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A numerical method for solving the space fractional Navier-Stokes equations

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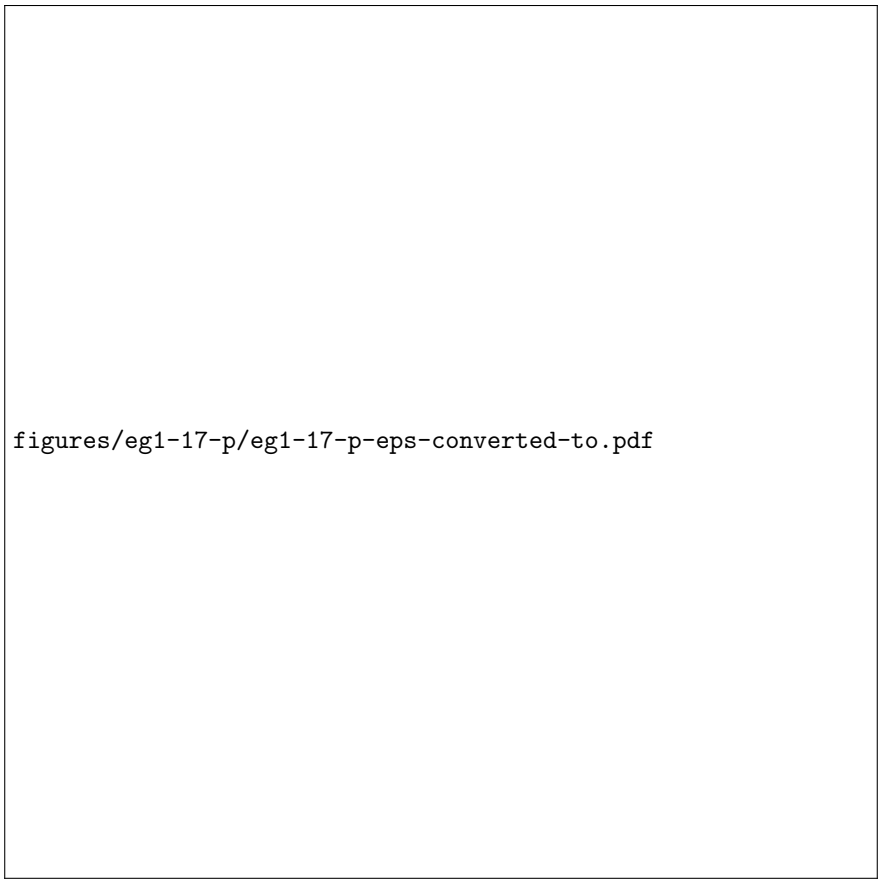
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Zihan Yue [1] · **Wei Jiang**[2] · **Zhong Chen**[2] · **Boying Wu**[1] · **Biao Zhang**[1]

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Abstract In this work, bases on the reproducing kernel theory and collocation method, we study the space Riesz fractional Navier-Stokes equations, and propose the numerical method to solve it. Firstly the new base space can be constructed by the spline and reproducing kernel space. The ε-approximate solution in binary spline space in the form of finite terms can be derived. Through using the collocation method, the approximate problem is solved. In addition, we provide analysis of the stability and convergence. In final, two numerical examples are provided to show the effectiveness of our method.

Keywords Fractional Navier-Stokes equations, Riesz fractional derivatives, Reproducing kernel space

Mathematics Subject Classification (2020) 65M60

1 Introduction

The Navier-Stokes (N-S) equation is established by the French scientist Navier in 1821 and the British scientist Stokes in 1845. It applies the laws of mass conservation, momentum conservation, and energy conservation to fluid motion. The Navier-Stokes equations describe the interaction between liquid and rigid body, which can be termed as the second law of Newton's motion for fluid, and play an indispensable role in many important practical problems [1–3]. The classic Navier-Stokes equation is denoted as:

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \Delta \mathbf{u}.$$

Recently, fractional calculus has received much attention due to its practical applications. The fractional calculus theory has found widespread applications in many fields. Based on the fractional theory, the fractional Navier-Stokes equations have been received much attention in recent years [4–6]. El-Shahed and Salem [7] firstly have replaced differential operator in the classic Navier-Stokes equation with a fractional differential operator, and obtained the time fractional Navier-Stokes equation. Meanwhile, the space fractional Navier-Stokes equation replaces Laplace operator with a fractional differential operator. As pointed out in [8], it was not feasible to use the traditional differential equation to describe the irregular wind flow, and the space fraction Navier-Stokes equation happened to be a turbulence model. Xu and Shen [9] have considered the stochastic time-space fractional incompressible Navier-Stokes equation driven by white noise. Except different fractional Navier-Stokes equation modeling, there are also many

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studies on fractional Navier-Stokes equation theory [10–12]. But it makes difficult to obtain the analytical solutions, so the numerical solution of fractional Navier-Stokes equations has attracted the interest of many authors. There are many numerical method applying to this kind of equations, such as finite element method, finite difference method and spectral method and so on. Yang [13] has investigated the fractional partial differential equation with Riesz fractional derivatives. Xu [14] has studied the space fractional Navier-Stokes equations by using the finite difference method. Sayevand [15] has obtained the numerical solution of the fractional Navier-Stokes equation through a non-standard finite difference method.

In this paper, we consider the following space fractional Navier-Stokes equations with Riesz derivatives, which are obtained from replacing Laplacian operator in the Navier-Stokes equations by Riesz fractional derivatives:

$$\frac{\partial}{\partial t}u(y,t) - \frac{1}{Re}\frac{\partial^{\alpha}}{\partial\,|y|^{\alpha}}u(y,t) = f(y,t), \tag{1}$$

where $(y,t) \in D := [0,1] \times [0,T]$, $f(y,t) \in C(D)$, Re represents the generalized Reynolds number, and $\frac{\partial^{\alpha}}{\partial|y|^{\alpha}}$ is denoted as the Riesz fractional derivative operator with $\alpha \in (1,2]$. The initial and boundary conditions are satisfied as follows:

$$\begin{aligned} &u(y,0) = 0, \quad 0 \le y \le 1, \\ &u(0,t) = u(1,t) = 0, \quad 0 \le t \le T \end{aligned} \tag{2}$$

Reproducing kernel methods are a case of spectral collocation method and reproducing kernel space is an ideal space framework for studying function approximation. In the recently year, it has been used to solve various differential equations owing to its good properties [16–18].

To this end, a novel numerical method is proposed to solve these kinds of equations. Using the reproducing kernel theory and spline space, the new basis can be constructed. Through the collocation method, we can get the ε-approximate solution by solving coefficients of approximation solution. The stability of the method is further proved, and the convergence of the method is analyzed. Finally, the theoretical results are verified by numerical examples. In this framework, the rest of the paper is organized as follows: Section 2 presents the preliminary concepts and notions. Section 3 develops the numerical method for solving the Eq.(1) with (2). The stability of the method and the convergence appears in Section 4. Section 5 gives numerical examples to illustrate the effectiveness of the proposed theory. Section 6 outlines a brief conclusion.

2 Preliminary concepts and notions

2.1 Risez fractional derivative operators

Definition 1 The left and right Riemann-Liouvile fractional derivatives operators with respect to order α are denoted as the following:

$$\begin{aligned} {}_0D_y^{\alpha}u(y,t) &= \frac{1}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial y^n}\int_0^y (y-\eta)^{n-\alpha-1}u(\eta,t)\mathrm{d}\eta, \\ {}_1D_y^{\alpha}u(y,t) &= \frac{(-1)^n}{\Gamma(n-\alpha)}\frac{\partial^n}{\partial y^n}\int_y^1 (\eta-y)^{n-\alpha-1}u(\eta,t)\mathrm{d}\eta, \end{aligned}$$

where $n-1 < \alpha \le n$, and $n = \lceil \alpha \rceil$.

