The effect of migration on the transmission of HIV/AIDS using a fractional model: local and global dynamics and numerical simulations.

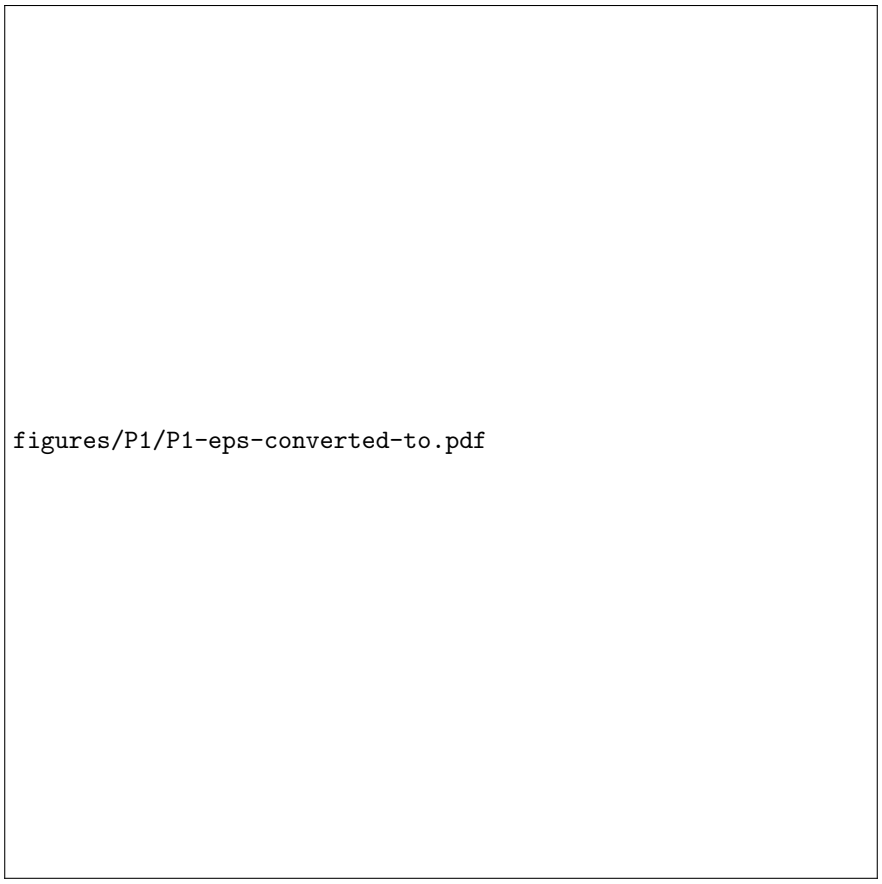
S. L'KIMA[1], Abdelouahed Allahamou[2], and E. AZROUL[1]

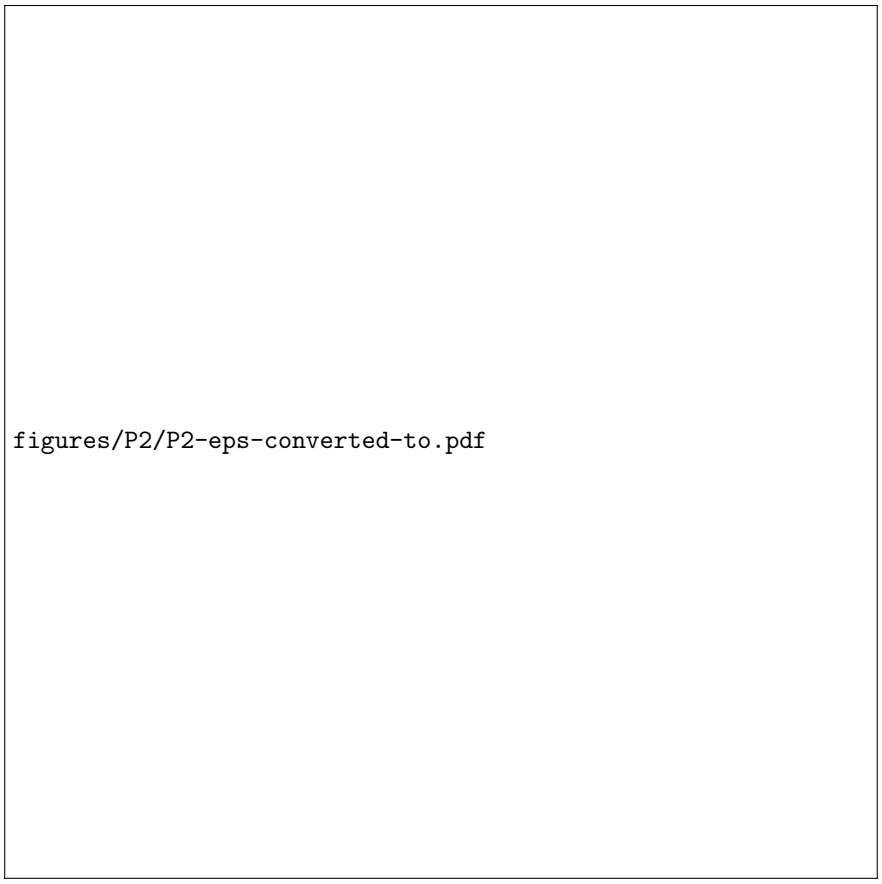
[1]Universite Sidi Mohamed Ben Abdellah Faculte des Sciences Dhar El Mahraz-Fes
[2]Universite Mohammed V de Rabat

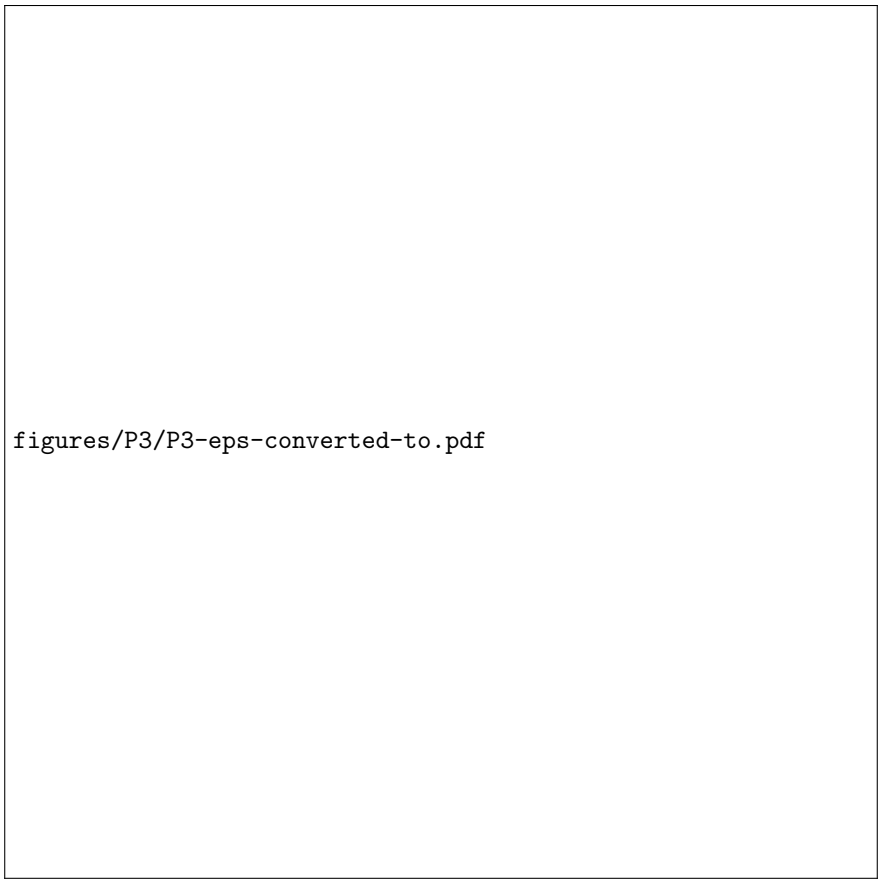
March 25, 2023

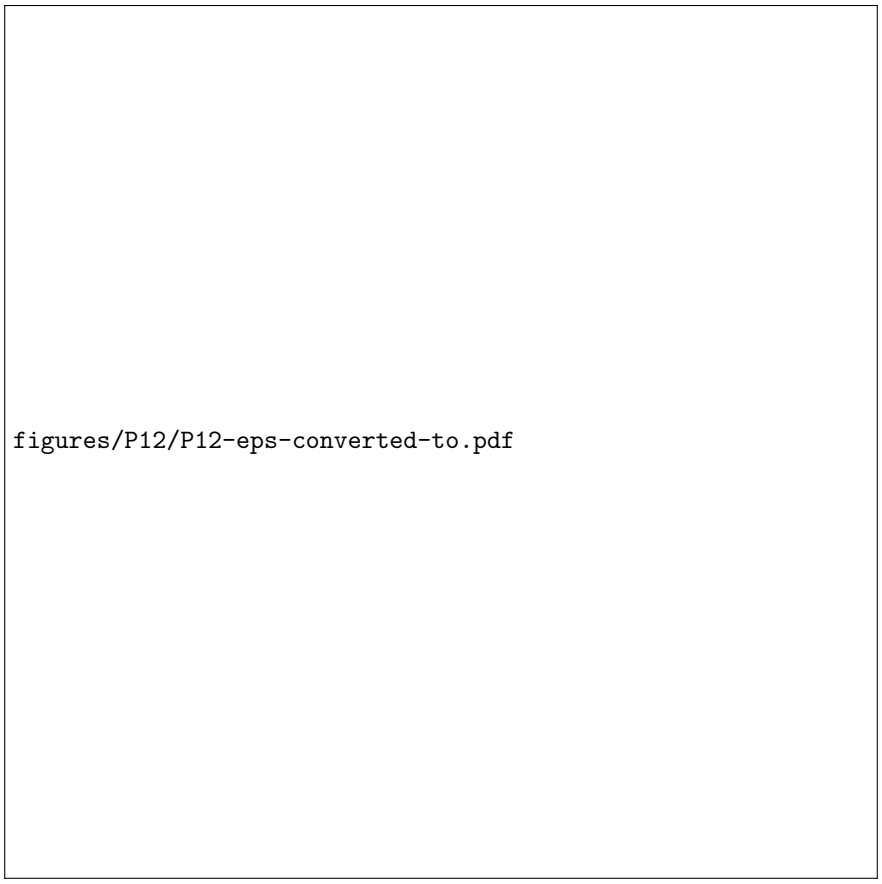
Abstract

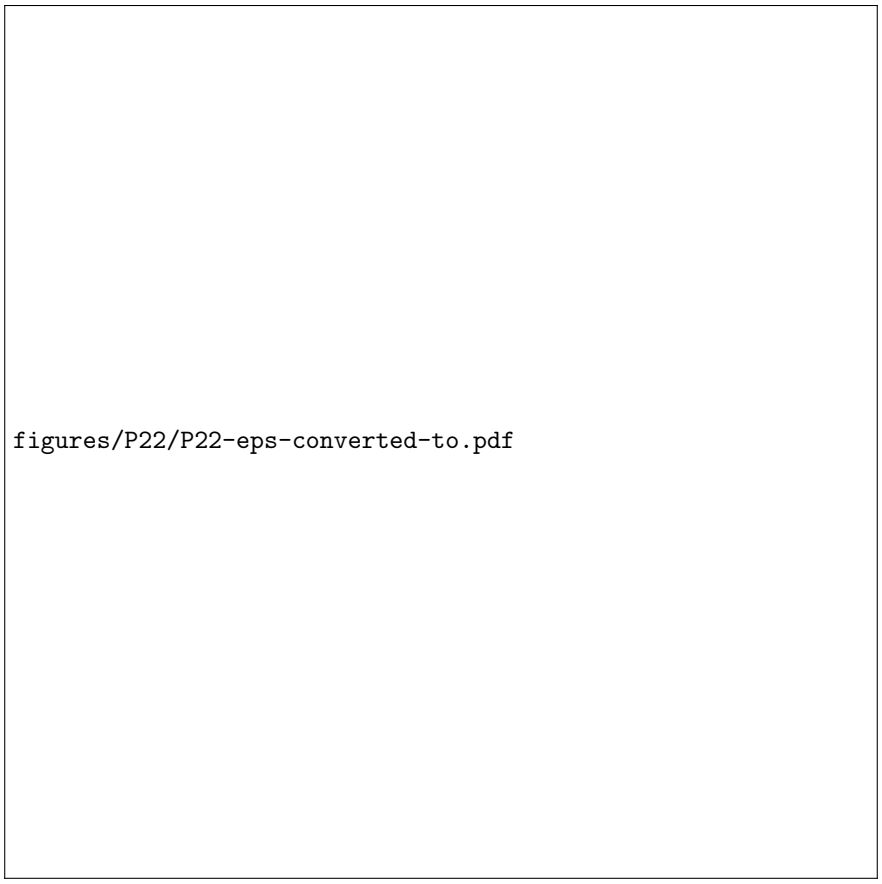
HIV is a serious disease that threatens and affects capital stock, population composition and economic growth. This research paper aims to study the mathematical modeling and disease dynamics of HIV/AIDS with memory effect. We propose two fractional models in the Caputo sense for HIV/AIDS with and without migration. First, we prove the existence and positivity of both models and calculate the basic reproduction number R 0 using the next generation method. Then, we study the local and global stability of the obtained equilibria. In addition, numerical simulations are provided for different values of the fractional order ϱ using the Adams-Bashforth-Moulton fractional scheme, to verify the theoretical results. Moreover, a sensitivity analysis of the parameters for the model with migration is carried out.

