Cartan equivalence method on fifth-order differential operator

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Abstract

In this paper, we carry out the equivalence problem for fifth-order differential operators on the line under general fiber-preserving transformation using the Cartan method of equivalence. Two versions of equivalence problems have been solved. We consider the direct equivalence problem and an equivalence problem to determine the sufficient and necessary conditions on fifth-order differential operators such that there exists a fiber–preserving transformation mapping one to the other according to gauge equivalence.

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Abstract

In this paper, we carry out the equivalence problem for fifth-order differential operators on the line under general fiber-preserving transformation using the Cartan method of equivalence. Two versions of equivalence problems have been solved. We consider the direct equivalence problem and an equivalence problem to determine the sufficient and necessary conditions on fifth-order differential operators such that there exists a fiber–preserving transformation mapping one to the other according to gauge equivalence.

Keywords: equivalence problem; fifth-order differential operators; gauge equivalence; fiber-preserving transformation.

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1 Introduction

The equivalence problem, [6], for the fifth-order operator means that two fifth-order differential operators on a real line can be transformed into each other by an appropriate change of variables, [2, 3, 4, 5]. We will treat both versions of equivalence problems that include the direct equivalence problem and an equivalence problem to determine conditions on two differential operators such that there exists a fiber-preserving transformation mapping one to the other according to gauge equivalence. We associate a collection of one-forms to an object under investigation in the original coordinates; the the corresponding object in the new coordinates will have its own collection of one-forms. Once an equivalence problem has been reformulated in the proper Cartan form, in terms of a coframe ω on the m-dimensional base manifold M, along with a structured group $G \subset \mathrm{GL}(m)$, we can apply the Cartan equivalence method. The goal is to normalize the structure group valued coefficients in a suitably invariant manner, and this is accomplished through the determination of a sufficient number of invariant combinations thereof, [10].

The classification of linear differential equations is a special case of the general problem of classifying differential operators, which has a variety of important applications, including quantum mechanics

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and the projective geometry of curves, [10]. In the last section, applications of this method for fifth-order differential operators are presented. S. S. Chern turned his attention to the problem under contact transformations [5] and Hajime Sato et all [11], but are specialized by linearity. Niky Kamran and Peter J. Olver have solved the equivalence problem for the second-order differential operator with two versions of the equivalence problem [7] and recently some papers have been written about the equivalence of differential operators [1, 9]. The fifth-order operators have different geometry and equivalence problem for fifth-order differential operators is a different challenge. In the end, we apply the equivalence problem for an illustrative example for a boundary value problem for the fifth-order differential operator with integrable potential.

2 The Cartan equivalence method

We first review Cartan's equivalence problem as an algorithmic method that include the structure equations, normalization, and absorption. The standard reference is [10].

Let $G \subset \mathrm{GL}(m)$ be a lie group and ω and $\overline{\omega}$ denote coframes defined on the m-dimensional manifolds M and $\overline{M}$. The G-valued equivalence problem for these coframes is to determine whether or not there exists a local diffeomorphism $\Phi : M \to \overline{M}$ and a G-valued function $g : M \to G$ with the property that

$$\Phi^*(\overline{\omega}) = g(x)\omega. \tag{1}$$

In full detail, the equivalence condition(1) has the form

$$\Phi^*(\overline{\omega}^i) = \sum_{j=1}^{m} g^i_j(x)\omega^j, \tag{2}$$

for $i = 1, \cdots, m$ where the functions $g^i_j(x)$ are the entries of the matrix $g(x)$, which is constrained to belong to the structure group G at each point$x \in M$. In view of the group property of G, it will be satisfied if and only if we can find a pair of G-valued functions $\overline{g}(\overline{x})$ and $g(x)$ such that omitting pull-back for clarity

$$\overline{g}(\overline{x})\overline{\omega} = g(x)\omega. \tag{3}$$

Our goal is to reduce a given G-equivalence problem to a standard equivalence problem for coframes, and the way to do that is to specify the matrix entries of $g = g(x)$ and $\overline{g}(\overline{x})$ as functions of their respective coordinate. The new coframes defined by

$$\theta^i = \sum_{j=1}^{m} g^i_j\omega^j, \qquad \overline{\theta}^i = \sum_{j=1}^{m} \overline{g}^i_j\overline{\omega}^j, \tag{4}$$

which will be invariant: $\Phi^*(\overline{\theta}^i) = \theta^i$. This motivates the preliminary step in the Cartan solution to the equivalence problem, which is to introduce the lifted coframe

$$\theta = g.\omega, \tag{5}$$

or, in full detail,

$$\theta^i = \sum_{j=1}^{m} g^i_j(x)\omega^j. \tag{6}$$

We compute the differentials of the lifted coframe elements:

$$d\theta^i = d\Big(\sum_{j=1}^{m} g^i_j \omega^j\Big) = \sum_{j=1}^{m}\{dg^i_j \wedge \omega^j + g^i_j d\omega^j\}. \tag{7}$$

Since the ω forms a coframe on M, one can rewrite the 2-forms $d\omega^j$ in terms of sums of wedge products of the ω^i's. Moreover, viewing (5), these can be rewritten as wedge products of the θ^k's, so that

$$d\theta^i = \sum_{j=1}^{m} \gamma^i_j \wedge \theta^j + \sum_{\substack{j,k=1\\ j<k}}^{m} T^i_{jk}(x,g)\theta^j \wedge \theta^k, \quad i = 1,\ldots,m. \tag{8}$$

The functions T^i_{jk} are called torsion coefficients. The torsion coefficients are constant, or depend on the base variables x or the group parameters g. Some of torsion coefficients may be invariants but they are typically not invariants for the problem. The γ^i_js in (8) are the 1-forms

$$\gamma^i_j = \sum_{k=1}^{m} dg^i_k (g^{-1})^k_j, \tag{9}$$

which have the following matrix notation

$$\gamma = dg \cdot g^{-1}. \tag{10}$$

The γ forms the matrix of Maurer-Cartan forms on the structure group G. Assume the set $\{\alpha^1,\ldots,\alpha^r\}$ is a basis for the space of Maurer-Cartan forms then each γ^i_j is a linear combination of the Maurer-Cartan basis:

$$\gamma^i_j = \sum_{l=1}^{r} A^i_{jl}\alpha^l, \qquad i,j = 1,\ldots,m. \tag{11}$$

