ON THE LAGRANGIAN STRUCTURE OF VLASOV-MAXWELL EQUATIONS FOR ELECTROMAGNETIC FIELD WITH BOUNDED VARIATION

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Abstract

We study the Lagrangian structure of Vlasov-Maxwell equations. We show that for sufficiently regular initial conditions, renormalized solutions of these systems are Lagrangian and that these notions of solution, in fact, coincide. As a consequence, finite-energy solutions are shown to be transported by a global flow. These results extend to our setting those obtained by Ambrosio, Colombo, and Figalli [3] for the Vlasov-Poisson system and by the first author and Marcon for relativistic Vlasov systems [5]; here, we analyze the electromagnetic fields with bounded variation under Maxwell equations.

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Keywords: Vlasov equation, transport equations, Lagrangian flows, renormalized solutions.

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1. Introduction

1.1. **Overview.** In this paper, we are interested in the Lagrangian structure of Vlasov-Maxwell equations. This system of equations describes the evolution of a nonnegative distribution function $f : (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow [0, \infty)$ under the action of a self-consistent Lorentz acceleration:

$$\begin{cases} \partial_t f_t + \hat{v} \cdot \nabla_x f_t + (E_t + \hat{v} \times H_t) \cdot \nabla_v f_t = 0 & \text{in} \quad (0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3; \\ \rho_t(x) = \int_{\mathbb{R}^3} f_t(x, v)\, \mathrm{d}v, \quad J_t(x) = \int_{\mathbb{R}^3} \hat{v} f_t(x, v)\, \mathrm{d}v & \text{in} \quad (0, \infty) \times \mathbb{R}^3; \\ \operatorname{div} E_t = \rho_t, \quad \operatorname{div} H_t = 0 & \text{in} \quad (0, \infty) \times \mathbb{R}^3; \\ \partial_t H_t + \operatorname{curl} E_t = 0, \quad \partial_t E_t - \operatorname{curl} H_t = -J_t & \text{in} \quad (0, \infty) \times \mathbb{R}^3. \end{cases} \tag{1.1}$$

Here, $f_t(x, v)$ denotes the distribution of particles with position x and velocity v at time t and $\hat{v}$ is the velocity of particles. We may assume the classical mechanics case, where $\hat{v} = v$ or the relativistic one, where $\hat{v} \coloneqq (1 + |v|^2)^{-1/2} v$ (we assume the speed of light is $c = 1$). Such system is very important in mathematical physics and appear in a variety of physical models, in particular at plasma physics. Typically, ρ_t and J_t represent the charge density and the current density and E_t and H_t the electric and magnetic fields, respectively.

Concerning the existence of classical solutions of (1.1), we refer to [8, 19, 21], where the existence of solutions with small velocities is proven. As mentioned in [23], very little is known regarding the existence of global solutions for general initial data. Existence results can be found, for instance, for Vlasov-Poisson, relativistic Vlasov-Darwin, and relativistic Vlasov-Maxwell equations, assuming further hypothesis on initial data; see [17, 18, 27, 28]. In the aforementioned results, higher integrability assumptions and moment conditions on the initial data are required. Nonetheless, global existence results are available for weak solution [14, 24]. More recently, a discontinuous Galerkin method were developed for numerical results [9]; see also [10, 29] and references therein. Regarding the hypothesis of bounded variation, this is not a novelty: in [6, 26], the authors obtained renormalization property for Vlasov system with general coefficients of bounded variation

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and conservation of total energy for weak solutions for electromagnetic field with local bounded variation, respectively. Moreover, regarding only the structure of the vector field, [11] obtained well-posedness of regular Lagrangian flows. However, since the most physically relevant hypothesis for f_t is to be integrable, we are not able to use the aforementioned results, since they also require boundedness of density distribution. We thus consider renormalized solutions, which allow us to establish a Lagrangian structure for (1.1) for $f_t \in L^1(\mathbb{R}^3)$. Moreover, under suitable (and physically relevant) integrability assumptions, we are able to prove the flow's global well-posedness. Such results were obtained for the Vlasov-Poisson system [3] and for relativistic Vlasov systems [5], not including Vlasov-Maxwell equations.

1.2. **Main results.** In order to use the machinery developed in [2], we write the first equation in (1.1) as

$$\partial_t f_t + \boldsymbol{b}_t \cdot \nabla_{x,v} f_t = 0, \tag{1.2}$$

where, for each fixed $t > 0$, the vector field $\boldsymbol{b}_t : \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ is given by $\boldsymbol{b}_t(x, v) = (\hat{v}, E_t + \hat{v} \times H_t)$. Since

$$\nabla_{x,v} \cdot \boldsymbol{b}_t = \nabla_v \cdot (\hat{v} \times H_t) = (\nabla_v \times \hat{v}) \cdot H_t - \hat{v} \cdot (\nabla_v \times H_t) = 0, \tag{1.3}$$

we also have a continuity form of (1.2), so it is expected that solutions have a Lagrangian structure, meaning that the initial condition f_0 is transported to f_t by an associated flow, as well as conservation of such quantity through the flow. In the weak regularity regime, however, the existence of such flow is not guaranteed by the classical Cauchy-Lipschitz theory if we only assume finite number of particles, i.e., $f_t \in L^1(\mathbb{R}^6)$. Indeed, Maxwell equations (formally) imply the wavelike behavior of the electromagnetic field:

$$\begin{aligned} (\partial_{tt} - \Delta) E_t &= -\nabla \rho_t - \partial_t J_t; \\ (\partial_{tt} - \Delta) H_t &= \operatorname{curl} J_t. \end{aligned} \tag{1.4}$$