Definition 2 The Riesz fractional derivative in Eq.(1) is denotes as:

$$\frac{\partial^{\alpha}}{\partial\,|y|^{\alpha}}u(y,t) = -\sigma({}_0D_y^{\alpha} + {}_1D_y^{\alpha})u(y,t),$$

where $\sigma = \frac{1}{2\cos(\frac{\pi\alpha}{2})}$, and $1 < \alpha \le 2$.

Definition 3 The left and right Caputo fractional differential operators ${}^{C}D_{0}^{\alpha}$ and ${}^{C}D_{1}^{\alpha}$ are denoted as the following:

$$
\begin{aligned}
{}^{C}D_{0}^{\alpha}u(y,t) &= \frac{1}{\Gamma(n-\alpha)}\int_{0}^{y}(y-\eta)^{n-\alpha-1}\frac{\partial^{n}u(\eta,t)}{\partial\eta^{n}}\mathrm{d}\eta,\\
{}^{C}D_{1}^{\alpha}u(y,t) &= \frac{(-1)^{n}}{\Gamma(n-\alpha)}\int_{y}^{1}(\eta-y)^{n-\alpha-1}\frac{\partial^{n}u(\eta,t)}{\partial\eta^{n}}\mathrm{d}\eta,
\end{aligned}
$$

where $n-1<\alpha\leq n$, and $n=\lceil\alpha\rceil$.

2.2 Reproducing kernel spaces

Definition 4 Setting B is an arbitrary nonempty abstract set and H is a Hilbert space, the element of H is the real or complex functions defined on B. Then, set $K(t,s):B\times B\to\mathbb{R}$ is an binary function and the conditions is as follows:

1. $K(t,s)\in H$ with respect to t, for $\forall s\in B$;
2. for $\forall s\in B$ and $f\in H$,

$$f(s)=(f(t),K(t,s))_{t},$$

therefore, H is called a reproducing kernel space, $K(t,s)$ is called a reproducing kernel.

Accronding to [19], absolutely continuous function space is denoted by the following form:

$$
\begin{aligned}
AC[a,b] &= \{f(x)|f(x)\text{ is an absolutely continuous function in }[a,b]\},\\
AC^{n}[a,b] &= \{f(x)|f^{(n-1)}(x)\text{ is an absolutely continuous function in }[a,b]\}.
\end{aligned}
$$

Definition 5 Define the Hilbert space as $W_{2}^{m}=W_{2}^{m}[a,b]=\{f(x)|f(x)\in AC^{m}[a,b],\ f^{m}(x)\in L^{2}[a,b]\}$. And we give the definition of the inner product in W_{2}^{m}:

$$(f(x),g(x))_{W_{2}^{m}}=\sum_{k=0}^{m-1}f^{(k)}(a)g^{(k)}(a)+\int_{a}^{b}f^{(m)}(x)g^{(m)}(x)\mathrm{d}x,\forall f(x),g(x)\in W_{2}^{m},$$

and $\|f(x)\|_{W_{2}^{m}}=\sqrt{(f(x),f(x))_{W_{2}^{m}}}$.

Similar to [19], we can define and prove that $W_{2}[0,T]$ and $W_{3}[0,1]$ are reproducing kernel spaces.

Definition 6 $W_{2}[0,T]=\{f(x)\in W_{2}^{2}[0,T]|f(0)=0\}$, equipped with the following inner product

$$(f(x),g(x))_{W_{2}[0,T]}=f'(0)g'(0)+\int_{0}^{T}f''(x)g''(x)\mathrm{d}x. \tag{3}$$

Definition 7 $W_{3}[0,1]=\{f(x)\in W_{2}^{3}[0,1]|f(0)=0,f(1)=0\}$, equipped with the following inner product

$$(f(x),g(x))_{W_{3}[0,1]}=f'(0)g'(0)+f''(0)g''(0)+\int_{0}^{1}f'''(x)g'''(x)\mathrm{d}x. \tag{4}$$

Meanwhile, the reproducing kernels of $W_{2}[0,T]$ and $W_{3}[0,1]$ can be denoted respectively by

$$
r_{2}(t,s)=\begin{cases}ts+\frac{ts^{2}}{2}-\frac{s^{3}}{6}, & s\leq t,\\ ts+\frac{st^{2}}{2}-\frac{t^{3}}{6}, & t<s.\end{cases}
\qquad
R_{3}(x,y)=\begin{cases}1+xy+\frac{x^{2}y^{2}}{4}+\frac{x^{2}y^{3}}{12}-\frac{xy^{4}}{24}+\frac{y^{5}}{120}, & y\leq x,\\ 1+\frac{x^{5}}{120}+(x-\frac{x^{4}}{24})y+\frac{1}{12}x^{2}(3+x)y^{2}, & x<y.\end{cases} \tag{5}
$$

Definition 8 Define the continuous space $C_{1}(\Omega)$, $\Omega=[0,1]\times[0,T]$:

$$C_{1}(\Omega)=\left\{\frac{\partial^{|i|}\partial^{|j|}}{\partial y^{|i|}\partial t^{|j|}}u(y,t)\in C(\Omega),|i|+|j|\leq 2,u(0,t)=u(1,t)=0,u(y,0)=0\right\},$$

the norm is denoted as:

$$\|u\|_{C_{1}(\Omega)}=\max\left\{\|\frac{\partial^{|i|}\partial^{|j|}}{\partial y^{|i|}\partial t^{|j|}}u(y,t)\|_{C(\Omega)},|i|+|j|\leq 2\right\}.$$

2.3 Spline spaces

Definition 9 Define the division of $[a,b]$, $\pi : a = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_{n-1} < \xi_n = b$. Set $k_l = [\xi_{l-1},\ \xi_l]$, $l = 1, \cdots, n$, so that the spline space can be expressed as

$$S_{k,\pi} = \left\{ \phi \in C^{k-1}[a,b] : \phi|_{k_j} \in P_k, j = 1,2,3,\cdots,n \right\},$$

where P_k is polynomial function space and its order is not greater than k over k_j.

Then due to the Definition 2.9, there are two spline spaces defined as:

$$\widetilde{S}_{5,\pi_y} = \left\{ \phi \in S_{5,\pi_y}[0,1] : \phi(0) = \phi(1) = 0 \right\}, \quad \widetilde{S}_{3,\pi_t} \quad = \left\{ \phi \in S_{3,\pi_t}[0,T] : \phi(0) = 0 \right\},$$

where π_y is a division of $[0,1]$ about space and π_t is a division of $[0,T]$ about time. Let $\Omega = \{(y,t) : 0 \le y \le 1,\ 0 \le t \le T\}$, $\pi_y : 0 = y_0 < y_1 < \cdots < y_n = 1$ and $\pi_t : 0 = t_0 < t_1 < \cdots < t_n = T$. The set of points $\pi_{yt} = \{(y_i, t_j) : 0 \le i \le n;\ 0 \le j \le n\} = \pi_y \cdot \pi_t$ represents the discretization of the domain Ω.

Definition 10 The space $S_{\pi_{yt}} \triangleq \widetilde{S}_{5,\pi_y} \otimes \widetilde{S}_{3,\pi_t} = \{u_n(y,t) | u_n(y,t) = \sum_{i,j=1}^{n} d_{i,j} B_i(y) B_j(t)\}$ is a bivariate spline space with base $\{B_i(y)B_j(t),\ i,j = 1,2,\cdots,n\}$, where $\otimes$ means the direct product, $\{B_i(y)\}_{i=1}^n$ and $\{B_j(t)\}_{j=1}^n$ are the bases of the spline space $\widetilde{S}_{5,\pi_y}$ and $\widetilde{S}_{3,\pi_t}$ respectively.

