Impulsive fractional partial differential system and its correct integral solution

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Abstract

It is found that there are two piecewise functions satisfy the conditions in the impulsive fractional partial differential system (IFrPDS), which deduce that the three different integral solutions of the IFrPDS given in the cited papers are inappropriate. Next, by applying two limit properties of the IFrPDS and the properties of piecewise function, the new formula of solution of the IFrPDS is discovered that is the integral equation with an arbitrary continuously differentiable function of t on $[0\ ,c]$ to reveal the non-uniqueness of the IFrPDE's solution. Finally, an example is provided to expound the computation of the solution of the IFrPDS.

Impulsive fractional partial differential system and its correct integral solution*

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Abstract: It is found that there are two piecewise functions satisfy the conditions in the impulsive fractional partial differential system (IFrPDS), which deduce that the three different integral solutions of the IFrPDS given in the cited papers are inappropriate. Next, by applying two limit properties of the IFrPDS and the properties of piecewise function, the new formula of solution of the IFrPDS is discovered that is the integral equation with an arbitrary continuously differentiable function of t on $[0, c]$ to reveal the non-uniqueness of the IFrPDE's solution. Finally, an example is provided to expound the computation of the solution of the IFrPDS.

Key Words: impulsive fractional partial differential equations; equivalent integral equation; non-uniqueness of solution

MSC2010: 26A33; 34A08; 34A37; 34A12

1 Introduction

Impulsive fractional order system (IFOS) has become a focus of research and several hundreds articles are found by searching the topic of impulsive fractional order system from Web of science. For the IFOS, its integral solution (or equivalent integral equation) is key in studying numerical solution [1, 2], existence of solution [3–11], oscillation behavior [12, 13], periodic motion [14], solvability [15], asymptotic behavior of solution [16], stability [17–19], integral solution [20–27] and general solution [28] etc.

Furthermore, the impulsive fractional partial differential order system (IFrPDOS) was firstly introduced in [29] by

$$\begin{cases} {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t) = F(s,t,\Upsilon(s,t)), & (s,t)\in\Theta, s\neq s_j\ (j=1,2,...,J), \\ \Upsilon(s_j^{+},t)-\Upsilon(s_j^{-},t) = U_j(\Upsilon(s_j^{-},t)), & t\in[0,c], j=1,2,...,J, \\ \Upsilon(s,0)=\eta(s), \Upsilon(0,t)=\xi(t), & s\in[0,b], t\in[0,c], \end{cases} \tag{1.1}$$

where ${}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}$ represents the Caputo fractional derivative with order $e=(e_1,e_2)$ $(e_1,e_2\in(0,1])$, $\Theta=[0,b]\times[0,c]$, $b=s_{J+1}>s_J>...>s_1>s_0=0$, $F:\Theta\times\mathbb{R}^n\to\mathbb{R}^n$, $U_j:\mathbb{R}^n\to\mathbb{R}^n$ $(j=1,2,...,J)$,

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$\eta : [0,b] \to \mathbb{R}^n$ and $\xi : [0,c] \to \mathbb{R}^n$ with $\eta(0) = \xi(0)$.

Remark 1.1. Let $\Theta_0 = [0, s_1] \times [0, c]$, $\Theta_j = (s_j, s_{j+1}] \times [0, c]$, $\overline{\Theta}_0 = \Theta_0$ and $\overline{\Theta}_j = [s_j, s_{j+1}] \times [0, c]$ $(j = 1, 2, ..., J)$, $\Lambda(s,t) = \eta(s) + \xi(t) - \eta(0)$ and $Fd\tau d\epsilon = F(\epsilon, \tau, w(\epsilon, \tau))d\tau d\epsilon$ throughout this paper.

For (1.1), it is found that there are three different integral solution in existing research. To study some properties of solution, the integral solution of (1.1) was given in [29–32] by

$$\Upsilon(s,t) = \begin{cases} \Lambda(s,t) + \int_0^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon, & (s,t) \in \Theta_0, \\ \Lambda(s_j^+, t) + \int_{s_j}^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon, & (s,t) \in \Theta_j, j = 1, ..., J, \end{cases} \tag{1.2}$$

or equivalently,

$$\Upsilon(s,t) = \begin{cases} \Lambda(s,t) + \int_0^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon, \quad (s,t) \in \Theta_0, \\ \Lambda(s,t) + \sum_{i=1}^j \int_{s_{i-1}}^{s_i} \int_0^t \frac{(s_i-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon \\ \quad + \sum_{i=1}^j \left[U_i(\Upsilon(s_i^-, t)) - U_i(\Upsilon(s_i^-, 0))\right] + \int_{s_j}^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon, \\ \quad (s,t) \in \Theta_j, j = 1, 2, ..., J. \end{cases} \tag{1.3}$$

In contrast, in [33], it was thought that the integral solution of (1.1) is another integral equality

$$\Upsilon(s,t) = \begin{cases} \Lambda(s,t) + \int_0^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon, \quad (s,t) \in \Theta_0, \\ \Lambda(s,t) + \sum_{i=1}^j \left[U_i(\Upsilon(s_i^-, t)) - U_i(\Upsilon(s_i^-, 0))\right] \\ \quad + \int_0^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} Fd\tau d\epsilon, \qquad (s,t) \in \Theta_j, j = 1, 2, ..., J. \end{cases} \tag{1.4}$$

Moreover, by combining the above two thoughts of integral solution, the third integral solution of (1.1) was given in [34].

Remark 1.2. By *Lemma 2.3*, in fact, (1.2) and (1.3) are the integral solution of the system

$$\begin{cases} {}^{C}_{(s_i^+, 0+)}\mathcal{D}^e_{(s,t)} \Upsilon(s,t) = F(s,t,\Upsilon(s,t)), & (s,t) \in \Theta_i, \text{here } i = 0, 1, ..., J, \\ \Upsilon(s_j^+, t) - \Upsilon(s_j^-, t) = U_j(\Upsilon(s_j^-, t)), & t \in [0, c], j = 1, 2, ..., J, \\ \Upsilon(s, 0) = \eta(s), \Upsilon(0, t) = \xi(t), & s \in [0, b], t \in [0, c], \end{cases} \tag{1.5}$$

However, recently it is found that there are two piecewise functions satisfying the conditions in (1.1), which derive that the given integral solutions in the cited papers are inappropriate. Therefore, in what follows, we will re-search the integral solution of (1.1).

We arrange the remainder of the paper as follows. Firstly, we review some definitions and properties of the fractional derivative and the fractional integration in Section 2. Secondly, we elaborate that there are two piecewise functions satisfying the conditions in (1.1), and then we re-explore the integral solution of (1.1) by some properties of piecewise function in Section 3. Finally, an example is provided to show the computation of integral solution of the IFrPDOS's solution in Section 4.

2 Preliminaries

Let $L^1(\Theta, \mathbb{R}^n)$ denote the space of the Lebesgue-integrable function $\Upsilon : \Theta \to \mathbb{R}^n$ with

$$\|\Upsilon(\mu,\nu)\|_1 = \int_0^b \int_0^c \|\Upsilon(\mu,\nu)\|\, d\nu d\mu$$

where $\|\cdot\|$ is an appropriate complete norm on $\mathbb{R}^n$.

