Saturated impulsive control of nonlinear systems using Sum of Squares approach

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Abstract

In this article, the stability and stabilization problems of saturated impulsive nonlinear control systems are investigated. With the use of a class of clock-dependent Lyapunov functions and polytopic representation approach, new sufficient conditions ensuring the local exponential stability (LES) are established in the framework of dwell time, which allow that both the continuous and discrete parts of the systems are destabilizing at the same time. Moreover, based on the sum of squares programming, an optimization algorithm is proposed to design the saturated impulsive controller with improvement of the allowable impulsive dwell-time and the size of the domain of attraction. Finally, the simulation results demonstrate the effectiveness of the results.

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Abstract

In this article, the stability and stabilization problems of saturated impulsive nonlinear control systems are investigated. With the use of a class of clock-dependent Lyapunov functions and polytopic representation approach, new sufficient conditions ensuring the local exponential stability (LES) are established in the framework of dwell time, which allow that both the continuous and discrete parts of the systems are destabilizing at the same time. Moreover, based on the sum of squares programming, an optimization algorithm is proposed to design the saturated impulsive controller with improvement of the allowable impulsive dwell-time and the size of the domain of attraction. Finally, the simulation results demonstrate the effectiveness of the results.

Keywords: actuator saturation, Impulsive systems, Stability, Sum of squares.

1. Introduction

Impulsive systems is an important class of mathematical model which have been widely used for describing real world process with abrupt state changes such as epidemiology[1], forestry[2], power electronic[3], sampled-data systems[4] and so on. Moreover, Impulsive control as a simple and effective control techniques have been intensively studied over the past decades [5]-[10]. On the other hand, due to the unavoidable magnitude or rate limits of the actuator in control systems, it has been recognized that the actuator saturation often occur in various control systems and may cause undesirable dynamical behaviour such as performance degradation and the instability [11]-[12]. Therefore, it is necessary to consider the actuator saturation effect when investigating the impulsive control systems.

In particular, the stability and stabilization problems of the saturated impulsive control systems have attracted considerable attention of many researchers during the last years. There are two main approach to deal with saturation term, polytopic representation approach and sector nonlinearity model approach [13]-[22]. The discrete-time impulsive systems with actuator saturation are considered in [13]-[14]. The time-delay systems with saturated impulses are investigated in [15]-[19].In paper [20], the saturated impulses controller with input disturbances is proposed to stabilize the nonlinear systems so as to enlarge the size of the estimation of attraction domain. In paper [21], the saturated impulsive control of nonlinear systems is considered. The sufficient conditions ensuring the local exponential stabilization and the optimization algorithm to get a less conservative estimate of domain of attraction are proposed. Paper [22] consider the synchronization problem of coupled delayed neural networks with the saturated impulses controller.

The main idea of those results is to use the Lyapunov function/functional depicting the worst-case divergence of the continuous-time dynamics and the worst-case convergence of the impulses, then use the stabilization property of impulsive behaviors to compensate the state divergence made by continuous-time dynamics. However, the states of systems often have different divergence rate, and even some times exhibit convergence. Then the impulsive controller designed by this approach is conservative. Recently, the looped-functional approach[5]-[6] and clock-dependent lyapunov approach [7]-[8] have been used to overcome those drawbacks. However, all those results only focus on the linear impulsive systems without considering the nonlinear dynamics and the actuator saturation effect.

Motivated by the above discussions, a new Lyapunov-based method is proposed to analysis the stability and stabilization problem of the saturated impulsive nonlinear control systems. The main contributions of this article are listed as follows:
(a) a class of clock-dependent Lyapunov functions is used to analysis the stability problem of the system when aperiodic and periodic saturated impulse is considered. The proposed method is advantageous in that it is able to make good use of the information on the interaction among continuous and discrete dynamics, especially when systems consist of unstable continuous dynamics and destabilizing impulses.
(b) an optimization algorithm is proposed to design the saturated impulsive controller with the goal of enlarging the size of the estimation of attraction domain. Then, we use the Sum of Square (SOS) approach to solve the optimization algorithm and get a larger estimation of attraction domain with given impulsive dwell-time compared with the common lyapunov method considered in the previously literature.

The structure of this paper is outlined as follows: In Section 2, problem formulation and preliminaries are given. In Section3, we derive the stability conditions of saturated impulsive control systems by constructing a clock-dependent lyapunov function. Then the optimization algorithm is provided in Section 4 to design the saturated impulsive controller. Illustrative examples are given to demonstrate the effectiveness of the proposed approach in Section 5. Finally, Section 6 concludes the article.

Notations: Let $\mathbb{R}$ denote the set of real numbers, $\mathbb{R}^n$ the n-dimensional Euclidean space with norm $\|\cdot\|$, I the unit matrix, and $\mathbb{N}_+$ the set of positive integers. For a matrix A, $\lambda_{\min}(A)$, $\lambda_{\max}(A)$ and A^T stand for the minimum eigenvalue, maximum eigenvalue and the transposition of matrix A, respectively. The set $\mathbb{R}^{n\times n}$ denote the set of $n \times n$ matrices. The set $\mathbb{S}^{n\times n}$ denote the set of $n \times n$ symmetric matrices. $A > 0$ means that matrix A is positive definite. Let $\mathcal{J}[a, b] = \{a, a + 1, ..., b\}$, $a < b \in \mathbb{N}_+$. Given matrix $P \in \mathbb{R}^{n\times n} > 0$ and constant $\epsilon > 0$, $\mathcal{B}(P, \epsilon) : \{x \in \mathbb{R}^{n\times n} : x^T Px < \epsilon\}$. Given matrix $H \in \mathbb{R}^{m\times n}$, let h_i be the ith row of the matrix H and define $\mathcal{L}(H) : \{x \in \mathbb{R}^n : |h_i x| \leq 1, i \in \mathcal{J}[1, m]\}$. $\mathcal{D}$ denotes the set of $m \times m$ diagonal matrices whose diagonal elements are either 1 or 0. Suppose that each element of $\mathcal{D}$ is labeled as D_i and denote $D_i^- = I - D_i, i \in J[1, 2^m]$.