3 The ε-approximate solution of equation

3.1 Constructing the new base

In this section, we will demonstrate that a new base is constructed in the spline space.

Lemma 1 *$\left\{t, t^2, r_2(t,t_1), \cdots, r_2(t,t_n)\right\}$ is linearly independent and is a base of $\widetilde{S}_{3,\pi_t}$.*

Proof. Assuming that the division of $[0,T]$ is $0 = t_0 < t_1 < t_2 < \cdots < t_n = T$, for every $1 \le l \le n$, select $q \in W_2[0,T]$, such that $q(t_l) = 1$ and $q(t_k) = 0$, $k \ne l$. Meanwhile, $q(t) = 0$ when t is close to 0 or T, so $q'(0) = q'(T) = 0$.

Taking any integer in $[1, n-1]$, assuming that $at + bt^2 + \sum_{i=1}^{n-1} c_i r_2(t,t_i) = 0$. From Eq.(3), so

$$\begin{aligned} 0 &= (at + bt^2 + \sum_{i=1}^{n-1} c_i r_2(t,t_i), q(t))_{W_2[0,T]} \\ &= a(t, q(t))_{W_2[0,T]} + b(t^2, q(t))_{W_2[0,T]} + \sum_{i=1}^{n-1} c_i (r_2(t,t_i), q(t))_{W_2[0,T]} \\ &= 0 + 0 + \sum_{i=1}^{n-1} c_i q(t_i) \\ &= c_l, \qquad l = 1,2,\cdots,n-1. \end{aligned}$$

On the other hand, when $l = n$, considering with Eq.(5), it follows that

$$at + bt^2 + c_n(tT + \frac{Tt^2}{2} - \frac{t^3}{6}) = 0,$$

it is obvious that $a = b = c_n = 0$. Therefore $\{B_{t_i}\}_{i=1}^{n+2} = \{t, t^2, r_2(t,t_1), \cdots, r_2(t,t_n)\}$ is linearly independent.

Next we will verify that $\{B_{t_i}\}_{i=1}^{n+2}$ is a base of $\widetilde{S}_{3,\pi_t}$. According to definition of the $r_2(t,t_i)$, we can get $r_2(t,t_i) \in C^2[0,T]$. Meanwhile, $r_2(t,t_i)$ is a piecewise cubic polynomial with $r_2(0,t_i) = 0$, which implies that $r_2(t,t_i) \in \widetilde{S}_{3,\pi_t}$, for $i \in [1,n]$. Also $t, t^2 \in \widetilde{S}_{3,\pi_t}$, then $\{B_{t_i}\}_{i=1}^{n+2} \in \widetilde{S}_{3,\pi_t}$. Since $\dim \widetilde{S}_{3,\pi_t} = n+2$, *and $\{B_{t_i}\}_{i=1}^{n+2}$ is a linear independent set, so that $\{B_{t_i}\}_{i=1}^{n+2}$ is a base of $\widetilde{S}_{3,\pi_t}$.*

Theorem 1 $\{y(1-y), y^2(1-y), y^3(1-y), y^4(1-y), R_3(y,y_1), \cdots, R_3(y,y_{n-1})\}$ *is linearly independent and is a base of* $\widetilde{S}_{5,\pi_y}$.

Proof. Assuming that the division of $[0,1]$ *is* $0 = y_0 < y_1 < y_2 < \cdots < y_{n-1} < y_n = 1$, *for every* $1 \le k \le n-1$, *select* $h \in W_3[0,1]$, *which satisfying* $h(y_k) = 1$ *and* $h(y_j) = 0$, *when* $j \ne k$. *It can be found that when* y *verges to* 0 *or* 1, $h(y) = 0$. *Then* $h'(0) = h'(1) = h''(0) = h''(1) = 0$.

Assuming that

$$ay(1-y) + by^2(1-y) + cy^3(1-y) + dy^4(1-y) + \sum_{i=1}^{n-1} e_i R_3(y,y_i) = 0.$$

So considering with the Eq.(5), and

$$\int_0^1 yh'''(y)\mathrm{d}y = h''(1) - h'(1) + h'(0),$$
$$\int_0^1 y^2h'''(y)\mathrm{d}y = h''(1) - 2h'(1) + 2h(1) - 2h(0),$$

it follows that,

$$\begin{aligned}
0 &= (ay(1-y) + by^2(1-y) + cy^3(1-y) + dy^4(1-y) + \sum_{i=1}^{n-1} e_i R_3(y,y_i), h(y))_{W_3[0,1]} \\
&= a(y(1-y,h(y))_{W_3[0,1]} + b(y^2(1-y),h(y))_{W_3[0,1]} + c(y^3(1-y),h(y))_{W_3[0,1]} \\
&+ d(y^4(1-y),h(y))_{W_3[0,1]} + \sum_{i=1}^{n-1} e_i(R_3(y,y_i),h(y))_{W_3[0,1]} \\
&= 0+0+0+0+\sum_{i=1}^{n-1} e_i h(y_i) \\
&= e_k, \qquad k = 1,2,\cdots,n-1.
\end{aligned}$$

On the other hand, assuming that

$$ay(1-y) + by^2(1-y) + cy^3(1-y) + dy^4(1-y) = 0,$$

it is obvious that $a = b = c = d = 0$.

Therefore $\{B_{y_i}\}_{i=1}^{n+3} = \{y(1-y), y^2(1-y), y^3(1-y), y^4(1-y), R_3(y,y_1), \cdots, R_3(y,y_{n-1})\}$ *is linearly independent. Next we will verify that it is a base of* $\widetilde{S}_{5,\pi_y}$. *Due to the definition of the* $R_3(y,y_i)$, *we can get* $R_3(y,y_i) \in C^5[0,1]$. *On the other hand,* $R_3(y,y_i)$ *is a piecewise quintic polynomial with* $R_3(0,y_i) = R_3(1,y_i) = 0$, *which implies that* $R_3(y,y_i) \in \widetilde{S}_{5,\pi_y}$, *for every* $i = 1,2,\cdots,n-1$.

Obviously, $y(1-y), y^2(1-y), y^3(1-y), y^4(1-y) \in \widetilde{S}_{5,\pi_y}$, *then* $\{B_{y_i}\}_{i=1}^{n+3} \in \widetilde{S}_{5,\pi_y}$. *Since* $\dim \widetilde{S}_{5,\pi_y} = n+3$, $\{B_{y_i}\}_{i=1}^{n+3}$ *is a linear independent set, it yields that* $\{B_{y_i}\}_{i=1}^{n+3}$ *is a base of* $\widetilde{S}_{5,\pi_y}$.