Definition 2.1 ([35, 36]). Let $z \in [0,b]$, $\Theta_z = [z,b] \times [0,c]$ and $u = (u_1, u_2)$ $(u_1, u_2 > 0)$. For $\Upsilon \in L^1(\Theta_z, \mathbb{R}^n)$, its mixed Riemann-Liouville fractional integral is defined by

$$_{(z+,0+)}\mathcal{I}^{u}_{(s,t)}\Upsilon(s,t) = \int_z^s \int_0^t \frac{(s-\epsilon)^{u_1-1}}{\Gamma(u_1)} \frac{(t-\tau)^{u_2-1}}{\Gamma(u_2)} \Upsilon(\epsilon,\tau) d\tau d\epsilon.$$

Definition 2.2 [29]. Let $e = (e_1, e_2)$ $(e_1, e_2 \in (0,1])$ and $\Upsilon : \Theta_z \to \mathbb{R}^n$ satisfy $\frac{\partial^2}{\partial s \partial t}\Upsilon(s,t) \in L^1(\Theta_z, \mathbb{R}^n)$. the Caputo fractional derivative of Υ with order e is defined by

$$^{C}_{(z+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t) = {}_{(z+,0+)}\mathcal{I}^{1-e}_{(s,t)}\left(\frac{\partial^2}{\partial s \partial t}\Upsilon(s,t)\right).$$

Lemma 2.3 [32]. Let the function $g : \Theta_z \to \mathbb{R}^n$ satisfy $\frac{\partial^2}{\partial s \partial t} g(s,t) \in C(\Theta_z, \mathbb{R}^n)$. If $\Upsilon(s,t) \in C(\Theta_z, \mathbb{R}^n)$, then $\Upsilon(s,t)$ satisfies

$$^{C}_{(z+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t) = g(s,t), \quad (s,t) \in \Theta_z,$$

iff $\Upsilon(s,t)$ satisfies

$$\begin{aligned}\Upsilon(s,t) &= \Upsilon(s,0) + \Upsilon(z+,t) - \Upsilon(z+,0) + {}_{(z+,0+)}\mathcal{I}^{e}_{(s,t)} g(s,t) \\ &= \Upsilon(s,0) + \Upsilon(z+,t) - \Upsilon(z+,0) + \int_z^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1} g(\epsilon,\tau)}{\Gamma(e_1)\Gamma(e_2)} d\tau d\epsilon, \\ &\text{for } (s,t) \in \Theta_z.\end{aligned}$$

Next, we use *Definitions 2.1-2.2* to consider the properties of piecewise function. Let

$$\begin{aligned} f(s,t) &= \begin{cases} f_0(s,t), & (s,t) \in \Theta_0, \\ f_1(s,t), & (s,t) \in \Theta_1, \\ \quad\vdots & \\ f_J(s,t), & (s,t) \in \Theta_J, \end{cases} \\ &= \begin{cases} f_0(s,t), & (s,t) \in \Theta_0, \\ 0, & (s,t) \in \cup_{k=1}^{J}\Theta_k, \end{cases} + \begin{cases} 0, & (s,t) \in \Theta_0, \\ f_1(s,t), & (s,t) \in \Theta_1, \\ 0, & (s,t) \in \cup_{k=2}^{J}\Theta_k, \end{cases} \\ &\quad + \ldots + \begin{cases} 0, & (s,t) \in \cup_{k=0}^{J-2}\Theta_k, \\ f_{J-1}(s,t), & (s,t) \in \Theta_{J-1}, \\ 0, & (s,t) \in \Theta_J, \end{cases} + \begin{cases} 0, & (s,t) \in \cup_{k=0}^{J-1}\Theta_k, \\ f_J(s,t), & (s,t) \in \Theta_J, \end{cases} \end{aligned} \tag{2.1}$$

Property 2.4. Let $e=(e_1,e_2)$ (here $e_1,e_2\in(0,1]$) and $f_k:\overline{\Theta}_k\to\mathbb{R}^n$ satisfy $\frac{\partial^2}{\partial s\partial t}f_k(s,t)\in C(\overline{\Theta}_k,\mathbb{R}^n)$ (here $k=0,1,...,J$), then the fractional derivative of (2.1) can be computed by

$$
\begin{aligned}
{}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}f(s,t) &= \begin{cases} {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}f_0(s,t), & (s,t)\in\Theta_0,\\ \displaystyle\int_0^{s_1}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2 f_0(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon, & (s,t)\in\cup_{k=1}^{J}\Theta_k,\end{cases}\\
&+\begin{cases} 0, & (s,t)\in\Theta_0,\\ {}^{C}_{(s_1^+,0+)}\mathcal{D}^{e}_{(s,t)}f_1(s,t), & (s,t)\in\Theta_1,\\ \displaystyle\int_{s_1}^{s_2}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2 f_1(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon, & (s,t)\in\cup_{k=2}^{J}\Theta_k,\end{cases}\\
&+\dots+\begin{cases} 0, & (s,t)\in\cup_{k=0}^{J-2}\Theta_k,\\ {}^{C}_{(s_{J-1}^+,0+)}\mathcal{D}^{e}_{(s,t)}f_{J-1}(s,t), & (s,t)\in\Theta_{J-1},\\ \displaystyle\int_{s_{J-1}}^{s_J}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2 f_{J-1}(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon, & (s,t)\in\Theta_J,\end{cases}\\
&+\begin{cases} 0, & (s,t)\in\cup_{k=0}^{J-1}\Theta_k,\\ {}^{C}_{(s_J^+,0+)}\mathcal{D}^{e}_{(s,t)}f_J(s,t), & (s,t)\in\Theta_J.\end{cases}
\end{aligned}\tag{2.2}
$$

Property 2.5. Let $e=(e_1,e_2)$ (here $e_1,e_2\in(0,1]$) and $f_k:\overline{\Theta}_k\to\mathbb{R}^n$ satisfy $f_k(s,t)\in C(\overline{\Theta}_k,\mathbb{R}^n)$ (here $k=0,1,...,J$), then the fractional integral of (2.1) can be computed by

$$
\begin{aligned}
{}_{(0+,0+)}\mathcal{I}^{e}_{(s,t)}f(s,t) &= \begin{cases} {}_{(0+,0+)}\mathcal{I}^{e}_{(s,t)}f_0(s,t), & (s,t)\in\Theta_0,\\ \displaystyle\int_0^{s_1}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}f_0(\epsilon,\tau)}{\Gamma(e_1)\Gamma(e_2)}d\tau d\epsilon, & (s,t)\in\cup_{k=1}^{J}\Theta_k,\end{cases}\\
&+\begin{cases} 0, & (s,t)\in\Theta_0,\\ {}_{(s_1^+,0+)}\mathcal{I}^{e}_{(s,t)}f_1(s,t), & (s,t)\in\Theta_1,\\ \displaystyle\int_{s_1}^{s_2}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}f_1(\epsilon,\tau)}{\Gamma(e_1)\Gamma(e_2)}d\tau d\epsilon, & (s,t)\in\cup_{k=2}^{J}\Theta_k,\end{cases}\\
&+\dots+\begin{cases} 0, & (s,t)\in\cup_{k=0}^{J-2}\Theta_k,\\ {}_{(s_{J-1}^+,0+)}\mathcal{I}^{e}_{(s,t)}f_{J-1}(s,t), & (s,t)\in\Theta_{J-1},\\ \displaystyle\int_{s_{J-1}}^{s_J}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}f_{J-1}(\epsilon,\tau)}{\Gamma(e_1)\Gamma(e_2)}d\tau d\epsilon, & (s,t)\in\Theta_J,\end{cases}\\
&+\begin{cases} 0, & (s,t)\in\cup_{k=0}^{J-1}\Theta_k,\\ {}_{(s_J^+,0+)}\mathcal{I}^{e}_{(s,t)}f_J(s,t), & (s,t)\in\Theta_J.\end{cases}
\end{aligned}\tag{2.3}
$$

3 The integral solution of (1.1)

We first present some limit properties of (1.1) and next illustrate that there are two piecewise functions satisfying the conditions in (1.1) to expound that the above integral solutions of (1.1) given in existing papers are inappropriate, and finally use the basic limit properties and *Properties 2.4-2.5* to re-explore the correct integral solution of (1.1) in this section.