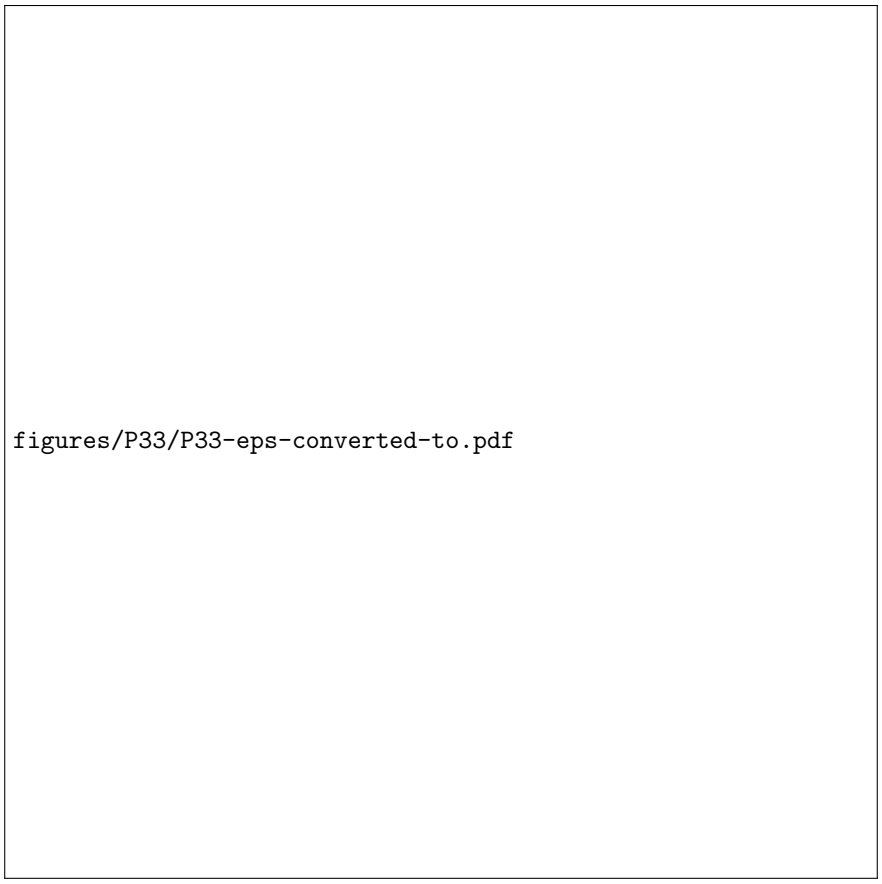












The effect of migration on the transmission of HIV/AIDS using a fractional model: local and global dynamics and numerical simulations.

A. ALLA HAMOU[a], E. AZROUL[b], S. L'KIMA[b,*]

[a] *Faculty of Sciences Mohammed V University in Rabat B.P. 1014 Rabat Morocco*
[b] *Sidi Mohamed Ben Abdellah University,Laboratory of Mathematical Analysis and Application, Faculty of Sciences Dhar Al Mahraz, B.P. 1796, Fez, 30000, Morocco*

Abstract

HIV is a serious disease that threatens and affects capital stock, population composition and economic growth. This research paper aims to study the mathematical modeling and disease dynamics of HIV/AIDS with memory effect. We propose two fractional models in the Caputo sense for HIV/AIDS with and without migration. First, we prove the existence and positivity of both models and calculate the basic reproduction number $\mathcal{R}_0$ using the next generation method. Then, we study the local and global stability of the obtained equilibria. In addition, numerical simulations are provided for different values of the fractional order ρ using the Adams-Bashforth-Moulton fractional scheme, to verify the theoretical results. Moreover, a sensitivity analysis of the parameters for the model with migration is carried out.

Keywords: Caputo derivative, HIV/AIDS disease, Fractional Adams-Bashforth method, Fractional differential equations.

1. Introduction

AIDS is a disease brought on by the human immunodeficiency virus (HIV) that results in a continuous deterioration of the immune system [19, 20]. It

*Correspondingauthor

Email addresses: abdelouahed.allahamou@gmail.com (A. ALLA HAMOU), elhoussine.azroul@usmba.ac.ma (E. AZROUL), safae.lkima@usmba.ac.ma (S. L'KIMA)

is extremely harmful. Once infected with HIV, the immune system become impaired, resulting in various diseases such as opportunistic infections and eventually leading to death.

In 1981, five young homosexual men in California who had been diagnosed with Pneumocystis carinii pneumonia were found to have the first cases of human immunodeficiency virus [10]. Since then, there has been a steady rise in cases throughout the world, which has had a significant impact on global public health. According to the World Health Organization (WHO), approximately 20 million people have already lost their lives due to acquired immunodeficiency syndrome(AIDS), and an estimated 38.4 million people are living with the virus through 2021; 25.6 million of them are in Africa. The virus kills CD4+ T cells, causing a loss of cell-mediated immunity and leaving the immune system vulnerable to cancer and other infectious diseases.AIDS is not always present in people with HIV. It can take many years for the virus to cause a major infection in the body. With the development of HIV/AIDS, many mathematicians have established a large number of mathematical models to assess potential effects and explanations for the spread of HIV/AIDS and to find out how these models can be used to prevent or reduce this disease ([1, 4, 14, 18, 21, 25, 28, 31]).

We refer to some research on the evolution of the HIV/AIDS pandemic and ways to prevent it. In [12], the authors have illustrated the influence of treatment on the progression of the HIV epidemic by including a compartment of the population on treatment in the epidemiological model. Y.-H. Hsieh and S.-P. Sheu [11] have proposed a mathematical model for the spread of HIV and AIDS through heterosexual relationships, with community-based systems for testing and enrollment in care. The authors in [13] develop a mathematical model to analyze the impact of awareness-raising and educational efforts on the spread of HIV epidemic. There are many reasons for the spread of HIV in the world, one of them is migration. We review some articles that deal with the issue of migration and its effects on the spread of HIV [7, 9, 32].

Most of the mathematical models used to explain the transmission of epidemics are differential equations (systems), which contain derivatives of functions with respect to time. Recently, the fractional derivative has been widely used in mathematical epidemiology[5, 15, 22, 30]. The memory effect is the primary characteristic of the fractional order derivative as opposed to the classical (integer order) derivative, which is a local operator, see [17, 29]. Furthermore, the extendement of the stability region of mathematical sys-

tems is considered an added benefit of the non-integer derivative. Saeedian et al.[24] constructed a SIR model to show the effects of memory on the evolution of the epidemic. Moreover, there are many definitions for the fractional derivative, including those from Riemann-Liouville, Hadamard, Caputo, Grnwald-Letnikov, Weyl, and Hifler. The characteristic that the derivative of a constant is zero is not satisfied by the majority of definitions of fractional derivatives, in contrast to Caputo's fractional derivative.

Inspired by this work [3], which established a mathematical model with population migration, and obtained that migration plays an essential role in the prevalence of AIDS. We have proposed two mathematical models of HIV/AIDS that identify , and describe HIV transmission using the fractional derivative in the sense of Caputo, motivated by existing work and the benefits of fractional derivative equations.

This work is organized as follows: in section 2, we will start with some preliminaries on fractional derivatives in the Caputo sense. Sections 3 and 4 provide the basic properties of the fractional model without migration and with migration . Then, the numerical simulation of the two proposed fractional models using the Adams-Bashforth-Moulton fractional technique is obtained in section 5. We conclude this article with a sensitivity analysis of the model parameters without migration.

2. Preliminaries

In this section, we will present some basic definitions and notations related to the Caputo fractional derivative that we will use in this paper (see for example [27, 23]).

Definition 2.1. *[23]. The Riemann-Liouville fractional integral of order $\rho > 0$ of a function $f : \mathbb{R}^+ \to \mathbb{R}$ is given by*

$$I^{\rho} f(t) = \frac{1}{\Gamma(\rho)} \int_0^t (t-x)^{\rho-1} f(x) dx,$$

where $\Gamma(\rho) = \int_0^{\infty} t^{\rho-1} e^{-t} dt$ is the Euler Gamma function.

Definition 2.2. *[23]. Let $\rho > 0, n = [\rho] + 1, n - 1 < \rho \leq n$, where $[\rho]$ denotes the integer part of ρ. The Caputo fractional derivative of order ρ for*

a function $f \in C^n([0,+\infty),\mathbb{R})$ *is defined by*

$$ {}^{C}D^{\rho}f(t) = I^{n-\rho}D^{n}f(t) = \frac{1}{\Gamma(n-\rho)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\rho+1-n}}ds, \quad t>0 $$

when $0<\rho\leq 1$ *we find*

$$ {}^{C}D^{\rho}f(t) = I^{1-\rho}Df(t) = \frac{1}{\Gamma(1-\rho)}\int_0^t \frac{f(s)}{(t-s)^{\rho}}ds, \quad t>0 $$

In this article, we propose two models of HIV/AIDS-infected population. The first is the model withoud migration and the second model with migration.

3. Fractional Model of HIV/AIDS whithout migration

In this model, we divide the hosts population into three compartiments, namely Susceptible (S), the infected (I) by HIV, and (A) the infected by AIDS. Where susceptible individuals can be infected at a rate of αI where α is HIV transmission rate for susceptible individuals. Individuals infected with HIV develop AIDS at a γ rate.

Finally, μ is the natural mortality rate, Λ is the recruitment birth rate, δ_1 and δ_2 are the HIV-related mortality rate and the AIDS-related mortality rate respectively. The flow diagram of HIV transmission without migration is shown in Figure 1.

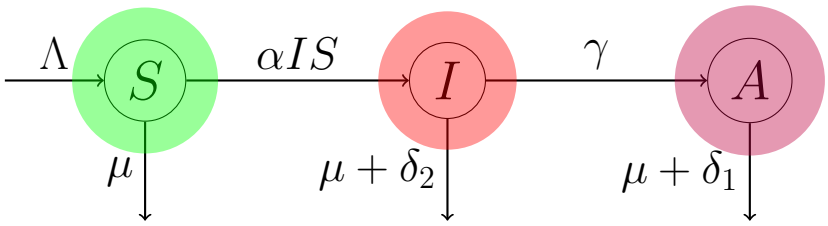


Figure 1: Schematic diagram of HIV transmission.