Therefore, by assuming initial data (E_0, H_0) with compatibility conditions

$$\operatorname{div} E_0 = \rho_0, \quad \operatorname{div} H_0 = 0, \tag{1.5}$$

we are able to explicitly write (E_t, H_t) as a sum of solutions depending on the densities (ρ_t, J_t)[1] and on initial conditions (E_0, H_0, f_0):

$$\begin{aligned} E_t(x) = &⨍_{\partial B_t(x)} E_0(y) + t(\omega(y-x)\cdot\nabla)E_0(y) + t \operatorname{curl} H_0(y) - tJ_0(y) - t\rho_0(y)\omega(y-x)\,\mathrm{d}S(y) \\ &- \frac{1}{4\pi}\int_{B_t(x)} [\rho_t(y)]_{\mathrm{ret}}(x)\frac{\omega(y-x)}{|y-x|^2} + [J_t(y)]_{\mathrm{ret}}(x)\cdot\omega(y-x)\frac{\omega(y-x)}{|y-x|^2} \\ &+ ([J_t(y)]_{\mathrm{ret}}(x)\times\omega(y-x))\times\frac{\omega(y-x)}{|y-x|^2} + ([\partial_t J_t(y)]_{\mathrm{ret}}(x)\times\omega(y-x))\times\frac{\omega(y-x)}{|y-x|}\,\mathrm{d}y; \\ H_t(x) = &⨍_{\partial B_t(x)} H_0(y) + t(\omega(y-x)\cdot\nabla)H_0(y) - t\operatorname{curl} E_0(y) - tJ_0(y)\times\omega(y-x)\,\mathrm{d}S(y) \\ &- \frac{1}{4\pi}\int_{B_t(x)} [J_t(y)]_{\mathrm{ret}}(x)\times\frac{\omega(y-x)}{|y-x|^2} + [\partial_t J_t(y)]_{\mathrm{ret}}(x)\times\frac{\omega(y-x)}{|y-x|}\,\mathrm{d}y, \end{aligned} \tag{1.6}$$

where $\omega(z) \coloneqq z/|z|$ and $[f_t(y)]_{\mathrm{ret}}(x) \coloneqq f_{t-|y-x|}(y)$. Notice that if the initial data is in Lebesgue spaces, the surface integrals do not make sense. Moreover, if we only assume finite number of particles for all times, i.e., $f \in L^\infty([0,T]; L^1(\mathbb{R}^6))$, then the time derivative of J is also ill-defined. Since the main results for Lagrangian structure in the quasistatic limits [3, 5] heavily use the

[1] Such a solution is known as Jefimenko's equations; see [20, 29].

explicit expression of the electromagnetic field, new ideas are needed, and we intend to work with the explicit expression (1.6) in a future work.

One may use the weak solution approach: since $\boldsymbol{b}_t$ is divergence-free, (1.2) can be rewritten as

$$\partial_t f_t + \nabla_{x,v} \cdot (\boldsymbol{b}_t f_t) = 0.$$

The latter can be interpreted in the distributional sense provided $\boldsymbol{b} f$ is locally integrable which, however, does not follow only from the assumption $f_t \in L^1(\mathbb{R}^6)$, since one cannot assure that $\boldsymbol{b}_t \in L^\infty_{\text{loc}}(\mathbb{R}^6)$. To treat this problem, we introduce a function $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$ such that

$$\partial_t \beta(f_t) + \nabla_{x,v} \cdot (\boldsymbol{b}_t \beta(f_t)) = 0 \tag{1.7}$$

whenever f_t is a smooth solution of (1.2); of course, such equality holds since $\boldsymbol{b}_t$ is divergence-free and due to the chain rule. Hence, $\boldsymbol{b}_t \beta(f_t) \in L^1_{\text{loc}}$, which leads to the concept of a renormalized solution; as in the celebrated results by DiPerna-Lions [15, 16].