2. Problem formulation and preliminaries

Consider the saturated impulsive nonlinear control system described by [21]

$$\begin{cases} \dot{x}(t) = Ax(t) + Wf(x(t)), t \neq t_k, t \geq t_0 \\ x(t) = x(t^-) + B\text{sat}(u(t)), t = t_k, k \in \mathbb{N}_+, \\ x(t_0) = x_0, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the system state, $A, W \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times m}$ are known constant matrices, $u \in \mathbb{R}^m$ is the control input, $x_0 \in \mathbb{R}^n$ is the initial state, and $f(x) = (f_1(x_1), ..., f_n(x_n)) : \mathbb{R}^n \to \mathbb{R}^n$ is the nonlinear vector function. The standard saturation function sat: $\mathbb{R}^m \to \mathbb{R}^m$ is defined as $\text{sat}(u) = [\text{sat}(u_1), ..., \text{sat}(u_m)]$, where $\text{sat}(u_j) = \text{sign}(u_j) \min\{1, |u_j|\}, j \in \mathcal{J}[1, m]$. The impulse instants sequence $\{t_k\}_{k\in\mathbb{N}_+}$, $t_k > 0$, is assumed to have positive increments $T_k = t_{k+1} - t_k > 0$ that are bounded away from 0. We assume that $x(t)$ is right continuous at $t = t_k, \forall k \in \mathbb{N}_+$, i.e., $x(t_k) = x(t_k^+)$.

We consider the state feedback $u(t) = Kx(t^-)$, where $K \in \mathbb{R}^{m\times n}$ is the control gain.

Let $x(t) = x(t, t_0, x_0)$ be the solution of system (1) through $(t_0, x_0), x_0 \in \mathbb{R}^n$, and the domain of attraction is defined as $\Psi =: \{x_0 \in \mathbb{R}^n : \lim_{t\to\infty} x(t, t_0, x_0) = 0\}$.

Definition 1: System (1) is said to be local exponential stability (LES) if there exist a set Ω and constants $M \geq 1$ and $\lambda > 0$ such that $\|x(t)\| \leq Me^{-\lambda(t-t_0)}\|x_0\|$, for all $t \geq t_0, x_0 \in \Omega$. Obviously, Ω is contained in Ψ.

Assumption 1. Assume that there exist constants $l_i > 0, i \in \mathcal{J}[1, n]$ such that f satisfies $f_i(0) = 0$ and $|f_i(s_1) - f_i(s_2)| \leq l_i|s_1 - s_2|$ for any $s_1, s_2 \in \mathbb{R}$. Denote $L = \mathrm{diag}(l_i)$ for latter use.

Lemma 1. Let $x, y \in \mathbb{R}^n$, and S a diagonal positive definite matrix with appropriate dimensions; then the following inequality holds:

$$x^T y + y^T x \leq x^T S x + y^T S^{-1} y.$$

Lemma 2. The linear matrix inequality

$$\begin{bmatrix} Q(x) & S(x) \\ S^T(x) & R(x) \end{bmatrix} > 0$$

where $Q(x) = Q^T(x), R(x) = R^T(x)$, is equivalent to either of the following conditions:

1) $Q(x) > 0, R(x) - S^T(x)Q(x)^{-1}S(x) > 0$

2) $R(x) > 0, Q(x) - S^T(x)R(x)^{-1}S(x) > 0$.

Lemma 3 [24]. Let P be a non-singular symmetric matrix in $\mathbb{R}^{n\times n}$, and let $\mathcal{U}, \mathcal{V}$ be two complementary subspaces whose sum equals $\mathbb{R}^n$. Then

$x^T Px < 0$ for all $x \in \mathcal{U} \setminus \{0\}$ and $x^T Px \geq 0$ for all $x \in \mathcal{V}$

is equivalent to

$x^T P^{-1}x > 0$ for all $x \in \mathcal{U}^\perp \setminus \{0\}$ and $x^T P^{-1}x \leq 0$ for all $x \in \mathcal{V}^\perp$.

Lemma 4 [20]-[21]. Given matrices $K, H \in \mathbb{R}^{m\times n}$, for any $x \in \mathbb{R}^n$, if $x \in \mathfrak{L}(H)$, then

$$\mathrm{sat}(Kx) = \sum_{i=1}^{2^m} v_i(D_i Kx + D_i^- Hx),$$

where $0 \leq v_i \leq 1$ and $\sum_{i=1}^{2^m} v_i = 1$. Denote $N(v) = \sum_{i=1}^{2^m} v_i(D_i K + D_i^- H)$ for latter use, where $v = [v_1, ..., v_{2^m}]$.

3. Main results

The sufficient local exponential stability conditions of system (1) with periodic and aperiodic saturated impulses are presented in this section.

Theorem 1. Suppose that the impulsive dwell-time satisfies $T_k \in [T_{\min}, T_{\max}]$, $k \in \mathbb{N}_+$. For some matrices $K, H \in \mathbb{R}^{m\times n}$, if there exist a continuously differentiable matrix function $P : [0, T_{\max}] \to \mathbb{S}^{n\times n} > 0$, $S \in \mathbb{R}^{n\times n} > 0$, $\mathcal{B}(P(\theta), 1) \subset \mathcal{L}(H)$, positive constants λ and

$$\begin{bmatrix} P(\tau)A + A^T P(\tau) + \dot{P}(\tau) + LSL + \lambda P(\tau) & P(\tau)W \\ * & -S \end{bmatrix} \leq 0, \tag{2}$$

$$\begin{bmatrix} -P(\theta) & (I + BD_i K + BD_i^- H)^T P(0) \\ * & -P(0) \end{bmatrix} \leq 0, i \in \mathcal{J}[1, 2^m], \tag{3}$$

hold for all $\tau \in [0, T_{\max}]$ and all $\theta \in [T_{\min}, T_{\max}]$, then system (1) with impulsive dwell-time $T_k \in [T_{\min}, T_{\max}]$, $k \in \mathbb{N}_+$, is LES. Moreover, $\mathcal{B}(P(0), 1)$ is contained in Ψ.

Proof: Construct the clock-dependent Lyapunov function in the form

$$V(t) = x^T(t)P(\tau)x(t), t \in [t_k, t_{k+1}), \tag{4}$$

where $\tau = t - t_k$, is the time elapsed since the last impulsive instant (i.e. a clock). Therefore, when $t = t_k$, $V(t) = x^T(t)P(0)x(t)$. when $t = t_{k+1}^-$, $V(t) = x^T(t)P(T_k)x(t)$. In the impulsive interval $[t_k, t_{k+1})$, $P(\tau)$ is varying from $P(0)$ to $P(T_k)$ continuously.