For simplifying some formulas, let

$$\begin{aligned}
&\Phi_i(s,t)=\int_{s_i}^{s}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, \qquad i=0,1,...,J;\\
&\Psi_j(s,t)=\int_0^{s_j}\int_0^t \frac{(s_j-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon+\int_{s_j}^{s}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon \\
&\qquad -\int_0^{s}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, \quad j=1,2,...,J.
\end{aligned} \tag{3.1}$$

3.1 The basic limit properties of (1.1)

Consider two limit cases of (1.1):

$$\begin{aligned}
&\lim_{U_j(\Upsilon(s_j^-,t))\to 0 \text{ for } \forall j\in\{1,2,...,J\}}\{\text{system (1.1)}\}\\
&=\begin{cases} {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t)=F(s,t,\Upsilon(s,t)), & (s,t)\in\Theta,\\ \Upsilon(s,0)=\eta(s),\Upsilon(0,t)=\xi(t), & s\in[0,b],t\in[0,c],\end{cases}\\
&\Leftrightarrow \Upsilon(s,t)=\Lambda(s,t)+\int_0^{s}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, \quad (s,t)\in\Theta;
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&\lim_{s_j\to s_l \text{ for } \forall j\in\{1,2,...,J\} \text{ and } l\in\{1,2,...,J\}}\{\text{system (1.1)}\}\\
&=\begin{cases} {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t)=F(s,t,\Upsilon(s,t)), & (s,t)\in\Theta, s\neq s_l,\\ \Upsilon(s_l^+,t)-\Upsilon(s_l^-,t)=\sum_{j=1}^{J}U_j(\Upsilon(s_l^-,t)), & t\in[0,c],\\ \Upsilon(s,0)=\eta(s),\Upsilon(0,t)=\xi(t), & s\in[0,b],t\in[0,c].\end{cases}
\end{aligned} \tag{3.3}$$

3.2 Two piecewise functions satisfy the conditions in (1.1)

We will expound that there are two piecewise functions satisfying the conditions in (1.1).

Because $\Lambda(s,t)+\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon$ satisfies ${}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t)=F(s,t,\Upsilon(s,t))$ on whole interval Θ, it is sure that the piecewise function (1.4) satisfies the conditions of (1.1).

Next we will discuss whether (1.2) meets the fractional derivative of (1.1). Use *Definition 2.1* to calculate the fractional derivative of (1.2):

$$\begin{aligned}
&{}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t)\Big|_{(s,t)\in\Theta_0}=F(s,t,\Upsilon(s,t)), \quad (s,t)\in\Theta_0,\\
&{}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t)\Big|_{(s,t)\in\Theta_j}=\int_0^{s_1}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\Phi_0(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon\\
&\qquad +\int_{s_1}^{s_2}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\Phi_1(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon+...\\
&\qquad +\int_{s_j}^{s}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\Phi_j(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon, \quad (s,t)\in\Theta_j\\
&\neq \int_0^{s}\int_0^t \frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\Phi_0(\epsilon,\tau)}{\partial\epsilon\partial\tau}d\tau d\epsilon, \qquad (s,t)\in\Theta_j\\
&=F(s,t,\Upsilon(s,t)), \quad (s,t)\in\Theta_j \quad (\text{here } j\in\{1,2,...,J\}),
\end{aligned} \tag{3.4}$$

which means that (1.2) doesn't satisfy the fractional derivative of (1.1).

However, we can use (1.2) to construct another piecewise function satisfying the conditions in (1.1) as follows

$$\Upsilon(s,t)=\begin{cases}\Lambda(s,t)+\displaystyle\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, & (s,t)\in\Theta_0,\\ \Lambda(s_j^+,t)+\displaystyle\int_{s_j}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, & (s,t)\in\Theta_j, j=1,2,...,J,\end{cases}$$
$$-\sum_{i=0}^{J-1}\begin{cases}0, \quad (s,t)\in[s_0,s_{i+1}]\times[0,c],\\ \displaystyle\int_{s_{i+1}}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}\left[\int_{s_i}^{s_{i+1}}\int_0^\tau\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\right.\\ \quad\left.\times\dfrac{\partial^2[\Phi_i(u,v)]}{\partial u\partial v}dvdu\right]d\tau d\epsilon, \quad (s,t)\in(s_{i+1},b]\times[0,c].\end{cases} \tag{3.5}$$

We use (2.2) to obtain the fractional derivative of (3.5)

$$\begin{aligned}&{}^C_{(0+,0+)}\mathcal{D}^e_{(s,t)}\Upsilon(s,t)\\ &=\begin{cases}F(s,t,\Upsilon(s,t)), \quad (s,t)\in\Theta_0,\\ \displaystyle\int_{s_0}^{s_1}\int_0^t\frac{(s-u)^{-e_1}(t-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\Phi_0(u,v)]}{\partial u\partial v}dvdu, \quad (s,t)\in(x_1,b]\times[0,c],\end{cases}\\ &\quad+\sum_{j=1}^{J-1}\begin{cases}0, & (s,t)\in[0,s_j]\times[0,c],\\ F(s,t,\Upsilon(s,t)), & (s,t)\in\Theta_j,\\ \displaystyle\int_{s_j}^{s_{j+1}}\int_0^t\frac{(s-u)^{-e_1}(t-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\Phi_j(u,v)]}{\partial u\partial v}dvdu, & (s,t)\in(s_{j+1},b]\times[0,c],\end{cases}\\ &\quad+\begin{cases}0, & (s,t)\in[0,s_J]\times[0,c],\\ F(s,t,\Upsilon(s,t)), & (s,t)\in\Theta_J,\end{cases}\\ &\quad-\sum_{i=0}^{J-1}\begin{cases}0, \quad (s,t)\in[0,s_{i+1}]\times[0,c],\\ \displaystyle\int_{s_i}^{s_{i+1}}\int_0^t\frac{(s-u)^{-e_1}(t-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\Phi_i(u,v)]}{\partial u\partial v}dvdu, \quad (s,t)\in(s_{i+1},b]\times[0,c],\end{cases}\\ &=F(s,t,\Upsilon(s,t)), \quad (s,t)\in\cup_{i=0}^J\Theta_i.\end{aligned} \tag{3.6}$$

Therefore, it can be verified that both (1.4) and (3.5) satisfy these conditions of the initial value, the impulses and the fractional derivative in (1.1).

Additionally, by the above discussion, (1.4) is only *a special solution* of (1.1) because it doesn't contain the key part $\Phi_j(s,t)$ $(j=1,2,...,J)$, but (3.5) is not the solution of (1.1) because it doesn't satisfy (3.2).

