HEAT CONDUCTED THROUGH FINS OF VARYING CROSS-SECTIONS

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Abstract

The conduction of heat takes place through the fins or spines from particle to particle due to a temperature gradient in the direction of decreasing temperature. Heat is not lost equally by each element of the fin but is lost mostly near the base of the fin. Thus there would be a waste of the material if a uniform fin is used. These aspects demand the construction of fins of varying cross-sections like triangular fin, hyperbolic fin, and parabolic fin. The paper analyzes a triangular fin and a parabolic fin to find the rate of conduction of heat through them via Laplace Transform means.

Index terms: Triangular fin, Parabolic fin, Laplace Transform.

1. Introduction

The conduction of heat takes place through the fins or spines from particle to particle due to a temperature gradient in the direction of decreasing temperature [1, 2]. Fins or spines are the extended surfaces that are mostly used in devices that exchange heat [3] like computer central processing units, power plants, radiators, heat sinks, etc. The paper analyzes a triangular fin and a parabolic fin to find the rate of conduction of heat through them via Laplace Transform means. Laplace Transform means comes out to be a very effective tool to find the temperature distribution along a triangular fin, and a parabolic fin and hence the rate of conduction of heat through them.

The Laplace transform [4, 5] of a function $y(x)$ is defined as $L[y(x)] = \int_0^\infty e^{-px}\, y(x)\, dx = Y(p)$, provided that the integral is convergent for some value of the real or complex parameter p. The Laplace transform of derivatives [5, 6] of $y(x)$ is given by

$$L\{y'(x)\} = pY(p) - y(0),$$

$$L\{y''(x)\} = p^2H(p) - py(0) - y'(0),$$

and so on.

2. Material and Method

Case I: Triangular Fin

The differential equation for analyzing a triangular fin [6] (assuming that heat flow pertains to one - dimensional conduction of heat) is given by

$$\theta''(x) + \frac{1}{x}\theta'(x) - \frac{D^2}{x}\theta(x) = 0 \ldots\ldots(1),$$

$where\ D = \sqrt{\frac{2hL}{tk}}$, L is the length of the fin between the base at x = L and the tip at x = 0, t is the thickness of the fin which increases uniformly from zero at the tip to t at the base, k is thermal conductivity, h is the coefficient of transfer of heat by convection, $\theta(x) = T(x) - T_s$, T_s is the temperature of the environment of the fin.

Multiplying both sides of (1) by x, we get

$$x\,\theta''(x) + \theta'(x) - D^2\theta(x) = 0 \ldots\ldots (2)$$

The Laplace transform [7, 8] of (2) gives

$$-\frac{d}{dp}[\,p^2\theta(p) - p\theta(0) - \theta'(0)] + p\,\theta(p) - \theta(0) - D^2\theta(p) = 0 \ldots\ldots. (3)$$

Put $\theta(0) = b\ and\ \theta'(0) = a$, and by simplifying and rearranging (3), we get

$$\frac{\theta'(p)}{\theta(p)} = -[\frac{1}{p} + \frac{D^2}{p^2}] \ldots\ldots (4)$$

Integrating both sides of (4) w.r.t. p and simplifying, we get

$$\log_e \theta(p) = -[\log_e p - D^2\frac{1}{p} - \log_e c] \ldots.. (5)$$

Simplifying (5), we get

$$p\,\theta(p) = ce^{(D^2\frac{1}{p})}$$

Expanding the exponential term, we get

$$p\ \theta(p) = \left[1 + D^2\frac{1}{p} + \frac{\left(D^2\frac{1}{p}\right)^2}{2!} + \frac{\left(D^2\frac{1}{p}\right)^3}{3!} + \frac{\left(D^2\frac{1}{p}\right)^4}{4!} \dots\dots\right]$$

$$or\ \ \theta(p) = \left[\frac{1}{p} + D^2\frac{1}{p^2} + \frac{D^4}{2!}\frac{1}{p^3} + \frac{D^6}{3!}\frac{1}{p^4} + \frac{D^8}{4!}\frac{1}{p^5} \dots\dots\right] \dots\dots\ (6)$$