Then, the derivative of $V(t)$ along the trajectory of system (1) can be obtained as follows:

$$\begin{aligned} &\dot{V}(t) = x^T(t)[P(\tau)A + A^T P(\tau)]x(t) + 2x^T(t)P(\tau)Wf(x(t))) + x^T(t)\dot{P}(\tau)x(t), \\ &t \in [t_{k-1}, t_k), k \in \mathbb{N}_+. \end{aligned} \tag{5}$$

By Lemma 1, one can get

$$2x^T(t)P(\tau)Wf(x(t))) \leq x^T(t)P(\tau)WS^{-1}W^T P(\tau)x(t) + f^T(x(t)))Sf(x(t))) \tag{6}$$

From Assumption 1, one have

$$f^T(x(t)))Sf(x(t))) \leq x^T(t)L^T SLx(t) \tag{7}$$

From (5)-(7), one can get

$$\dot{V}(t) \leq x^T(t)[P(\tau)A + AP(\tau) + \dot{P}(\tau) + P(\tau)WS^{-1}W^T P(\tau) + LSL]x(t) \tag{8}$$

Applying Lemma 2 to (2), one can get

$$P(\tau)A + AP(\tau) + \dot{P}(\tau) + P(\tau)WS^{-1}W^T P(\tau) + LSL \leq -\lambda P(\tau) \tag{9}$$

Then, from (8) and (9), the following holds:

$$\dot{V}(t) \leq -\lambda V(t), \forall t \in [t_{k-1}, t_k), k \in \mathbb{N}_+. \tag{10}$$

Therefore,

$$V(t) \leq e^{-\lambda(t-t_k)}V(t_{k-1}), \forall t \in [t_{k-1}, t_k), k \in \mathbb{N}_+. \tag{11}$$

Suppose now that $x(t_0) \in \mathcal{B}(P(0), 1)$, then from (11) it follows that

$$V(t_1^-) \leq e^{-\lambda(t_1-t_0)}V(t_0) \leq e^{-\lambda(t_1-t_0)}x^T(t_0)P(0)x(t_0) < 1. \tag{12}$$

From the definition of $V(t)$, we known that

$$V(t_1^-) = x^T(t_1^-)P(T_1)x(t_1^-), T_1 \in [T_{\min}, T_{\max}]. \tag{13}$$

From (12) and (13), one can get

$$x(t_1^-) \in \mathcal{B}(P(T_1), 1), T_1 \in [T_{\min}, T_{\max}]. \tag{14}$$

From the condition $\mathcal{B}(P(\theta), 1) \in \mathcal{L}(H), \forall\theta \in [T_{\min}, T_{\max}]$, we known that $x(t_1^-)$ also belong to $\mathcal{L}(H)$.

Then based on Lemma 4, the polytopic representation approach, one can get that

$$\mathrm{sat}(Kx(t_1^-)) = N(v)x(t_1^-). \tag{15}$$

It from (3) that for all $\theta \in [T_{\min}, T_{\max}]$

$$\sum_{i=1}^{2^m} v_i \begin{bmatrix} -P(\theta) & (I + BD_iK + BD_i^- H)^T P(0) \\ * & -P(0) \end{bmatrix} \leq 0, i \in \mathcal{J}[1, 2^m] \tag{16}$$

where $0 \leq v_i \leq 1$ and $\sum_{i=1}^{2^m} v_i = 1$.

Then, one can get

$$\begin{bmatrix} -P(\theta)I & (I + BN(v))^T P(0) \\ * & -P(0) \end{bmatrix} \leq 0, \forall\theta \in [T_{\min}, T_{\max}]. \tag{17}$$

By Lemma 2, the following inequality is satisfied.

$$(I + BN(v))^T P(0)(I + BN(v)) - P(T_k) \leq 0, k \in \mathbb{N}_+. \tag{18}$$

By (15), (18) and the fact that $T_1 \in [T_{\min}, T_{\max}]$, one obtain that

$$\begin{aligned} & V(x(t_1)) - V(x(t_1^-)) \\ & = x^T(t_1)P(0)x(t_1) - x^T(t_1^-)P(T_1)x(t_1^-) \\ & = x^T(t_1^-)[(I + BN(v))^T P(0)(I + BN(v))]x(t_1^-) - x^T(t_1^-)P(T_1)x(t_1^-) \\ & \leq 0 \end{aligned} \tag{19}$$

Together with (11) and (19), one obtain that

$$\begin{aligned} & V(t) \leq e^{-\lambda(t-t_1)}V(t_1) \\ & \leq e^{-\lambda(t-t_1)}V(t_1^-) \\ & \leq e^{-\lambda(t-t_1)}e^{-\lambda(t_1-t_0)}V(t_0) \\ & \leq e^{-\lambda(t-t_0)}V(t_0), t \in [t_1, t_2). \end{aligned} \tag{20}$$

By mathematical deduction, we assume that

$$V(t) \leq e^{-\lambda(t-t_0)}V(t_0), t \in [0, t_k). \tag{21}$$

Then, if $x(t_0) \in \mathcal{B}(P(0), 1)$, it follows that

$$V(t_k^-) = x^T(t_k^-)P(T_k)x(t_k^-) \le e^{-\lambda(t-t_0)}V(t_0) < 1. \tag{22}$$

Therefore, $x(t_k^-) \in \mathcal{B}(P(T_k), 1)$, $T_k \in [T_{\min}, T_{\max}]$. From the condition $\mathcal{B}(P(\theta), 1) \in \mathcal{L}(H), \forall\theta \in [T_{\min}, T_{\max}]$, we known that $x(t_k^-)$ also belong to $\mathcal{L}(H)$.