Definition 1.1 (Renormalized solution)**.** *For a Borel vector field $\boldsymbol{b} \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^6; \mathbb{R}^6)$, we say that a Borel function $f \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^6)$ is a renormalized solution of (1.2) starting from f_0 if (1.7) holds in the sense of distributions, that is,*

$$\int_{\mathbb{R}^6} \phi_0(x,v)\beta(f_0(x,v))\,\mathrm{d}x\,\mathrm{d}v + \int_0^T \int_{\mathbb{R}^6} \Big[\partial_t \phi_t(x,v) + \nabla_{x,v}\phi_t(x,v) \cdot \boldsymbol{b}_t(x,v)\Big]\beta(f_t(x,v))\,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = 0 \tag{1.8}$$

for all $\phi \in C^1_c([0,T) \times \mathbb{R}^6)$ and $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Moreover, $f \in L^\infty((0,T); L^1(\mathbb{R}^6))$ (and $v f_t \in L^\infty((0,T); L^1(\mathbb{R}^6))$ if $\hat{v} = v$) is called a renormalized solution of (1.1) starting from f_0 and electromagnetic field starting from $(E_0,\, H_0)$ satisfying (1.5) if, by setting

$$\rho_t(x) := \int_{\mathbb{R}^3} f_t(x,v)\,\mathrm{d}v, \quad J_t(x) := \int_{\mathbb{R}^3} \hat{v} f_t(x,v)\,\mathrm{d}v, \quad \boldsymbol{b}_t(x,v) := (\hat{v}, E_t(x) + \hat{v} \times H_t(x)),$$

where $(E_t, H_t) \in L^1_{\text{loc}}([0,T] \times \mathbb{R}^3; \mathbb{R}^3)^2$ satisfy in a weak sense

$$\operatorname{div} E_t = \rho_t, \quad \operatorname{div} H_t = 0, \quad \partial_t H_t + \operatorname{curl} E_t = 0, \quad \partial_t E_t - \operatorname{curl} H_t = -J_t, \tag{1.9}$$

we have that f_t satisfies (1.8), for every $\phi \in C^1_c([0,T) \times \mathbb{R}^6)$ and $\beta \in C^1(\mathbb{R}) \cap L^\infty(\mathbb{R})$.

Observe that the integrability assumption of f_t is used so that ρ_t, J_t are well defined. Moreover, (1.6) satisfies (1.9) for initial condition sufficiently regular.

Our first main result shows that distributional or renormalized solutions of (1.1) are in fact Lagrangian solutions. This gives a characterization of solutions of (1.1), since Lagrangian solutions are generally stronger than renormalized or distributional solutions.

Theorem 1.1. *Let $T > 0$, $f_0 \in L^1_+(\mathbb{R}^6)$ (and $v f_0 \in L^1(\mathbb{R}^6)$, that is, $J_0 \in L^1(\mathbb{R}^3)$ if $\hat{v} = v$), and $E_0,\, H_0 \in L^2(\mathbb{R}^3)$ satisfying (1.5). Moreover, assume that (E, H) are in $L^1([0,T]; \mathrm{BV}(\mathbb{R}^3;\mathbb{R}^3))^2$. Assume $f \in L^\infty([0,T); L^1_+(\mathbb{R}^6))$ is weakly continuous in the sense that*

$$t \longmapsto \int_{\mathbb{R}^6} f_t\,\varphi\,\mathrm{d}x\,\mathrm{d}v \quad \textit{is continuous for any } \varphi \in C_c(\mathbb{R}^6),$$

Assume further that:

(i) either $f \in L^\infty((0,T); L^\infty(\mathbb{R}^6))$ and f_t is a distributional solution of (1.1) starting from f_0; or
(ii) f_t is a renormalized solution of (1.1) starting from f_0.

Then, f_t is a Lagrangian solution transported by the Maximal Regular Flow $\boldsymbol{X}(t,x)$ associated to $\boldsymbol{b}_t(x,v) = (\hat{v}, E_t(x) + \hat{v} \times H_t(x))$ (see Definition 2.1 and Definition 2.2), starting from 0. In particular, f_t is renormalized.

Remark 1.1. As a simple example of an electromagnetic field which is not regular enough for Cauchy-Lipschitz theorem, one may look at solutions with only regularity on initial condition $J_0 \in W^{1,1}(\mathbb{R}^3)$ and $(E_0, H_0) \in W^{2,1}(\mathbb{R}^3)^2$, which by wave equation kernel regularity (see [22]), we have that homogeneous solutions are in $L^1((0,T); W^{1,1}(\mathbb{R}^3))^2$, and it follows that homogeneous solutions are in $L^1((0,T); \mathrm{BV}(\mathbb{R}^3))$ by the embedding $W^{1,1} \subset \mathrm{BV}$. Therefore, even if one mimics the proof of Glassey-Strauss theorem [19] for inhomogeneous terms, by the above reasoning, one cannot infer the existence of classical flow; nevertheless, Theorem 1.1 provides a suitable structure of Lagrangian solution.

Our second main result provides conditions to obtain a globally defined flow, so that there is no finite-time blow up; namely, if one has time integrability of the total energy of the system. Since relativistic and kinetic energies provide different moment control on $v-$marginals of f, we split the statement for each case.