Thus the final structure equations for our lifted coframe, in terms of the Maurer-Cartan forms, have the following general form

$$d\theta^i = \sum_{l=1}^{r}\sum_{j=1}^{m} A^i_{jl}\alpha^l \wedge \theta^j + \sum_{\substack{j,k=1\\ j<k}}^{m} T^i_{jk}(x,g)\theta^j \wedge \theta^k, \quad i = 1,\ldots,m. \tag{12}$$

Now one can reduce the Maurer-Cartan forms α^l back to the base manifold M by replacing them by general linear combinations of coframe elements

$$\alpha^l \mapsto \sum_{l=1}^{r} z^l_j \theta^j, \tag{13}$$

where the z^l_j are as yet unspecified coefficients, whose explicit dependence on x. By substituting (13) into the structure equations (12), one can obtain a system of 2-forms

$$\Theta^i = \sum_{\substack{j,k=1\\ j<k}}^{m} \{B^i_{jk}[\mathbf{z}] + T^i_{jk}(x,g)\}\theta^j \wedge \theta^k, \quad i,j = 1,\ldots,m, \tag{14}$$

where

$$B^i_{jk}[\mathbf{z}] = \sum_{l=1}^{r}(A^i_{kl} z^l_j - A^i_{jl} z^l_k), \tag{15}$$

are linear functions of the coefficients $\mathbf{z} = (z^l_k)$, whose constant coefficients are determined by the specific representation of the structure group $G \subset \mathrm{GL}(m)$, and so do not depend on the coordinate system.

The general process of determining the unknown coefficients $\mathbf{z}$ from the full torsion coefficients is known as *absorption of torsion* and the other being *normalization* of the resulting invariant torsion coefficients, as described above. Replacing each Maurer-Cartan form α^l by the modified 1-form

$$\pi^l = \alpha^l - \sum_{i=1}^{p} z_i^l \theta^i, \qquad l = 1, \ldots, r, \tag{16}$$

leads to absorb the inessential torsion in the structure equation (12). Here the $z_i^l = z_i^l(x, g)$ are the solutions to the absorption equations. Thus the structure equations change to the simpler absorbed form

$$d\theta^i = \sum_{l=1}^{r} \sum_{j=1}^{m} A_{jl}^i \pi^l \wedge \theta^j + \sum_{\substack{j,k=1 \\ j<k}}^{m} U_{jk}^i \theta^j \wedge \theta^k, \quad i, j = 1, \ldots, m. \tag{17}$$

where the remaining nonzero coefficients U_{jk}^i consist only of essential torsion. We write the linear system of absorption equations

$$\sum_{l=1}^{r} (A_{jl}^i z_k^l - A_{kl}^i z_j^l) = -T_{jk}^i. \tag{18}$$

and solve for the unknowns $\mathbf{z}$ using the standard Gaussian elimination method.

3 Equivalence of fifth order differential operators

Consider the fifth order differential operator applied on a scalar-valued function $u(x)$

$$\mathcal{D}[u] = \sum_{i=0}^{5} f_i(x)\, D^i u \tag{19}$$

and another fifth order differential operator applied on a scalar-valued function $\bar{u}(\bar{x})$

$$\bar{\mathcal{D}}[\bar{u}] = \sum_{i=0}^{5} \bar{f}_i(\bar{x})\, \bar{D}^i \bar{u}. \tag{20}$$

where f_i and $\bar{f}_i$, $i = 1, 2, 3, 4, 5$, are analytic functions of the real variable x and $\bar{x}$ respectively. For simplicity we let $f_5 = \bar{f}_5 = 1$. Further, $D^i = d/dx^i$, $\bar{D}^i = d/d\bar{x}^i$ and $D^0 = \bar{D}^0 = \mathrm{Id}$ are the identity operators.

The appropriate space to work in will be the fifth jet space J^5, which has local coordinates

$$\Upsilon = \{(x, u, p, q, r, s, t) \in \mathrm{J}^5 : p = u_x, q = u_{xx}, r = u_{xxx}, s = u_{xxxx}, t = u_{xxxxx}\}$$

and the goal is to know whether there exists a suitable transformation of variables $(x, u, p, q, r, s, t) \longrightarrow (\bar{x}, \bar{u}, \bar{p}, \bar{q}, \bar{r}, \bar{s}, \bar{t})$ which brings (19) to (20). Several types of such transformations are of particular importance. Here we consider fiber preserving transformations, which are of the form

$$\bar{x} = \xi(x), \qquad \bar{u} = \varphi(x)\, u, \tag{21}$$

where $\varphi(x) \neq 0$. Using the chain rule formula we find following relation between the total derivative operators

$$\bar{D} = \frac{d}{d\bar{x}} = \frac{1}{\xi'(x)} \frac{d}{dx} = \frac{1}{\xi'(x)} D. \tag{22}$$

First we consider the *direct equivalence problem*, which identifies the two linear differential functions

$$\mathcal{D}[u] = \bar{\mathcal{D}}[\bar{u}]. \tag{23}$$

under change of variables (21). This induces the transformation rule

$$\bar{\mathcal{D}} = \mathcal{D} \cdot \frac{1}{\varphi(x)} \qquad \text{when} \qquad \bar{x} = \xi(x), \tag{24}$$

on the differential operators themselves, and solving local direct equivalence problem is to find explicit conditions on the coefficients of the two differential operators that guarantee they satisfy (23) for some change of variables of the form (21).