Then based on Lemma 4, one get

$$\mathrm{sat}(Kx(t_k^-)) = N(v)x(t_k^-). \tag{23}$$

Therefore, one can deduced that

$$\begin{aligned}
&V(x(t_k)) - V(x(t_k^-)) \\
&= x^T(t_k)P(0)x(t_k) - x^T(t_k^-)P(T_k)x(t_k^-) \\
&= x^T(t_k^-)[(I + BN(v))^T P(0)(I + BN(v))]x(t_k^-) \\
&- x^T(t_k^-)P(T_k)x(t_k^-)
\end{aligned} \tag{24}$$

By (18), (24) and the fact that $T_k \in [T_{\min}, T_{\max}]$, it follows that

$$V(t_k) \le V(t_k^-). \tag{25}$$

Therefore, by (11), (21) and (25), the following inequality holds:

$$\begin{aligned}
V(t) &\le e^{-\lambda(t-t_k)}V(t_k) \\
&\le e^{-\lambda(t-t_k)}V(t_k^-) \\
&\le e^{-\lambda(t-t_k)}e^{-\lambda(t_k-t_0)}V(t_0) \\
&\le e^{-\lambda(t-t_0)}V(t_0), t \in [t_k, t_{k+1}).
\end{aligned} \tag{26}$$

Then from (20), (26) and mathematical deduction, it is satisfied that

$$V(t) \le e^{-\lambda(t-t_0)}V(t_0), \forall t \ge t_0. \tag{27}$$

From the definition of $V(t)$, one can see that

$$V(t_0) = x^T(t_0)P(0)x(t_0) \le \lambda_{\max}(P(0))\|x(t_0)\|^2 \tag{28}$$

For $\forall t \ge t_0$, there always exists an impulsive interval $[t_{k-1}, t_k), k \in \mathbb{N}_+$ such that t will be local in. Therefore,

$$V(t) = x^T(t)P(\tau)x(t) \ge \lambda_{\min}(P(\tau))\|x(t)\|^2, \tau \in [0, T_k] \tag{29}$$

Because the differentiable matrix function $P(\tau)$ is continuous in $[0, T_{\max}]$, then there is a minimum eigenvalue $\lambda_{\min}$ such that $\lambda_{\min} \le \lambda_{\min}(P(\tau))$ for all $\tau \in [0, T_{\max}]$. Therefore,

$$V(t) \ge \lambda_{\min}\|x(t)\|^2 \text{ hold for } \forall t \ge t_0. \tag{30}$$

Then based on (27), (28) and (30), one obtains that

$$\|x(t)\| \le \sqrt{\frac{\lambda_{\max}(P(0))}{\lambda_{\min}}} e^{-\frac{\lambda}{2}(t-t_0)}\|x(t_0)\|. \tag{31}$$

Hence, according to Definition 1, system (1) is LES. This completes the proof.

Remark 1. The conditions of Theorem 1 guarantee that the system states starting from $\mathcal{B}(P(0), 1)$ will convergence to the origin exponentially under the aperiodic impulses with impulsive dwell-time $T_k \in [T_{\min}, T_{\max}]$. In general, the states of the system have different divergence rate, and some of them even exhibit convergence. Therefore, the stability conditions derived by worst convergence analysis will be conservative. The condition (2) in Theorem 1 guarantee that the clock-dependent lyapunov function is exponentially decreasing in any impulsive interval $[t_{k-1}, t_k), k \in \mathbb{N}_+$ with the differentiable matrix function $P(\tau)$ varying from $P(0)$ to $P(T_k)$. Then the destabilizing effect of the continuous systems have been absorbed by the $P(\tau)$. Therefore, at impulsive instants t_k the saturated-impulse need to ensure the decrement of the clock-dependent lyapunov function from $x^T(t_k^-)P(T_k)x(t_k^-)$ to $x^T(t_k)P(0)x(t_k)$ as the condition in (3). The different divergence rate of the systems states will lead to different changing of the $P(\tau)$. If there is some stabilizing effect in the continuous part, the $P(\tau)$ will be also able to capture it. Therefore, the method proposed in Theorem 1 is more effective in stability analysis.

Remark 2. The $P(\tau)$ can be chosen as any differentiable matrix function with respect to τ as long as it satisfy the conditions in Theorem 1. However, we don't know what's the best one in advance. Fortunately, we can use some function to approximate the those matrix function, for example, piecewise linear function [23], polynomial function [7] and so on. In [20]-[21], the the constant matrix $P(\tau) = P$ is used to deal with the stability problem based on worst convergence analysis .Therefore, our results are more general compared with those results.

When the impulsive dwell-time $T_k = T, \forall k \in \mathbb{N}_+$, the following corollary for periodic saturation impulsive system is derived.

Corollary 1. For some matrices $K, H \in \mathbb{R}^{m\times n}$, if there exist a differentiable matrix function $P : [0, T] \to \mathbb{S}^{n\times n} > 0$, $S \in \mathbb{R}^{n\times n} > 0$, $\mathcal{B}(P(\theta), 1) \subset \mathcal{L}(H)$, positive constants λ, ε and

$$\begin{bmatrix} P(\tau)A + A^T P(\tau) + \dot{P}(\tau) + LSL + \lambda P(\tau) & P(\tau)W \\ * & -S \end{bmatrix} \leq 0, \tag{32}$$

$$\begin{bmatrix} -P(T) & (I + BD_iK + BD_i^- H)^T P(0) \\ * & -P(0) \end{bmatrix} \leq 0, i \in \mathcal{J}[1, 2^m], \tag{33}$$

hold for all $\tau \in [0, T]$, then system (1) with with impulsive dwell-time $T_k = T$, $k \in \mathbb{N}_+$, is LES. Moreover, $\mathcal{B}(P(0), 1)$ is contained in Ψ.

Proof: Based on Theorem 1, if $T_k = T_{\min} = T_{\max}$, then one can get Corollary 1 immediately.

4. Optimization problems

In this section, We will give the optimization algorithm for impulsive controller designing with estimation of the domain of attraction as large as possible.

Firstly, similar to Theorem 1, the following Theorem is derived which is more convenience in saturated impulsive controller design.