In terms of $S_{\pi_{yt}} \triangleq \widetilde{S}_{5,\pi_y} \otimes \widetilde{S}_{3,\pi_t}$, combining the above Lemma 1 and Theorem 1, it yields that $K = \{B_{y_i}(y)\}_{i=1}^{n+3} \otimes \{B_{t_i}(t)\}_{i=1}^{n+2}$ is a new base in $S_{\pi_{yt}}$. Then the approximate solution $u_n(y,t)$ can be expressed by

$$\begin{aligned}
u_n(y,t) &= \sum_{i=1}^{n+3}\sum_{j=1}^{n+2} \Lambda_{ij} B_{y_i}(y) \otimes B_{t_j}(t) \\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij} R_3(y,y_i) r_2(t,t_j) + \sum_{i=1}^{n-1}(b_i t + c_i t^2) R_3(y,y_i) + \sum_{j=1}^{n}(d_j y(1-y) \\
&+ e_j y^2(1-y) + p_i y^3(1-y) + q_i y^4(1-y)) r_2(t,t_j) + (z_1 y(1-y) \\
&+ z_2 y^2(1-y) + z_3 y^3(1-y) + z_4 y^4(1-y))t + (z_5 y(1-y) \\
&+ z_6 y^2(1-y) + z_7 y^3(1-y) + z_8 y^4(1-y))t^2.
\end{aligned} \tag{6}$$

3.2 Implementation of a method to approximate the solution of the equation

According to the above reproducing kernel theory, this section will introduce a new numerical method to solve the Eq.(1) with (2) based on collocation method. Because of the Definition 2, the Eq.(1) is expanded as:

$$\frac{\partial}{\partial t}u(y,t)+\frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}({}_0D_y^{\alpha}+{}_1D_y^{\alpha})u(y,t)=f(y,t). \tag{7}$$

Let $m=\lceil\alpha\rceil$, then converting Riemann-Liouvile fractional derivatives operator ${}_0D_y^{\alpha}$ and ${}_1D_y^{\alpha}$ to Caputo fractional derivatives operator ${}^CD_0^{\alpha}$ and ${}^CD_1^{\alpha}$, the Eq.(7) is rewritten as:

$$\begin{aligned}&\frac{\partial}{\partial t}u(y,t)+\frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}\frac{1}{\Gamma(2-\alpha)}((1-\alpha)y^{-\alpha}u(0,t)+y^{1-\alpha}\partial_y u(0,t)+(1-\alpha)(1-y)^{-\alpha}u(1,t)\\&-(1-y)^{1-\alpha}\partial_y u(1,t))+\frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}({}^CD_0^{\alpha}u(y,t)+{}^CD_1^{\alpha}u(y,t))=f(y,t),\end{aligned}$$

where $1<\alpha\leq 2$. Since the boundary condition is $u(0,t)=u(1,t)=0$, so define the operator $\mathbb{L}$ as the follow:

$$\begin{aligned}\mathbb{L}u(y,t)=&\frac{\partial}{\partial t}u(y,t)+\frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}\frac{1}{\Gamma(2-\alpha)}(\int_0^y(y-\eta)^{1-\alpha}\partial_\eta^2u(\eta,t)\mathrm{d}\eta+\int_y^1(\eta-y)^{1-\alpha}\partial_\eta^2u(\eta,t)\mathrm{d}\eta\\&+y^{1-\alpha}\partial_y u(0,t)-(1-y)^{1-\alpha}\partial_y u(1,t))=f(y,t),\end{aligned} \tag{8}$$

Then the Eq.(1) is turned to an equation:

$$\mathbb{L}u(y,t)=f(y,t). \tag{9}$$

Lemma 2 *$\mathbb{L}:C_1(\Omega)\to L^1(\Omega)$ is a bouned operator.*

Proof. Since $\mathbb{L}$ is denoted by Eq.(8):

$$\begin{aligned}\mathbb{L}u(y,t)=&\frac{\partial}{\partial t}u(y,t)+\frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}\frac{1}{\Gamma(2-\alpha)}(\int_0^y(y-\eta)^{1-\alpha}\partial_\eta^2u(\eta,t)\mathrm{d}\eta+\int_y^1(\eta-y)^{1-\alpha}\partial_\eta^2u(\eta,t)\mathrm{d}\eta\\&+y^{1-\alpha}\partial_y u(0,t)-(1-y)^{1-\alpha}\partial_y u(1,t))\end{aligned}$$

Let $M_0=\frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}\frac{1}{\Gamma(2-\alpha)}$, so that in the sense of L^1 norm:

$$\|\mathbb{L}u\|_{L^1(\Omega)}=\int_0^T\int_0^1|\frac{\partial}{\partial t}u(y,t)+M_0(\int_0^y(y-\eta)^{1-\alpha}\partial_\eta^2u(\eta,t)\mathrm{d}\eta+\int_y^1(\eta-y)^{1-\alpha}\partial_\eta^2u(\eta,t)\mathrm{d}\eta$$

$$
\begin{aligned}
&+ y^{1-\alpha}\partial_y u(0,t) - (1-y)^{1-\alpha}\partial_y u(1,t))|\mathrm{d}y\mathrm{d}t \\
&\le \int_0^T\int_0^1 |\frac{\partial}{\partial t}u(y,t)|\mathrm{d}y\mathrm{d}t + M_0(\int_0^T\int_0^1 |\int_0^y (y-\eta)^{1-\alpha}\partial_\eta^2 u(\eta,t)\mathrm{d}\eta|\mathrm{d}y\mathrm{d}t + \int_0^T\int_0^1 |y^{1-\alpha}\partial_y u(0,t)|\mathrm{d}y\mathrm{d}t \\
&\quad + \int_0^T\int_0^1 |\int_y^1 (\eta-y)^{1-\alpha}\partial_\eta^2 u(\eta,t)\mathrm{d}\eta|\mathrm{d}y\mathrm{d}t + \int_0^T\int_0^1 |(1-y)^{1-\alpha}\partial_y u(1,t)|\mathrm{d}y\mathrm{d}t) \\
&\le \int_0^T\int_0^1 \|u\|_{C_1(\Omega)}\mathrm{d}y\mathrm{d}t + M_0(\int_0^T\int_0^1\int_0^y (y-\eta)^{1-\alpha}\|u\|_{C_1(\Omega)}\mathrm{d}\eta\mathrm{d}y\mathrm{d}t + \int_0^T\int_0^1 y^{1-\alpha}\|u\|_{C_1(\Omega)}\mathrm{d}y\mathrm{d}t \\
&\quad + \int_0^T\int_0^1\int_y^1 (\eta-y)^{1-\alpha}\|u\|_{C_1(\Omega)}\mathrm{d}\eta\mathrm{d}y\mathrm{d}t + \int_0^T\int_0^1 (1-y)^{1-\alpha}\|u\|_{C_1(\Omega)}\mathrm{d}y\mathrm{d}t) \\
&\le (T + M_0(\int_0^T\int_0^1\int_0^y (y-\eta)^{1-\alpha}\mathrm{d}\eta\mathrm{d}y\mathrm{d}t + \int_0^T\int_0^1\int_y^1 (\eta-y)^{1-\alpha}\mathrm{d}\eta\mathrm{d}y\mathrm{d}t + \int_0^T\int_0^1 y^{1-\alpha}\mathrm{d}y\mathrm{d}t \\
&\quad + \int_0^T\int_0^1 (1-y)^{1-\alpha}\mathrm{d}y\mathrm{d}t))\|u\|_{C_1(\Omega)} \\
&= (T + M_0 T(\frac{1}{2-\alpha}\frac{1}{3-\alpha} + \frac{1}{2-\alpha}\frac{1}{3-\alpha} + \frac{1}{2-\alpha} + \frac{1}{2-\alpha}))\|u\|_{C_1(\Omega)} \\
&\triangleq M\|u\|_{C_1(\Omega)},
\end{aligned}
$$

where $M \triangleq T + 2M_0T(\frac{1}{2-\alpha}\frac{1}{3-\alpha} + \frac{1}{2-\alpha})$ *is a constant. So,* $\|\mathbb{L}u\|_{L^1(\Omega)} \le M\|u\|_{C_1(\Omega)}$*, it means that* $\mathbb{L} : C_1(\Omega)$ *to* $L^1(\Omega)$ *is a bounded operator.*

Lemma 3 *$u(y,t)$ and $u_n(y,t)$ are the true solution and the approximate solution in $S_{\pi_{yt}}$ of the Eq.(9) respectively, there are the following conditions according to [20] and [21]:*

$$
\begin{aligned}
&\|u - u_n\|_{C(\Omega)} \le \lambda_0 h^4, \\
&\|D^{|1|}u - D^{|1|}u_n\|_{C(\Omega)} \le \lambda_1 h^3, \\
&\|D^{|2|}u - D^{|2|}u_n\|_{C(\Omega)} \le \lambda_2 h^2,
\end{aligned}
$$

where $h = \min\{h_y, h_t\}$, h_y and h_t are the partition of the space $\widetilde{S}_{5,\pi_y}$ and $\widetilde{S}_{3,\pi_t}$ respectively, and $p = (i,j)$, $|p| = i + j$, $D^{|p|}u = \frac{\partial^{|p|}u}{\partial y^i \partial t^j}$. λ_0, λ_1, λ_2 represent only depending on the C norms of mixed partial of no more than p order, $p = 0, 1, 2$.