The transformation rule (24) doesn't preserve either the eigenvalue problem $\mathcal{D}[u] = \lambda u$ or the Schrödinger equation $iu_t = \mathcal{D}[u]$, since we are missing a factor of $\varphi(x)$. To solve the problem, we consider the *gauge equivalence* with the following transformation rule

$$\bar{\mathcal{D}} = \varphi(x) \cdot \mathcal{D} \cdot \frac{1}{\varphi(x)} \qquad \text{when} \qquad \bar{x} = \xi(x). \tag{25}$$

Proposition 1 *Let $\mathcal{D}$ and $\bar{\mathcal{D}}$ denote fifth-order differential operators. There are two coframes $\Omega = \{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$ and $\bar{\Omega} = \{\bar{\omega}^1, \bar{\omega}^2, \bar{\omega}^3, \bar{\omega}^4, \bar{\omega}^5, \bar{\omega}^6, \bar{\omega}^7\}$ on open subsets Γ and $\bar{\Gamma}$ of the fifth jet space, respectively, such that the differential operators are equivalent under the pseudogroup (21) according to the respective transformation rules (24) and (25) that coframes Ω and $\bar{\Omega}$ satisfy in following relation*

$$\begin{pmatrix} \bar{\omega}^1 \\ \bar{\omega}^2 \\ \bar{\omega}^3 \\ \bar{\omega}^4 \\ \bar{\omega}^5 \\ \bar{\omega}^6 \\ \bar{\omega}^7 \end{pmatrix} = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & a_2 & a_3 & 0 & 0 & 0 & 0 \\ 0 & a_4 & a_5 & a_6 & 0 & 0 & 0 \\ 0 & a_7 & a_8 & a_9 & a_{10} & 0 & 0 \\ 0 & a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \omega^1 \\ \omega^2 \\ \omega^3 \\ \omega^4 \\ \omega^5 \\ \omega^6 \\ \omega^7 \end{pmatrix} \tag{26}$$

where $a_i \in \mathbb{R}$ for $i = 1, \cdots, 15$ and $a_1 a_3 a_6 a_{10} a_{15} \neq 0$.

Proof: Note that a point transformation will be in the desired linear form (21) if and only if, for pair of functions $\alpha = \xi_x$ and $\beta = \varphi_x/\varphi$, one-form equations

$$d\bar{x} = \alpha\, dx, \tag{27}$$

$$\frac{d\bar{u}}{\bar{u}} = \frac{du}{u} + \beta\, dx. \tag{28}$$

hold on the subset of J^5 where $u \neq 0$. In order that the derivative variables p, q, r, s and t transform correctly, we need to preserve the contact ideal $\mathcal{I}$ on J^5, which is

$$\mathcal{I} = \langle du - p\, dx, dp - q\, dx, dq - r\, dx, dr - s\, dx, ds - t\, dx \rangle. \tag{29}$$

Generally, a diffeomorphism $\Phi : \mathrm{J}^5 \to \mathrm{J}^5$ determines a contact transformation if and only if

$$d\bar{u} - \bar{p}\, d\bar{x} = a_1(du - p\, dx), \tag{30}$$

$$d\bar{p} - \bar{q}\, d\bar{x} = a_2(du - p\, dx) + a_3(dp - q\, dx), \tag{31}$$

$$d\bar{q} - \bar{r}\, d\bar{x} = a_4(du - p\, dx) + a_5(dp - q\, dx) + a_6(dq - r\, dx), \tag{32}$$

$$d\bar{r} - \bar{s}\, d\bar{x} = a_7(du - p\, dx) + a_8(dp - q\, dx) + a_9(dq - r\, dx) + a_{10}(dr - s\, dx), \tag{33}$$

$$d\bar{s} - \bar{t}\, d\bar{x} = a_{11}(du - p\, dx) + a_{12}(dp - q\, dx) + a_{13}(dq - r\, dx) + a_{14}(dr - s\, dx) + a_{15}(ds - t\, dx), \tag{34}$$

where a_i are functions on J^5. The combination of the first contact condition (30) with the linearity conditions (27) and (28) constitutes part of an overdetermined equivalence problem. Taking $\beta = -p/u,\ a_1 = 1/u$, in (28) and (30), it is found the one-form

$$\frac{d\bar{u} - \bar{p}\, d\bar{x}}{\bar{u}} = \frac{du - p\, dx}{u}, \tag{35}$$

which is invariant, and (35) can replace both (28) and (30). Therefore, we may choose five elements of our coframe the one-forms

$$\omega^1 = dx,\ \omega^2 = \frac{du - p\, dx}{u},\ \omega^3 = dp - q\, dx,\ \omega^4 = dq - r\, dx,\ \omega^5 = dr - s\, dx,\ \omega^6 = ds - t\, dx, \tag{36}$$

which are defined on the fourth jet space J^4 locally parameterized by (x, u, p, q, r, s, t), with the transformation rules

$$\begin{aligned}
\bar{\omega}^1 &= a_1\omega^1, \\
\bar{\omega}^2 &= \omega^2, \\
\bar{\omega}^3 &= a_2\omega^2 + a_3\omega^3, \\
\bar{\omega}^4 &= a_4\omega^2 + a_5\omega^3 + a_6\omega^4, \\
\bar{\omega}^5 &= a_7\omega^2 + a_8\omega^3 + a_9\omega^4 + a_{10}\omega^5, \\
\bar{\omega}^6 &= a_{11}\omega^2 + a_{12}\omega^3 + a_{13}\omega^4 + a_{14}\omega^5 + a_{15}\omega^6.
\end{aligned} \tag{37}$$

According to (23), the function $I(x, u, p, q, r, s, t) = \mathcal{D}[u] = t + f_4(x)s + f_3(x)r + f_2(x)q + f_1(x)p + f_0(x)u$ is an invariant for the problem, and thus its differential

$$\omega^7 = dI = dt + f_4 ds + f_3 dr + f_2 dq + f_1 dp + f_0 du + (f_4' s + f_3' r + f_2' q + f_1' p + f_0' u)dx, \tag{38}$$

is an invariant one-form, thus one can take it as a final element of our coframe.