3.3 The integral solution of (1.1)

We will use *Properties 2.4-2.5*, two limit properties ((3.2) and (3.3)) and the thoughts of (1.4) and (3.5) to re-explore the integral solution of (1.1). Define the space of functions

$$
\begin{aligned}
IC_P(\Theta,\mathbb{R}^n) := \Bigg\{ & \Upsilon(s,t) : \cup_{i=0}^{J}\Theta_i \to \mathbb{R}^n \text{ and } \frac{\partial^2}{\partial s\partial t}\Upsilon(s,t) \in C(\cup_{i=0}^{J}\Theta_i,\mathbb{R}^n), \\
& \Upsilon(s_j^-,t) = \lim_{s\to s_j^-}\Upsilon(s,t) = \Upsilon(s_j,t) < \infty,\ \Upsilon(s_j^+,t) = \lim_{s\to s_j^+}\Upsilon(s,t) < \infty, \\
& \lim_{s\to s_j^-}\left[\frac{\partial^2}{\partial s\partial t}\Upsilon(s,t)\right] < \infty, \lim_{s\to s_j^+}\left[\frac{\partial^2}{\partial s\partial t}\Upsilon(s,t)\right] < \infty,\ \text{here } j = 1,2,...,J \Bigg\}.
\end{aligned}
$$

Lemma 3.1. Let $F : \Theta \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy $\frac{\partial^2}{\partial s\partial t}F(s,t,\Upsilon(s,t)) \in C(\overline{\Theta}_i \times \mathbb{R}^n,\mathbb{R}^n)$ for $\frac{\partial^2}{\partial s\partial t}\Upsilon(s,t) \in C(\overline{\Theta}_i,\mathbb{R}^n)$ (here $i = 0,1,...,J$), and let $\rho(t)$ be an arbitrary continuously differentiable function of t on $[0,c]$, then

$$
{}_{(0+,0+)}^{\ \ C}\mathcal{D}^{e}_{(s,t)}\begin{cases} 0, \quad (s,t) \in [0,s_j]\times[0,c], \\ \rho(t)\Psi_j(s,t) - \displaystyle\int_{s_j}^{s}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} \\ \quad\times \left[\displaystyle\int_0^{s_j}\int_0^{\tau}\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\left[\rho(v)\Phi_0(u,v)\right]}{\partial u\partial v}dvdu\right]d\tau d\epsilon, \\ \quad (s,t) \in (s_j,b]\times[0,c],\ \text{here } j = 1,2,...,J; \end{cases} = 0. \tag{3.7}
$$

The proof will be given in the *Appendix*.

Theorem 3.2. Let $F : \Theta \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy $\frac{\partial^2}{\partial s\partial t}F(s,t,\Upsilon(s,t)) \in C(\overline{\Theta}_i \times \mathbb{R}^n,\mathbb{R}^n)$ for $\frac{\partial^2}{\partial s\partial t}\Upsilon(s,t) \in C(\overline{\Theta}_i,\mathbb{R}^n)$ (here $i = 0,1,...,J$), and let $\lambda(t)$ be an arbitrary continuously differentiable function of t on $[0,c]$.

If $\Upsilon(s,t) \in IC_P(\Theta,\mathbb{R}^n)$, then $\Upsilon(s,t)$ satisfies (1.1) iff $\Upsilon(s,t)$ satisfies

$$
\begin{aligned}
\Upsilon(s,t) = & \begin{cases} \Lambda(s,t) + \displaystyle\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, \quad (s,t)\in\Theta_0, \\ \Lambda(s,t) + \displaystyle\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon \\ \quad + \displaystyle\sum_{k=1}^{j}\left[U_k(\Upsilon(s_k^-,t)) - U_k(\Upsilon(s_k^-,0))\right], \quad (s,t)\in\Theta_j, j=1,2,...,J, \end{cases} \\
& + \sum_{j=1}^{J}\begin{cases} 0, \quad (s,t)\in[0,s_j]\times[0,c], \\ \lambda(t)U_j\left(\Upsilon(s_j^-,t)\right)\Psi_j(s,t) - \displaystyle\int_{s_j}^{s}\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} \\ \quad\times\left[\displaystyle\int_0^{s_j}\int_0^{\tau}\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\lambda(v)U_j(\Upsilon(s_j^-,v))\Phi_0(u,v)]}{\partial u\partial v}dvdu\right]d\tau d\epsilon, \\ \quad (s,t)\in(s_j,b]\times[0,c], \end{cases}
\end{aligned} \tag{3.8}
$$

Proof. 'Necessity'. It will be proved that the integral solution of (1.1) is (3.8). Let us consider (1.1) with an impulse:

$$
\begin{cases} {}_{(0+,0+)}^{\ \ C}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t) = F(s,t,\Upsilon(s,t)), \quad (s,t)\in\Theta, s\neq s_j, \\ \Upsilon(s_j^+,t) - \Upsilon(s_j^-,t) = U_j(\Upsilon(s_j^-,t)), \qquad t\in[0,c], \\ \Upsilon(s,0)=\eta(s), \Upsilon(0,t)=\xi(t), \qquad s\in[0,b], t\in[0,c], \end{cases} \tag{3.9}
$$

By *Lemma 2.3* and the limit property (3.2), the integral solution of (3.9) satisfies

$$\Upsilon(s,t)=\Lambda(s,t)+\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon,\quad (s,t)\in[0,s_j]\times[0,c]; \tag{3.10}$$

and

$$\lim_{U_j(\Upsilon(s_j^-,t))\to 0}\Upsilon(s,t)=\Lambda(s,t)+\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon,\quad (s,t)\in\Theta. \tag{3.11}$$

Next, by using (3.10), it is similar with (3.5) to construct a piecewise function $\widetilde{\Upsilon}(s,t)$

$$\begin{aligned}
&\widetilde{\Upsilon}(s,t)\\
&=\begin{cases}\Lambda(s,t)+\displaystyle\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon,\quad (s,t)\in[0,s_j]\times[0,c],\\
\Lambda(s,t)+U_j(\Upsilon(s_j^-,t))-U_j(\Upsilon(s_j^-,0))+\displaystyle\int_0^{s_j}\int_0^t\frac{(s_j-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon\\
\quad+\displaystyle\int_{s_j}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon,\quad (s,t)\in(s_j,b]\times[0,c],\end{cases}\\
&\quad-\begin{cases}0,\quad (s,t)\in[0,s_j]\times[0,c],\\
\displaystyle\int_{s_j}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}\left[\int_0^{s_j}\int_0^\tau\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\Phi_0(u,v)}{\partial u\partial v}dvdu\right]d\tau d\epsilon,\\
\quad (s,t)\in(s_j,b]\times[0,c],\end{cases}
\end{aligned} \tag{3.12}$$

Remark 3.3. (3.12) satisfies these conditions of the initial value, the impulses and fractional derivative in (3.9), but (3.12) doesn't satisfy (3.11) to be only as an approximate solution with $e(s,t)=\Upsilon(s,t)-\widetilde{\Upsilon}(s,t)$ for $(s,t)\in\Theta$.