Theorem 2. For some matrices $K, H \in \mathbb{R}^{m\times n}$, if there exist a differentiable matrix function $R : [0, T_{\max}] \to \mathbb{S}^{n\times n} > 0$, $S \in \mathbb{R}^{n\times n} > 0$, $\mathcal{B}(R(0), 1) \subset \mathcal{L}(H)$, positive constants λ and

$$\begin{bmatrix} R(\tau)A + A^T R(\tau) - \dot{R}(\tau) + LSL + \lambda R(\tau) & R(\tau)W \\ * & -S \end{bmatrix} \leq 0, \tag{34}$$

$$\begin{bmatrix} -R(0) & (I + B_i K + BD_i^- H)^T R(\theta) \\ * & -R(\theta) \end{bmatrix} \leq 0, i \in \mathcal{J}[1, 2^m] \tag{35}$$

hold for all $\tau \in [0, T_{\max}]$ and all $\theta \in [T_{\min}, T_{\max}]$, then system (1) with impulsive dwell-time $T_k \in [T_{\min}, T_{\max}]$, $k \in \mathbb{N}_+$, is LES. Moreover, $\Omega = \{x \in \mathbb{R}^n : x \in \mathcal{B}(R(\theta), 1) \text{ for all } \theta \in [T_{\min}, T_{\max}]\}$ is contained in Ψ.

Proof: The proof follows the same lines as the one of Theorem 1 by using the clock-dependent Lyapunov function in the form of $V(t) = x^T(t)R(T_k - \tau)x(t), \tau = t - t_k$.

With all the ellipsoids satisfying the set invariance condition, we would like to choose from among them the "largest" one to get a least conservative estimation of the domain of attraction. Let $X_{\bar{R}} \subset \mathbb{R}^n$ be a prescribed bounded convex set. For a set $S \subset \mathbb{R}^n$, define $\alpha_{\bar{R}}(S) := \sup\{\alpha > 0 : \alpha X_{\bar{R}} \subset S\}$. If $\alpha_{\bar{R}}(S) \geq 1$, then $X_{\bar{R}} \subset S$. Then based on Theorem 2, the convex optimization problem for estimating the largest α in the following:

$\max_{R,K,H,S,T_{\min},T_{\max}} \alpha$

s.t. (a1) $\alpha X_{\bar{R}} \subset \mathcal{B}(R(\theta), 1), \theta \in [T_{\min}, T_{\max}]$,

(a2) (34) in Theorem 2,

(a3) (35) in Theorem 2,

(a4) $\mathcal{B}(R(0), 1) \subset \mathcal{L}(H)$.

Now we transform the constraints into LMIs. Let $\gamma = 1/\alpha^2$, $M = S^{-1}$, $U = R^{-1}$, $Y = KU(0)$ and $Z = HU(0)$. Also let the ith row of Z be z_i. Here we choose the $X_{\bar{R}}$ to be the ellipsoid defined as: $X_{\bar{R}} = \mathcal{B}(\bar{R}, 1) = \{x \in \mathbb{R}^n : x^T \bar{R} x \leq 1\}$. then form (a1)-(a4) optimization problem can be rewritten as:

$\min_{U,T,Y,Z,T_{\min},T_{\max}} \gamma$

$(b1) \begin{bmatrix} \gamma\bar{R} & I \\ I & U(\theta) \end{bmatrix} \geq 0, \theta \in [T_{\min}, T_{\max}]$,

$(b2) \begin{bmatrix} U(\tau)A + A^T U(\tau) + \dot{U}(\tau) + WMW^T - \lambda U(\tau) & U(\tau)L \\ * & -M \end{bmatrix} \leq 0, \tau \in [0, T_{\max}]$

$(b3) \begin{bmatrix} -U(\theta) & U(0) + BD_i Y + BD_i^- Z \\ \star & -U(0) \end{bmatrix} \leq 0, i \in \mathcal{J}[1, 2^m], \theta \in [T_{\min}, T_{\max}]$

$(b4) \begin{bmatrix} 1 & z_i \\ z_i^T & U(0) \end{bmatrix} \geq 0, i \in [1, m].$

Proof: (a1)$\Rightarrow$ (b1): By Lemma 2, (a1) is equivalent to

$$\alpha^2 R(\theta) \leq \bar{R} \Leftrightarrow \begin{bmatrix} \gamma\bar{R} & I \\ I & U(\theta) \end{bmatrix} \geq 0 \tag{36}$$

for all $\theta \in [T_{\min}, T_{\max}]$.

(a2)⇒ (b2): By Lemma 2, (a2) is equivalent to

$$R(\tau)A + A^T R(\tau) - \dot{R}(\tau) + LSL - \lambda R(\tau) + R(\tau)WS^{-1}W^T R(\tau) \le 0. \tag{37}$$

Pre-and post-multiplying by $U(\tau)$ and using the fact that $U(\tau)\dot{R}(\tau)U(\tau) = -\dot{U(\tau)}$, one can get from (37) that

$$AU(\tau) + U(\tau)A^T + \dot{U}(\tau) + U(\tau)LSLU(\tau) - \lambda U(\tau) + WS^{-1}W^T \le 0, \tag{38}$$

which is equivalent to (b2) by Lemma 2.

(a3)⇒ (b3): Looking now at the LMI in (a3), one can easily see that it is equivalent to

$$-R(0) + (I + B_iK + BD_i^- H)^T R(\theta)(I + B_iK + BD_i^- H) \le 0 \tag{39}$$

which can be rewritten as the following form

$$\begin{bmatrix} (I + B_iK + BD_i^- H) \\ I \end{bmatrix}^T \begin{bmatrix} R(\theta) & 0 \\ 0 & -R(0) \end{bmatrix} \begin{bmatrix} (I + B_iK + BD_i^- H) \\ I \end{bmatrix} \le 0. \tag{40}$$

Notice then that the central matrix has n positive and n negative eigenvalues, and the outer-factors are of rank n, one observes that

$$\begin{bmatrix} R(\theta) & 0 \\ 0 & -R(0) \end{bmatrix} \le 0, \text{ on im} \begin{pmatrix} (I + B_iK + BD_i^- H) \\ I \end{pmatrix} \tag{41}$$

and

$$\begin{bmatrix} R(\theta) & 0 \\ 0 & -R(0) \end{bmatrix} > 0, \text{ on im} \begin{pmatrix} I \\ 0 \end{pmatrix} \tag{42}$$

Since the direct sum of im$\begin{pmatrix} (I + B_iK + BD_i^- H) \\ I \end{pmatrix}$ and im$\begin{pmatrix} I \\ 0 \end{pmatrix}$ spans the whole $\mathbb{R}^{n\times n}$, one can apply the Lemma 3 and get the equivalent LMI

$$\begin{bmatrix} I \\ -(I + B_iK + BD_i^- H)^T \end{bmatrix}^T \begin{bmatrix} U(\theta) & 0 \\ 0 & -U(0) \end{bmatrix} \begin{bmatrix} I \\ -(I + B_iK + BD_i^- H)^T \end{bmatrix} \ge 0 \tag{43}$$

which can be rewritten as the following form

$$-U(\theta) + (I + B_iK + BD_i^- H)U(0)(I + B_iK + BD_i^- H)^T \le 0 \tag{44}$$

by Lemma 2, which is equivalent to (b3).