In the second problem (25), for the extra factor of φ, the invariant is

$$I(x, u, p, q, r, s, t) = \frac{\mathcal{D}[u]}{u} = \frac{f_5(x)dt + f_4(x)ds + f_3(x)r + f_2(x)q + f_1(x)p}{u} + f_0(x). \tag{39}$$

Thus, it is found

$$\omega^7 = dI = \frac{1}{u}\, dt + \frac{f_4}{u}\, ds + \frac{f_3}{u}\, dr + \frac{f_2}{u}\, dq + \frac{f_1}{u}\, dp - \frac{t + f_4 s + f_3 r + f_2 q + f_1 p}{u^2}\, du \tag{40}$$

$$+\Big\{\frac{f_4' s + f_3' r + f_2' q + f_1' p}{u} + f_0'\Big\}\, dx, \tag{41}$$

as a final element of coframe for the equivalence problem (25). The set of one-forms

$$\Omega = \{\omega^1, \omega^2, \omega^3, \omega^4, \omega^5, \omega^6, \omega^7\}$$

is a coframe on the subset

$$\Gamma^* = \Big\{(x, u, p, q, r, s, t) \in \mathrm{J}^5 \,\Big|\, u \neq 0 \text{ and } f_5(x) \neq 0\Big\}. \tag{42}$$

All of attention is restricted to a connected component $\Gamma \subset \Gamma^*$ of the subset (42) that the signs of $f_0(x)$ and u are fixed. It means, the last coframe elements agree up to contact

$$\bar{\omega}^7 = \omega^7. \tag{43}$$

Viewing (37) and (43) relations, one can find the structure group associated with the equivalence problems (24) and (25) that is a 15-dimensional matrix group G such that $\bar{\Omega} = G\Omega$ which leads to

(26) and then the *lifted coframe* on the space $\mathrm{J}^5 \times G$ has the form

$$\begin{aligned}
\theta^1 &= a_1\omega^1,\\
\theta^2 &= \omega^2,\\
\theta^3 &= a_2\omega^2 + a_3\omega^3,\\
\theta^4 &= a_4\omega^2 + a_5\omega^3 + a_6\omega^4,\\
\theta^5 &= a_7\omega^2 + a_8\omega^3 + a_9\omega^4 + a_{10}\omega^5,\\
\theta^6 &= a_{11}\omega^2 + a_{12}\omega^3 + a_{13}\omega^4 + a_{14}\omega^5 + a_{15}\omega^6,\\
\theta^7 &= \omega^7.
\end{aligned} \tag{44}$$

In the following, two important results will be presented in the form of two theorems:

Theorem 1 *The final structure equations for direct equivalence with (36) and (38) coframes are*

$$\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1\wedge\theta^2,\\
d\theta^2 &= \theta^1\wedge\theta^3,\\
d\theta^3 &= \theta^1\wedge\theta^4 + \frac{1}{5}\theta^2\wedge\theta^3,\\
d\theta^4 &= I_1\theta^1\wedge\theta^4 + \theta^1\wedge\theta^5 + \frac{2}{5}\theta^2\wedge\theta^4,\\
d\theta^5 &= I_2\theta^1\wedge\theta^4 + \theta^1\wedge\theta^6 + \frac{3}{5}\theta^2\wedge\theta^5 + \frac{17}{5}\theta^3\wedge\theta^4,\\
d\theta^6 &= I_3\theta^1\wedge\theta^2 + I_4\theta^1\wedge\theta^3 + I_5\theta^1\wedge\theta^4 + \theta^1\wedge\theta^7 + \frac{4}{5}\theta^2\wedge\theta^6 + I_6\theta^3\wedge\theta^4 + 4\theta^3\wedge\theta^5,\\
d\theta^7 &= 0,
\end{aligned} \tag{45}$$

where the coefficients I_1, I_2, I_3, I_4, I_5 and I_6 are

$$I_1 = -\frac{1}{\sqrt[5]{u^4}}\left[f_4u + 3p\right], \tag{46}$$

$$\begin{aligned}
I_2 &= \frac{1}{5\sqrt[5]{u^8}}\left[(10\dot{f}_4u^2 - 12f_4pu - 5f_3u^2 - 9p^2 - 10qu\right],\\
I_3 &= -(f_0u + f_1p + f_2q + f_3r + f_4s + t),\\
I_4 &= -\frac{1}{625\sqrt[5]{u^{16}}}\left[625u^4f_1 - 800u^2f_4pq + 2375u^3f_4r + 1770p^2qu - 1275pru^2 + 3000su^3\right.\\
&\quad \left.+270f_4p^3u - 225u^2f_3p^2 + 1750u^3f_3q + 1125u^3f_2p - 594p^4 - 800q^2u^2\right],
\end{aligned} \tag{47}$$

$$\begin{aligned}
I_5 &= 7 - \frac{1}{25\sqrt[5]{u^{12}}}\left[25u^3f_2 + 6up^2f_4 + 65u^2qf_4 - 55pu^2\dot{f}_4 + 50f_3pu^2 - 25u^3\dot{f}_3\right.\\
&\quad \left.+25u^3\dot{f}_4 + 33p^3 - 45pqu + 100ru^2\right],
\end{aligned} \tag{48}$$

$$I_6 = -\frac{1}{\sqrt[5]{u^4}}(f_4u + 3p).$$

Theorem 2 *The final structure equations for gauge equivalence with (36) and (40) coframes are*

$$\begin{aligned}
d\theta^1 &= 0,\\
d\theta^2 &= \theta^1\wedge\theta^3,\\
d\theta^3 &= \theta^1\wedge\theta^4,\\
d\theta^4 &= L_1\theta^1\wedge\theta^4 + \theta^1\wedge\theta^5,\\
d\theta^5 &= L_2\theta^1\wedge\theta^4 + \theta^1\wedge\theta^6 + 5\theta^3\wedge\theta^4,\\
d\theta^6 &= L_3\theta^1\wedge\theta^3 + L_4\theta^1\wedge\theta^4 + \theta^1\wedge\theta^7 + L_5\theta^3\wedge\theta^4 + 5\theta^3\wedge\theta^5,\\
d\theta^7 &= 0,
\end{aligned} \tag{49}$$

where the coefficients $L_1, \ldots, L_5$ are

$$
\begin{aligned}
L_1 &= -\frac{1}{u}\left[f_4 u + 5p\right], \\
L_2 &= \frac{1}{u^2}\left[2\dot{f}_4 u^2 - f_3 u^2 - 4 f_4 p u - 10p^2\right], \\
L_3 &= -\frac{1}{u}\left[2pf_2 + 3f_3 q + 4f_4 r + f_1 u + 5s\right], \\
L_4 &= \frac{1}{u^3}\left[4\dot{f}_4 p u^2 - f_2 u^3 + \dot{f}_3 u^3 - 3f_3 p u^2 - 4f_4 p^2 u - 2u^2 f_4 q - \ddot{f}_4 u^3 - 10p^3 + 5pqu - 5ru^2\right], \\
L_5 &= -\frac{1}{u}(f_4 u + 5p),
\end{aligned}
\tag{50}
$$