Theorem 1.2. *Fix $T > 0$ and let f be a nonnegative renormalized solution as in Theorem 1.1. If $\hat{v} = v$, assume that*

$$\int_0^T \int_{\mathbb{R}^6} |v|^3 f_t(x,v)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} |H_t|^2\, \mathrm{d}x\, \mathrm{d}t < \infty, \tag{1.10}$$

that is, the transport kinetic and the magnetic energies are integrable in time. Then,

(i) The maximal regular flow $\boldsymbol{X}(t,\cdot)$ associated to $\boldsymbol{b}_t = (\hat{v}, E_t + \hat{v} \times H_t)$ and starting from 0 is globally defined on $[0,T]$ for f_0-a.e. (x,v);

(ii) f_t is the image of f_0 through this flow, that is, $f_t = \boldsymbol{X}(t,\cdot)_\# f_0 = f_0 \circ \boldsymbol{X}^{-1}(t,\cdot)$ for all $t \in [0,T]$;

(iii) the map

$$[0,T] \ni t \longmapsto \int_{\mathbb{R}^6} \psi(f_t(x,v))\, \mathrm{d}x\, \mathrm{d}v$$

is constant in time for all Borel $\psi : [0,\infty) \longrightarrow [0,\infty)$.

Moreover, if $\hat{v} = (1+|v|^2)^{-1/2} v$, by changing (1.10) hypothesis to

$$\int_0^T \int_{\mathbb{R}^6} \sqrt{1+|v|^2} f_t(x,v)\, \mathrm{d}x\, \mathrm{d}v\, \mathrm{d}t + \int_0^T \int_{\mathbb{R}^3} \tfrac{1}{2}|E_t|^2 + \tfrac{1}{2}|H_t|^2\, \mathrm{d}x\, \mathrm{d}t < \infty$$

that is, the relativistic and electromagnetic energies are integrable in time, then properties (i)-(iii) hold.

We remark that for the relativistic case $\hat{v} = (1+|v|^2)^{-1/2} v$, the computation done in [5, Corollary 2.1] is applicable to Vlasov-Maxwell system; we merely state the results Theorem 1.2 in such case for the sake of completeness.

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2. Lagrangian solution and associated flow

In this section, we prove Theorem 1.1 which states that Lagrangian and renormalized solutions of (1.1) are equivalent. For this, we use the machinery developed in [3, Sections 4 and 5], which is a application of the theory created by the same authors [2, Sections 5, 6, and 7]. More precisely, the existence, uniqueness, and a semigroup property for the maximal regular flow (see Definition 2.1) follow at once from [2, Theorems 5.7, 6.1, 7.1], which only requires three main properties of the vector field $\boldsymbol{b}$:

(H1) $\int_0^T \int_{B_R} |\boldsymbol{b}_t(x)|\, \mathrm{d}x\, \mathrm{d}t < \infty$ for all $R > 0$;

(H2) for any $\bar{\rho} \in L^\infty_+(\mathbb{R}^3)$ with compact support and any closed interval $[a,b] \subset [0,T]$, the continuity equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t + \nabla_{x,v} \cdot (\boldsymbol{b}_t \rho_t) = 0 \quad \text{in } (a,b) \times \mathbb{R}^6$$

have at most one solution in the class of all nonnegative weakly* continuous functions starting at $\rho_a = \bar{\rho}$ and $\cup_{t\in[a,b]} \operatorname{supp} \rho_t \Subset \mathbb{R}^6$;

(H3) $\operatorname{div} \boldsymbol{b}_t \geq m(t)$ in $\mathbb{R}^6$, where $m(t) \in L^1((0,T))$.

Since for our vector field, property (H1) holds and it is diverge-free, so that (H3) hold for $m(t) \equiv 0$, one expects that if we reversed the time variable, which is equivalent to property (H2) be strengthened for continuity equation with both vector fields $\pm \boldsymbol{b}_t$, then volume is conserved by the flow, and the incompressibility constant (ii) in Definition 2.1 equals 1; see [3, Theorem 4.3] for a concise statement. Therefore, we aim to prove (H2) for our case adapting [3, Theorem 4.4], [5, Lemma 2.1], which in turn are adaptations of the main result of [4].

From now on, we denote by $\mathscr{M}(\mathbb{R}^d)$ the space of (signed) measures in $\mathbb{R}^d$ with finite total mass, by $\mathscr{M}_+(\mathbb{R}^d)$ the space of nonnegative measures with finite total mass, by $\mathrm{AC}(I;\mathbb{R}^d)$ the space of absolutely continuous curves on the interval I with values in $\mathbb{R}^d$, and by $\mathscr{L}^d$ the Lebesgue measure in $\mathbb{R}^d$. We begin with the preliminary definition of maximal regular flow:

Definition 2.1 (Maximal regular flow)**.** For every $s \in (0,T)$, a Borel map $\mathbf{X}(\cdot,s,\cdot)$ is said to be a maximal regular flow (starting at s) if there exist two Borel maps $T^+_{s,\mathbf{X}} : \mathbb{R}^6 \longrightarrow (s,T]$, $T^-_{s,\mathbf{X}} : \mathbb{R}^6 \longrightarrow [0,s)$ such that $\mathbf{X}(\cdot,s,x)$ is defined in $(T^-_{s,\mathbf{X}}(x), T^+_{s,\mathbf{X}}(x))$ and