(a4)⇒ (b4): Applies Lemma 2, and one can get the equivalent LMI

$$h_i R(0)^{-1} h_i^T \le 1 \Leftrightarrow \begin{bmatrix} 1 & h_i R(0)^{-1} \\ R(0)^{-1} h_i^T & R(0)^{-1} \end{bmatrix} \ge 0 \tag{45}$$

which is equivalent to (b4). This completes the proof.

Remark 3. Note that if the conditions of Theorem 1 had been used, we would have obtained a controller matrix depending on the dwell-time T_k, which may not be implementable. This fact emphasizes the importance of Theorem 2.

Remark 4. The conditions in (b1)-(b4) having the matrix function $U(\tau)$ can not solved directly. Here we use the polynomials matrix functions to approximate the function $U(\tau)$, which have been proved to have lower numerical complexity compared with the piecewise-wise linear function in [7]. Then the conditions in (b1)-(b4) can be solved by the Sum of Square(SOS) programming [7]. For example, we can use the monomial basis such as $\{\tau, \tau^2, ..., \tau^d\}$ to construct the polynomials matrix functions $U(\tau) = \tau P_0 + \tau^2 P_1 + ... + \tau^d P_d$, where the $P_0, P_1, ..., P_d$ is the coefficient matrix and d is the order of the polynomials matrix functions.

Next, we present the SOS program associated with the conditions of (b1)-(b4) as follows:

Algorithm 1: Computation on admissible K with prescribed $T_{\max} \geq T_{\min} > 0$, $\bar{R} \in \mathbb{R}^{n\times n}$, and $\lambda > 0$.

1) Initialize $U \in \mathbb{S}^{n\times n}, G_1 \in \mathbb{S}^{2n\times 2n}, G_2 \in \mathbb{S}^{2n\times 2n}, Q_i \in \mathbb{S}^{2n\times 2n}, i \in \mathcal{J}[1, 2^m]$ as polynomial matrix functions with monomial basis of order d, $Y \in \mathbb{R}^{m\times n}, Z \in \mathbb{R}^{m\times n}, M \in \mathbb{S}^{n\times n}$,

2) Add restrictions: $U, G_1, G_2, Q_i, i \in \mathcal{J}[1, 2^m]$ are SOS, $M > 0$,

$$\begin{bmatrix} \gamma\bar{R} & I \\ I & U(\tau) \end{bmatrix} - G_1(T_{\max} - \tau)(\tau - T_{\min}) \geq 0, \tag{46}$$

$$\begin{bmatrix} U(\tau)A + A^T U(\tau) + \dot{U}(\tau) + WMW^T - \lambda U(\tau) & U(\tau)L \\ * & -M \end{bmatrix} - G_2\tau(T_{\max} - \tau) \leq 0, \tag{47}$$

$$\begin{bmatrix} -U(\tau) & U(0) + BD_iY + BD_i^- Z \\ \star & -U(0) \end{bmatrix} - Q_i(T_{\max} - \tau)(\tau - T_{\min}) \leq 0, i \in \mathcal{J}[1, 2^m], \tag{48}$$

$$\begin{bmatrix} 1 & z_i \\ z_i^T & U(0) \end{bmatrix} \geq 0, i \in [1, m]. \tag{49}$$

3) Call for the SeDuMi solver to minimize γ. Then the impulsive control gain $K = Y * U(0)^{-1}$ can be used to stabilize system (1) with the domain of attraction $\Omega = \{x \in \mathbb{R}^n : x \in \mathcal{B}(U^{-1}(\theta), 1) \text{ for all } \theta \in [T_{\min}, T_{\max}]\}$.

For initial state x_0, we need to check whether it stay in the domain of attraction $\Omega = \{x \in \mathbb{R}^n : x \in \mathcal{B}(U^{-1}(\theta), 1) \text{ for all } \theta \in [T_{\min}, T_{\max}]\}$. This constraint can be transformed into the following form:

$$\begin{bmatrix} 1 & x_0^T \\ x_0 & U(\theta) \end{bmatrix} \geq 0, \theta \in [T_{\min}, T_{\max}].$$

Then, the following algorithm can be used to check whether those LMIs holds or not.

Algorithm 2: Assume that the sum of squares program

Find polynomials $M : [0, T_{\max}] \rightarrow \mathbb{S}^{2n}$,

such that $M(\tau)$ is SOS,

$\begin{bmatrix} 1 & x_0^T \\ x_0 & U \end{bmatrix} - M(\tau)(T_{\max} - \tau)(\tau - T_{\min})$ is SOS,

is feasible. Then, the initial state x_0 is contained in the domain of attraction Ω.

5. Numerical examples

In order to verify the effectiveness of the theoretical results, we will present numerical examples. All the results are solved based on SOSTOOLs box of version 303 and SeDuMi of version 1.3.

Example 1: Consider the following saturated impulsive linear system

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 0.1 \\ 0 & 1.2 \end{bmatrix} x(t), t \neq t_k, \\ x(t_k) = x(t_k^-) + B\text{sat}(Kx(t_k^-)), t = t_k, \end{cases} \tag{50}$$

Note that the continuous-time dynamics of the first state is stable while the second is unstable. Let $T = 0.1$, $B = I$, $\bar{R} = I$, $\lambda = 0.01$. By using the Algorithm 1 with a matrix polynomial $U(\tau)$ of order 2, we obtain $\gamma = 0.0151$, $K = \begin{bmatrix} -0.66901 & -9.88313 \\ -0.03652 & -0.81307 \end{bmatrix}$, $R(T) = \begin{bmatrix} 0.00004 & 0.00067 \\ 0.00067 & 0.01506 \end{bmatrix}$. Choosing the initial state $x_0 = [5, -5]$ in the attraction domain $B(R(T), 1)$, the state trajectory of (50) convergence to the origin as shown in Fig 1. As shown in Fig 1, the impulse is allowed to be destabilizing in some state space due to the existence of stabilizing effect in the continuous-time dynamics which is ignored in [20]-[21].