4 The proof of Theorem 1

We start with the help of the initial six one-forms (36) and (38) are taken as our final coframe constituent. So equivalence problem turns into G-equivalence problem. We normalize the problem then (44) are obtained. We use (44) instead of six one-forms (36) and (38). The goal is the basis manifold $M \times G$ turns into $M \times G'$ which dim $G' <$ dim G and finally G' turns into e. F or the problem, this algorithm is done in five stage. We compute right-invariant Maurer-Cartan on lie group G. Thus we calculate $dg \cdot g^{-1}$. To calculate structure equations differential of (44) and write the result based on right-invariant Maurer-Cartan and (44). Then we use absorption algorithm and find coefficient of (44) which does not depend on z. The corresponding torsion coefficient will be invariant. Now we calculate the differentials of lifted coframe elements (44). An explicit computation leads to the structure equations

$$
\begin{aligned}
d\theta^1 &= \alpha^1 \wedge \theta^1, \\
d\theta^2 &= T^2_{12}\theta^1 \wedge \theta^2 + T^2_{13}\theta^1 \wedge \theta^3, \\
d\theta^3 &= \alpha^2 \wedge \theta^2 + \alpha^3 \wedge \theta^3 + T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + T^3_{14}\theta^1 \wedge \theta^4, \\
d\theta^4 &= \alpha^4 \wedge \theta^2 + \alpha^5 \wedge \theta^3 + \alpha^6 \wedge \theta^4 + T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + T^4_{15}\theta^1 \wedge \theta^5, \\
d\theta^5 &= \alpha^7 \wedge \theta^2 + \alpha^8 \wedge \theta^3 + \alpha^9 \wedge \theta^4 + \alpha^{10} \wedge \theta^5 + T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 + T^5_{15}\theta^1 \wedge \theta^5 \\
&\quad + T^5_{16}\theta^1 \wedge \theta^6, \\
d\theta^6 &= \alpha^{11} \wedge \theta^2 + \alpha^{12} \wedge \theta^3 + \alpha^{13} \wedge \theta^4 + \alpha^{14} \wedge \theta^5 + \alpha^{15} \wedge \theta^6 + T^6_{12}\theta^1 \wedge \theta^2 + T^6_{13}\theta^1 \wedge \theta^3 + T^6_{14}\theta^1 \wedge \theta^4 \\
&\quad + T^6_{15}\theta^1 \wedge \theta^5 + T^6_{16}\theta^1 \wedge \theta^6 + T^6_{17}\theta^1 \wedge \theta^7, \\
d\theta^7 &= 0,
\end{aligned}
\tag{51}
$$

which α^i $i=1,\cdots,15$ are forming a basis for the right-invariant *Maurer-Cartan forms* on the Lie group G:

$$
\begin{aligned}
\alpha^1 &= \frac{da_1}{a_1},\\
\alpha^2 &= \frac{a_3da_2-a_2da_3}{a_3},\\
\alpha^3 &= \frac{da_3}{a_3},\\
\alpha^4 &= \frac{a_3a_6da_4-a_2a_6da_5+(a_2a_5-a_3a_4)da_6}{a_3a_6},\\
\alpha^5 &= \frac{a_6da_5-a_5da_6}{a_3a_6},\\
\alpha^6 &= \frac{da_6}{a_6},\\
\alpha^7 &= \frac{a_3a_6a_{10}da_7-a_2a_6a_{10}da_8+a_{10}(a_2a_5-a_3a_4)da_9-(a_3a_6a_7-a_3a_4a_9-a_2a_6a_8+a_2a_5a_9)da_{10}}{a_3a_6a_{10}},\\
\alpha^8 &= \frac{a_6a_{10}da_8-a_5a_{10}da_9+(a_5a_9-a_6a_8)da_{10}}{a_3a_6a_{10}},
\end{aligned}
$$

$$
\begin{aligned}
\alpha^9 &= \frac{a_{10}da_9-a_9da_{10}}{a_6a_{10}},\\
\alpha^{10} &= \frac{da_{10}}{a_{10}},\\
\alpha^{11} &= \frac{1}{a_3a_6a_{10}a_{15}}\Big[a_3a_6a_{10}a_{15}da_{11}-a_2a_6a_{10}a_{15}da_{12}+a_{10}a_{15}(a_2a_5-a_3a_4)da_{13}-a_{15}(a_2a_5a_9-a_2a_6a_8\\
&\qquad -a_3a_4a_9+a_3a_6a_7)da_{14}-(a_3a_6a_{10}a_{11}-a_2a_6a_{10}a_{12}+a_2a_5a_{10}a_{13}-a_3a_4a_{10}a_{13}\\
&\qquad -a_2a_5a_9a_{14}+a_2a_6a_8a_{14}+a_3a_4a_9a_{14}-a_3a_6a_7a_{14})da_{15}\Big],\\
\alpha^{12} &= \frac{1}{a_3a_6a_{10}a_{15}}\Big[a_6a_{10}a_{15}da_{12}-a_5a_{10}a_{15}da_{13}+a_{15}(a_5a_9-a_6a_8)da_{14}-(a_6a_{10}a_{12}-a_5a_{10}a_{13}\\
&\qquad +a_5a_9a_{14}-a_6a_8a_{14})\Big],\\
\alpha^{13} &= \frac{a_{10}a_{15}da_{13}-a_9a_{15}da_{14}-(a_{10}a_{13}-a_9a_{14})da_{15}}{a_6a_{10}a_{15}},\\
\alpha^{14} &= \frac{a_{15}da_{14}-a_{14}da_{15}}{a_{10}a_{15}},\\
\alpha^{15} &= \frac{da_{15}}{a_{15}},
\end{aligned}
$$

In first loop the essential torsion coefficients are

$$
T^2_{12}=-\frac{a_2+a_3p}{a_1a_3u},\quad T^2_{13}=\frac{1}{a_1a_3u},\quad T^3_{14}=\frac{a_3}{a_1a_6},\quad T^4_{15}=\frac{a_6}{a_1a_{10}},\quad T^5_{16}=\frac{a_{10}}{a_1a_{15}},\quad T^6_{17}=\frac{a_{15}}{a_1}. \tag{52}
$$