Based on the above assumptions, we obtain the classical formula for this model as follows:

$$\begin{cases} \dfrac{dS}{dt} = \Lambda - \alpha IS - \mu S \\ \dfrac{dI}{dt} = \alpha IS - (\mu + \delta_1 + \gamma)I \\ \dfrac{dA}{dt} = \gamma I - (\mu + \delta_2)A \end{cases} \tag{1}$$

Fractional Model

Now by changing the classical derivative in model 1 to the fractional derivative in the sense of Caputo, we get the following system of fractional ODEs:

$$\begin{cases} {}^{C}D_t^{\rho} S = \Lambda - \alpha IS - \mu S \\ {}^{C}D_t^{\rho} I = \alpha IS - (\mu + \delta_1 + \gamma)I \\ {}^{C}D_t^{\rho} A = \gamma I - (\mu + \delta_2)A \end{cases} \tag{2}$$

With non-negative initial conditions $S(0) = S_0 \geq 0, I(0) = I_0 \geq 0, A(0) = A_0 \geq 0$ and the operator ${}^{C}D$ is the Caputo fractional derivatives of order $\rho \in (0, 1]$.

3.1. Positivity and boundedness of solution

Lemma 3.1. ***(Generalized Mean Value Theorem [26]).*** *Suppose that* $f \in \mathcal{C}[a, b]$, *and* ${}^{C}D_t^{\rho} f(t) \in \mathcal{C}(a, b]$, $0 < \rho \leq 1$. *Then,*

$$f(t) = f(0) + \frac{1}{\Gamma(\rho)} \left({}^{C}D_t^{\rho} f\right)(\xi)(t - a)^{\rho},$$

Where $0 \leq \xi \leq t, \forall t \in (a, b]$.

Corollary 3.1. *[26] Suppose that* $f \in \mathcal{C}[a, b]$, *and* ${}^{C}D_t^{\rho} f(t) \in \mathcal{C}(a, b]$, $0 < \rho \leq 1$. *Then,*

1. *If* ${}^{C}D_t^{\rho} f(t) \geq 0$, $\forall t \in (a, b]$, *then* ${}^{C}D_t^{\rho} f(t)$ *is non-decreasing.*
2. *If* ${}^{C}D_t^{\rho} f(t) \leq 0$, $\forall t \in (a, b]$, *then* ${}^{C}D_t^{\rho} f(t)$ *is non-increasing.*

Theorem 3.1. *The region* $\Theta_+ = \{(S, I, A) \in \mathbb{R}^+; S \geq 0,\ I \geq 0, A \geq 0\}$ *is positivity invariant set for the system* (2).

Proof. From *Theorem 3.1* and *the remark 3.2* in [16] the solution of system (2) is exist and unique. Now we prove the positivity of the solution, since

$$\begin{aligned} {}^{C}D_t^{\rho} S\backslash_{S=0} &= \Lambda \geq 0, \\ {}^{C}D_t^{\rho} I\backslash_{I=0} &= 0, \\ {}^{C}D_t^{\rho} A\backslash_{A=0} &= \gamma I \geq 0, \end{aligned}$$

By using the Corollary (3.1), we get that is, the solution will stay in $\mathbb{R}^4_+$ and hence,
$\Theta_+ = \{(S, I, A) \in \mathbb{R}^+; S \geq 0,\ I \geq 0, A \geq 0\}$ is positivity invariant set for the system (2). □

Next we establish the boundedness of solution by the following theorem.

Theorem 3.2. *The region* $\Theta = \{(S, I, A) \in \mathbb{R}^+; S \geq 0,\ I \geq 0, A \geq 0,$ $0 \leq N(t) \leq \dfrac{\Lambda}{\mu},$ *where* $N(t) = S(t) + I(t) + A(t)\}$ *is positivity invariant set for the system* (2).

Proof. Summing all the equations of system (2), we have

$$ {}^{C}D_t^{\rho} N(t) = \Lambda - \mu N(t) - \delta_1 I(t) - \delta_2 A(t), \tag{3}$$

In the absence of disease the equation (3) becames

$$ {}^{C}D_t^{\rho} N(t) \leq \Lambda - \mu N(t), \tag{4}$$

Solving the equation (4), we get

$$\begin{aligned} N(t) &\leq E_{\rho,1}(-\mu\, t^{\rho}) N(0) + \Lambda\, t^{\rho}\, E_{\rho,\rho+1}(-\mu\, t^{\rho}), \\ &\leq \frac{\Lambda}{\mu}\left[\, E_{\rho,1}(-\mu\, t^{\rho}) + \mu\, t^{\rho}\, E_{\rho,\,\rho+1}(-\mu\, t^{\rho})\right] = \frac{\Lambda}{\mu}, \end{aligned}$$

Then, the feasible region Θ is positivity invariant. this imply the boundedness of solution . □

3.2. Equilibrium and basic reproduction number

To calculate the equilibrium of model (2) we solve :

$$^{C}D_t^{\rho}S() = {}^{C}D_t^{\rho}I(t) = {}^{C}D_t^{\rho}A(t) = 0,$$

Then, we obtain the disease free equilibrium DFE which is given by

$$\mathcal{E}_0 = (S_0, I_0, A_0), \tag{5}$$

Where $S_0 = \dfrac{\Lambda}{\mu}, I_0 = 0, A_0 = 0$. The basic reproduction number $\mathcal{R}_0$ obtained by the next generation which is given as follows :

1. There are two infected states: A, I, and one non infected state: S.
2. Equilibrium without infection is $S = \dfrac{\Lambda}{\mu}, I = A = 0$. Linearized system around the equilibrium point associated with the infected states,

$$^{C}D_t^{\rho}I = \alpha IS - (\mu + \delta_1 + \gamma)I, \qquad {}^{C}D_t^{\rho}A = \gamma I - (\mu + \delta_2)A,$$

The transmission and transition matrices are given by

$$T = \begin{pmatrix} \alpha\, S_0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} -(\mu + \delta_1 + \gamma) & 0 \\ \gamma & -(\mu + \delta_2) \end{pmatrix},$$

Finally the basic reproduction number for the SIA model is given by

$$\mathcal{R}_0 = \varrho\left(-T\Sigma^{-1}\right) = \frac{\alpha S_0}{(\mu + \delta_1 + \gamma)} = \frac{\alpha\,\Lambda}{\mu(\mu + \delta_1 + \gamma)}. \tag{6}$$

Theorem 3.3. *If $\mathcal{R}_0 > 1$, then there exists a unique endemic equlibrium $\mathcal{E}_1 = (S^*, I^*, A^*)$ of system (2).*

Proof. To find the endemic equlibrium, we solve the system

$$^{C}D_t^{\rho}S(t) = {}^{C}D_t^{\rho}I(t) = {}^{C}D_t^{\rho}A(t) = 0,$$

The model (2) has a unique endemic equilibrium denoted by $\mathcal{E}_1 = (S^*, I^*, A^*)$ With,

$$S^* = \frac{\mu + \delta_1 + \gamma}{\alpha},\ I^* = \frac{\alpha\,\Lambda - \mu\,(\mu + \delta_1 + \gamma)}{\alpha\,(\mu + \delta_1 + \gamma)},\ A^* = \frac{\gamma\, I^*}{\mu + \delta_2}, \tag{7}$$

As we have $\mathcal{R}_0 > 1$, then $I^* > 0$, which implies $S^* > 0$, and $A^* > 0$. □

3.3. Stability analysis

Theorem 3.4. *If $\mathcal{R}_0 < 1$, then the disease free equilibrium $\mathcal{E}_0$ of system (2) is locally asymptotically stable. If $\mathcal{R}_0 > 1$, then $\mathcal{E}_0$ is unstable.*

Proof. The Jacobian matrix of system (2) at $\mathcal{E}_0$ is

$$\mathcal{J}(\mathcal{E}_0) = \begin{pmatrix} -\mu & -\alpha & 0 \\ 0 & \alpha S_0 - (\mu + \delta_1 + \gamma) & 0 \\ 0 & \gamma & -(\mu + \delta_2) \end{pmatrix}.$$

By some calculations, we obtain the characteristic roots are:

$$\lambda_1 = -\mu, \qquad \lambda_2 = -(\mu + \delta_2), \qquad \lambda_3 = \alpha\frac{\Lambda}{\mu} - (\mu + \delta_1 + \gamma).$$

Evident, $\lambda_1 < 0$ and $\lambda_2 < 0$. If $\mathcal{R}_0 < 1$, then $\lambda_3 < 0$. Thus, we have $|\arg(\lambda_i)| > \dfrac{\rho\,\pi}{2}$ $(i = 1, 2, 3)$. Hence the stability local of disease-free equilibrium. □

Theorem 3.5. *If $\mathcal{R}_0 > 1$ then the endemic equilibrium $\mathcal{E}_1$ of system (2) is locally asymptotically stable. If $\mathcal{R}_0 < 1$, then $\mathcal{E}_1$ is unstable.*

Proof. The Jacobian matrix of system (2) at $\mathcal{E}_1$ is

$$\mathcal{J}(\mathcal{E}_1) = \begin{pmatrix} -\alpha I^* - \mu & -\alpha S^* & 0 \\ \alpha I^* & \alpha S^* - (\mu + \delta_1 + \gamma) & 0 \\ 0 & \gamma & -(\mu + \delta_2) \end{pmatrix}.$$

We replace I^* and S^* by their values we obtain

$$\mathcal{J}(\mathcal{E}_1) = \begin{pmatrix} -\mu\,\mathcal{R}_0 & -(\mu + \delta_1 + \gamma) & 0 \\ \mu\,(\mathcal{R}_0 - 1) & 0 & 0 \\ 0 & \gamma & -(\mu + \delta_2) \end{pmatrix}.$$

To determine the eigenvalues, we solve the equation

$$\det(\mathcal{J}(\mathcal{E}_1) - \lambda I_3) = 0.$$

$$\begin{aligned} \det(\mathcal{J}(\mathcal{E}_1) - \lambda I_3) =& det \begin{pmatrix} -\mu\,\mathcal{R}_0 - \lambda & -(\mu + \delta_1 + \gamma) & 0 \\ \mu\,(\mathcal{R}_0 - 1) & -\lambda & 0 \\ 0 & \gamma & -(\mu + \delta_2) - \lambda \end{pmatrix}, \\ &= (\mu + \delta_2 + \lambda)((\mu\,\mathcal{R}_0 + \lambda)\lambda + (\mu + \delta_1 + \gamma)\mu\,(\mathcal{R}_0 - 1)), \\ &= (\mu + \delta_2 + \lambda)(\lambda^2 + \mu\,\mathcal{R}_0\lambda + (\mu + \delta_1 + \gamma)\mu\,(\mathcal{R}_0 - 1)). \end{aligned}$$

We pose $\Delta = (\mu\,\mathcal{R}_0)^2 - 4\mu(\mu+\delta_1+\gamma)(\mathcal{R}_0-1)$.

We can see if $\Delta > 0$, we find

$$\lambda_1 = \frac{-(\mu\,\mathcal{R}_0) - \sqrt{\Delta}}{2}, \qquad \lambda_2 = \frac{-(\mu\,\mathcal{R}_0) + \sqrt{\Delta}}{2}, \qquad \lambda_3 = -(\mu+\delta_2).$$

If $\mathcal{R}_0 > 1$, then $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus all eigenvalues have negative real parts. Hence the endemic equlibrium of system (2) is locally asymptotically stable for all $\alpha \in (0, 1]$. □

Theorem 3.6. *The disease-free equilibrium $\mathcal{E}_0$, of the fractional model (2) is globally asymptotically stable when $\mathcal{R}_0 \leq 1$.*

Proof. Consider the Lyapunov function $\mathcal{L}_0(S, I, A)$ defined as follows:

$$\mathcal{L}_0(t) = \frac{1}{2}\left(S(t) - S_0 + I(t)\right)^2 + \frac{2\,\mu+\delta_1+\gamma}{\alpha}\,I(t).$$

Using Lemma 1 in [2], we can calculate the derivative of $\mathcal{L}_0$ along the solution of model (2), we have got that

$$\begin{aligned}
{}^{C}D_t^{\rho}\mathcal{L}_0(t) \leq{}& \left(S(t) - S_0 + I(t)\,\right)\left({}^{C}D_t^{\rho}S(t) + {}^{C}D_t^{\rho}I(t)\right) + \frac{2\,\mu+\delta_1+\gamma}{\alpha}\,{}^{C}D_t^{\rho}I(t),\\
={}&(S(t) - S_0 + I(t))[-\mu(S(t) - S_0) - (\mu+\delta_1+\gamma)I(t)]\\
&+ \frac{2\,\mu+\delta_1+\gamma}{\alpha}\left[\alpha(S(t) - S_0)I(t) - (\mu+\delta_1+\gamma-\alpha S_0)\,I\right],\\
={}&-\,\mu\,(S(t) - S_0)^2 - (\mu+\delta_1+\gamma)I^2(t) + \frac{2\,\mu+\delta_1+\gamma}{\alpha}\,(\mathcal{R}_0-1)\,I(t).
\end{aligned}$$

If $\mathcal{R}_0 \leq 1$, then we get

$${}^{C}D_t^{\rho}\mathcal{L}_0 = -\mu(S(t) - S_0)^2 - (\mu+\delta_1+\gamma)I^2 + \frac{2\,\mu+\delta_1+\gamma}{\alpha}\,(\mathcal{R}_0-1) < 0.$$

The globally asymptotic stability of the disease-free equilibrium is demonstrated by the Lyapunov stability theorem[34].