(i) for a.e. $x \in \mathbb{R}^6$, we have that $\mathbf{X}(\cdot,s,x) \in \mathrm{AC}((T^-_{s,\mathbf{X}}, T^+_{s,\mathbf{X}});\mathbb{R}^6)$ and that it solves the equation $\dot{x}(t) = \boldsymbol{b}_t(x(t))$ a.e. in $(T^-_{s,\mathbf{X}}, T^+_{s,\mathbf{X}})$ with $\mathbf{X}(s,s,x) = x$;

(ii) there exists a constant $C > 0$ such that $\mathbf{X}(t,s,\cdot)_\#(\mathscr{L}^6 \llcorner \{T^-_{s,\mathbf{X}} < t < T^+_{s,\mathbf{X}}\}) \leq C\mathscr{L}^6$ for all $t \in [0,T]$. As before, this constant C can depend of $\mathbf{X}$ and s;

(iii) for a.e. $x \in \mathbb{R}^6$, either $T^+_{s,\mathbf{X}} = T$ and $\mathbf{X}(\cdot,s,x) \in C([s,T];\mathbb{R}^6)$, or $\lim_{t\uparrow T^+_{s,\mathbf{X}}} |\mathbf{X}(t,s,x)| = \infty$. Analogousy, either $T^-_{s,\mathbf{X}} = 0$ and $\mathbf{X}(\cdot,s,x) \in C([0,s];\mathbb{R}^6)$, or $\lim_{t\downarrow T^-_{s,\mathbf{X}}} |\mathbf{X}(t,s,x)| = \infty$.

Here, we denote by $\boldsymbol{X}_\#\mu$ the pushforward of a measure μ by $\boldsymbol{X}$ and by $\nu \llcorner B$ the restriction of the measure ν to the set B.

We now define Lagrangian solutions, which are by [3, Theorem 4.10] a stronger notion of solution than renormalized one.

Definition 2.2 (Lagrangian solutions)**.** *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ be a Borel vector field having a maximal regular flow $\boldsymbol{X}$. We say that u is a Lagrangian solution of the continuity equation with vector field $\boldsymbol{b}$ starting from u_0 if $u_t = \boldsymbol{X}(t,0,\cdot)_\#(u_0 \llcorner \{T^+_{0,\boldsymbol{X}} > t\})$.*

We now wish to prove that the vector field associated to Vlasov-Maxwell's equations satifies assumption (H2). We remark, however, that no $W^{1,1}$ estimates are proven in order to use the standard techniques, as pointed out by DiPerna-Lions [13]. Nevertheless, by imposing more regularity on the electromagnetic field:

$$(E,H) \in L^1([0,T];\mathrm{BV}(\mathbb{R}^3;\mathbb{R}^3))^2. \tag{2.1}$$

The main idea in imposing (2.1) is to use the following: for $u \in BV(\mathbb{R}^3;\mathbb{R}^3)$, there exists a constant $C > 0$ and a $\mathscr{L}^3-$negligible set L such that

$$|u(x) - u(y)| \leq C|x-y|(M(Du)(x) + M(Du)(y)) \quad \forall x, y \in \mathbb{R}^3 \setminus L, \tag{2.2}$$

where Du is the distributional derivative of u (which is also a finite Radon measure) and M is the maximal operator; see [25].

Remark 2.1. It is known that (H2) holds for a class of vector fields with space bounded variation [1, 6]. Nevertheless, in [12] it has proven that there exists time-dependent space BV −vector fields that does not satisfy (H2).

Proposition 2.1. *Let $\boldsymbol{b} : (0,T) \times \mathbb{R}^6 \longrightarrow \mathbb{R}^6$ be given by $\boldsymbol{b}_t(x,v) = (\boldsymbol{b}_{1t}(v), \boldsymbol{b}_{2t}(x,v))$, where*

$$\boldsymbol{b}_1 \in L^\infty((0,T); W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^3;\mathbb{R}^3)),$$

$$\boldsymbol{b}_{2t}(x,v) = E_t(x) + \boldsymbol{b}_{1t}(v) \times H_t(x),$$

where (E_t, H_t) satisfy (2.1). Then, the vector field $\boldsymbol{b}$ satisfies property (H2).