Thus, by (3.11) and (3.12), we have

$$\begin{aligned}
&\lim_{U_j(\Upsilon(s_j^-,t))\to 0}e(s,t)=\lim_{U_j(\Upsilon(s_j^-,t))\to 0}\left\{\Upsilon(s,t)-\widetilde{\Upsilon}(s,t)\right\}\quad\text{for } (s,t)\in\Theta,\\
&=\begin{cases}0,\quad (s,t)\in[0,s_j]\times[0,c],\\
\displaystyle\int_{s_j}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}\left[\int_0^{s_j}\int_0^\tau\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\Phi_0(u,v)}{\partial u\partial v}dvdu\right]d\tau d\epsilon\\
\quad-\Psi_j(s,t),\quad (s,t)\in(s_j,b]\times[0,c],\end{cases}
\end{aligned} \tag{3.13}$$

From (3.13), we think that $e(s,t)$ connects with $U_j(\Upsilon(s_j^-,t))$ and $\lim_{U_j(\Upsilon(s_j^-,t))\to 0}e(s,t)$ and assume

$$\begin{aligned}
&e(s,t)=g\left(U_j(\Upsilon(s_j^-,t))\right)\lim_{U_j(\Upsilon(s_j^-,t))\to 0}e(s,t)\quad\text{(here } g(\cdot)\text{ is an undetermined function)}\\
&=-\begin{cases}0,\quad (s,t)\in[0,s_j]\times[0,c],\\
g\left(U_j(\Upsilon(s_j^-,t))\right)\Psi_j(s,t)-\displaystyle\int_{s_j}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}\\
\quad\times\left[\displaystyle\int_0^{s_j}\int_0^\tau\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\left[g(U_j(\Upsilon(s_j^-,v)))\Phi_0(u,v)\right]}{\partial u\partial v}dvdu\right]d\tau d\epsilon,\\
(s,t)\in(s_j,b]\times[0,c],\end{cases}
\end{aligned} \tag{3.14}$$

By (3.12) and (3.14), we get

$$
\begin{aligned}
\Upsilon(s,t) &= \widetilde{\Upsilon}(s,t) + e(s,t) \quad \text{for } (s,t)\in\Theta, \\
&= \begin{cases}
\Lambda(s,t) + \displaystyle\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} F d\tau d\epsilon, \quad (s,t)\in[0,s_j]\times[0,c], \\
\Lambda(s,t) + \displaystyle\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} F d\tau d\epsilon \\
\quad + U_j(\Upsilon(s_j^-,t)) - U_j(\Upsilon(s_j^-,0)), \qquad (s,t)\in(s_j,b]\times[0,c],
\end{cases} \\
&+ \begin{cases}
0, \quad (s,t)\in[0,s_j]\times[0,c], \\
\left[1-g\left(U_j(\Upsilon(s_j^-,t))\right)\right]\Psi_j(s,t) - \displaystyle\int_{s_j}^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} \\
\times \left[\displaystyle\int_0^{s_j}\int_0^\tau \frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)} \frac{\partial^2\left[(1-g(U_j(\Upsilon(s_j^-,v))))\Phi_0(u,v)\right]}{\partial u\partial v} dvdu\right] d\tau d\epsilon \\
(s,t)\in(s_j,b]\times[0,c],
\end{cases}
\end{aligned} \tag{3.15}
$$

Furthermore, by (3.2) and (3.3), it shows that the impulsive effects have the linear additivity. Therefore, by combining with the special solution (1.4), we obtain the solution of (1.1) as

$$
\begin{aligned}
\Upsilon(s,t) &= \begin{cases}
\Lambda(s,t) + \displaystyle\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} F d\tau d\epsilon, \quad (s,t)\in\Theta_0, \\
\Lambda(s,t) + \displaystyle\int_0^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} F d\tau d\epsilon \\
\quad + \displaystyle\sum_{i=1}^{j}\left[U_i(\Upsilon(s_i^-,t)) - U_i(\Upsilon(s_i^-,0))\right], \quad (s,t)\in\Theta_j, j=1,2,...,J,
\end{cases} \\
&+ \sum_{j=1}^{J}\begin{cases}
0, \quad (s,t)\in[0,s_j]\times[0,c], \\
\left[1-g\left(U_j(\Upsilon(s_j^-,t))\right)\right]\Psi_j(s,t) - \displaystyle\int_{s_j}^s\int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} \\
\times \left[\displaystyle\int_0^{s_j}\int_0^\tau \frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)} \frac{\partial^2\left[(1-g(U_j(\Upsilon(s_j^-,v))))\Phi_0(u,v)\right]}{\partial u\partial v} dvdu\right] d\tau d\epsilon \\
(s,t)\in(s_j,b]\times[0,c],
\end{cases}
\end{aligned} \tag{3.16}
$$

Moreover, (3.16) satisfies (3.3) such that

$$
\begin{aligned}
1-g[U_i(\Upsilon(s_i^-,t)) + U_j(\Upsilon(s_j^-,t))] &= 1-g(U_i(\Upsilon(s_i^-,t))) + 1-g(U_j(\Upsilon(s_j^-,t))) \\
&\quad \text{for } \forall U_i(\Upsilon(s_i^-,t)), U_j(\Upsilon(s_j^-,t))\in\mathbb{R}^n.
\end{aligned} \tag{3.17}
$$

Then

$$
1-g(U_j(\Upsilon(s_j^-,t))) = \lambda(t)U_j(\Upsilon(s_j^-,t)) \tag{3.18}
$$

here $\lambda(t)$ is an arbitrary continuously differentiable function, and then (3.16) is (3.8).

'Sufficiency'. By *Lemma 3.1*, we obtain the fractional derivative of a part of (3.8):

$$
{}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\begin{cases} 0, \quad (s,t)\in[0,s_j]\times[0,c], \\ \lambda(t)U_j(\Upsilon(s_j^-,t))\Psi_j(s,t)-\displaystyle\int_{s_j}^{s}\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} \\ \quad\times\left[\displaystyle\int_0^{s_j}\int_0^{\tau}\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2\left[\lambda(v)U_j(\Upsilon(s_j^-,v))\Phi_0(u,v)\right]}{\partial u\partial v}dvdu\right]d\tau d\epsilon, \\ \quad(s,t)\in(s_j,b]\times[0,c], \text{ here } j=1,2,...,J, \end{cases} \tag{3.19}
$$

$$= 0.$$

Then the fractional derivative of (3.8) satisfies

$$
\begin{aligned}
&{}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t) \\
&= {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\begin{cases}\Lambda(s,t)+\displaystyle\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, \quad (s,t)\in\Theta_0, \\ \Lambda(s,t)+\displaystyle\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon \\ \quad+\displaystyle\sum_{i=1}^{j}\left[U_i(\Upsilon(s_i^-,t))-U_i(\Upsilon(s_i^-,0))\right], \quad (s,t)\in\Theta_j, j=1,2,...,J,\end{cases} \\
&= F(s,t,\Upsilon(s,t)) \quad \text{for } (s,t)\in\cup_{i=0}^{J}\Theta_i.
\end{aligned} \tag{3.20}
$$

For each s_j $(j=1,2,...,J)$ in (3.8), we have

$$
\begin{aligned}
\Upsilon(s_j^+,t)-\Upsilon(s_j^-,t) &= \Lambda(s_j^+,t)-\Lambda(s_j^-,t)+U_j(\Upsilon(s_j^-,t))-U_j(\Upsilon(s_j^-,0)) \\
&= \eta(s_j^+)-\eta(s_j^-)+U_j(\Upsilon(s_j^-,t))-U_j(\Upsilon(s_j^-,0)) \\
&= U_j(\Upsilon(s_j^-,t)).
\end{aligned} \tag{3.21}
$$

Finally, letting $U_j(\Upsilon(s_j^-,t))\to 0$ for $\forall j\in\{1,2,...,J\}$ in (3.8), we obtain

$$
\begin{aligned}
&\lim_{U_j(\Upsilon(s_j^-,t))\to 0 \text{ for all } j\in\{1,2,...,J\}}\{\text{equation (3.8)}\} \\
&\Leftrightarrow \Upsilon(s,t)=\Lambda(s,t)+\int_0^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon, \quad (s,t)\in\Theta, \\
&\Leftrightarrow \begin{cases}{}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)}\Upsilon(s,t)=F(s,t,\Upsilon(s,t)), & (s,t)\in\Theta, \\ \Upsilon(s,0)=\eta(s), \Upsilon(0,t)=\xi(t), & s\in[0,b], t\in[0,c],\end{cases} \\
&= \lim_{U_j(\Upsilon(s_j^-,t))\to 0 \text{ for all } j\in\{1,2,...,J\}}\{\text{system (1.1)}\}.
\end{aligned} \tag{3.22}
$$

Moreover, it is obvious that (3.8) satisfies the property (3.3). Thus the proof is now accomplished. □

4 Applications

We will offer an example to show the computation of solution of the IFrPDOS and illustrate that there are many solutions for the IFrPDOS in this section.