The inverse Laplace transform of (6) provides

$$\theta(x) = c\left[1 + D^2x + \frac{D^4x^2}{2!\,2!} + \frac{D^6x^3}{3!\,3!} + \frac{D^8x^4}{4!\,4!} \dots\right]$$

$$or\ \ \theta(x) = c\begin{bmatrix}1 + \frac{1}{4}\left(2D\sqrt{x}\right)^2 + \frac{1}{2!2!}\left(\frac{2D\sqrt{x}}{2}\right)^4 \\ + \frac{1}{3!3!}\left(\frac{2D\sqrt{x}}{2}\right)^6 + \frac{1}{4!4!}\left(\frac{2D\sqrt{x}}{2}\right)^8 \dots\end{bmatrix} \dots..(7)$$

The modified Bessel function [5] of the first kind of order n and its first order derivative is given by

$$I_n(z) = \sum_{r=0}^{\infty}\frac{1}{r!(n+r)!}\left(\frac{z}{2}\right)^{n+2r} \dots\dots.\ (8)$$

Also $\frac{d}{dx}\left(I_n(z)\right) = I_{n+1}(z)\frac{d}{dx}(z) \dots.\ (9)$

Put z = $2D\sqrt{x}$ and n = 0, we get

$$I_0\left(2D\sqrt{x}\right) = \sum_{r=0}^{\infty}\frac{1}{r!\,r!}\left(\frac{2D\sqrt{x}}{2}\right)^{2r}$$

Or

$$I_0\left(2D\sqrt{x}\right) = 1 + \left(\frac{2D\sqrt{x}}{2}\right)^2 + \frac{1}{2!2!}\left(\frac{2D\sqrt{x}}{2}\right)^4 + \frac{1}{3!3!}\left(\frac{2D\sqrt{x}}{2}\right)^6 + \frac{1}{4!4!}\left(\frac{2D\sqrt{x}}{2}\right)^8 \dots\dots$$

Hence (7) can be rewritten as

$$\theta(x) = cI_0\left(2D\sqrt{x}\right) \dots.\ (10)$$

To find the constant c, at x = L, $\Theta(L) = \Theta_0$ [1, 3], therefore, c = $\frac{\Theta_0}{I_0(2D\sqrt{L})}$

Hence (10) can be written as

$$\Theta(x) = \frac{\Theta_0}{I_0(2D\sqrt{L})}I_0\left(2D\sqrt{x}\right) \dots\ (11)$$

Equation (11) gives the temperature distribution along the length of the triangular fin.

The heat conducted through the triangular is given by Fourier's Law [6] of heat conduction as

$$H = kA\left(\Theta'(x)\right)_{x=L} = kbt\left(\Theta'(x)\right)_{x=L}$$

Using (11), we get

$$H = kbt\ \frac{\Theta_0}{I_0(2D\sqrt{L})}I_1\left(2D\sqrt{L}\right)\ \left(\frac{d}{dx}\left(2D\sqrt{x}\right)\right)_{x=L}$$

On simplifying, we get

$$H = kbtD\ \frac{\Theta_0}{\sqrt{L}\,I_0(2D\sqrt{L})}I_1\left(2D\sqrt{L}\right) \dots.\ (12)$$

Put the value of D, we get

$$H = b\sqrt{2hkt}\frac{\Theta_0}{I_0(2D\sqrt{l})}I_1\left(2D\sqrt{L}\right) \dots\dots (13)$$

Equation (13) gives the expression for the rate of conduction of heat through the triangular fin.

Case II: Parabolic fin

The differential equation for analyzing a parabolic fin [9] (assuming that heat flow pertains to one-dimensional conduction of heat) is given by

$$x^2\theta''(x) + 2x\theta'(x) - M^2l^2\theta(x) = 0 \dots\dots.(14),$$

$where\ M = \sqrt{\frac{2h}{tk}}$, l is the length of the fin between the base at x = l and the tip at x = 0, t is the thickness of the fin which increases uniformly from zero at the tip to t at the base, k is thermal conductivity, h is the coefficient of transfer of heat by convection, $\theta(x) = T(x) - T_s$ is the excess temperature, T_s is the temperature of the environment of the fin. Substituting $x = e^z$, the equation (14) can be rewritten into a form:

$$\theta''(x) + \theta'(x) - M^2l^2\theta(x) = 0 \dots\dots\ (15)$$

$$\theta' \equiv \frac{d}{dz}$$

The Laplace transform [10, 11] of (15) gives

$$[\,p^2\theta(p) - p\theta(0) - \theta'(0)] + p\,\theta(p) - \theta(0) - M^2l^2\theta(p) = 0 \dots..\ (16)$$