Proof. We follow the same strategy as [3, Theorem 4.4], [5, Lemma 2.1]. We begin by setting $\mathscr{P}(X)$ as the set of probability measures on X and

$$e_t : C([0,T];\mathbb{R}^6) \longrightarrow \mathbb{R}^6$$

the evaluation map at time t, which means $e_t(\eta) \coloneqq \eta(t)$. By the same argument as in [3] (which heavily uses the extended superposition principle [3, Theorem 5.1]), it is enough to show that given $\boldsymbol{\eta} \in \mathscr{P}(C([0,T]; B_R \times B_R))$ for some $B_R \subset \mathbb{R}^3$ concentrated on integral curves of $\boldsymbol{b}$ such that $(e_t)_\#\boldsymbol{\eta} \le C_0\mathscr{L}^6$ for all $t \in [0,T]$, the disintegration $\boldsymbol{\eta}_z$ of $\boldsymbol{\eta}$ with respect to e_0 is a Dirac delta for $(e_0)_\#\boldsymbol{\eta}$−a.e. $z = (x,v) \in B_R \times B_R$. Recall that the disintegration of $\boldsymbol{\eta}$ with respect to e_0 is a family of measures $\boldsymbol{\eta}_z$ such that, for all $E \in C([0,T]; B_R \times B_R)$,

$$\boldsymbol{\eta}(E) = \int_{\mathbb{R}^6} \boldsymbol{\eta}_z(E \cap e_0^{-1}(x))\,\mathrm{d}z.$$

For this purpose, the authors of [3] consider the function

$$\Phi_{\delta,\zeta}(t) \coloneqq \iiint \log\left(1 + \frac{|\gamma^1(t) - \eta^1(t)|}{\zeta\delta} + \frac{|\gamma^2(t) - \eta^2(t)|}{\delta}\right) \mathrm{d}\mu(\eta,\gamma,z),$$

where $\mathrm{d}\mu(\eta,\gamma,z) \coloneqq \mathrm{d}\boldsymbol{\eta}_z(\gamma)\mathrm{d}\boldsymbol{\eta}_z(\eta)\mathrm{d}(e_0)_\#\boldsymbol{\eta}(z)$[2], δ, $\zeta \in (0,1)$ are small constants to be chosen later, $t \in [0,T]$, with notation

$$\eta(t) = (\eta_1(t), \eta_2(t)) \in \mathbb{R}^3 \times \mathbb{R}^3,$$

and assume by contradiction that $\boldsymbol{\eta}_z$ is not a Dirac delta for $(e_0)_\#\boldsymbol{\eta}$−a.e. z, which means that there exists a constant $a > 0$ such that

$$\iiint \left(\int_0^T \min\{|\gamma(t) - \eta(t)|, 1\}\mathrm{d}t\right) \mathrm{d}\mu(\eta,\gamma,z) \ge a.$$

Indeed, if $\boldsymbol{\eta}_z$ is a Dirac delta for $(e_0)_\#\boldsymbol{\eta}$−a.e. z, the integrand above would vanish.

Moreover, they show that, without loss of generality, by assuming $a \le 2T$, there exists $t_0 \in [0,T]$ such that

$$\Phi_{\delta,\zeta}(t_0) \ge \frac{a}{2T} \log\left(1 + \frac{a}{2\delta T}\right). \tag{2.3}$$

Of course, we wish to show that such bound is impossible. In order to prove it, we compute the time derivative of $\Phi_{\delta,\zeta}$ and conclude that

$$\begin{aligned}\frac{\mathrm{d}\Phi_{\delta,\zeta}}{\mathrm{d}t}(t) \le \iiint \Bigg(&\frac{|\boldsymbol{b}_{1t}(\gamma^2(t)) - \boldsymbol{b}_{1t}(\eta^2(t))|}{\zeta(\delta + |\gamma^2(t) - \eta^2(t)|)} + \frac{\zeta|\boldsymbol{b}_{1t}(\gamma^2(t)) \times (H_t(\gamma^1(t)) - H_t(\eta^1(t)))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \\ &+ \frac{|(\boldsymbol{b}_{1t}(\gamma^2(t)) - \boldsymbol{b}_{1t}(\eta^2(t))) \times H_t(\eta^1(t))|}{\delta + |\gamma^2(t) - \eta^2(t)|} + \frac{\zeta|E_t(\gamma^1(t)) - E_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|}\Bigg) \mathrm{d}\mu(\eta,\gamma,z).\end{aligned} \tag{2.4}$$

[2]Note that $\mu \in \mathscr{P}(C([0,T]; B_R)^2 \times B_R)$ and $\Phi_{\delta,\zeta}(0) = 0$.

By our assumption on $\boldsymbol{b}_{1t}$, the first summand is easily estimated using the Lipschitz regularity of $\boldsymbol{b}_{1t}$ in B_R:

$$\iiint \frac{|\boldsymbol{b}_{1t}(\gamma^2(t)) - \boldsymbol{b}_{1t}(\eta^2(t))|}{\zeta(\delta + |\gamma^2(t) - \eta^2(t)|)} \mathrm{d}\mu(\eta,\gamma,z) \le \frac{\|\nabla \boldsymbol{b}_1\|_{L^\infty((0,T)\times B_R)}}{\zeta}. \tag{2.5}$$