Example 2: Consider the following Hopfield-type dynamic neural network

$$\dot{x}(t) = Ax(t) + Wf(x(t)) \tag{51}$$

where $x(t) = [x_1(t), x_2(t)]^T$, $f(x(t)) = [f_1(x(t)), f_2(x(t))]^T$ and $f_i(x_i(t)) = tanh(x_i(t)), i = 1, 2$, $A = \begin{bmatrix} -0.53 & 0 \\ 0 & -0.53 \end{bmatrix}$, $W = \begin{bmatrix} 0.56 & 0.17 \\ -0.32 & 0.83 \end{bmatrix}$. When initial state $x_0 = [0.3733, -0.3411]^T$, the state trajectory of (35) are shown in Fig 2.

We consider the saturated impulsive controller, the system can be modeled as

$$\begin{cases} \dot{x}(t) = Ax(t) + Wf(x(t)), t \neq t_k, \\ x(t_k) = x(t_k^-) + B\text{sat}(Kx(t_k^-)), t = t_k, \end{cases} \tag{52}$$

where $B = 1.2I$, $\bar{R} = I$, $\lambda = 0.01$ are choosen in advance .

Periodic impulses case.

Let $T = 0.2$. By using optimization 1 with a matrix polynomial $U(\tau)$ of order 2, we obtain $K = \begin{bmatrix} -0.71439 & 0.35079 \\ 0.33123 & -0.75767 \end{bmatrix}$, $R(T) = \begin{bmatrix} 0.001083 & -0.00054 \\ -0.00054 & 0.00121 \end{bmatrix}$. Selecting the initial state $x_0 = [15, -15]$ in the attraction domain $B(R(T), 1)$, the state trajectory of (52) convergence to the origin as shown in Fig 3.

The estimation of attraction domain solved with different order of matrix polynomial $U(\tau)$ is considered here. when $d = 0$ or 1, the optimization problem is not solved. This case cannot be analyzed by the method proposed in [20]-[21]. The estimation of attraction domain with degree of $d = 2, 4$ are shown in Fig 4. we can see that the higher order we choose, the larger attraction domain we will get. Therefore, we need to choose the proper order of $U(\tau)$ in order to balance the computational complexity and the accuracy of the results.

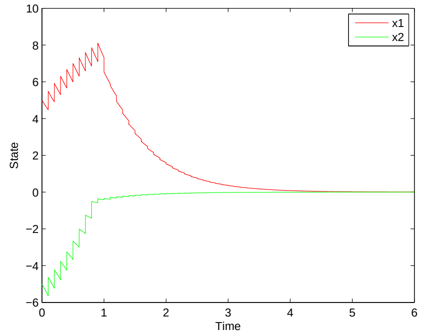

Figure 1: State trajectories of systems (50).

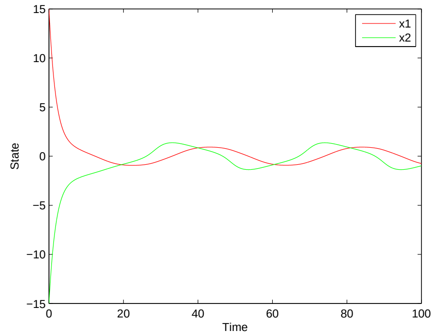

Figure 2: State trajectories of neural network (51) without impulses.

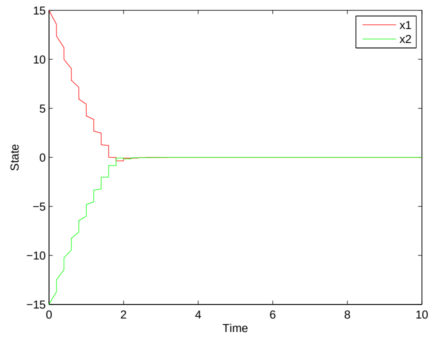

Figure 3: State trajectories of neural network (52) with periodic saturated impulsive controller.

The estimation of attraction domain solved with different choosing of $T = 0.01, 0.05, 0.1, 0.2$ and $d = 2$ are shown in Fig 5. In order to get the larger attraction domain, we can choose the impulsive dwell-time T sufficient small.

Aperiodic impulses case.

Let $T_k \in [0.001, 0.2]$. Solving the optimization problem with $d = 2$, we obtain

$$K = \begin{bmatrix} -0.72323 & 0.35471 \\ 0.32816 & -0.75239 \end{bmatrix}, U(\tau) = \begin{bmatrix} 60.57\tau^2 - 391.26\tau + 1271.7 & -8.95\tau^2 - 10.45\tau + 540.21 \\ -8.95\tau^2 - 10.45\tau + 540.21 & 106.3\tau^2 - 378.41\tau + 1132.1 \end{bmatrix}.$$

Choosing the initial state $x_0 = [15, -15]$ which satisfied the Algorithm 2, we obtain that the states for system (52) convergence to the origin as depicted in Fig 6, where T_k is picked randomly from $[0.001, 0.2]$.