Example 4.1. We consider a IFrPDOS

$$\begin{cases} {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)} w(s,t) = st, & (s,t) \in [0,3] \times [0,3], s \neq 1, \\ w(1^+,t) - w(1^-,t) = t, & t \in [0,3], \\ w(s,0) = w(0,t) \equiv 0, & s \in [0,3], t \in [0,3], \end{cases} \tag{4.1}$$

where $e = (e_1, e_2) = (\frac{1}{2}, \frac{1}{2})$.

Using *Theorem 3.2*, the solution of (4.1) is

$$w(s,t) = \begin{cases} \frac{16}{9\pi} s^{\frac{3}{2}} t^{\frac{3}{2}}, & (s,t) \in [0,1] \times [0,3], \\ \frac{16}{9\pi} s^{\frac{3}{2}} t^{\frac{3}{2}} + t, & (s,t) \in (1,3] \times [0,3], \end{cases}$$
$$+ \begin{cases} 0, \quad (s,t) \in [0,1] \times [0,3], \\ \frac{8\lambda(t)}{9\pi} \left[2t^{\frac{5}{2}} + (s-1)^{\frac{1}{2}}(2s+1)t^{\frac{5}{2}} - 2s^{\frac{3}{2}} t^{\frac{5}{2}} \right] - \int_1^s \int_0^t \frac{(s-\epsilon)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \\ \quad \times \left[\int_0^1 \int_0^\tau \frac{(\epsilon-u)^{-\frac{1}{2}}(\tau-v)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\partial^2}{\partial u \partial v} \left(\lambda(v) \frac{16}{9\pi} u^{\frac{3}{2}} v^{\frac{5}{2}} \right) dv du \right] d\tau d\epsilon, \\ (s,t) \in (1,3] \times [0,3], \end{cases} \tag{4.2}$$

where $\lambda(t)$ is an arbitrary continuously differentiable function of t on $[0,3]$.

Because of

$$\begin{aligned} \Phi_0(s,t) &= \int_0^s \int_0^t \frac{(s-\epsilon)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \epsilon\tau d\tau d\epsilon = \frac{16}{9\pi} s^{\frac{3}{2}} t^{\frac{3}{2}}; \\ \Psi_1(s,t) &= \int_0^1 \int_0^t \frac{(s-\epsilon)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \epsilon\tau d\tau d\epsilon + \int_1^s \int_0^t \frac{(s-\epsilon)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \epsilon\tau d\tau d\epsilon \\ &\quad - \int_0^s \int_0^t \frac{(s-\epsilon)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \epsilon\tau d\tau d\epsilon \\ &= \frac{16}{9\pi} t^{\frac{3}{2}} + \frac{8}{9\pi}(s-1)^{\frac{1}{2}}(2s+1)t^{\frac{3}{2}} - \frac{16}{9\pi} s^{\frac{3}{2}} t^{\frac{3}{2}}, \end{aligned} \tag{4.3}$$

by *Lemma 3.1*, we have

$$\begin{aligned} {}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)} & \begin{cases} 0, \quad (s,t) \in [0,1] \times [0,3], \\ \frac{8\lambda(t)[2t^{\frac{5}{2}} + (s-1)^{\frac{1}{2}}(2s+1)t^{\frac{5}{2}} - 2s^{\frac{3}{2}} t^{\frac{5}{2}}]}{9\pi} - \int_1^s \int_0^t \frac{(s-\epsilon)^{-\frac{1}{2}}(t-\tau)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \\ \quad \times \left[\int_0^1 \int_0^\tau \frac{(\epsilon-u)^{-\frac{1}{2}}(\tau-v)^{-\frac{1}{2}}}{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})} \frac{\partial^2}{\partial u \partial v} \left(\lambda(v) \frac{16}{9\pi} u^{\frac{3}{2}} v^{\frac{5}{2}} \right) dv du \right] d\tau d\epsilon, \\ (s,t) \in (1,3] \times [0,3], \end{cases} \\ &= 0 \end{aligned} \tag{4.4}$$

Thus, we come to that (4.2) satisfies ${}^{C}_{(0+,0+)}\mathcal{D}^{e}_{(s,t)} w(s,t) = st$ for $(s,t) \in ([0,3] \times [0,3]$ and $s \neq 1)$ in (4.1). Moreover, (4.2) also satisfies $w(1^+,t) - w(1^-,t) = t$ for $t \in [0,3]$ and $w(s,0) = w(0,t) \equiv 0$ for $s \in [0,3], t \in [0,3]$ in (4.1). Because of the arbitrariness of $\lambda(t)$, (4.1) has many solutions.

5 Conclusion

It is found that there are two piecewise functions to satisfy the conditions in (1.1). And then, by constructing piecewise function and combining with the limit properties of (1.1), the integral solution of (1.1) is found that is an integral equality with an arbitrary continuously differentiable function $\lambda(t)$ to reveal the non-uniqueness for solution of (1.1) due to the arbitrariness of $\lambda(t)$.

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7 Appendix A

We will give the proof of *Lemma 3.1*.

Proof. Using *Definitions 2.1-2.2* and *Property 2.4*, we rewrite (3.7) into

$$
\begin{aligned}
{}_{(0+,0+)}^{\ \ C}\mathcal{D}_{(s,t)}^{e}&\begin{cases} 0, \quad (s,t)\in[0,s_j]\times[0,c],\\ \rho(t)\Psi_j(s,t)-\displaystyle\int_{s_j}^{s}\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}\\ \quad\times\left[\displaystyle\int_0^{s_j}\int_0^{\tau}\frac{(\epsilon-u)^{-e_1}(\tau-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\rho(v)\Phi_0(u,v)]}{\partial u\partial v}dvdu\right]d\tau d\epsilon,\\ \quad(s,t)\in(s_j,b]\times[0,c],\end{cases}\\
&=\begin{cases} 0, \quad (s,t)\in[0,s_j]\times[0,c],\\ \displaystyle\int_{s_j}^{s}\int_0^t\frac{(s-u)^{-e_1}(t-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\rho(v)\Phi_j(u,v)-\rho(v)\Phi_0(u,v)]}{\partial u\partial v}dvdu\\ -\displaystyle\int_0^{s_j}\int_0^t\frac{(s-u)^{-e_1}(t-v)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\frac{\partial^2[\rho(v)\Phi_0(u,v)]}{\partial u\partial v}dvdu,\\ (s,t)\in(s_j,b]\times[0,c],\end{cases}\\
&=\begin{cases} 0, \quad (s,t)\in[0,s_j]\times[0,c],\\ {}_{(s_j^+,0+)}\mathcal{I}_{(s,t)}^{1-e}\dfrac{\partial^2[\rho(t)\Phi_j(s,t)]}{\partial s\partial t}-{}_{(0+,0+)}\mathcal{I}_{(s,t)}^{1-e}\dfrac{\partial^2[\rho(t)\Phi_0(s,t)]}{\partial s\partial t}, \quad (s,t)\in(s_j,b]\times[0,c].\end{cases}
\end{aligned}
\tag{A.1}
$$