Analogously, the third summand is estimated using that H_t is locally integrable (since $\mathrm{BV}(\mathbb{R}^3) \subset L^{3/2}(\mathbb{R}^3) \subset L^1_{\mathrm{loc}}(\mathbb{R}^3)$), the condition $(e_t)_\# \boldsymbol{\eta} \le C_0 \mathscr{L}^6$, and the Lipschitz regularity of $\boldsymbol{b}_1$ in B_R:

$$\begin{aligned}\iiint \frac{|(\boldsymbol{b}_{1t}(\gamma^2(t)) - \boldsymbol{b}_{1t}(\eta^2(t))) \times H_t(\eta^1(t))|}{\delta + |\gamma^2(t) - \eta^2(t)|} \mathrm{d}\mu(\eta,\gamma,z) &\le \|\nabla \boldsymbol{b}_1\|_{L^\infty((0,T)\times B_R)} \iiint |H_t(\eta^1(t))| \,\mathrm{d}\mu \\ &\le \|\nabla \boldsymbol{b}_1\|_{L^\infty((0,T)\times B_R)} C_0 |B_R| \|H_t\|_{L^1(B_R)}.\end{aligned} \tag{2.6}$$

For the second term, we have

$$\begin{aligned}&\iiint \frac{\zeta |\boldsymbol{b}_{1t}(\gamma^2(t)) \times (H_t(\gamma^1(t)) - H_t(\eta^1(t)))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z) \\ &\qquad \le C\|\boldsymbol{b}_1\|_{L^\infty((0,T)\times B_R)} \iiint \frac{\zeta |H_t(\gamma^1(t)) - H_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z).\end{aligned}$$

By the result proven in [7, Lemma 2.2], for $p \in (1,\infty]$, there exists constant $C_p > 0$ such that

$$\|u\|_{L^1(\mu)} \le C_p \|u\|_{L^1_w(\mu)} \left[1 + \log\left(\frac{\|u\|_{L^p_w(\mu)}}{\|u\|_{L^1_w(\mu)}}\right)\right] \quad \text{if } p < \infty; \tag{2.7}$$

where L^p_w stands for the weak L^p space. We now claim that

$$\int_0^{t_0} \iiint \frac{\zeta |E_t(\gamma^1(t)) - E_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z)\, \mathrm{d}t \le C\zeta \left(1 + \log\left(\frac{C}{\zeta\delta}\right)\right). \tag{2.8}$$

Of course, the same proof will provide the same estimate for the magnetic field H. By (2.2), we have that

$$\iiint \frac{\zeta |E_t(\gamma^1(t)) - E_t(\eta^1(t))|}{\zeta\delta + |\gamma^1(t) - \eta^1(t)|} \mathrm{d}\mu(\eta,\gamma,z) \le C\zeta \iiint g(t,\eta,\gamma,z) \mathrm{d}\mu(\eta,\gamma,z), \tag{2.9}$$

where

$$g(t,\eta,\gamma,z) := \min\left\{ M(DE_t(\gamma^1(t))) + M(DE_t(\eta^1(t))), \frac{|E_t(\eta^1(t))|}{\zeta\delta} + \frac{|E_t(\gamma^1(t))|}{\zeta\delta} \right\}.$$

By recalling that $\mathrm{d}\mu(\eta,\gamma,z) = \mathrm{d}\boldsymbol{\eta}_z(\gamma)\mathrm{d}\boldsymbol{\eta}_z(\eta)\mathrm{d}(e_0)_\#\boldsymbol{\eta}(z)$, the condition $(e_t)_\#\boldsymbol{\eta} \le C_0\mathscr{L}^6$, and the estimate $\|M(u)\|_{L^1_w(\mathbb{R}^3)} \le C\|u\|_{L^1(\mathbb{R}^3)}$, we have

$$\begin{aligned}\|g_t\|_{L^1_w(\mu)} &\le 2\|M(DE_t(\eta^1(t)))\|_{L^1_w(\boldsymbol{\eta})} \le 2C_0 \|M(DE_t)\|_{L^1_w(B_R\times B_R, \mathscr{L}^6)} \\ &= 2C_0|B_R| \|M(DE_t)\|_{L^1_w(B_R)} \le 2CC_0|B_R||DE_t|(\mathbb{R}^3) \le C_R \|E_t\|_{\mathrm{BV}(\mathbb{R}^3)}.\end{aligned}$$

By similar argument, using the estimate $\|M(u)\|_{L^{3/2}(\mathbb{R}^3)} \le C\|u\|_{L^{3/2}(\mathbb{R}^3)}$ and the embedding $\mathrm{BV} \subset L^{3/2}$, we also have

$$\|g_t\|_{L^{3/2}_w(\mu)} \le 2C_0(\zeta\delta)^{-1}\|E_t\|_{L^{3/2}(B_R\times B_R)} \le C_R(\zeta\delta)^{-1}\|E_t\|_{\mathrm{BV}(\mathbb{R}^3)}.$$

By integrating with respect to in time $[0,t_0]$, the claim is proven for C constant depending only on R, and the norms of $\boldsymbol{b}_1$, E, B. Now, by (2.5), (2.6) and (2.8), we have that

$$\Phi_{\delta,\zeta}(t_0) \le C\left(\frac{t_0}{\zeta} + 1 + \zeta + \zeta\log\left(\frac{C}{\zeta}\right)\right) + C\zeta\left(\log\left(\frac{1}{\delta}\right)\right)$$

Choosing first $\zeta > 0$ small enough in order to have $C\zeta < a/(2T)$ and then taking δ small enough, we find a contradiction with (2.3), concluding the proof. $\square$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1. Notice that the vector field $\boldsymbol{b}$ satisfies properties (H1)-(H3). Therefore by [3, Theorem 5.1], we deduce that: if (i) holds, then f is a Lagrangian solution; if (ii) holds and it is not bounded, then $\beta(f_t)$ is a Lagrangian solution, where we choose $\beta(s) := \arctan(s)$. In particular, by [3, Theorem 4.10] we have that f_t is a renormalized solution. □

We have a direct corollary that provides conditions to obtain a globally defined flow, that is, to avoid a finite-time blow up.