One can normalize the group parameters by setting

$$
a_1=\frac{1}{\sqrt[5]{u}},\quad a_2=-\frac{p}{\sqrt[5]{u^4}},\quad a_3=\frac{1}{\sqrt[5]{u^4}},\quad a_6=\frac{1}{\sqrt[5]{u^3}},\quad a_{10}=\frac{1}{\sqrt[5]{u^2}},\quad a_{15}=\frac{1}{\sqrt[5]{u}}. \tag{53}
$$

In the second loop, the normalization (53) is substituted in the lifted coframe (44) and calculate the differentials of new invariant coframe to obtain revised structure equations. Now, the essential torsion components (52) are normalized by the parameters

$$
a_4=-\frac{q}{\sqrt[5]{u^3}},\quad a_5=-\frac{9p}{5\sqrt[5]{u^4}},\quad a_9=\frac{5f_4u+3p}{5\sqrt[5]{u^4}},\quad a_{14}=\frac{5f_4u+p}{5\sqrt[5]{u^4}}. \tag{54}
$$

In third loop, substituting the normalization (54) in the lifted coframe (44) and determine parameters a_4, a_7, a_8. we recalculate the differentials. Therefore, the new structure equations are

$$\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1\wedge\theta^2,\\
d\theta^2 &= \theta^1\wedge\theta^3,\\
d\theta^3 &= \theta^1\wedge\theta^4+\frac{1}{5}\theta^2\wedge\theta^3,\\
d\theta^4 &= T^4_{12}\theta^1\wedge\theta^2+T^4_{13}\theta^1\wedge\theta^3+T^4_{14}\theta^1\wedge\theta^4+\theta^1\wedge\theta^5+\frac{2}{5}\theta^2\wedge\theta^4,\\
d\theta^5 &= T^5_{12}\theta^1\wedge\theta^2+T^5_{13}\theta^1\wedge\theta^3+T^5_{14}\theta^1\wedge\theta^4+\theta^1\wedge\theta^6+T^5_{23}\theta^2\wedge\theta^3-\frac{2}{5}\theta^2\wedge\theta^5+\frac{3}{5}\theta^3\wedge\theta^4+\alpha^7\wedge\theta^2+\alpha^8\wedge\theta^3\\
d\theta^6 &= T^6_{12}\theta^1\wedge\theta^2+T^6_{13}\theta^1\wedge\theta^3+T^6_{14}\theta^1\wedge\theta^4+T^6_{15}\theta^1\wedge\theta^5+\theta^1\wedge\theta^7+T^6_{23}\theta^2\wedge\theta^3+\frac{3}{5}\alpha^{13}\theta^2\wedge\theta^4\\
&\quad -\frac{1}{5}\theta^2\wedge\theta^6+T^6_{34}\theta^3\wedge\theta^4+\frac{1}{5}\theta^3\wedge\theta^5+\alpha^{11}\wedge\theta^2+\alpha^{12}\wedge\theta^3+\alpha^{13}\wedge\theta^4,\\
d\theta^7 &= 0.
\end{aligned} \tag{55}$$

where $\alpha^7, \alpha^8, \alpha^{11}, \alpha^{12}$ and α^{13} are the Maurer-Cartan forms on G and the essential torsion coefficients are

$$\begin{aligned}
T^4_{12} &= -\frac{5a_7u^{27/5}+5u^5f_4q+3pqu^4+5u^5r}{5u^{27/5}},\\
T^4_{13} &= -\frac{25a_8u^{13/5}+45u^2f_4p+18p^2u+70qu^2}{25u^{13/5}},\\
T^6_{15} &= \frac{-25f_3u^2+25\dot{f}_4u^2+25a_{13}u^{8/5}-5f_4pu-6p^2+5qu}{25u^{8/5}}.
\end{aligned} \tag{56}$$

We find the following parameters:

$$\begin{aligned}
a_7 &= -\frac{5f_4qu+3pq+5ru}{5\sqrt[5]{u^7}},\\
a_8 &= -\frac{45f_4pu+18p^2+70qu}{25\sqrt[5]{u^8}},\\
a_{13} &= \frac{5f_4pu+25f_3u^2-25\dot{f}_4u^2+6p^2-5qu}{25\sqrt[5]{u^8}}.
\end{aligned} \tag{57}$$

Substituting (57) in (44) and recomputing the differentials leads to

$$\begin{aligned}
d\theta^1 &= \frac{1}{5}\theta^1\wedge\theta^2,\\
d\theta^2 &= \theta^1\wedge\theta^3,\\
d\theta^3 &= \frac{1}{5}\theta^2\wedge\theta^3+\theta^1\wedge\theta^4,\\
d\theta^4 &= T^4_{14}\theta^1\wedge\theta^4+\theta^1\wedge\theta^5+\frac{2}{5}\theta^2\wedge\theta^4,\\
d\theta^5 &= T^5_{12}\theta^1\wedge\theta^2+T^5_{13}\theta^1\wedge\theta^3+T^5_{14}\theta^1\wedge\theta^4+\theta^1\wedge\theta^6+\frac{3}{5}\theta^2\wedge\theta^5+\frac{17}{5}\theta^3\wedge\theta^4,\\
d\theta^6 &= T^6_{12}\theta^1\wedge\theta^2+T^6_{13}\theta^1\wedge\theta^3+T^6_{14}\theta^1\wedge\theta^4\\
&\quad +\theta^1\wedge\theta^7+T^6_{23}\theta^2\wedge\theta^3+T^6_{24}\theta^2\wedge\theta^4-\frac{1}{5}\theta^2\wedge\theta^6+\frac{1}{5}\theta^3\wedge\theta^5+\alpha^{11}\wedge\theta^2+\alpha^{12}\wedge\theta^3,\\
d\theta^7 &= 0.
\end{aligned} \tag{58}$$

In final loop, we find the remaining parameters a_{11}, a_{12} which is as follows

$$\begin{aligned}
a_{11} &= -\frac{5f_4ru+pr+5su}{5u^{6/5}},\\
a_{12} &= \frac{9f_4p^2u-70u^2f_4q-9p^3+18pqu-95u^2r}{25u^{12/5}},
\end{aligned} \tag{59}$$

and then it leads to final structure equations (45).