□

Theorem 3.7. *The endemic equilibrium $\mathcal{E}_1$, of the fractional model (2) is globally asymptotically stable when $\mathcal{R}_0 > 1$.*

Proof. We define the Lyapunov function as follows:

$$\mathcal{L}_1(t) = \frac{1}{2}\left(S - S^* + I - I^*\right)^2 + \frac{(2\mu + \delta_1 + \gamma)}{\alpha}\left(I - I^* - I^* \ln \frac{I}{I^*}\right) + \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)^2.$$

Using Lemma 1 in [2], and Lemma 3.1 in [33] , we find

$$\begin{aligned}
{}^C D_t^\rho \mathcal{L}_1 \leq & \left(S - S^* + I - I^*\right)\left({}^C D_t^\alpha S + {}^C D_t^\alpha I\right) + \frac{2\mu + \delta_1 + \gamma}{\alpha}\left(1 - \frac{I^*}{I}\right) {}^C D_t^\alpha I \\
& + 2 \times \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right) {}^C D_t^\alpha A, \\
= & \left(S - S^* + I - I^*\right)\left[\Lambda - \mu S - (\mu + \delta_1 + \gamma) I\right] \\
& + \frac{(2\mu + \delta_1 + \gamma)}{\alpha}\left(1 - \frac{I^*}{I}\right)\left[\alpha S I - (\mu + \delta_1 + \gamma) I\right] \\
& + 2 \times \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)\left[\gamma I - (\mu + \delta_2) A\right], \\
= & \left(S - S^* + I - I^*\right)\left[\mu S^* + (\mu + \delta_1 + \gamma) I^* - \mu S - (\mu + \delta_1 + \gamma) I\right] \\
& + 2 \times \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)\left[\gamma I - \gamma I^* + (\mu + \delta_2) A^* - (\mu + \delta_2) A\right] \\
& + \frac{2\mu + \delta_1 + \gamma)}{\alpha}\left(I - I^*\right)\left(\alpha S - \alpha S^*\right), \\
\leq & -\mu\left(S - S^*\right)^2 - (\mu + \delta_1 + \gamma)\left(I - I^*\right)^2 - 2 \times \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)^2}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)^2 \\
& + 2 \times \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)}{\gamma \mathcal{R}_0}\left(A - A^*\right)\left(I - I^*\right) \\
\leq & -\mu\left(S - S^*\right)^2 - (\mu + \delta_1 + \gamma)\left(I - I^*\right)^2 - 2 \times \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)^2}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)^2 \\
& + \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)^2}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)^2 + \frac{\mu + \delta_1 + \gamma}{\mathcal{R}_0}\left(I - I^*\right)^2, \\
\leq & -\mu\left(S - S^*\right)^2 - (\mu + \delta_1 + \gamma)(1 - \frac{1}{\mathcal{R}_0})\left(I - I^*\right)^2 - \frac{(\mu + \delta_1 + \gamma)(\mu + \delta_2)^2}{\gamma^2 \mathcal{R}_0}\left(A - A^*\right)^2.
\end{aligned}$$

Since $\mathcal{R}_0 > 1$, then

$${}^C D_t^\rho \mathcal{L}_1 < 0.$$

Therefore, $\mathcal{E}_1$ is globally asymptotically stable . □

4. Fractional model of HIV/AIDS whith migration

The HIV/AIDS model is extended in this part to account for the impact of migration, as shown in the following figure 2.

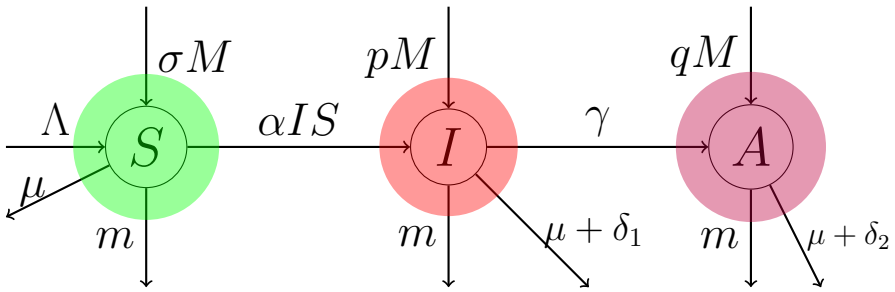


Figure 2: HIV/AIDS transmission pattern with migration

We use the same parameters as the previous model and add others:

- M represents the immigration of people into the host country.
- σ rate of migrants engaged in S.
- p rate of migrants enrolled in I.
- q rate of migrants enrolled in A.
- m migration rate at which the population leaves.

We assumed that a proportion of σ, p, and q were recruited from the $S(t)$, $I(t)$, and $A(t)$ populations, respectively. In addition, We assumed that immigrants in populations $S(t)$, $I(t)$, and $A(t)$ left the host country at a rate of m. Under these assumptions, we have the model described below:

$$\begin{cases} \dfrac{dS}{dt} = \sigma M + \Lambda - \alpha IS - (\mu + m)S \\ \dfrac{dI}{dt} = pM + \alpha IS - (\mu + m + \delta_1 + \gamma)I \\ \dfrac{dA}{dt} = qM + \gamma I - (\mu + m + \delta_2)A \end{cases} \tag{8}$$

Fractional model

By replacing the classical derivative in model (8) by the fractional derivative of Caputo, we obtain the following system

$$\begin{cases} {}^{C}D_t^{\rho} S = \sigma M + \Lambda - \alpha I S - (\mu + m) S \\ {}^{C}D_t^{\rho} I = pM + \alpha I S - (\mu + m + \delta_1 + \gamma) I \\ {}^{C}D_t^{\rho} A = qM + \gamma I - (\mu + m + \delta_2) A \end{cases} \tag{9}$$

4.1. Positivity and boundedness of solution

Theorem 4.1. *The region* $\Theta_1 = \{(S, I, A) \in \mathbb{R}^+; S \geq 0,\ I \geq 0, A \geq 0\}$, *is positivity invariant set for the system* (9).

Proof. Similar to the Theorem (3.1), we obtain that Θ_1 is positivity invariant set for the system (9). □

Theorem 4.2. *The region* $\Theta_2 = \{(S, I, A) \in \mathbb{R}^+; S \geq 0,\ I \geq 0, A \geq 0,$ $0 \leq N(t) \leq \dfrac{\Lambda + \sigma M}{\mu + m}$, *where* $N(t) = S(t) + I(t) + A(t)\ \}$ *is positivity invariant set for the system* (9).

Proof. Similar to the Theorem (3.2), we obtain that Θ_2 is positivity invariant set for the system (9).

□

Next, we will analyze the local and global stability of the equilibrium points associated with System 9 with the help of the basic reproduction number. As transportation has advanced, leading to an increase in migration and the movement of individuals from different areas contributing to the overall population movement, we will examine three scenarios of individual migration and their impact on disease transmission.

4.2. The absence of migration in the infected population

In this subsection, we will focus on the presence of migration only in the susceptible population. The migration within infected individuals is ignored ($p = q = 0$).

4.2.1. The equilibrium points

To obtain the equilibria, we solve the following equations;

$$\begin{cases} \Lambda + \sigma M - \alpha IS - (\mu + m)S = 0 \\ \alpha IS - (\mu + m + \delta_1 + \gamma)I = 0 \\ \gamma I - (\mu + m + \delta_2)A = 0 \end{cases} \tag{10}$$

Then, we obtain the disease free equilibrium DFE which is given by

$$\mathbb{E}_0 = (S_0, I_0, A_0), \tag{11}$$

Where $S_0 = \dfrac{\Lambda + \sigma M}{\mu + m}, I_0 = 0, A_0 = 0$. The basic reproduction number $\mathcal{R}_0$ obtained by the next generation which is given as follows :

$$\mathcal{R}_0 = \frac{\alpha S_0}{(\mu + m + \delta_1 + \gamma)} = \frac{\alpha\,(\Lambda + \sigma M)}{(\mu + m)(\mu + m + \delta_1 + \gamma)}. \tag{12}$$

The model (9) has a unique endemic equilibrium denoted by $\mathbb{E}_1 = (S^*, I^*, A^*)$ with,

$$S^* = \frac{\mu + m + \delta_1 + \gamma}{\alpha},\ I^* = \frac{\alpha\,(\sigma M + \Lambda) - (\mu + m)\,(\mu + m + \delta_1 + \gamma)}{\alpha\,(\mu + m + \delta_1 + \gamma)},\ A^* = \frac{\gamma\, I^*}{\mu + m + \delta_2}. \tag{13}$$

Theorem 4.3. *When $p = q = 0$, we have*

1. *If $\mathcal{R}_0 < 1$, then the fractional SIA model (9) has a unique disease-free equilibrium $\mathbb{E}_0$.*
2. *If $\mathcal{R}_0 \geq 1$, then the fractional SIA model (9) also has a unique endemic equilibrium $\mathbb{E}_1 = (S^*, I^*, R^*)$, where S^*, I^* and A^* are given in (13).*

4.2.2. The local stability analysis of the equilibrium points

The local stability of equilibria can be determined by examining the Jacobian matrix.

Theorem 4.4. *For model SIA fractional with $p = 0$ and $q = 0$, we get*

1 - *The disease-free equilibrium $\mathbb{E}_0$ is locally asymptotically stable when $\mathcal{R}_0 < 1$, and it is unstable when $\mathcal{R}_0 > 1$.*

2 - *The unique endemic equilibrium $E^* = (S^*, I^*, R^*)$ is locally asymptotically stable when $\mathcal{R}_0 > 1$.*

Proof. **1 -** The Jacobian matrix at equilibrium $\mathbb{E}_0$ of fractional model (10) can be presented as

$$\mathcal{J}(\mathbb{E}_0) = \begin{pmatrix} -(\mu+m) & -\alpha & 0 \\ 0 & \alpha S_0 - (\mu+m+\delta_1+\gamma) & 0 \\ 0 & \gamma & -(\mu+m+\delta_2) \end{pmatrix}$$

The eigenvalues of the matrix can be easily obtained:

$$\lambda_1 = -(\mu+m+\delta_2), \quad \lambda_2 = -(\mu+m), \quad \lambda_3 = \alpha S_0 - (\mu+m+\delta_1+\gamma).$$

If $\mathcal{R}_0 < 1$, the eigenvalues are all less than zero. This suggests that $\mathbb{E}_0$ is locally asymptotically stable.