Definition 11 For any $\varepsilon > 0$, if $\|\mathbb{L}v - f\|_{L^1(\Omega)} = \int_0^T\int_0^1 |\mathbb{L}v(y,t) - f(y,t)|\mathrm{d}y\mathrm{d}t < \varepsilon$, then $v(y,t)$ is an ε-approximate solution for Eq.(9).

Theorem 2 *Supposed that $u_n(y,t)$ is the solution from Eq.(6), if for any $\varepsilon > 0$, there exists $N \in \mathbb{N}^+$, when each $n > N$, $\widetilde{u}_n(y,t) = \sum_{i=1}^{n+3}\sum_{j=1}^{n+2}\widetilde{\Lambda_{ij}}B_{y_i}(y)\otimes B_{t_j}(t)$ is an ε-approximate solution of Eq.(7), whose coefficient variation $\widetilde{\Lambda_{ij}}$ satisfies*

$$
\|\mathbb{L}\tilde{u}_n(y,t) - f(y,t)\|_{L^1(\Omega)} = \min_{\widetilde{\Lambda_{ij}}}\|\mathbb{L}u_n(y,t) - f(y,t)\|_{L^1(\Omega)},
$$

and

$$
\begin{aligned}
\widetilde{u}_n(y,t) &= \sum_{i=1}^{n+3}\sum_{j=1}^{n+2}\widetilde{\Lambda_{ij}}B_y(y_i)\otimes B_t(t_j) \\
&= \sum_{i=1}^{n-1}\sum_{j=1}^{n}\widetilde{a}_{ij}R_3(y,y_i)r_2(t,t_j) + \sum_{i=1}^{n-1}(\widetilde{b}_i t + \widetilde{c}_i t^2)R_3(y,y_i) + \sum_{j=1}^{n}(\widetilde{d}_j y(1-y) \\
&+ \widetilde{e}_j y^2(1-y) + \widetilde{p}_i y^3(1-y) + \widetilde{q}_i y^4(1-y))r_2(t,t_j) + (\widetilde{z}_1 y(1-y) \\
&+ \widetilde{z}_2 y^2(1-y) + \widetilde{z}_3 y^3(1-y) + \widetilde{z}_4 y^4(1-y))t + (\widetilde{z}_5 y(1-y) \\
&+ \widetilde{z}_6 y^2(1-y) + \widetilde{z}_7 y^3(1-y) + \widetilde{z}_8 y^4(1-y))t^2.
\end{aligned}
$$

Proof. Supposed that $u(y,t)$ is the true solution. As Lemma 3 points out, there exists a positive number γ such that

$$\|u-u_n\|_{C_1(\Omega)} \leq \gamma h^2.$$

Since Lemma 2 shows $\|\mathbb{L}\| \leq M$, division h becomes decreasing as n increasing, then it can be found that for $\forall \varepsilon > 0$, there exists $N \in \mathbb{N}^+$, such that for every $n > N$,

$$\|u-u_n\|_{C_1(\Omega)} \leq \gamma h^2 \leq \frac{\varepsilon}{M},$$

we obtain that

$$\|\mathbb{L}u_n - f\|_{L^1(\Omega)} = \|\mathbb{L}(u_n-u)\|_{L^1(\Omega)} \leq \|\mathbb{L}\|\|u-u_n\|_{C_1(\Omega)} \leq M\|u-u_n\|_{C_1(\Omega)} \leq \varepsilon.$$

Therefore, $\|\mathbb{L}\widetilde{u}_n - f\|_{L^1(\Omega)} \leq \|\mathbb{L}u_n - f\|_{L^1(\Omega)} \leq \varepsilon$. Owing to the definition of ε-approximate solution, it obviously that $\widetilde{u}_n(y,t)$ is an ε-approximate solution of Eq.(7).

As mentioned above, we can derive $\|\mathbb{L}\widetilde{u}_n - f\|_{L^1(\Omega)} \leq \gamma M h^2 \leq \varepsilon$. So u_n can be an $\varepsilon-$approximate solution by choosing different n. To this end, we can use the numerical experiments to estimate the value of γ by selecting the value h. As long as letting $h \leq \sqrt{\frac{\varepsilon}{\gamma M}}$, the estimation of n can be obtained.

Now, we give the procedure of this method in order to find the minimum of

$$\|\mathbb{L}u_n(y,t) - f(y,t)\|_{L^1(\Omega)}. \tag{10}$$

We put $u_n(y,t) = \sum\limits_{i=1}^{n+3}\sum\limits_{j=1}^{n+2} \Lambda_{ij} B_{y_i}(y) \otimes B_{t_j}(t)$, then it can derive that

$$\begin{aligned}
\mathbb{L}u_n(y,t) &= \sum_{i=1}^{n+3}\sum_{j=1}^{n+2} \Lambda_{ij}\mathbb{L}B_{y_i}(y)\otimes B_{t_j}(t) \\
&= \sum_{i=1}^{n+3}\sum_{j=1}^{n+2} \Lambda_{ij}\Big(B_{y_i}(y)\frac{\partial}{\partial t}B_{t_j}(t) + \frac{1}{Re}\frac{1}{2\cos(\frac{\pi\alpha}{2})}\frac{1}{\Gamma(2-\alpha)}\Big(\int_0^y (y-\eta)^{1-\alpha}\partial_\eta^2 B_{y_i}(\eta)\mathrm{d}\eta \\
&+ \int_y^1 (\eta-y)^{1-\alpha}\partial_\eta^2 B_{y_i}(\eta)\mathrm{d}\eta + \frac{\partial}{\partial y}B_{y_i}(0)y^{1-\alpha} - \frac{\partial}{\partial y}B_{y_i}(1)(1-y)^{1-\alpha}\Big)B_{t_j}(t)\Big).
\end{aligned}$$

In further,

$$\begin{aligned}
&\mathbb{L}u_n(y,t) = \sum_{i=1}^{n-1}\sum_{j=1}^{n} a_{ij}\mathbb{L}R_3(y,y_i)r_2(t,t_j) + \sum_{i=1}^{n-1}\mathbb{L}(b_i t + c_i t^2)R_3(y,y_i) + \sum_{j=1}^{n}\mathbb{L}(d_j y(1-y) + e_j y^2(1-y) \\
&+ p_i y^3(1-y) + q_i y^4(1-y))r_2(t,t_j) + \mathbb{L}(z_1 y(1-y) + z_2 y^2(1-y) + z_3 y^3(1-y) + z_4 y^4(1-y))t \\
&+ \mathbb{L}(z_5 y(1-y) + z_6 y^2(1-y) + z_7 y^3(1-y) + z_8 y^4(1-y))t^2 \\
&\triangleq \sum_{i=1}^{N_1}\mu_i g_i(y,t),
\end{aligned}$$

where $N_1 = n^2+5n+6$, $\{\mu_i\}_{i=1}^{N_1} = \{\{\Lambda_{ij}\}_{j=1}^{n+2}\}_{i=1}^{n+3} = \{\{a_{ij}\}_{j=1}^{n}\}_{i=1}^{n-1} \cup \{b_i\}_{i=1}^{n-1} \cup \{c_i\}_{i=1}^{n-1} \cup \{d_i\}_{i=1}^{n} \cup \{e_i\}_{i=1}^{n} \cup \{p_i\}_{i=1}^{n} \cup \{q_i\}_{i=1}^{n} \cup \{z_i\}_{i=1}^{8}$.