5 The proof of Theorem 2

The calculations of this problem are similar to the previous section except that we use the initial six coframes (36) and the 1-form element is (40). In the first loop through the second equivalence problem procedure, according to Proposition 1, the structure group G in (26) relation is exactly the structure group of direct equivalence, and then the equivalence method has the same intrinsic structure (51) by the essential torsion coefficients

$$T^2_{12} = -\frac{a_2 + a_3 p}{a_1 a_3 u}, \quad T^2_{13} = \frac{1}{a_1 a_3 u}, \quad T^3_{14} = \frac{a_3}{a_1 a_6}, \quad T^4_{15} = \frac{a_6}{a_1 a_{10}}, \quad T^5_{16} = \frac{a_{10}}{a_1 a_{15}}, \quad T^6_{17} = \frac{a_{15} u}{a_1}. \tag{60}$$

We can normalize the group parameters by setting

$$a_1 = 1, \quad a_2 = -\frac{p}{u}, \quad a_3 = a_6 = a_{10} = a_{15} = \frac{1}{u}. \tag{61}$$

In the second loop of the equivalence problem, we substitute the normalization (61) in lifted coframe (44) and calculate differentials of new invariant coframe to determining following revised structure equations:

$$\begin{aligned}
d\theta^1 &= 0, \\
d\theta^2 &= \theta^1 \wedge \theta^3, \\
d\theta^3 &= T^3_{12}\theta^1 \wedge \theta^2 + T^3_{13}\theta^1 \wedge \theta^3 + \theta^1 \wedge \theta^4, \\
d\theta^4 &= \alpha^4 \wedge \theta^2 + \alpha^5 \wedge \theta^3 + T^4_{12}\theta^1 \wedge \theta^2 + T^4_{13}\theta^1 \wedge \theta^3 + T^4_{14}\theta^1 \wedge \theta^4 + \theta^1 \wedge \theta^5 \\
&\quad + \alpha^5 \theta^2 \wedge \theta^3 - \theta^2 \wedge \theta^4, \\
d\theta^5 &= \alpha^7 \wedge \theta^2 + \alpha^8 \wedge \theta^3 + \alpha^9 \wedge \theta^4 + T^5_{12}\theta^1 \wedge \theta^2 + T^5_{13}\theta^1 \wedge \theta^3 + T^5_{14}\theta^1 \wedge \theta^4 \\
&\quad + T^5_{15}\theta^1 \wedge \theta^5 + T^5_{23}\theta^2 \wedge \theta^3 - \theta^2 \wedge \theta^5 + \theta^1 \wedge \theta^6, \\
d\theta^6 &= 0,
\end{aligned} \tag{62}$$

where $\alpha^4, \alpha^5, \alpha^7, \alpha^8$ and α^9 are the Maurer-Cartan forms and the essential torsion components of structure equations (62) are

$$\begin{aligned}
T^3_{12} &= -\frac{a_4 u + q}{u}, \\
T^3_{13} &= -\frac{a_5 u + 2p}{u}, \\
T^5_{15} &= -\frac{a_{14} u - a_9 u + p}{u}, \\
T^6_{16} &= \frac{a_{14} u - f_4 u - p}{u}.
\end{aligned} \tag{63}$$

and so the normalization is

$$a_4 = -\frac{q}{u}, \quad a_5 = -\frac{2p}{u}, \quad a_9 = \frac{f_4 u + 2p}{u}, \quad a_{14} = \frac{f_4 u + p}{u}. \tag{64}$$

Putting (64) into (44) and then recomputing the differential of new 1-forms leads to

$$
\begin{aligned}
d\theta^1 &= 0,\\
d\theta^2 &= \theta^1\wedge\theta^3,\\
d\theta^3 &= \theta^1\wedge\theta^4,\\
d\theta^4 &= T^4_{12}\theta^1\wedge\theta^2 + T^4_{13}\theta^1\wedge\theta^3 + T^4_{14}\theta^1\wedge\theta^4 + \theta^1\wedge\theta^5,\\
d\theta^5 &= \alpha^7\wedge\theta^2 + \alpha^8\wedge\theta^3 + T^5_{12}\theta^1\wedge\theta^2 + T^5_{13}\theta^1\wedge\theta^3 + T^5_{14}\theta^1\wedge\theta^4 + T^5_{23}\theta^2\wedge\theta^3 - \theta^2\wedge\theta^5\\
&\quad +2\theta^3\wedge\theta^4 + \theta^1\wedge\theta^6,\\
d\theta^6 &= \alpha^{11}\wedge\theta^2 + \alpha^{12}\wedge\theta^3 + \alpha^{13}\wedge\theta^4 + T^6_{12}\theta^1\wedge\theta^2 + T^6_{13}\theta^1\wedge\theta^3 + T^6_{14}\theta^1\wedge\theta^4 + T^6_{15}\theta^1\wedge\theta^5 + \theta^1\wedge\theta^7\\
&\quad +T^6_{23}\theta^2\wedge\theta^3 + T^6_{34}\theta^3\wedge\theta^4 + \theta^3\wedge\theta^5 + \alpha^{13}\theta^2\wedge\theta^4 - \theta^2\wedge\theta^6,\\
d\theta^7 &= 0.
\end{aligned}
\tag{65}
$$

This immediately implies following normalization

$$
\begin{aligned}
a_7 &= -\frac{f_4qu + 2pq + ru}{u^2},\\
a_8 &= -\frac{2f_4pu + 4p^2 + 3qu}{u^2},\\
a_{11} &= -\frac{f_4ru + pr + su}{u^2},\\
a_{12} &= -\frac{3f_4qu + 3pq + 4ru}{u^2},\\
a_{13} &= -\frac{\dot f_4u^2 - f_4pu - f_3u^2 - 2p^2 + qu}{u^2}.
\end{aligned}
\tag{66}
$$