To calculate ${}_{(s_j^+,0+)}\mathcal{I}_{(s,t)}^{1-e}\frac{\partial^2[\rho(t)\Phi_j(s,t)]}{\partial s\partial t}={}_{(s_j^+,0+)}\mathcal{I}_{(s,t)}^{1-e}\frac{\partial^2}{\partial s\partial t}\left[\rho(t)\int_{s_j}^s\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon\right]$, we transform $\int_{s_j}^s\int_0^t(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}Fd\tau d\epsilon$ into

$$
\begin{aligned}
&\int_{s_j}^{s}\int_0^t(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}Fd\tau d\epsilon=\frac{1}{e_1e_2}\int_{s_j}^{s}\left[\int_0^t Fd(t-\tau)^{e_2}\right]d(s-\epsilon)^{e_1}\\
&=\frac{1}{e_1e_2}\left\{(s-s_j)^{e_1}t^{e_2}F(s_j,0,\Upsilon(s_j,0))+(s-s_j)^{e_1}\int_0^t(t-\tau)^{e_2}\frac{\partial F(s_j,\tau,\Upsilon(s_j,\tau))}{\partial\tau}d\tau\right.\\
&\left.+t^{e_2}\int_{s_j}^{s}(s-\epsilon)^{e_1}\frac{\partial F(\epsilon,0,\Upsilon(\epsilon,0))}{\partial\epsilon}d\epsilon+\int_{s_j}^{s}(s-\epsilon)^{e_1}\left[\int_0^t(t-\tau)^{e_2}\frac{\partial^2F(\epsilon,\tau,\Upsilon(\epsilon,\tau))}{\partial\epsilon\partial\tau}d\tau\right]d\epsilon\right\}.
\end{aligned}
$$

Then

$$
\begin{aligned}
&{}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\frac{\partial^2}{\partial s\partial t}\left[\rho(t)\int_{s_j}^{s}\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon\right]\\
&=\frac{1}{\Gamma(e_1)\Gamma(e_2)}{}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\left\{(s-s_j)^{e_1-1}\left[t^{e_2-1}\rho(t)+\frac{1}{e_2}t^{e_2}\rho'(t)\right]F(s_j,0,\Upsilon(s_j,0))\right.\\
&\quad+(s-s_j)^{e_1-1}\int_0^t\left[(t-\tau)^{e_2-1}\rho(t)+\frac{1}{e_2}(t-\tau)^{e_2}\rho'(t)\right]\frac{\partial F(s_j,\tau,\Upsilon(s_j,\tau))}{\partial\tau}d\tau\\
&\quad+\left[t^{e_2-1}\rho(t)+\frac{1}{e_2}t^{e_2}\rho'(t)\right]\int_{s_j}^{s}(s-\epsilon)^{e_1-1}\frac{\partial F(\epsilon,0,\Upsilon(\epsilon,0))}{\partial\epsilon}d\epsilon\\
&\quad\left.+\int_{s_j}^{s}(s-\epsilon)^{e_1-1}\int_0^t\left[(t-\tau)^{e_2-1}\rho(t)+\frac{1}{e_2}(t-\tau)^{e_2}\rho'(t)\right]\frac{\partial^2F(\epsilon,\tau,\Upsilon(\epsilon,\tau))}{\partial\epsilon\partial\tau}d\tau d\epsilon\right\}.
\end{aligned}
\tag{A.2}
$$

We divide the right side of (A.2) into four parts (I)-(IV) as follows

$$
\begin{aligned}
\text{(I)}\quad&\frac{1}{\Gamma(e_1)\Gamma(e_2)}{}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\left\{(s-s_j)^{e_1-1}\left[t^{e_2-1}\rho(t)+\frac{1}{e_2}t^{e_2}\rho'(t)\right]F(s_j,0,\Upsilon(s_j,0))\right\}\\
&=\int_{s_j}^{s}\int_0^t\frac{(s-\epsilon)^{1-e_1-1}(\epsilon-s_j)^{e_1-1}(t-\tau)^{1-e_2-1}}{\Gamma(e_1)\Gamma(1-e_1)\Gamma(e_2)\Gamma(1-e_2)}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau d\epsilon\\
&\quad\times F(s_j,0,\Upsilon(s_j,0))\quad\text{(let }\epsilon-s_j=\phi)\\
&=\int_0^{s-s_j}\frac{(s-s_j-\phi)^{1-e_1-1}\phi^{e_1-1}}{\Gamma(e_1)\Gamma(1-e_1)}d\phi\int_0^t\frac{(t-\tau)^{1-e_2-1}}{\Gamma(e_2)\Gamma(1-e_2)}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau\\
&\quad\times F(s_j,0,\Upsilon(s_j,0))\\
&=F(s_j,0,\Upsilon(s_j,0))\int_0^t\frac{(t-\tau)^{1-e_2-1}}{\Gamma(e_2)\Gamma(1-e_2)}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau;
\end{aligned}
$$

$$
\begin{aligned}
\text{(II)}\quad&{}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\left\{\frac{(s-s_j)^{e_1-1}}{\Gamma(e_1)\Gamma(e_2)}\int_0^t\left[(t-v)^{e_2-1}\rho(t)+\frac{(t-v)^{e_2}}{e_2}\rho'(t)\right]\frac{\partial F(s_j,v,\Upsilon(s_j,v))}{\partial v}dv\right\}\\
&=\int_{s_j}^{s}\int_0^t\frac{(s-\epsilon)^{-e_1}(t-\tau)^{-e_2}}{\Gamma(1-e_1)\Gamma(1-e_2)}\left\{\frac{(\epsilon-s_j)^{e_1-1}}{\Gamma(e_1)\Gamma(e_2)}\int_0^\tau\left[(\tau-v)^{e_2-1}\rho(\tau)+\frac{1}{e_2}(\tau-v)^{e_2}\rho'(\tau)\right]\right.\\
&\quad\left.\times\frac{\partial F(s_j,v,\Upsilon(s_j,v))}{\partial v}dv\right\}d\tau d\epsilon\quad\text{(let }\epsilon-s_j=\phi)\\
&=\int_0^{s-s_j}\frac{(s-s_j-\phi)^{1-e_1-1}\phi^{e_1-1}}{\Gamma(e_1)\Gamma(1-e_1)}d\phi\int_0^t\frac{(t-\tau)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\\
&\quad\times\left(\int_0^\tau\left[(\tau-v)^{e_2-1}\rho(\tau)+\frac{1}{e_2}(\tau-v)^{e_2}\rho'(\tau)\right]\frac{\partial F(s_j,v,\Upsilon(s_j,v))}{\partial v}dv\right)d\tau\\
&=\int_0^t\int_v^t\frac{(t-\tau)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\left[(\tau-v)^{e_2-1}\rho(\tau)+\frac{1}{e_2}(\tau-v)^{e_2}\rho'(\tau)\right]\frac{\partial F(s_j,v,\Upsilon(s_j,v))}{\partial v}d\tau dv\\
&\quad\text{(let }\tau-v=\varphi)\\
&=\int_0^t\int_0^{t-v}\frac{(t-v-\varphi)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\left[\varphi^{e_2-1}\rho(\varphi+v)+\frac{1}{e_2}\varphi^{e_2}\rho'(\varphi+v)\right]\frac{\partial F(s_j,v,\Upsilon(s_j,v))}{\partial v}d\varphi dv;
\end{aligned}
$$