Next finding the minimum value of Eq.(10) is equivalent to solving the following equations which is linear and at least has one solution.

$$\min_{\mu_i\in\mathbb{R}} \sum_{j=1}^{m}\Big(\sum_{i=1}^{N_1}\mu_i g_i(y_j,t_j) - f(y_j,t_j)\Big)^2, \tag{11}$$

where $\{(y_j = \frac{1}{m}j, t_j = \frac{T}{m}j)\}_{j=1}^{m}$, m is large enough.

Putting $g_i = (g_i(y_1, t_1), \cdots, g_i(y_m, t_m))^{\mathrm{T}} \in \mathbb{R}^m$, $F = (f(y_1, t_1), \cdots, f(y_m, t_m))^{\mathrm{T}} \in \mathbb{R}^m$, where $\mathbb{R}^m$ represents a Euclidean space of dimension m. Eq.(11) is equivalent to

$$\min_{\mu_i \in \mathbb{R}} \| \sum_{i=1}^{N_1} \mu_i g_i - F \|_{\mathbb{R}^m}^2.$$

Definition 12 Set $F \in \mathbb{R}^m$, $\Gamma = \mathrm{span}\{g_i\}_{i=1}^{N_1}$,

$$\|\Phi - F\|_{\mathbb{R}^m} = \min_{S^* \in \Gamma} \|S^* - F\|_{\mathbb{R}^m},$$

consequently, the best approximation element of F in Γ is Φ.

Theorem 3 *Supposed that the vector $\Phi \in \Gamma$ satisfying*

$$\|\Phi - F\|_{\mathbb{R}^m} = \min_{S^* \in \Gamma} \left\|S^* - F\right\|_{\mathbb{R}^m},$$

then Φ is the unique vector in Γ.

Theorem 4 *Set $F \in \mathbb{R}^m$, when $k \in [1, N_1]$, $z \in \Gamma$, there exists $(F - z, g_k)_{\mathbb{R}^m} = 0$ if and only if z is the best approximation element for F in Γ.*

Theorem 5 $z = \sum\limits_{i=1}^{N_1} \mu^* g$ *is best element of approximation F in Γ if and only if the solution of the normal equation of Eq.(11) is $\{\mu_i^*\}_{i=1}^{N_1}$, which is obtained from solving the following equation.*

$$\frac{\partial}{\partial \mu_k} \sum_{j=1}^{m} [\sum_{i=1}^{N_1} \mu_i g_i(y_j, t_j) - f(y_j, t_j)]^2 = 0, \qquad k = 1, 2, \cdots, N_1. \tag{12}$$

Proof. The system Eq.(12) is

$$2 \sum_{j=1}^{m} [\sum_{i=1}^{N_1} \mu_i g_i(y_j, t_j) - f(y_j, t_j)] \cdot g_k(y_j, t_j) = 0, \qquad k = 1, 2, \cdots, N_1,$$

$$\sum_{j=1}^{m} \sum_{i=1}^{N_1} \mu_i g_i(y_j, t_j) \cdot g_k(y_j, t_j) = \sum_{j=1}^{m} f(y_j, t_j) \cdot g_k(y_j, t_j), \qquad k = 1, 2, \cdots, N_1,$$

it can be rewritten as

$$\sum_{j=1}^{m} \mu_i (g_i, g_k)_{\mathbb{R}^m} = \sum_{j=1}^{m} (f, g_k)_{\mathbb{R}^m}, \qquad i, k = 1, 2, \cdots, N_1,$$

it is then following as that

$$\sum_{j=1}^{m} (f - \sum_{i=1}^{N_1} \mu_i g_i, g_k)_{\mathbb{R}^m} = 0, \qquad k = 1, 2, \cdots, N_1,$$

$$\sum_{j=1}^{m} (f - \sum_{i=1}^{N_1} \mu_i g_i, g_k)_{\mathbb{R}^m} = 0, \qquad k = 1, 2, \cdots, N_1,$$

$$\sum_{j=1}^{m} (f - z, g_k)_{\mathbb{R}^m} = 0, \qquad k = 1, 2, \cdots, N_1,$$

using the Theorem 4 derives the conclusion.

4 Stability and convergence analysis

In further, we will discuss the stability of the numerical scheme and show the convergence in C_1 norm sense.

Denoting that $D(\mathbb{L}) = \{\mathbb{L}u | u \in C_1(\Omega)\} \subset L^1(\Omega)$.

Theorem 6 *$\mathbb{L}$ is the closed operator and inverse operator $\mathbb{L}^{-1}$ is bounded.*

Proof. First of all, we will show that $D(\mathbb{L})$ is a closed operator. It can be sufficiently proved from that for any $z_0 \in \overline{D(\mathbb{L})}$. When $z_0 = 0$, the conclusion is obvious derived. Assuming that $z_0 \neq 0$, then there has a sequence $z_n \in D(\mathbb{L})$ such that $\|z_n - z_0\|_{L^1(\Omega)}$ tends to zero and has a sequence $y_n \in C_1(\Omega)$ satisfying $\mathbb{L}y_n = z_n$.

Let $\mathcal{N} = \{y \in C_1(\Omega) | \mathbb{L}y = 0\}$, then it is equivalent to $\mathcal{N} = \{0\}$. Otherwise, assuming that there exists other nonzero solution y, then there exists a infinite sequence $y_{n_k} \subseteq \mathcal{N}$, so $y_{n_k} \in \mathcal{N}$, it follows that $\mathbb{L}y_{n_k} = z_{n_k} = 0$, then $z_0 = 0$, it derives a contradiction.

Next we will illustrate that $\{y_n\}$ is bounded. Conversely, supposing that $\|y_n\|_{C_1(\Omega)} \to \infty$.

Set $u_n = \frac{y_n}{\|y_n\|_{C_1(\Omega)}}$, then $\|u_n\|_{C_1(\Omega)} = 1$, so $\{u_n\}$ is bounded. Since $\mathbb{L}$ is bounded, so far it has convergence subsequence u_{n_k}, it has

$$\lim_{k\to\infty} u_{n_k} = u_0, \qquad \lim_{k\to\infty} \mathbb{L}u_{n_k} = \mathbb{L}u_0,$$

so,

$$\mathbb{L}u_0 = \lim_{k\to\infty} \mathbb{L}u_{n_k} = \lim_{k\to\infty} \mathbb{L}\frac{y_{n_k}}{\|y_{n_k}\|_{C_1(\Omega)}} = \lim_{k\to\infty} \frac{z_{n_k}}{\|y_{n_k}\|_{C_1(\Omega)}} = 0,$$

then $u_0 \in \mathcal{N}$, $u_0 = 0$,

$$\|u_{n_k} - u_0\|_{L^1(\Omega)} = \left\| \frac{y_{n_k}}{\|y_{n_k}\|_{C_1(\Omega)}} - 0 \right\|_{L^1(\Omega)} = 1,$$

which implies the contradiction, so $\{y_n\}$ is bounded. Then there is a subsequence $\{y_{n_l}\} \subset \{y_n\}$ such that $y_{n_l} \to y_n$. Note that

$$z_{n_l} = \mathbb{L}y_{n_l}.$$

Let $l \to \infty$,

$$\|\mathbb{L}y_0 - z_0\|_{L^1(\Omega)} = \|\mathbb{L}y_0 - \mathbb{L}y_{n_l} + \mathbb{L}y_{n_l} - z_{n_l} + z_{n_l} - z_0\|_{L^1(\Omega)} \to 0,$$

then, $\mathbb{L}y_0 = z_0$, which implies $z_0 \in R(\mathbb{L})$, so $R(\mathbb{L})$ is closed.