Put $\theta(0) = P\ and\ \theta'(0) = Q,$ and by simplifying and rearranging (16), we get

$$\theta(p) = \frac{pP + Q}{p^2 + p - M^2l^2}$$

Or

$$\theta(p) = \frac{pP + Q}{(p - c_1)(p + c_2)} \ldots (17)$$

$where\ c_1 = \frac{-1+(1+4M^2l^2)^{1/2}}{2}\ and\ c_1 = \frac{-1-(1+4M^2l^2)^{1/2}}{2}$.

This equation (17) can be rewritten as

$$\theta(p) = \frac{c_1P + Q}{c_1 + c_2}\frac{1}{(p - c_1)} + \frac{c_2P + Q}{c_2 - c_1}\frac{1}{(p + c_2)} \ldots . (18)$$

The inverse Laplace transform [12] of (18) provides

$$\Theta(\mathrm{x}) = \frac{c_1P + Q}{c_1 + c_2}\, e^{c_1z} + \frac{c_2P + Q}{c_2 - c_1} e^{-c_2z}$$

Or

$$\Theta(\mathrm{x}) = \frac{c_1P+Q}{c_1+c_2}\, x^{c_1} + \frac{c_2P+Q}{c_2-c_1} x^{-c_2} \ \ldots . (19)$$

As $\Theta(0)\ is\ finite\ [1, 3], therefore,$ $the\ term\ \frac{c_2P+Q}{c_2-c_1} x^{-c_2}\ is\ equated\ to\ zero$

$$i.e. \frac{c_2P + Q}{c_2 - c_1} x^{-c_2} = 0, which\ gives$$

$$c_2P + Q = 0$$

or $\mathrm{Q} = -c_2P$.

From (19), we have

$$\Theta(\mathrm{x}) = \frac{c_1-c_2}{c_1+c_2} P\, x^{c_1} \ \ldots . (20)$$

To find the constant P, at x = l, $\Theta(\,l) = \Theta_0$ [1, 3], therefore, from (20), P = $\frac{c_1+c_2}{c_1-c_2}\,\Theta_0 l^{-c_1}$

Hence (20) can be rewritten as

$$\Theta(\mathrm{x}) = \Theta_0 l^{-c_1} x^{c_1}$$

Or

$$\Theta(\mathrm{x}) = \Theta_0 (x/l)^{c_1} \ldots (21)$$

Equation (21) gives the temperature distribution along the length of the parabolic fin.

The heat conducted through the parabolic fin is given by Fourier's Law of heat conduction [9, 13, 14,] as

$$H = \mathrm{kA}\left(\Theta'(\mathrm{x})\right)_{\mathrm{x}=l} = \mathrm{kbt}\left(\Theta'(\mathrm{x})\right)_{\mathrm{x}=l}$$

Using equation (21), we get

$$H = \mathrm{kbt}\ \Theta_0 c_1/l$$

Or

$$H = \mathrm{kbt}\ \Theta_0 \frac{-1+(1+4M^2l^2)^{1/2}}{2l} \ \ldots .. (22)$$

Equation (22) gives the expression for the rate of conduction of heat through the parabolic fin.

3. Result and Conclusion

We have found find the temperature distribution along the lengths of the triangular fin as well as the parabolic fin and hence the rate of conduction of heat through them via Laplace Transform means. It is found that with the increase in the length of the triangular fin or parabolic fin, temperature increases, and hence the rate of conduction of heat at any cross-section of the triangular fin or parabolic fin increases. The method has come out to be a very effective tool to find temperature distribution along the lengths of the triangular fin as well as parabolic fin and hence the rate of conduction of heat through them.

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