G^T E\Delta Z_t|^2 + \sum_{j=1}^{\infty} |G^T\Delta R_t^{(j)}|^2 \Big\} dt + \delta E\int_0^T \Big\{ \langle G\Delta b_t, \Delta Y_t\rangle - \langle G\Delta X_t, -\beta_1 G\Delta x_t + \Delta f_t\rangle \\ &\quad + \langle \Delta\sigma_t, G^T\Delta Z_t\rangle + \sum_{j=1}^{\infty} \langle \Delta g_t^{(j)}, G^T\Delta R_t^{(j)}\rangle \Big\} dt. \end{aligned}$$

where $\Delta h(X_T, EX_T) = h(X_T, EX_T) - h(X'_T, EX'_T)$,

$\Delta A(t,\Lambda_t,E\Lambda_t,R_t) = A(t,\Lambda_t,E\Lambda_t,R_t) - A(t,\Lambda'_t,E\Lambda'_t,R'_t)$,

$\Delta g^{(j)}(t,\Lambda_t,E\Lambda_t,R_t) = \bar{g}^{(j)}(t,\Lambda_t,E\Lambda_t,R_t) - \bar{g}^{(j)}(t,\Lambda'_t,E\Lambda'_t,R'_t)$,

$\Delta b_t = \bar{b}(t,\lambda_t,E\lambda_t,r_t) - \bar{b}(t,\lambda'_t,E\lambda'_t,r'_t)$, $\Delta f_t = \bar{f}(t,\lambda_t,E\lambda_t,r_t) - \bar{f}(t,\lambda'_t,E\lambda'_t,r'_t)$,

$\Delta\sigma_t = \bar{\sigma}(t,\lambda_t,E\lambda_t,r_t) - \bar{\sigma}(t,\lambda'_t,E\lambda'_t,r'_t)$, $\Delta g_t^{(j)} = \bar{g}^{(j)}(t,\lambda_t,E\lambda_t,r_t) - \bar{g}^{(j)}(t,\lambda'_t,E\lambda'_t,r'_t)$.

By the assumptions (H1), (H2), we use the Jensen's inequality and the fact that $Ex^2 \geq (Ex)^2$, then

$$\begin{aligned} &\big(\alpha_0\mu + (1-\alpha_0)\big)E|G\Delta X_T|^2 + \beta_1 E\int_0^T |G\Delta X_t|^2 dt \\ &\leq \delta CE\int_0^T (|\Delta\lambda_t|^2 + |\Delta r_t|^2 + |\Delta\Lambda_t|^2 + |\Delta R_t|^2)dt + \delta C\big(E|G\Delta X_T|^2 + E|G\Delta x_T|^2\big). \end{aligned}$$

By applying the usual technique to the BSDE part, we can obtain

$$\begin{aligned} E\int_0^T (|\Delta\Lambda_t|^2 + |\Delta R_t|^2)dt &\leq \delta C\Big(E\int_0^T (|\Delta\lambda_t|^2 + |\Delta r_t|^2)dt + E|G\Delta x_T|^2\Big) \\ &\quad + C\Big(E\int_0^T |G\Delta X_t|^2)dt + E|G\Delta X_T|^2\Big). \end{aligned}$$

Combing the above estimates, it follows that

$$E\int_0^T (|\Delta\Lambda_t|^2+|\Delta R_t|^2)dt+E|\Delta X_T|^2\leq \delta C'\big(E\int_0^T(|\Delta\lambda_t|^2+|\Delta r_t|^2)dt+E|\Delta x_T|^2\big).$$

Here the constant $C'$ depends on the Lipschitz constants $G,\beta_1,\beta_2$ and $T$. If we take $\delta_0=\frac{1}{2C'}$, then for each $\delta\in[0,\delta_0]$, we have

$$E\int_0^T (|\Delta\Lambda_t|^2+|\Delta R_t|^2)dt+E|\Delta X_T|^2\leq \frac{1}{2}\big(E\int_0^T(|\Delta\lambda_t|^2+|\Delta r_t|^2)dt+E|\Delta x_T|^2\big).$$

It follows that $I_{\alpha_0+\delta}$ is a contraction. Hence $I_{\alpha_0+\delta}$ has a unique fixed point $(x^{\alpha_0+\delta},y^{\alpha_0+\delta},z^{\alpha_0+\delta},r^{\alpha_0+\delta})$ which is the unique solution of mean-field FBSDE. Similarly, the case for $m<n$ can be proved by the same technique. Then the proof of Lemma2.1 is complete.

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The authors declare no potential conflict of interests.

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Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

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