Finally, take into account $D(\mathbb{L})$ is closed in the Banach space $L^1(\Omega)$, obviously $\mathbb{L}^{-1} : D(\mathbb{L}) \to C_1(\Omega)$ is bounded through inverse operator theorem.

Theorem 7 *The approximate scheme of the above numerical method is stable.*

Proof. Suppose $\theta(x, y)$ is the disturbing function, then $\mathbb{L}\overline{u}(x, y) = F(x, y) + \theta(x, y)$. $u(x, y)$ is the true solution, and the ε-approximate solution of $\overline{u}(x, y)$ is $\overline{u}_n(x, y)$. Then, we can get that

$$\begin{aligned}
\|u - \overline{u}_n\|_{C_1(\Omega)} &= \|u - \overline{u} + \overline{u} - \overline{u}_n\|_{C_1(\Omega)} \\
&\leq \|u - \overline{u}\|_{C_1(\Omega)} + \|\overline{u} - \overline{u}_n\|_{C_1(\Omega)} \\
&\leq \|\mathbb{L}^{-1}\| \cdot \|\mathbb{L}u - \mathbb{L}\overline{u}\|_{L^1(\Omega)} + \|\mathbb{L}^{-1}\| \cdot \|\mathbb{L}\overline{u} - \mathbb{L}\overline{u}_n\|_{L^1(\Omega)} \\
&\leq \|\mathbb{L}^{-1}\|(\|\theta\|_{C_1(\Omega)} + \|\mathbb{L}\| \cdot \varepsilon).
\end{aligned}$$

Using the Theorem 6, $\mathbb{L}^{-1}$ is bounded, ε is arbitrarily and θ is the disturbing function, so the approximate scheme in this paper is stable.

Theorem 8 *In the sense of C_1 norm, the numerical solution $\widetilde{u}_n(y,t)$ converges to the true solution $u(y,t)$.*

Proof. Noting that

$$\|u-\widetilde{u}_n\|_{C_1(\Omega)}=\left\|\mathbb{L}^{-1}(\mathbb{L}u-\mathbb{L}\widetilde{u}_n)\right\|_{C_1(\Omega)}\leq\left\|\mathbb{L}^{-1}\right\|\cdot\|\mathbb{L}u-\mathbb{L}\widetilde{u}_n\|_{L^1(\Omega)}$$

From Theorem2 and Theorem 6, we can get that $\|u-\widetilde{u}_n\|_{C_1(\Omega)}\leq\left\|\mathbb{L}^{-1}\right\|\cdot\varepsilon<\varepsilon$.

5 Numerical examples

In this section, we will give two examples to demonstrate the effectiveness of our theoretical analysis. Examples of different values of α are discussed, different values of the number of grids are showed and their true solutions are available for comparison in the literature. Our numerical is computed by Mathematics 11 software and all calculations run on a lenovo computer with Intel i5-5200U, 2.20 GHz CPU and 8GB RAM.

Smooth Process Considering the base of our method is integer order polynomial, when the exact solution contains the fractional terms t^α, we can do some transformation to solve. Let $t=\rho^{\frac{2}{\alpha}}$, then $\rho=t^{\frac{\alpha}{2}}$, the new exact solution is $v(y,\rho)=y^2(1-y)^2\rho^2$. Denoting the left end term of Eq.(9) as $\mathbb{L}u(y,t)$ and the right end term of Eq.(9) as $f(y,t)$, then the Eq.(9) becomes:

$$\mathbb{L}v(y,\rho)=f(y,\rho^{\frac{2}{\alpha}}),\tag{13}$$

using our method to solve the new Eq.(13) obtaining the numerical solution $v_n(y,\rho)$, then $u_n(y,t)=v_n(y,t^{\frac{\alpha}{2}})$. We define this process as the smooth process of the solution.

Example 1 Consider the following equation from Reference [14]:

$$\begin{aligned}\frac{\partial}{\partial t}u(y,t)-\frac{1}{Re}\frac{\partial^\alpha}{\partial|y|^\alpha}u(y,t)=\frac{1}{Re}\frac{t^\alpha}{2\cos(\frac{\pi\alpha}{2})}(\frac{2}{\Gamma(3-\alpha)}(y^{2-\alpha}+(1-y)^{2-\alpha})-\frac{12}{\Gamma(4-\alpha)}(y^{3-\alpha}+\\(1-y)^{3-\alpha})+\frac{24}{\Gamma(5-\alpha)}(y^{4-\alpha}+(1-y)^{4-\alpha}))+\alpha t^{\alpha-1}y^2(1-y)^2,\end{aligned}\tag{14}$$

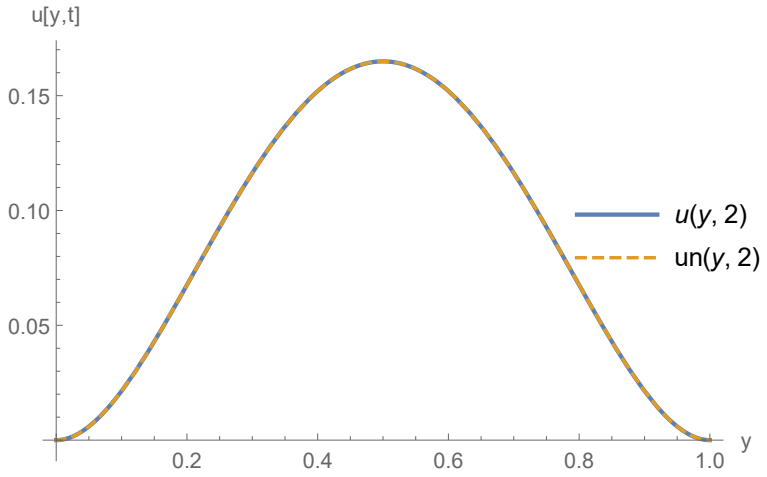
with

$$\begin{aligned}u(y,0)=0,\qquad&0\leq y\leq 1,\\u(0,t)=u(1,t)=0,\qquad&t>0,\quad 1<\alpha\leq 2.\end{aligned}\tag{15}$$