2 - For the endemic equilibrium E^*, the Jacobian matrix is given by

$$\mathcal{J}(\mathbb{E}_1) = \begin{pmatrix} -(\mu+m)\,\mathcal{R}_0 & -(\mu+m+\delta_1+\gamma) & 0 \\ (\mu+m)\,(\mathcal{R}_0-1) & 0 & 0 \\ 0 & \gamma & -(\mu+m+\delta_2) \end{pmatrix}$$

To calculate the eigenvalues, we solve the equation

$$\det(\mathcal{J}(\mathbb{E}_1) - \lambda I_3) = 0.$$

We obtain,

$$(\mu+m+\delta_2+\lambda)(\lambda^2+(\mu+m)\,\mathcal{R}_0\lambda+(\mu+\delta_1+m+\gamma)(\mu+m)\,(\mathcal{R}_0-1)) = 0.$$

If $\Delta = ((\mu+m)\,\mathcal{R}_0)^2 - 4(\mu+m)(\mu+m+\delta_1+\gamma)(\mathcal{R}_0-1) > 0$, then

$$\lambda_1 = \frac{-((\mu+m)\,\mathcal{R}_0) - \sqrt{\Delta}}{2}; \quad \lambda_2 = \frac{-((\mu+m)\,\mathcal{R}_0) + \sqrt{\Delta}}{2}; \quad \lambda_3 = -(\mu+m+\delta_2).$$

If $\mathcal{R}_0 > 1$, then $\lambda_1 < 0$, $\lambda_2 < 0$ and $\lambda_3 < 0$. Thus all eigenvalues have negative real parts. Hence the endemic equlibrium of system (9) is locally asymptotically stable for all $\rho \in (0, 1]$.

□

4.2.3. The global stability analysis of the equilibrium points

Theorem 4.5. *If $p = q = 0$ and $\mathcal{R}_0 < 1$, then the disease-free equilibrium $\mathbb{E}_0$ of model (9) is globally asymptotic stable.*

Proof. We consider the following Lyapunov function

$$\mathbb{L}_0(t)) = \frac{1}{2}\left(S(t) - S_0 + I(t)\right)^2 + \frac{2(\mu+m)+\delta_1+\gamma}{\alpha} I(t).$$

Using Lemma 1 in [2] ,we obtain,

$$\begin{aligned}
{}^{C}D_t^{\rho}\mathbb{L}_0(t) \leq & \left(S(t) - S_0 + I(t)\right)\left({}^{C}D_t^{\rho}S(t) + {}^{C}D_t^{\rho}I(t)\right) + \frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\,{}^{C}D_t^{\rho}I(t), \\
=&(S(t) - S_0 + I(t))[-(\mu+m)(S(t)-S_0) - (\mu+m+\delta_1+\gamma)I(t)] \\
&+ \frac{2\left(\mu+m\right)+\delta_1+\gamma}{\alpha}\left[\alpha(S(t)-S_0)I(t) - (\mu+\delta_1+\gamma-\alpha S_0)\, I\right], \\
=&-(\mu+m)(S(t)-S_0)^2 - (\mu+m+\delta_1+\gamma)I^2 + \frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(\mathcal{R}_0 - 1\right) I(t),
\end{aligned}$$

If $\mathcal{R}_0 < 1$, then we find

$${}^{C}D_t^{\rho}\mathbb{L}_0(t) = -(\mu+m)(S(t)-S_0)^2 - (\mu+m+\delta_1+\gamma)I^2(t) + \frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(\mathcal{R}_0 - 1\right) I(t) < 0.$$

as the theorem is established. □

The following theorem is used to determine the global stability of the endemic equilibrium $\mathbb{E}_1$.

Theorem 4.6. *If $p = q = 0$ and $\mathcal{R}_0 > 1$, then the endemic equilibrium $\mathbb{E}_1$ of the model (9) is globally asymptotically stable.*

Proof. We define the Lyapunov function as follows:

$$\begin{aligned}
\mathcal{L}_1(t) =& \frac{1}{2}\left(S - S^* + I - I^*\right)^2 + \frac{\left(2(\mu+m)+\delta_1+\gamma\right)}{\alpha}\left(I - I^* - I^* \ln\frac{I}{I^*}\right) \\
&+ \frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)}{\gamma^2\,\mathcal{R}_0}\left(A - A^*\right)^2.
\end{aligned}$$

Using Lemma 1 in [2], and Lemma 3.1 in [33] , we find

$$
\begin{aligned}
{}^{C}D_t^{\rho}\mathcal{L}_1(t) \leq\ & (S-S^*+I-I^*)\left({}^{C}D_t^{\rho}S+{}^{C}D_t^{\rho}I\right)+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(1-\frac{I^*}{I}\right){}^{C}D_t^{\rho}I\\
&+2\times\frac{(\mu+\delta_1+\gamma)(\mu+\delta_2)}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)\,{}^{C}D_t^{\rho}A,\\
=\ & (S-S^*+I-I^*)\left[\Lambda+\sigma M-(\mu+m)\,S-(\mu+m+\delta_1+\gamma)I\right]\\
&+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(1-\frac{I^*}{I}\right)\left[\alpha SI-(\mu+m+\delta_1+\gamma)I\right]\\
&+2\times\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)\,\left[\gamma\,I-(\mu+m+\delta_2)\,A\right],\\
=\ & (S-S^*+I-I^*)\left[\,\mu\,S^*+(\mu+m+\delta_1+\gamma)I^*-\mu\,S-(\mu+m+\delta_1+\gamma)I\,\right]\\
&+2\times\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)\,\left[\gamma\,I-\gamma\,I^*+(\mu+m+\delta_2)A^*-(\mu+m+\delta_2)\,A\right]\\
&+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\,(I-I^*)\,(\alpha S-\alpha S^*)\,,\\
\leq\ & -(\mu+m)\,(S-S^*)^2-(\mu+m+\delta_1+\gamma)\,(I-I^*)^2\\
&-2\times\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)^2}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)^2\\
&+2\times\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)}{\gamma\,\mathcal{R}_0}\,(A-A^*)\,(I-I^*)\,,\\
\leq\ & -(\mu+m)\,(S-S^*)^2-(\mu+m+\delta_1+\gamma)\,(I-I^*)^2+\frac{\mu+m+\delta_1+\gamma}{\mathcal{R}_0}\,(I-I^*)^2\\
&-2\times\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)^2}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)^2+\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)^2}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)^2\,,\\
\leq\ & -(\mu+m)\,(S-S^*)^2-(\mu+m+\delta_1+\gamma)(1-\frac{1}{\mathcal{R}_0})\,(I-I^*)^2\\
&-\frac{(\mu+m+\delta_1+\gamma)(\mu+m+\delta_2)^2}{\gamma^2\,\mathcal{R}_0}\,(A-A^*)^2\,.
\end{aligned}
$$

Since $\mathcal{R}_0>1$, then

$$
{}^{C}D_t^{\rho}\mathcal{L}_1(t)<0.
$$

Therefore, $\mathcal{E}_1$ is globally asymptotically stable . □

4.3. The presence of migration in the HIV-infected population

We limit our selves to the case where individuals infected with HIV virus are able to migrate with the rate ($p>0$). The infected individuals with AIDS are less likely to migrate or travel $q=0$.

4.3.1. The equilibrium points

If $p = 0$ and $q > 0$. In this case, the existence of an equilibrium for the model (9) are given as follows.

Theorem 4.7. *When $p = 0$ and $q > 0$, the model (9) has one endemic equilibrium $E^* = (S^*, I^*, A^*)$, and also If $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m) > 0$, the model (9) has two endemic equilibriums $E^* = (S^*, I^*, A^*)$ and $E^{**} = (S^{**}, I^{**}, A^{**})$, where*

$$S^* = \frac{\Lambda+\sigma M}{\mu+m}, \qquad I^* = 0, \qquad A^* = \frac{qM}{\mu+m+\delta_2},$$

$$S^{**} = \frac{\Lambda+\sigma M-(\mu+\delta_1+m+\gamma)I^*}{\mu+m}, I^{**} = \frac{\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m)}{\alpha(\mu+\delta_1+m+\gamma)},$$
$$A^{**} = \frac{qM+\gamma\, I^{**}}{\mu+m+\delta_2}.$$

Proof. Any equilibrium of model (9) can be obtained as

$$\begin{aligned} \Lambda+\sigma M-\alpha IS-(\mu+m)S &= 0,\\ \alpha IS-(\mu+m+\delta_1+\gamma)I &= 0,\\ qM+\gamma I-(\mu+m+\delta_2)A &= 0, \end{aligned} \tag{14}$$

All solutions of equation (14) satisfy $I \neq 0$ and $A \neq 0$, indicating that model (9) does not have a disease-free equilibrium. When we add the first two equations in the system (14), we get

$$S = \frac{\Lambda+\sigma M-(\mu+\delta_1+m+\gamma)I}{\mu+m}, \tag{15}$$

We replace (15) in equation two of the system (17), we obtain

$$-\alpha(\mu+\delta_1+m+\gamma)I^2+(\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m))I = 0, \tag{16}$$

If $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m) > 0$, then the equation (16) admits two positive solutions which we note $E^* = (S^*, I^*, A^*)$ and $E^{**} = (S^{**}, I^{**}, A^{**})$. □

4.3.2. The local stability analysis of the equilibrium points

Theorem 4.8. *When $p = 0$ and $q > 0$,*

1- *If $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) > 0$, the endemic equilibrium $E^* = (S^*, I^*, A^*)$ of model (9) is locally asymptotically stable*

2- *If $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) > 0$, the endemic equilibrium $E^{**} = (S^{**}, I^{**}, A^{**})$ is locally asymptotically stable*

Proof. **1-** For the endemic equilibrium E^*, the Jacobian matrix is given by

$$\mathcal{J}(E^*) = \begin{pmatrix} -(\mu + m) & -\alpha S^* & 0 \\ 0 & \alpha S^* - (\mu + \delta_1 + m + \gamma) & 0 \\ 0 & \gamma & -(\mu + m + \delta_2) \end{pmatrix}$$

By some calculations, we obtain the characteristic roots are:

$$\lambda_1 = -(\mu + m), \quad \lambda_2 = \alpha S^* - (\mu + \delta_1 + m + \gamma), \quad \lambda_3 = -(\mu + m + \delta_2).$$

Evident, $\lambda_1 < 0$ and $\lambda_3 < 0$.
If $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) > 0$, then $\lambda_2 < 0$. Thus, we have $|\arg(\lambda_i)| > \dfrac{\rho\,\pi}{2}\,(i = 1, 2, 3)$. Hence the stability of the endemic equilibrium E^{**}.

2- For the endemic equilibrium E^{**}, the Jacobian matrix is given by

$$\mathcal{J}(E^{**}) = \begin{pmatrix} -\alpha I^{**} - (\mu + m) & -\alpha S^{**} & 0 \\ \alpha I^{**} & \alpha S^{**} - (\mu + \delta_1 + m + \gamma) & 0 \\ 0 & \gamma & -(\mu + m + \delta_2) \end{pmatrix}$$

$$\det(\lambda - \mathcal{J}(E^{**})) = \begin{pmatrix} -\alpha I^{**} - (\mu + m) - \lambda & -\alpha S^{**} \\ \alpha I^{**} & \alpha S^{**} - (\mu + \delta_1 + m + \gamma) - \lambda \\ 0 & \gamma \end{pmatrix}$$

We search for the eigenvalues of the Jacobian matrix $\mathcal{J}(E^{**})$ as follows:

$$\det(\lambda - \mathcal{J}(E^{**})) = 0.$$

$$\lambda^2 + \lambda(\alpha I^* + \mu + m + (\mu + \delta_1 + m + \gamma) - \alpha S^*) + \alpha^2 S^* I^* + (\alpha I^* + \mu + m)(\mu + \delta_1 + m + \gamma - \alpha S^*) = 0,$$

If $\Delta_1 = (\alpha I^* + \mu + m + (\mu + \delta_1 + m + \gamma) - \alpha S^*)^2 - 4(\alpha^2 S^* I^* + (\alpha I^* + \mu + m)(\mu + \delta_1 + m + \gamma - \alpha S^*)) > 0$, then we have

$$\lambda_1 = \frac{-(\alpha I^* + \mu + m + (\mu + \delta_1 + m + \gamma) - \alpha S^*) - \sqrt{\Delta_1}}{2},$$
$$\lambda_2 = \frac{-(\alpha I^* + \mu + m + (\mu + \delta_1 + m + \gamma) - \alpha S^*) + \sqrt{\Delta_1}}{2},$$

If $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m) > 0$, then $\lambda_1 < 0$ and $\lambda_2 < 0$.Thus, we have $|\arg(\lambda_i)| > \frac{\rho\,\pi}{2}$ $(i = 1, 2)$. Hence, the stability of the endemic equilibrium E^{***}.