Proof of Theorem 1.2. By Theorem 1.1, the solution is transported by the maximal regular flow associated to $\boldsymbol{b}_t(x,v) = (\hat{v}, E_t(x) + \hat{v} \times H_t(x))$. Moreover, since f_t is a renormalized solution, $g_t := \frac{2}{\pi}\arctan f_t : (0,T) \times \mathbb{R}^3 \longrightarrow [0,1]$ is a solution of the continuity equation with vector field $\boldsymbol{b}$. Assuming $\hat{v} = v$, we have $|\boldsymbol{b}_t(x,v)| \le |v| + |E_t(x)| + |v||H_t(x)|$, so that

$$\begin{aligned} I &:= \int_0^T \int_{\mathbb{R}^6} \frac{|\boldsymbol{b}_t(x,v)| g_t(x,v)}{(1 + (|x|^2 + |v|^2)^{1/2}) \log(2 + (|x|^2 + |v|^2)^{1/2})} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t \\ &\le \frac{1}{\log 2} \int_0^T \int_{\mathbb{R}^6} f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t + \int_0^T \int_{\mathbb{R}^6} \frac{(|E_t| + |v||H_t|) g_t}{(1+|v|)\log(2+|v|)} \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t. \end{aligned}$$

Notice that the electric field term is simpler to compute, since $g_t^3 \le g_t^2 \le f_t$ and by the embedding $\mathrm{BV}(\mathbb{R}^3) \subset L^{3/2}(\mathbb{R}^3)$, so it can be estimated by

$$\left(\int_{\mathbb{R}^3} \frac{1}{(1+|v|)^{5/2} \log^{3/2}(2+|v|)} \mathrm{d}v \right) \left(\int_0^T \int_{\mathbb{R}^3} |E_t|^{3/2} \,\mathrm{d}x\,\mathrm{d}t \right) + \int_0^T \int_{\mathbb{R}^6} (1+|v|)^2 f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

We remark that so far we did not use hypothesis (1.10). For the magnetic field term, we have

$$\left(\int_{\mathbb{R}^3} \frac{1}{(1+|v|)^3 \log^2(2+|v|)} \mathrm{d}v \right) \left(\int_0^T \int_{\mathbb{R}^3} |H_t|^2 \,\mathrm{d}x\,\mathrm{d}t \right) + C \int_0^T \int_{\mathbb{R}^6} (1+|v|)^3 f_t \,\mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t.$$

By (1.10), we conclude I is bounded.

Now, by the no blow-up criterion in [3, Proposition 4.11] we obtain that the maximal regular flow $\boldsymbol{X}$ of $\boldsymbol{b}$ is globally defined on $[0,T]$, whence (i) follows. Moreover, the trajectories $\boldsymbol{X}(\cdot, x, v)$ belong to $\mathrm{AC}([0,T];\mathbb{R}^6)$ for g_0-a.e. $(x,v) \in \mathbb{R}^6$, and $g_t = \boldsymbol{X}(t,\cdot)_\# g_0 = g_0 \circ \boldsymbol{X}^{-1}(t,\cdot)$. Since $f_t = \tan\left(\frac{\pi}{2} g_t\right)$ and the map $[0,1) \ni s \longrightarrow \tan\left(\frac{\pi}{2} s\right) \in [0,\infty)$ is a diffeomorphism, we obtain that $f_t = \boldsymbol{X}(t,\cdot)_\# f_0 = f_0 \circ \boldsymbol{X}^{-1}(t,\cdot)$, whence (ii) follows. In particular, for all Borel functions $\psi : [0,\infty) \longrightarrow [0,\infty)$ we have

$$\int_{\mathbb{R}^6} \psi(f_t) \,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^6} \psi(f_0) \circ \boldsymbol{X}^{-1}(t,\cdot) \,\mathrm{d}x\,\mathrm{d}v = \int_{\mathbb{R}^6} \psi(f_0) \,\mathrm{d}x\,\mathrm{d}v,$$

where the second equality follows by the incompressibility of the flow, which gives (iii). □

Remark 2.2. As in [3, Remark 2.4], given $0 \le s \le t \le T$, with our previouus results it is possible to reconstruct f_t from f_s by using the flow, that is, $f_t = \boldsymbol{X}(t,s,\cdot)_\#(f_s)$.

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