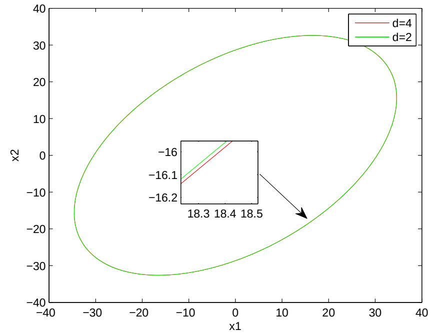

Figure 4: The estimation of attraction domain of (52) with different order of matrix polynomial $U(\tau)$

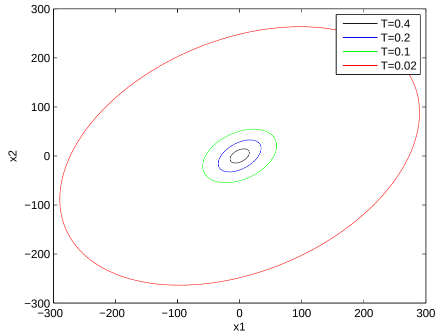

Figure 5: The estimation of attraction domain of (52) with different impulsive periodic

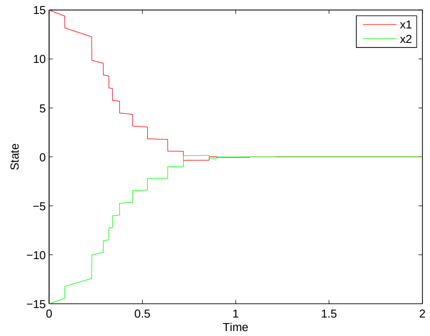

Figure 6: State trajectories of neural network (52) with aperiodic saturated impulsive controller

6. Conclusions

In this paper, new sufficient stability conditions of saturated impulsive nonlinear control systems are proposed. Based on the stability results, the designed impulsive controller is more effective comparing with the existing results not only in the admissible impulsive dwell-time but also in the estimation of attraction domain. However, the $P(\tau)$ is constructed by using the monomial basis. Using another method to construct the $P(\tau)$ to analysis the systems need to be further investigated, for example, Orthogonal Polynomial Bases, Chebyshev Polynomials, Legendre Polynomials, Jacobi Polynomials and so on.

References

[1] Stone L, Shulgin B, Agur Z. Theoretical examination of the pulse vaccination policy in the SIR epidemic model[J]. Mathematical and computer modelling, 2000, 31(4-5): 207-215.

[2] Verriest E I, Pepe P. Time optimal and optimal impulsive control for coupled differential difference point delay systems with an application in forestry[M]//Topics in time delay systems. Springer, Berlin, Heidelberg, 2009: 255-265.

[3] Loxton R C, Teo K L, Rehbock V, et al. Optimal switching instants for a switched-capacitor DC/DC power converter[J]. Automatica, 2009, 45(4): 973-980.

[4] Sivashankar N, Khargonekar P P. Characterization of the L_2-induced norm for linear systems with jumps with applications to sampled-data systems[J]. SIAM Journal on Control and Optimization, 1994, 32(4): 1128-1150.

[5] Briat C, Seuret A. A looped-functional approach for robust stability analysis of linear impulsive systems[J]. Systems & Control Letters, 2012, 61(10): 980-988.

[6] Shao H, Zhao J. Dwell-time-dependent stability results for impulsive systems[J]. IET Control Theory & Applications, 2017, 11(7): 1034-1040.

[7] Briat C. Theoretical and numerical comparisons of looped functionals and clock-dependent Lyapunov functions-The case of periodic and pseudo-periodic systems with impulses[J]. International Journal of Robust and Nonlinear Control, 2016, 26(10): 2232-2255.

[8] Briat C. Convex conditions for robust stability analysis and stabilization of linear aperiodic impulsive and sampled-data systems under dwell-time constraints[J]. Automatica, 2013, 49(11): 3449-3457.

[9] Zhang X, Li C, Li H. Finite-time stabilization of nonlinear systems via impulsive control with state-dependent delay[J]. Journal of the Franklin Institute, 2022, 359(3): 1196-1214..

[10] Xu H, Zhu Q. Stability analysis of impulsive stochastic delayed differential systems with infinite delay or finite delay and average-delay impulses[J]. Journal of the Franklin Institute, 2021, 358(16): 8593-8608.

[11] Hu T, Lin Z, Chen B M. An analysis and design method for linear systems subject to actuator saturation and disturbance[J]. Automatica, 2002, 38(2): 351-359.

[12] Tarbouriech S, Garcia G, da Silva Jr J M G, et al. Stability and stabilization of linear systems with saturating actuators[M]. Springer Science & Business Media, 2011.

[13] Shen Z, Li C, Li H, et al. Estimation of the domain of attraction for discrete-time linear impulsive control systems with input saturation[J]. Applied Mathematics and Computation, 2019, 362: 124502.

[14] Li L, Li C, Zhang W. Time-delayed impulsive control for discrete-time nonlinear systems with actuator saturation[J]. Nonlinear Analysis: Modelling and Control, 2019, 24(5): 804¨C818-804¨C818.

[15] Wei Y, Liu G P. Composite control for switched impulsive time-delay systems subject to actuator saturation and multiple disturbances[J]. Nonlinear Analysis: Hybrid Systems, 2020, 35: 100825.

[16] Ouyang D, Shao J, Jiang H, et al. Impulsive synchronization of coupled delayed neural networks with actuator saturation and its application to image encryption[J]. Neural Networks, 2020, 128: 158-171.

[17] Li H, Li C, Ouyang D, et al. Impulsive stabilization of nonlinear time-delay system with input saturation via delay-dependent polytopic approach[J]. IEEE Transactions on Systems, Man, and Cybernetics: Systems, 2020, 51(11): 7087-7098.

[18] He Z, Li C, Li Y, et al. Local synchronization of nonlinear dynamical networks with hybrid impulsive saturation control inputs[J]. Applied Mathematics and Computation, 2021, 410: 126452.

[19] Wu H, Li C, He Z, et al. Lag synchronization of nonlinear dynamical systems via asymmetric saturated impulsive control[J]. Chaos, Solitons & Fractals, 2021, 152: 111290.

[20] Wu H, Li C, He Z, et al. Estimation of attraction domain of nonlinear systems with saturated impulses and input disturbances[J]. International Journal of Robust and Nonlinear Control, 2021, 31(18): 9466-9482.

[21] Li X, Zhu C. Saturated impulsive control of nonlinear systems with applications[J]. Automatica, 2022: 110375.

[22] Wu S, Li X, Ding Y. Saturated impulsive control for synchronization of coupled delayed neural networks[J]. Neural Networks, 2021, 141: 261-269.

[23] Xiang W, Xiao J. Stabilization of switched continuous-time systems with all modes unstable via dwell time switching[J]. Automatica, 2014, 50(3): 940-945.

[24] Scherer C, Weiland S. Linear matrix inequalities in control[J]. Lecture Notes, Dutch Institute for Systems and Control, Delft, The Netherlands, 2000, 3(2).