□

4.3.3. The global stability analysis of the equilibrium points

Theorem 4.9. *When $p > 0$ and $q = 0$, and $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m) > 0$ the endemic equilibrium E^* of model (9) is globally asymptotically stable.*

Proof. We define the Lyapunov function as follows:

$$\mathcal{L}_2(t) = \frac{1}{2}(S - S^* + I - I^*)^2 + \frac{2(\mu + m) + \delta_1 + \gamma}{\alpha}\left(I - I^* - I^* \ln \frac{I}{I^*}\right) + C_1 (A - A^*)^2,$$

With

$$C_1 = \frac{(\mu + m + \delta_2)(\mu + m + \delta_1 + \gamma)^2(\mu + m)}{\alpha\gamma^2(\Lambda + \sigma M)}.$$

By using lemma 1 in [2], and lemma 3.1 in [33] we obtain

$$
\begin{aligned}
{}^{C}D_t^{\rho}\mathcal{L}_2 \leq & \left(S-S^*+I-I^*\right)\left({}^{C}D_t^{\rho}S+{}^{C}D_t^{\rho}I\right)+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(1-\frac{I^*}{I}\right){}^{C}D_t^{\rho}I \\
& +2\,C_1\left(A-A^*\right){}^{C}D_t^{\rho}A, \\
= & \left(S-S^*+I-I^*\right)\left[\sigma M+pM+\Lambda-(\mu+m)\,S-(\mu+m+\delta_1+\gamma)I\right] \\
& +\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(1-\frac{I^*}{I}\right)\left[pM+\alpha SI-(\mu+m+\delta_1+\gamma)I\right] \\
& +2\,C_1\left(A-A^*\right)\left[\gamma\,I-(\mu+m+\delta_2)\,A\right] \\
= & \left(S-S^*+I-I^*\right)\left[\,(\mu+m)(S^*-S)+(\mu+m+\delta_1+\gamma)(I^*-I)\right] \\
& +2\,C_1\left(A-A^*\right)\left[\gamma\,I-\gamma\,I^*+(\mu+m+\delta_2)A^*-(\mu+m+\delta_2)\,A\right] \\
& +\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(I-I^*\right)\left(\alpha S-\alpha S^*\right)-pM\frac{2(\mu+m)+\delta_1+\gamma}{\alpha I\,I^*}\left(I-I^*\right)^2, \\
\leq & -(\mu+m)\,\left(S-S^*\right)^2-(\mu+m+\delta_1+\gamma)\left(I-I^*\right)^2-pM\frac{2(\mu+m)+\delta_1+\gamma}{\alpha I\,I^*}\left(I-I^*\right)^2 \\
& +2\times C_1\gamma\,\left(A-A^*\right)\left(I-I^*\right)-2\times C_1(\mu+m+\delta_2)\left(A-A^*\right)^2, \\
\leq & -(\mu+m)\,\left(S-S^*\right)^2-(\mu+m+\delta_1+\gamma)\left(I-I^*\right)^2-2\times C_1(\mu+m+\delta_2)\left(A-A^*\right)^2 \\
& -pM\frac{2(\mu+m)+\delta_1+\gamma}{\alpha I\,I^*}\left(I-I^*\right)^2 \\
& +C_1(\mu+m+\delta_2)\left(A-A^*\right)^2+\frac{C_1\gamma^2}{(\mu+m+\delta_2)}\left(I-I^*\right)^2, \\
\leq & -(\mu+m)\,\left(S-S^*\right)^2-C_1(\mu+m+\delta_2)\left(A-A^*\right)^2 \\
& -(\mu+m+\delta_1+\gamma)\frac{\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m)}{\alpha(\Lambda+\sigma M)}\left(I-I^*\right)^2,
\end{aligned}
$$

If we have $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m)\geq 0$, then E^* is globally asymptotically stable. □

Theorem 4.10. *When $p>0$ and $q=0$, and $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m)>0$ the endemic equilibrium E^{**} of model (9) is globally asymptotically stable.*

Proof. As in the proof of the previous theorem, it is sufficient to replace the Lyapunov function with this function.

$$
\mathcal{L}_3(t)=\frac{1}{2}\left(S-S^{**}+I-I^{**}\right)^2+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(I-I^{**}-I^{**}\ln\frac{I}{I^{**}}\right)+C_2\left(A-A^{**}\right)^2.
$$

With

$$
C_2=\frac{(\mu+m+\delta_2)(\mu+m+\delta_1+\gamma)^2(\mu+m)}{\alpha\gamma^2(\Lambda+\sigma M)}.
$$

□

4.4. The presence of migration in the both HIV and AIDS infected populations

Here we extend the migration to the whole model and reveal its influence on the transmission of HIV and AIDS viruses ($p > 0, q > 0$).

4.4.1. The equlibrium points

If $p > 0$ and $q > 0$. In the latter case, the existence of equilibrium for model (9) is obtained as follows.

Theorem 4.11. *When $p > 0$, $q > 0$ and if $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) > 0$, the model (9) admits a unique endemic equilibrium, denoted by $E^* = (S^*, I^*, A^*)$, where*

$$S^* = \frac{(\mu + \delta_1 + m + \gamma)(\mu + m)(B + 1) + \alpha\, p\, m - \sqrt{C}}{2\,\alpha\,(\mu + m)},$$

$$I^* = \frac{(\mu + \delta_1 + m + \gamma)(\mu + m)(B - 1) + \alpha\, p\, m + \sqrt{C}}{2\,\alpha\,(\mu + \delta_1 + m + \gamma)}, \quad A^* = \frac{qM + \gamma I^*}{\mu + m + \delta_2}.$$

with

$$B = \frac{\alpha(\Lambda + \sigma M)}{(\mu + \delta_1 + m + \gamma)(\mu + m)} > 1,$$

$$C = (\alpha(\Lambda + (p + \sigma)m) - (\mu + \delta_1 + m + \gamma)(\mu + m))^2 + 4\alpha p m(\mu + \delta_1 + m + \gamma)(\mu + m).$$

Proof. whene Any equilibrium of model (9) can be found as follows

$$\begin{aligned} \sigma M + \Lambda - \alpha IS - (\mu + m)S &= 0, \\ pM + \alpha IS - (\mu + m + \delta_1 + \gamma)I &= 0, \\ qM + \gamma I - (\mu + m + \delta_2)A &= 0. \end{aligned} \tag{17}$$

It is simple to see that none of the solutions of equation (17) can have $I = 0$ and $A = 0$, indicating that model (9) has no disease-free equilibrium. This solution is noted by $E^* = (S^*, I^*, A^*)$. After some calculations, we find

$$\begin{aligned} -\alpha(\mu + \delta_1 + m + \gamma){I^*}^2 + (\alpha(\Lambda + (\sigma + p)m) - (\mu + \delta_1 + m + \gamma)(\mu + m))I^* \\ + 4\,\alpha p\, m(\mu + \delta_1 + m + \gamma)(\mu + m) = 0, \end{aligned} \tag{18}$$

If $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) > 0$, the equation (18) admits one positive solution I^*. □

4.4.2. The local stability analysis of the equilibrium points

The following theorem allows to study the local stability of the endemic equilibrium of model (9) in the case when $p > 0$ and $q > 0$.

Theorem 4.12. *When $p > 0$, $q > 0$, and if $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) > 0$, the endemic equilibrium $E^* = (S^*, I^*, A^*)$ of model (9) is locally asymptotically stable.*

Proof. As the second part of the theorem (4.8), we obtain that $E^* = (S^*, I^*, A^*)$ of model (9) is locally asymptotically stable. (9). □

4.4.3. The global stability analysis of the equilibrium points

Theorem 4.13. *When $p > 0$ and $q > 0$, and $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m) > 0$ the endemic equilibrium E^* of model (9) is globally asymptotically stable.*

Proof. We define the Lyapunov function as follows:

$$\mathcal{L}_4(t) = \frac{1}{2}\left(S - S^* + I - I^*\right)^2 + \frac{2(\mu + m) + \delta_1 + \gamma}{\alpha}\left(I - I^* - I^* \ln \frac{I}{I^*}\right) + C_3\left(A - A^*\right)^2.$$

With

$$C_3 = \frac{(\mu + m + \delta_2)(\mu + m + \delta_1 + \gamma)^2(\mu + m)}{\alpha\gamma^2(\Lambda + \sigma M)}.$$

By using lemma 1 in [2], and lemma 3.1 in [33] we have

$$
\begin{aligned}
{}^{C}D_t^{\rho}\mathcal{L}_4(t) \leq\ & (S-S^*+I-I^*)\left({}^{C}D_t^{\rho}S+{}^{C}D_t^{\rho}I\right)+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(1-\frac{I^*}{I}\right){}^{C}D_t^{\rho}I\\
&+2\,C_1\,(A-A^*)\,{}^{C}D_t^{\rho}A,\\
=\ & (S-S^*+I-I^*)\,[\sigma M+pM+\Lambda-(\mu+m)\,S-(\mu+m+\delta_1+\gamma)I]\\
&+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\left(1-\frac{I^*}{I}\right)[pM+\alpha SI-(\mu+m+\delta_1+\gamma)I]\\
&+2\,C_1\,(A-A^*)\,[\gamma\,I-(\mu+m+\delta_2)\,A],\\
=\ & (S-S^*+I-I^*)\,[\,(\mu+m)(S^*-S)+(\mu+m+\delta_1+\gamma)(I^*-I)]\\
&+2\,C_1\,(A-A^*)\,[\gamma\,I-\gamma\,I^*+(\mu+m+\delta_2)A^*-(\mu+m+\delta_2)\,A]\\
&+\frac{2(\mu+m)+\delta_1+\gamma}{\alpha}\,(I-I^*)\,(\alpha S-\alpha S^*)-pM\frac{2(\mu+m)+\delta_1+\gamma}{\alpha I\,I^*}\,(I-I^*)^2\,,\\
\leq\ & -(\mu+m)\,(S-S^*)^2-(\mu+m+\delta_1+\gamma)\,(I-I^*)^2-pM\frac{2(\mu+m)+\delta_1+\gamma}{\alpha I\,I^*}\,(I-I^*)^2\\
&+2\times C_1\gamma\,(A-A^*)\,(I-I^*)-2\times C_1(\mu+m+\delta_2)\,(A-A^*)^2\,,\\
\leq\ & -(\mu+m)\,(S-S^*)^2-(\mu+m+\delta_1+\gamma)\,(I-I^*)^2-2\times C_1(\mu+m+\delta_2)\,(A-A^*)^2\\
&-pM\frac{2(\mu+m)+\delta_1+\gamma}{\alpha I\,I^*}\,(I-I^*)^2\\
&+C_1(\mu+m+\delta_2)\,(A-A^*)^2+\frac{C_1\gamma^2}{(\mu+m+\delta_2)}\,(\,I-I^*)^2\,,\\
\leq\ & -(\mu+m)\,(S-S^*)^2-C_1(\mu+m+\delta_2)\,(A-A^*)^2\\
&-(\mu+m+\delta_1+\gamma)\frac{\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m)}{\alpha(\Lambda+\sigma M)}\,(I-I^*)^2\,.
\end{aligned}
$$

Since $\alpha(\Lambda+\sigma M)-(\mu+\delta_1+m+\gamma)(\mu+m)\geq 0$, then

$$
{}^{C}D_t^{\rho}\mathcal{L}_4(t)<0.
$$

Therefore, E^* is globally asymptotically stabe . □

5. Numerical resolution of FDEs using the Adams-Bashforth-Moulton fractional technique.

In this section, we are interested in the fractional Adams-Bashforth-Moulton numerical technique that we will use when simulating fractional order equations of the Caputo type. This technique was introduced and discussed by K.Diethelm and A. D. Freed [8].