Thus the final invariant coframe is now given by

$$
\begin{aligned}
\theta^1 &= \frac{dx}{\sqrt[4]{f_4}},\\
\theta^2 &= \frac{du - p\,dx}{u},\\
\theta^3 &= \frac{\sqrt[4]{f_4}}{u^2}\Big[(p^2 - qu)\,dx - p\,du + u\,dp\Big],\\
\theta^4 &= -\frac{1}{4\sqrt{f_4}\,u^3}\Big[(4f_4u^2r + \dot f_4u^2q - \dot f_4up^2 - 12f_4upq + 8f_4p^3)\,dx\\
&\qquad +(\dot f_4up + 4f_4uq - 8f_4p^2)\,du + (8f_4p - \dot f_3u)u\,dp - 4f_4u^2\,dq\Big],\\
\theta^5 &= \frac{1}{4\sqrt{f_4}\,u^3}\Big[(8f_4p^2q - 4f_4uq^2 - 4f_4u^2s + 4f_3u^2r(4f_3 - 3\dot f_4)upq + 3\dot f_4u^2r)\,dx\\
&\quad +(8f_4pq + 4f_4ur + (4f_3 - 3\dot f_4)uq)\,du + (4f_4uq)\,dp + (4f_4p + (4f_3 - \dot f_4)u)u\,dq\\
&\quad -4f_4u^2\,dr\Big],\\
\theta^6 &= \frac{f_4's + f_3'r + f_2'q + f_1p + f_0'u}{u}\,dx - \frac{f_4r + f_3r + f_2q + f_1p}{u^2}\,du + \frac{f_1}{u}\,dp\\
&\qquad +\frac{f_2}{u}\,dq + \frac{f_3}{u}dr + \frac{f_4}{u}ds.
\end{aligned}
\tag{67}
$$

Then the final structure equations (49) with fundamental invariant coefficients (50) are obtained.

6 An example

Consider boundary value problem for the fifth-order differential operator, [8],

$$
D^5u(x) + (Q(x) - \lambda a^5)u(x) = 0, \quad 0 \le x \le \pi,\ a > 0, \tag{68}
$$

with separated boundary conditions

$$\begin{aligned} &y^{(m_1)}(0)=y^{(m_2)}(0)=y^{(m_3)}(0)=y^{(m_4)}(0)=y^{(n_1)}(\pi)=0,\\ &m_1<m_2<m_3<m_4,\ m_k,n_1\in\{0,1,2,3,4\},\ k=1,2,3,4, \end{aligned} \tag{69}$$

where potential $Q(x)$ is a integrable function in interval $[0,\pi]$ and λ is spectral parameter. We are accomplishing Cartan equivalence method on the fifth-order differential operator (68). In direct method, consider one-forms (36) and the following one-form

$$\omega^7=Q'udx=(Q-\lambda a^5)du+dt \tag{70}$$

as a coframe. The final structure equations are

$$\begin{aligned} d\theta^1&=\frac{1}{5}\theta^1\wedge\theta^2,\\ d\theta^2&=\theta^1\wedge\theta^3,\\ d\theta^3&=\theta^1\wedge\theta^4+\frac{1}{5}\theta^2\wedge\theta^3,\\ d\theta^4&=I_1\theta^1\wedge\theta^4+\theta^1\wedge\theta^5+\frac{2}{5}\theta^2\wedge\theta^4,\\ d\theta^5&=I_2\theta^1\wedge\theta^4+\theta^1\wedge\theta^6+\frac{3}{5}\theta^2\wedge\theta^5+\frac{17}{5}\theta^3\wedge\theta^4,\\ d\theta^6&=I_3\theta^1\wedge\theta^2+I_4\theta^1\wedge\theta^3+I_5\theta^1\wedge\theta^4\\ &\quad+\theta^1\wedge\theta^7+\frac{4}{5}\theta^2\wedge\theta^6+I_6\theta^3\wedge\theta^4+4\theta^3\wedge\theta^5,\\ d\theta^7&=0, \end{aligned} \tag{71}$$

where the coefficients I_1,I_2,I_3,I_4,I_5 and I_6 are

$$I_1 = -\frac{3p}{\sqrt[5]{u^4}}, \tag{72}$$

$$I_2 = \frac{1}{5\sqrt[5]{u^8}}\big[-9p^2-10qu\big], \tag{73}$$

$$I_3 = a^5\lambda u-Q(x)u-t, \tag{74}$$

$$I_4 = -\frac{1}{625\sqrt[5]{u^{16}}}\left[1770p^2qu-1275pru^2+3000su^3-594p^4-800q^2u^2\right], \tag{75}$$

$$I_5 = -\frac{1}{25\sqrt[5]{u^{12}}}\left[33p^3-45pqu+100ru^2\right], \tag{76}$$

$$I_6 = -\frac{3p}{\sqrt[5]{u^4}}. \tag{77}$$

Now we solve this example by gauge equivalence method. The one-forms (36) and

$$\omega^7=Q'dx-\frac{tdu}{u^2}+\frac{dt}{u}, \tag{78}$$

are chosen as element of coframes and the final structure equations with above coframes are

$$\begin{aligned} d\theta^1&=0,\\ d\theta^2&=\theta^1\wedge\theta^3,\\ d\theta^3&=\theta^1\wedge\theta^4,\\ d\theta^4&=L_1\theta^1\wedge\theta^4+\theta^1\wedge\theta^5,\\ d\theta^5&=L_2\theta^1\wedge\theta^4+\theta^1\wedge\theta^6+5\theta^3\wedge\theta^4,\\ d\theta^6&=L_3\theta^1\wedge\theta^3+L_4\theta^1\wedge\theta^4+\theta^1\wedge\theta^7+L_5\theta^3\wedge\theta^4+5\theta^3\wedge\theta^5,\\ d\theta^7&=0, \end{aligned}$$

where the coefficients $L_1, \ldots, L_5$ are

$$\begin{aligned}
L_1 &= -\frac{5p}{u}, \\
L_2 &= -\frac{10p^2}{u^2}, \\
L_3 &= -\frac{5s}{u}, \\
L_4 &= \frac{1}{u^3}\left[-10p^3 + 5pqu - 5ru^2\right], \\
L_5 &= -\frac{5p}{u}.
\end{aligned} \tag{79}$$

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