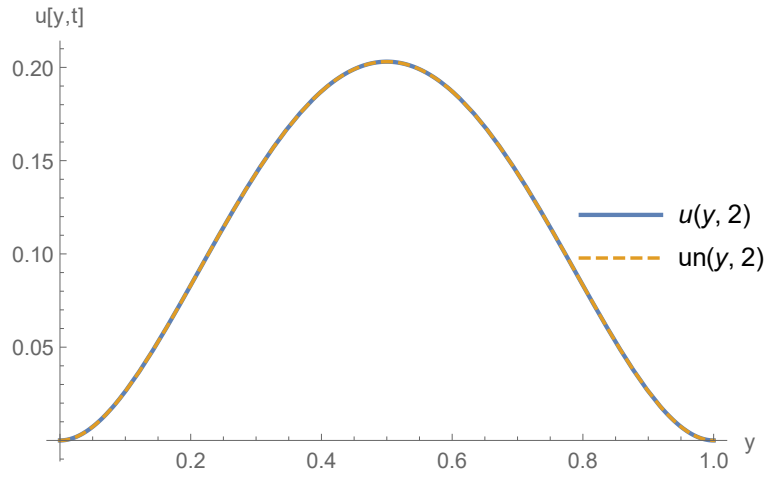
In the paper [14], it is shown that the exact solution of Eq.(14) and Eq.(15) is $u(y,t)=y^2(1-y)^2t^\alpha$. Using the smooth process, we get the following experimental results. First, Table 1 shows the comparison between the results of the non-smooth process and the smooth process. It shows the effectiveness of smooth processes. Figure 1 (a) and (b) compare the numerical and exact solution for $\alpha=1.4$ and $\alpha=1.7$ with N=5 in our method respectively. We calculate the approximate solution at $T=2.0$, $\alpha=1.6$, $Re=5$, and give numerical solutions for different values of y, meanwhile compare them with Reference [14]. The numerical results are shown in Table 2. Table 3 shows the absolute errors of our method at $T=2.0$, $Re=5$ with different values of α and the different numbers of grids, it shows that the more accurate approximate solution can be obtained by using the proposed method with the less number grids.

Example 2 Consider the following equation of the Eq.(1):

$$\begin{aligned}f(y,t)=&\frac{1}{Re}\frac{t^2+t^\alpha}{2\cos(\frac{\pi\alpha}{2})}(\frac{1}{\Gamma(2-\alpha)}(y^{1-\alpha}+(1-y)^{1-\alpha})-\frac{2}{\Gamma(3-\alpha)}(y^{2-\alpha}+(1-y)^{2-\alpha}))\\&+(2t+\alpha t^{\alpha-1})y(1-y),\end{aligned}\tag{16}$$



(a) $\alpha = 1.4$, Re=5



(b) $\alpha = 1.7$, Re=5

Fig. 1: Numerical and exact solutions for Eq.(14), when $\alpha = 1.4, 1.7$, $t = 2$, $Re = 5$, $y \in [0, 1]$.

Table 1: Absolute errors E in Example 1 at $\alpha = \frac{5}{3}, t = 2.0,\ Re = 5,\ N = 5$

y	*Exact solute* u	smooth u_N	non-smooth Errors	smooth Errors error
0.1	0.0257158970	0.0257158970	3.14808×10^{-4}	3.46945×10^{-18}
0.3	0.1400087728	0.1400087728	7.48720×10^{-4}	2.77556×10^{-17}
0.5	0.1984251315	0.1984251315	9.06256×10^{-4}	5.55112×10^{-17}
0.7	0.1400087728	0.1400087728	7.48720×10^{-4}	8.32667×10^{-17}
0.9	0.0257158970	0.0257158970	3.14808×10^{-4}	1.38778×10^{-17}

Table 2: Numerical and exact solutions of Example 1, when $N = 5, \alpha = 1.6,\ t = 2.0,\ Re = 5$

y	u_{Exact}	$u_{OurSmooth}$	Ref [15]	Absolute error
0.1	0.024554608	0.024554608	$3.68931 \times E^{-6}$	$3.46945 \times E^{-17}$
0.3	0.133686201	0.133686201	$5.40072 \times E^{-6}$	$8.32667 \times E^{-17}$
0.5	0.189464571	0.189464571	$2.02411 \times E{-}6$	$8.32667 \times E^{-17}$
0.7	0.133686201	0.133686201	$5.92651 \times E^{-6}$	$2.77556 \times E^{-17}$
0.9	0.024554608	0.024554608	$1.70567 \times E^{-6}$	$5.20417 \times E^{-17}$

Table 3: Max absolute errors E in Example 1 at $t = 2.0,\ Re = 5$

n	$\alpha = 1.1$	$\alpha = 1.3$	$\alpha = 1.5$	$\alpha = 1.7$	$\alpha = 1.9$
2	5.54030×10^{-3}	6.95651×10^{-3}	8.72048×10^{-2}	1.09855×10^{-2}	1.40833×10^{-2}
3	1.35456×10^{-4}	9.10695×10^{-5}	1.92272×10^{-4}	2.23553×10^{-4}	3.02558×10^{-4}
4	1.40363×10^{-15}	3.21556×10^{-16}	2.28002×10^{-16}	1.66599×10^{-16}	5.53226×10^{-16}
5	8.96865×10^{-16}	4.84510×10^{-16}	1.77189×10^{-16}	2.15629×10^{-17}	3.41824×10^{-16}
6	8.68382×10^{-16}	1.51252×10^{-16}	2.92916×10^{-16}	1.63766×10^{-16}	2.69174×10^{-16}

Table 4: Absolute errors E in Example 2 at $\alpha = \frac{5}{3}, t = 2.0,\ Re = 5,\ N = 5$

y	*Exact solute* u	smooth u_N	non-smooth Errors	smooth Errors error
0.1	0.64573219	0.64571560	4.46542×10^{-5}	1.65856×10^{-5}
0.3	1.50670844	1.50666925	1.07371×10^{-4}	3.91946×10^{-5}
0.5	1.79370053	1.79365306	1.30029×10^{-4}	4.74696×10^{-5}
0.7	1.50670844	1.50666925	1.07371×10^{-4}	3.91946×10^{-5}
0.9	0.64573219	0.64571560	4.46542×10^{-5}	1.65856×10^{-5}

Table 5: Max absolute errors E in Example 2 at $t = 2.0,\ Re = 5,\ N = 5$

n	$\alpha = 1.2$	$\alpha = 1.4$	$\alpha = 1.6$	$\alpha = 1.8$
3	3.71443×10^{-1}	2.34533×10^{-2}	7.18462×10^{-3}	2.08833×10^{-1}
4	2.01559×10^{-4}	5.98739×10^{-5}	9.35679×10^{-5}	3.87268×10^{-5}
5	6.97333×10^{-5}	3.46344×10^{-5}	5.71383×10^{-5}	2.34353×10^{-5}
10	5.58921×10^{-6}	4.48243×10^{-6}	1.02532×10^{-5}	5.22770×10^{-6}

with

$$\begin{aligned} u(y,0) &= 0, \qquad 0 \le y \le 1, \\ u(0,t) = u(1,t) &= 0, \qquad t > 0, \quad 1 < \alpha \le 2. \end{aligned} \tag{17}$$

Its exact solution is $u(y,t) = y(1-y)(t^2 + t^\alpha)$. We consider the two process for the time order, firstly we use our method to solve the Eq.(16) without discussing the smooth process, and get some numerical results. On the other hand, we use our method and the smooth process to solve the Eq.(16), then get some numerical results. The numerical results for $\alpha = \frac{5}{3},\ T = 2,\ Re = 5,\ N = 5$ are presented in Table 4. We can find that the order of absolute errors is 4 or 5, meanwhile the numerical results of using the smooth process is better than the other one, but is not good as the Example 1. Table 5 shows that the max absolute errors obtained by the method with the Smooth process for different α at $y \in [0,1],\ t = 2,\ Re = 5$. It can be found that with the number of grids increasing, the error reducing at the different value of α.

6 Conclusion

We proposed a novel basis based on reproducing kernel theory in the spline space, then through using the collocation method, we advance a new numerical method for solving the Riesz space fractional Navier-Stokes equations. On this basis, we analyze the stability of the method, and we solve the system of equations by means of $\varepsilon-$approximate solution theory. Finally, the numerical examples indicate the feasibility and effectiveness of the method.

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Confict of interest statement

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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