To be specific, we first consider the fractional differential equation.

$$
\begin{cases}
{}^{C}D_t^{\rho}(y(t))=h(t,y(t)), \quad 0<\rho<1\\
y^{(0)}(0)=y_0,
\end{cases}
\tag{19}
$$

Equation (19) is equivalent to the Volterra integral equation in the Caputo sense.

$$y(x) = \sum_{k=0}^{\lceil \rho \rceil - 1} y_0^{(k)} \frac{x^k}{k!} + \frac{1}{\Gamma(\rho)} \int_0^x (x-t)^{\rho-1} h(t, y(t)) \mathrm{d}t. \tag{20}$$

The method is explained in the manner below.

Let $h = \dfrac{T}{\hat{m}}, t_n = nh, n = 0, 1, 2, \ldots, \hat{m}$.

The formula for the Adams Bashforth fractional method corrector is given as follows

$$y_{n+1} = \sum_{k=0}^{\lceil \rho \rceil - 1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{h^\rho}{\Gamma(\rho+2)} \sum_{j=0}^{n} a_{j,k+1} \, h\left(x_j, y_j\right) + \frac{h^\rho}{\Gamma(\rho+2)} \, h\left(x_{k+1}, y_{k+1}^p\right), \tag{21}$$

Predictor formulae for $y_h^{\mathrm{P}}\left(t_{n+1}\right)$ is given by the Adams Bashforth fractional method

$$y_{n+1}^{\mathrm{P}} = \sum_{k=0}^{\lceil \alpha \rceil - 1} \frac{t_{n+1}^k}{k!} y_0^{(k)} + \frac{1}{\Gamma(\rho)} \sum_{j=0}^{n} b_{j,n+1} \, h\left(x_j, y_j\right), \tag{22}$$

where

$$a_{j,n+1} = \begin{cases} n^{\rho+1} - (n-\rho)(n+1)^\rho, & \text{if } j = 0 \\ (n-j+2)^{\rho+1} + (n-j)^{\rho+1} - 2(n-j+1)^{\rho+1}, & \text{if } 0 < j < n, \\ 1, & \text{if } j = n \end{cases}$$

and

$$b_{j,n+1} = \frac{h^\rho}{\rho} \left[(n+1-j)^\rho - (n-j)^\rho\right], 0 \leq j \leq n.$$

5.1. Numerical method for model without migration

In this subsection, we study the numerical solution of a fractional order SIA model without migration using the Adam-Bashforth-Moulton predictor-

corrector scheme. We obtain the following scheme;

$$\begin{aligned}
S_{n+1} =& S_0 + \frac{h^\rho}{\Gamma(\rho+2)}\left(\Lambda - \alpha S^p_{n+1} I^p_{n+1} - \mu S^p_{n+1}\right) + \frac{h^\rho}{\Gamma(\rho+2)}\sum_{j=0}^{n}\alpha_{j,n+1}\left(\Lambda - \alpha S_j I_j - \mu S_j\right),\\
I_{n+1} =& I_0 + \frac{h^\rho}{\Gamma(\rho+2)}\left(\alpha S^p_{n+1} I^p_{n+1} - (\mu+\delta_1+\gamma) I^p_{n+1}\right)\\
&+ \frac{h^\rho}{\Gamma(\rho+2)}\sum_{j=0}^{n}\alpha_{j,n+1}\left(\alpha S_j I_j - (\mu+\delta_1+\gamma) I_j\right),\\
A_{n+1} =& A_0 + \frac{h^\rho}{\Gamma(\rho+2)}\left(\gamma I^p_{n+1} - (\mu+\delta_2) A^p_{n+1}\right) + \frac{h^\rho}{\Gamma(\rho+2)}\sum_{j=0}^{n}\alpha_{j,n+1}\left(\gamma I_j - (\mu+\delta_2) A_j\right),
\end{aligned} \tag{23}$$

Where

$$\begin{aligned}
S^p_{n+1} &= S_0 + \frac{1}{\Gamma(\rho)}\sum_{j=0}^{n} b_{j,n+1}\left(\Lambda - \alpha S_j I_j - \mu S_j\right),\\
I^p_{n+1} &= I_0 + \frac{1}{\Gamma(\rho)}\sum_{j=0}^{n} b_{j,n+1}\left(\alpha S_j I_j - (\mu+\delta_1+\gamma) I_j\right),\\
A^p_{n+1} &= A_0 + \frac{1}{\Gamma(\rho)}\sum_{j=0}^{n} b_{j,n+1}\left(\gamma I_j - (\mu+\delta_2) A_j\right),
\end{aligned} \tag{24}$$

5.2. Numerical method for model with migration

We use the Adam-Bashforth-Moulton predictor-corrector scheme and with the same notations of the previous section we find the numerical scheme of

the SIA model with migration

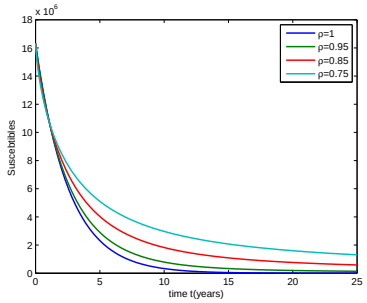
$$
\begin{aligned}
S_{n+1} =& S_0 + \frac{h^\rho}{\Gamma(\rho+2)}\left(\Lambda + \sigma M - \alpha S^p_{n+1} I^p_{n+1} - (\mu+m) S^p_{n+1}\right) \\
&+ \frac{h^\rho}{\Gamma(\rho+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(\Lambda + \sigma M - \alpha S_j I_j - (\mu+m) S_j\right), \\
I_{n+1} =& I_0 + \frac{h^\rho}{\Gamma(\rho+2)}\left(pM + \alpha S^p_{n+1} I^p_{n+1} - (\mu+m+\delta_1+\gamma) I^p_{n+1}\right) \\
&+ \frac{h^\rho}{\Gamma(\rho+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(pM + \alpha S_j I_j - (\mu+m+\delta_1+\gamma) I_j\right), \\
A_{n+1} =& A_0 + \frac{h^\rho}{\Gamma(\rho+2)}\left(qM + \gamma I^p_{n+1} - (\mu+m+\delta_2) A^p_{n+1}\right) \\
&+ \frac{h^\rho}{\Gamma(\rho+2)} \sum_{j=0}^{n} \alpha_{j,n+1}\left(qM + \gamma I_j - (\mu+m+\delta_2) A_j\right),
\end{aligned}
\tag{25}
$$

With

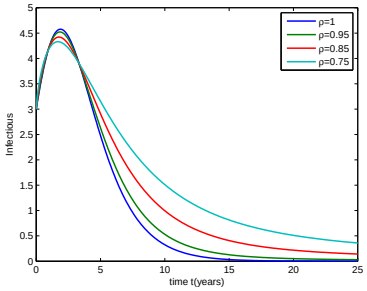
$$
\begin{aligned}
S^p_{n+1} &= S_0 + \frac{1}{\Gamma(\rho)} \sum_{j=0}^{n} b_{j,n+1}\left(\Lambda + \sigma M - \alpha S_j I_j - (\mu+m) S_j\right), \\
I^p_{n+1} &= I_0 + \frac{1}{\Gamma(\rho)} \sum_{j=0}^{n} b_{j,n+1}\left(pM + \alpha S_j I_j - (\mu+m+\delta_1+\gamma) I_j\right), \\
A^p_{n+1} &= A_0 + \frac{1}{\Gamma(\rho)} \sum_{j=0}^{n} b_{j,n+1}\left(qM + \gamma I_j - (\mu+m+\delta_2) A_j\right),
\end{aligned}
\tag{26}
$$

5.3. Numerical sumulations and discussions

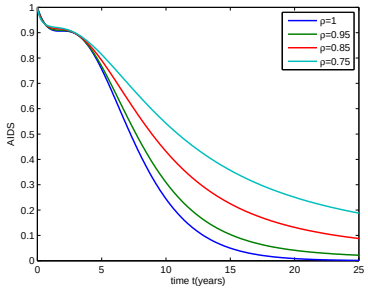
In this section, we perform numerical simulations to evaluate and validate the theoretical results of the two models presented in (2) and (9). We used the schemes obtained in (23) and (25), to show the influence of the fractional order. We took the initial values $N = 16329400$, $S(0) = 16329396$, $I(0) = 3$, $A(0) = 1$, $M = 435408$ [3], and the parameters values indicated in Tables 1 and 2.The graphs presented in Figures 3a,3b and 3c, show the curves of model solutions S(t), I(t), and A(t) taking different values of fractional order. We can see from Figures 3a and 3c, that when ρ is close to 1, the number of the people susceptible and those infected with AIDS increases.



(a) Susceptible population



(b) HIV-affected population



(c) Population affected by AIDS

Figure 3: Dynamics behavior of model without migration 2 as a function of time (years) for different ρ via Adams-Bashforth-Moulton fractional method.

We notice through Figure 3b that the increase of the fractional order allows to make the peak of the disease larger and takes a maximum value higher than those in the cases where the fractional order ρislessthan1.

The graphical representations of Figure 3b lead us to conclude that it is crucial to use fractional derivatives since, contrary to the cases where $\rho = 0.99, 0.85, 0.75$, convergence is rapid when classical derivatives are used. Therfore differential equations with fractional order derivative exhibit rich dynamics and reflect bilogical systems more accurately than conventional integer order models.

In model (2), we have $\mathcal{R}_0 < 1$, and the equilibrium point is $\mathbb{E}_0 = (1.0816 \times 10^{-09}, 0, 0)$. Figure 3 shows that the solution of model (2) converge to its equilibrium point for all orders derivatives. However, the disease-free equilibrium $\mathbb{E}_0$ is golobally asymptotically stable, which verifies the theoretical results.

Parameters	Value	Source
α	5.9921e-08	[3]
γ	7.7416e-02	[3]
μ	0.0069	[3]
Λ	4.2249e-10	[3]
δ_1	1.7396e-12	[3]
δ_2	4.8233e-07	[3]

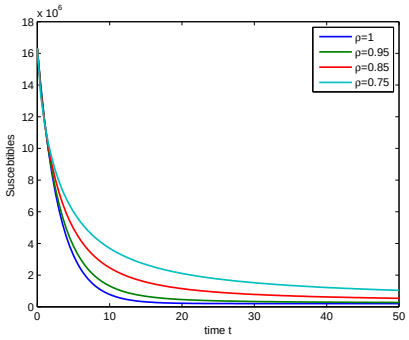
Table 1: The parameters values used in the model without migration.

Parameters	Value	Source
α	9.7847e-01	[3]
γ	7.7416e-02	[3]
μ	3.9060e-01	[3]
Λ	4.2249e-10	[3]
δ_1	1.7396e-12	[3]
δ_2	4.8233e-07	[3]
σ	1.5572 e-01	[3]
p	6.3773 e-04	[3]
q	1.6611 e-08	[3]
m	3.0110 e-02	[3]

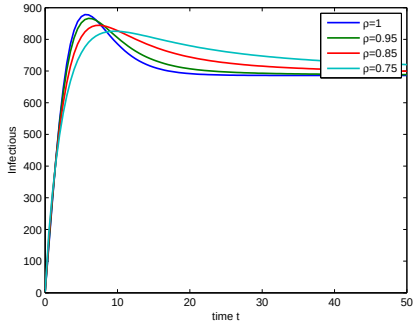
Table 2: The parameters values used in the model with migration

In Figure 4, the sub-plots show the population of susceptible individuals, HIV-infected individuals, and individuals with AIDS with migration. By reducing the value of ρ, the population of HIV-infected and AIDS-infected individuals increases effectively while the population of susceptible individuals decreases.

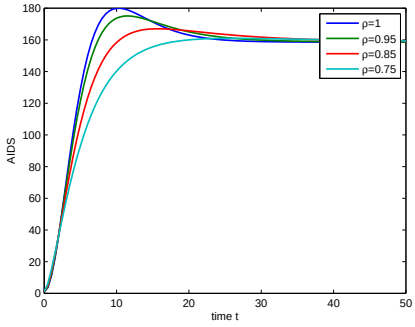
We calculate the condition of stability indicated in Theorem 7 for the model with migration, we find that $\alpha(\Lambda + \sigma M) - (\mu + \delta_1 + m + \gamma)(\mu + m) = -0.1355 < 0$ which shows the instability of the endemic equilibrium. In Figures (4a), (4c) and (4b), we see that model (9) does not converge toward the equilibrium that validates the previous theory. The illustrations above show that for different values of ρ, the curves converge to other points $Em = (2.0248 \times 10^5, 675.9010, 156.2623)$ than the equilibrium point, which confirms our analytical results.