$$
\begin{aligned}
&\text{(III)}\quad {}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\left\{\left[t^{e_2-1}\rho(t)+\frac{1}{e_2}t^{e_2}\rho'(t)\right]\int_{s_j}^{s}\frac{(s-u)^{e_1-1}}{\Gamma(e_1)\Gamma(e_2)}\frac{\partial F(u,0,\Upsilon(u,0))}{\partial u}du\right\}\\
&\quad=\int_{s_j}^{s}\frac{\partial F(u,0,\Upsilon(u,0))}{\partial u}\int_{u}^{s}\frac{(s-\epsilon)^{-e_1}(\epsilon-u)^{e_1-1}}{\Gamma(e_1)\Gamma(1-e_1)\Gamma(e_2)\Gamma(1-e_2)}d\epsilon du\\
&\qquad\times\int_0^t(t-\tau)^{-e_2}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau\quad(\text{let }\epsilon-u=\phi)\\
&\quad=\int_{s_j}^{s}\frac{\partial F(u,0,\Upsilon(u,0))}{\partial u}\left[\int_0^{s-u}\frac{(s-u-\phi)^{-e_1}\phi^{e_1-1}}{\Gamma(e_1)\Gamma(1-e_1)}d\phi\right]du\\
&\qquad\times\int_0^t\frac{(t-\tau)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau\\
&\quad=\frac{F(s,0,\Upsilon(s,0))-F(s_j,0,\Upsilon(s_j,0))}{\Gamma(e_2)\Gamma(1-e_2)}\int_0^t(t-\tau)^{-e_2}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau;
\end{aligned}
$$

$$
\begin{aligned}
&\text{(IV)}\quad {}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\left\{\int_{s_j}^{s}\frac{(s-u)^{e_1-1}}{\Gamma(e_1)\Gamma(e_2)}\int_0^t\left[(t-v)^{e_2-1}\rho(t)+\frac{(t-v)^{e_2}}{e_2}\rho'(t)\right]\frac{\partial^2F(u,v,\Upsilon(u,v))}{\partial u\partial v}dvdu\right\}\\
&\quad=\int_{s_j}^{s}\frac{(s-\epsilon)^{-e_1}}{\Gamma(e_1)\Gamma(1-e_1)}\int_{s_j}^{\epsilon}(\epsilon-u)^{e_1-1}\left\{\int_0^t\frac{\partial^2F(u,v,\Upsilon(u,v))}{\partial u\partial v}\int_v^t\frac{(t-\tau)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\right.\\
&\qquad\times\left.\left[(\tau-v)^{e_2-1}\rho(\tau)+\frac{1}{e_2}(\tau-v)^{e_2}\rho'(\tau)\right]d\tau dv\right\}dud\epsilon\quad(\text{let }\tau-v=\varphi)\\
&\quad=\int_{s_j}^{s}\int_u^s\frac{(s-\epsilon)^{-e_1}(\epsilon-u)^{e_1-1}}{\Gamma(e_1)\Gamma(1-e_1)}\left\{\int_0^t\frac{\partial^2F(u,v,\Upsilon(u,v))}{\partial u\partial v}\int_0^{t-v}\frac{(t-v-\varphi)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\right.\\
&\qquad\times\left.\left[\varphi^{e_2-1}\rho(\varphi+v)+\frac{1}{e_2}\varphi^{e_2}\rho'(\varphi+v)\right]d\varphi dv\right\}d\epsilon du\quad(\text{let }\epsilon-u=\phi)\\
&\quad=\int_{s_j}^{s}\int_0^{s-u}\frac{(s-u-\phi)^{-e_1}\phi^{e_1-1}}{\Gamma(e_1)\Gamma(1-e_1)}\left\{\int_0^t\frac{\partial^2F(u,v,\Upsilon(u,v))}{\partial u\partial v}\int_0^{t-v}\frac{(t-v-\varphi)^{-e_2}}{\Gamma(e_2)\Gamma(1-e_2)}\right.\\
&\qquad\times\left.\left[\varphi^{e_2-1}\rho(\varphi+v)+\frac{1}{e_2}\varphi^{e_2}\rho'(\varphi+v)\right]d\varphi dv\right\}d\phi du\\
&\quad=\int_0^t\int_0^{t-v}\frac{(t-v-\varphi)^{-e_2}\left[\varphi^{e_2-1}\rho(\varphi+v)+\frac{1}{e_2}\varphi^{e_2}\rho'(\varphi+v)\right]}{\Gamma(e_2)\Gamma(1-e_2)}\\
&\qquad\times\frac{\partial F(s,v,\Upsilon(s,v))-\partial F(s_j,v,\Upsilon(s_j,v))}{\partial v}d\varphi dv.
\end{aligned}
$$

Substituting (I)-(IV) into (A.2), we have

$$
\begin{aligned}
&{}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)}\frac{\partial^2}{\partial s\partial t}\left[\rho(t)\int_{s_j}^{s}\int_0^t\frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)}Fd\tau d\epsilon\right]\\
&\quad=\int_0^t\int_0^{t-v}\frac{(t-v-\varphi)^{-e_2}\left[\varphi^{e_2-1}\rho(\varphi+v)+\frac{1}{e_2}\varphi^{e_2}\rho'(\varphi+v)\right]}{\Gamma(e_2)\Gamma(1-e_2)}\frac{\partial F(s,v,\Upsilon(s,v))}{\partial v}d\varphi dv\\
&\qquad+\frac{F(s,0,\Upsilon(s,0))}{\Gamma(e_2)\Gamma(1-e_2)}\int_0^t(t-\tau)^{-e_2}\left[\tau^{e_2-1}\rho(\tau)+\frac{1}{e_2}\tau^{e_2}\rho'(\tau)\right]d\tau.
\end{aligned}
\tag{A.3}
$$

In addition, letting $s_j = 0$ in (A.3), we obtain

$$\begin{aligned} &{}_{(0+,0+)}\mathcal{I}^{1-e}_{(s,t)} \frac{\partial^2}{\partial s \partial t} \left[\rho(t) \int_0^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} F d\tau d\epsilon \right] \\ &= {}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)} \frac{\partial^2}{\partial s \partial t} \left[\rho(t) \int_{s_j}^s \int_0^t \frac{(s-\epsilon)^{e_1-1}(t-\tau)^{e_2-1}}{\Gamma(e_1)\Gamma(e_2)} F d\tau d\epsilon \right] \end{aligned} \tag{A.4}$$

then

$${}_{(s_j^+,0+)}\mathcal{I}^{1-e}_{(s,t)} \frac{\partial^2 [\rho(t)\Phi_j(s,t)]}{\partial s \partial t} - {}_{(0+,0+)}\mathcal{I}^{1-e}_{(s,t)} \frac{\partial^2 [\rho(t)\Phi_0(s,t)]}{\partial s \partial t} = 0. \tag{A.5}$$

This proof is completed. □

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