(a) Susceptible population



(b) HIV-affected population



(c) Population affected by AIDS

Figure 4: Dynamics behavior of model with migration (9) as a function of time (years) for different ρ via Adams-Bashforth-Moulton fractional method.

6. Sensitivity analysis

To identify the variables that have the most significant impact on the reproduction number $\mathcal{R}_0$, we computed the sensitivity indices of the reproduction number $\mathcal{R}_0$ for the model (2). To decrease the prevalence of the disease, we utilize the sensitivity index, which is defined as described in [6].

Definition 6.1. *[6] The normalized forward sensitivity index of a variable, $\mathcal{R}_0$, that depends differentiably on a parameter p is defined as:*

$$\Upsilon_p^{\mathcal{R}_0} := \frac{\partial \mathcal{R}_0}{\partial p} \times \frac{p}{\mathcal{R}_0}$$

With respect to the parameters α, Λ, μ, δ_1 and γ , we calculate the

sensitivity index for $\mathcal{R}_0$ obtained in (6).

$$\Upsilon_{\alpha}^{\mathcal{R}_0} := \frac{\partial \mathcal{R}_0}{\partial \alpha} \times \frac{\alpha}{\mathcal{R}_0} = \frac{\Lambda\,\alpha}{\mu\,(\mu+\delta_1+\gamma)} \times \frac{1}{\mathcal{R}_0} = 1,$$
$$\Upsilon_{\lambda}^{\mathcal{R}_0} := \frac{\partial \mathcal{R}_0}{\partial \Lambda} \times \frac{\Lambda}{\mathcal{R}_0} = \frac{\Lambda\,\alpha}{\mu\,(\mu+\delta_1+\gamma)} \times \frac{1}{\mathcal{R}_0} = 1,$$
$$\Upsilon_{\mu}^{\mathcal{R}_0} := \frac{\partial \mathcal{R}_0}{\partial \mu} \times \frac{\mu}{\mathcal{R}_0} = -\frac{2\mu+\delta_1+\gamma}{\mu+\delta_1+\gamma},$$
$$\Upsilon_{\delta_1}^{\mathcal{R}_0} := \frac{\partial \mathcal{R}_0}{\partial \delta_1} \times \frac{\delta_1}{\mathcal{R}_0} = -\frac{\delta_1}{\mu+\delta_1+\gamma},$$
$$\Upsilon_{\gamma}^{\mathcal{R}_0} := \frac{\partial \mathcal{R}_0}{\partial \gamma} \times \frac{\gamma}{\mathcal{R}_0} = -\frac{\gamma}{\mu+\delta_1+\gamma}.$$

Table 3: Sensitivity anlysis of $\mathcal{R}_0$ of the parameters described in table 1

Parameters	Sensitivity index
α	+1
γ	-0.9182
μ	-1.0818
Λ	+1
δ_1	-2.0632e-11
δ_2	0

Based on the information presented in Table 1, the most sensitive parameters are the HIV transmission rate α, and the birth rate Λ. Specifically, since $\Upsilon_{\alpha}^{\mathcal{R}0} = 1$, a slight increase in α or Λ will lead to a decrease of 10% in $\mathcal{R}_0$. Similarly, a small decrease in α or Λ will result in an increase of 10% in $\mathcal{R}_0$. The situation is the same when the HIV mortality rate δ_1, the rate at which HIV progresses to AIDS γ, or the rate at which people die naturally μ are high; consequently, the reproduction rate is low. Our analysis shows that the γ, δ_2, δ_1 and μ parameters do not affect the transmission of HIV, whereas the α and Λ parameters have a more significant effect.

Conclusion

In this work, we have proposed two fractional models of HIV/AIDS. We have proved that the fractional models are well posed by using the generalized mean value theorem. In addition, we have studied the local stability of each equilibrium for the two proposed models. We have also computed the basic

reproduction number. Furthermore, we have proved the global stability of the models by using adequate Lyapunov functions. Finally, the theoretical results have been confirmed by numerical simulations. We observed that the variation of the fractional order derivative ρ does not affect the stability of the equilibrium but has an impact on the time needed to reach the stable states, as we found that the convergence toward stable states takes longer with fractional values. This results perfectly simulates the nature of the HIV disease.

References

[1] Mansour A Abdulwasaa, Mohammed S Abdo, Kamal Shah, Taher A Nofal, Satish K Panchal, Sunil V Kawale, and Abdel-Haleem Abdel-Aty. Fractal-fractional mathematical modeling and forecasting of new cases and deaths of covid-19 epidemic outbreaks in india. *Results in Physics*, 20:103702, 2021.

[2] Norelys Aguila-Camacho, Manuel A Duarte-Mermoud, and Javier A Gallegos. Lyapunov functions for fractional order systems. *Communications in Nonlinear Science and Numerical Simulation*, 19(9):2951–2957, 2014.

[3] Ofosuhene O. Apenteng, Prince P. Osei, Noor Azina Ismail, and Aline Chiabai. Analysing the impact of migration on hiv/aids cases using epidemiological modelling to guide policy makers. *Infectious Disease Modelling*, 7:252–261, 3 2022.

[4] Stavros Busenberg, Kenneth Cooke, and Horst Thieme. Demographic change and persistence of hiv/aids in a heterogeneous population. *SIAM Journal on Applied Mathematics*, 51(4):1030–1052, 1991.

[5] LC Cardoso, FLP Dos Santos, and RF Camargo. Analysis of fractional-order models for hepatitis b. *Computational and Applied Mathematics*, 37:4570–4586, 2018.

[6] Nakul Chitnis, James M Hyman, and Jim M Cushing. Determining important parameters in the spread of malaria through the sensitivity analysis of a mathematical model. *Bulletin of mathematical biology*, 70(5):1272–1296, 2008.

[7] Kevin D Deane, Justin O Parkhurst, and Deborah Johnston. Linking migration, mobility and hiv. *Tropical Medicine & International Health*, 15(12):1458–1463, 2010.

[8] Kai Diethelm, Neville J Ford, and Alan D Freed. A predictor-corrector approach for the numerical solution of fractional differential equations. *Nonlinear Dynamics*, 29(1):3–22, 2002.

[9] Joakim Esbjörnsson, Mattias Mild, Fredrik Månsson, Hans Norrgren, and Patrik Medstrand. Hiv-1 molecular epidemiology in guinea-bissau, west africa: origin, demography and migrations. *PloS one*, 6(2):e17025, 2011.

[10] Michael S Gottlieb, Howard M Schanker, Peng Thim Fan, Andrew Saxon, Joel D Weisman, Irving Pozalski, et al. Pneumocystis pneumonialos angeles. *Mmwr*, 30(21):250–2, 1981.

[11] Ying-Hen Hsieh and Shin-Pyng Sheu. The effect of density-dependent treatment and behavior change on the dynamics of hiv transmission. *Journal of Mathematical Biology*, 43(1):69–80, 2001.

[12] Hai-Feng Huo, Rui Chen, and Xun-Yang Wang. Modelling and stability of hiv/aids epidemic model with treatment. *Applied Mathematical Modelling*, 40(13-14):6550–6559, 2016.

[13] Hem Joshi, Suzanne Lenhart, Kendra Albright, and Kevin Gipson. Modeling the effect of information campaigns on the hiv epidemic in uganda. *Math. Biosci. Eng*, 5(4):757–770, 2008.

[14] Moatlhodi Kgosimore and Edward M Lungu. The effects of vertical transmission on the spread of hiv/aids in the presence of treatment. *Mathematical Biosciences and Engineering*, 3(2):297, 2006.

[15] Muhammad Altaf Khan, Saif Ullah, and Muhammad Farhan. The dynamics of zika virus with caputo fractional derivative. *AIMS Math*, 4(1):134–146, 2019.

[16] Wei Lin. Global existence theory and chaos control of fractional differential equations. *Journal of Mathematical Analysis and Applications*, 332(1):709–726, 2007.

[17] N ZERAICK MONTEIRO and S RODRIGUES MAZORCHE. Fractional derivatives applied to epidemiology. *Trends in Computational and Applied Mathematics*, 22:157–177, 2021.

[18] Z Mukandavire, W Garira, and JM Tchuenche. Modelling effects of public health educational campaigns on hiv/aids transmission dynamics. *Applied Mathematical Modelling*, 33(4):2084–2095, 2009.

[19] Parvaiz Ahmad Naik, Jian Zu, and Kolade M Owolabi. Global dynamics of a fractional order model for the transmission of hiv epidemic with optimal control. *Chaos, Solitons & Fractals*, 138:109826, 2020.

[20] Hashem Najafi, Sina Etemad, Nichaphat Patanarapeelert, Joshua Kiddy K Asamoah, Shahram Rezapour, and Thanin Sitthiwirattham. A study on dynamics of cd4+ t-cells under the effect of hiv-1 infection based on a mathematical fractal-fractional model via the adams-bashforth scheme and newton polynomials. *Mathematics*, 10(9):1366, 2022.

[21] F Nyabadza, Z Mukandavire, and SD Hove-Musekwa. Modelling the hiv/aids epidemic trends in south africa: Insights from a simple mathematical model. *Nonlinear Analysis: Real World Applications*, 12(4):2091–2104, 2011.

[22] Weiqiu Pan, Tianzeng Li, and Safdar Ali. A fractional order epidemic model for the simulation of outbreaks of ebola. *Advances in Difference Equations*, 2021(1):1–21, 2021.

[23] Igor Podlubny. *Fractional differential equations: an introduction to fractional derivatives, fractional differential equations, to methods of their solution and some of their applications*. Elsevier, 1998.

[24] Meghdad Saeedian, Moein Khalighi, Nahid Azimi-Tafreshi, GR Jafari, and Marcel Ausloos. Memory effects on epidemic evolution: The susceptible-infected-recovered epidemic model. *Physical Review E*, 95(2):022409, 2017.

[25] O Sharomi and AB Gumel. Dynamical analysis of a multi-strain model of hiv in the presence of anti-retroviral drugs. *Journal of biological dynamics*, 2(3):323–345, 2008.

[26] Moulay Rchid Sidi Ammi, Mostafa Tahiri, and Delfim FM Torres. Global stability of a caputo fractional sirs model with general incidence rate. *Mathematics in Computer Science*, 15(1):91–105, 2021.

[27] Ram P Sigdel and C Connell McCluskey. Global stability for an sei model of infectious disease with immigration. *Applied Mathematics and Computation*, 243:684–689, 2014.

[28] Cristiana J Silva and Delfim FM Torres. A tb-hiv/aids coinfection model and optimal control treatment. *arXiv preprint arXiv:1501.03322*, 2015.

[29] Jagdev Singh, Devendra Kumar, Zakia Hammouch, and Abdon Atangana. A fractional epidemiological model for computer viruses pertaining to a new fractional derivative. *Applied mathematics and computation*, 316:504–515, 2018.

[30] NH Sweilam, SM Al-Mekhlafi, ZN Mohammed, and Dumitru Baleanu. Optimal control for variable order fractional hiv/aids and malaria mathematical models with multi-time delay. *Alexandria Engineering Journal*, 59(5):3149–3162, 2020.

[31] Yanni Tian and Xianning Liu. Global dynamics of a virus dynamical model with general incidence rate and cure rate. *Nonlinear Analysis: Real World Applications*, 16:17–26, 2014.

[32] Naveen K Vaidya and Jianhong Wu. Hiv epidemic in far-western nepal: effect of seasonal labor migration to india. *BMC Public Health*, 11(1):1–11, 2011.

[33] Cruz Vargas-De-León. Volterra-type lyapunov functions for fractional-order epidemic systems. *Communications in Nonlinear Science and Numerical Simulation*, 24(1-3):75–85, 2015.

[34] Ferdinand Verhulst. *Nonlinear differential equations and dynamical systems*. Springer Science & Business Media, 2006.