THE EXTREMALITY OF DISORDERED PHASES FOR THE MIXED SPIN-(1,1/2) ISING MODEL ON CAYLEY TREE OF ARBITRARY ORDER

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Abstract

The aim of this paper is to continue the investigation into the set of translation-invariant splitting Gibbs measures (TISGMs) for Ising model having the mixed spin (1,1/2) (shortly, (1,1/2)-MSIM) on a Cayley tree of arbitrary order. In our previous work [Akın and Mukhamedov, J. Stat. Mech. (2022) 053204], we provided a thorough explanation of the TISGMs, and studied the extremality of disordered phases using a Markov chain with a tree index on a semi-finite Cayley tree with order two. In this paper, we construct the TISGMs and tree-indexed Markov chains associated with to the model. Considering a tree-indexed Markov chain on a Cayley tree of any order, we clarify the extremality of the related disordered phases. By using the Kesten-Stigum condition (KSC), we investigate non-extremality of the disordered phases by means of the eigenvalues of the stochastic matrix associated with (1,1/2)-MSIM on a CT of order k[?]2.

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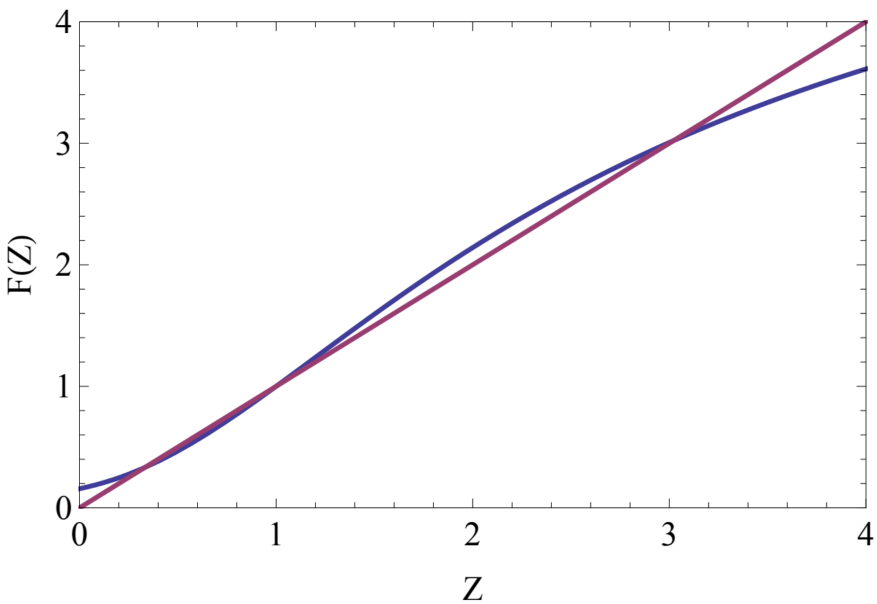

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Abstract. The aim of this paper is to continue the investigation into the set of translation-invariant splitting Gibbs measures (TISGMs) for Ising model having the mixed spin (1,1/2) (shortly, (1,1/2)-MSIM) on a Cayley tree of arbitrary order. In our previous work [Akın and Mukhamedov, J. Stat. Mech. (2022) 053204], we provided a thorough explanation of the TISGMs, and studied the extremality of disordered phases using a Markov chain with a tree index on a semi-finite Cayley tree with order two. In this paper, we construct the TISGMs and tree-indexed Markov chains associated with to the model. Considering a tree-indexed Markov chain on a Cayley tree of any order, we clarify the extremality of the related disordered phases. By using the Kesten-Stigum condition (KSC), we investigate non-extremality of the disordered phases by means of the eigenvalues of the stochastic matrix associated with (1,1/2)-MSIM on a CT of order $k \geq 2$.

Keywords: Ising model with mixed spin-(1,1/2), disordered phase, Gibbs measure, phase transition.

MSC: 82B23, 82B26, 60G70, 37J25.

1. Introduction

A broad variety of probabilistic models investigated in applied probability, statistical physics, artificial intelligence and other fields are captured by spin systems such as hard-core model, Ising model, Potts model [1, 2]. In addition to being a helpful simplification of its more traditional counterparts on the lattice $\mathbb{Z}^d$, spin systems on Bethe lattice (BL) and Cayley tree (CT) have lately received a lot of interest as the standard illustration of models on "non-amenable" graphs (see [3] for details). There are two rigorous approaches to dealing with BLs that are based on the Kolmogorov consistency condition (KCC) and self-similarity, combining the first with techniques more often used in physics like the Cavity and Belief Propagation [4]. Therefore, the investigation of lattice models on the CT and BL has recently increased its relevance for both mathematicians and researchers working in other sciences.

The (1,1/2)-MSIM on CT is a nice alternative for researches on mixed-spins [5]. Since spin-1 has a higher spin than spin-1/2, it is generally known that higher spins have higher critical temperatures. Recently, Ising model with the mixed spins has been investigated in several papers to characterize a broad range of systems. [6, 7, 8]. Unlike the spin-1/2 system, the authors [3, 9] showed that the spin-1 system exhibited the first-order phase transitions for some given parameters. Silva and Salinas [10] developed some precise calculations for a Curie-Weiss or mean-field form of the Ising Hamiltonian with mixed spin. They obtained the phase diagram of a ferromagnetic mixed-spin $(1, S)$ Ising model. Espriella *et al.* [7] figured out ground state configurations of the (1,1/2)-MSIM on a square lattice. Using the explicit recursion relations for a set of provided coordination numbers, Albayrak [5] elucidated isothermal entropy change and the entropy of (1,1/2)-MSIM on BL (see [11]).

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For a finite number of spin values, the theory of GMs on lattices is well established. Each GM is known to correlate to phase of a physical system. As a result, the existence and non-uniqueness of GMs is a major issue in the theory of GM. At a constant temperature, non-uniqueness of corresponding GMs indicates that the physical system's phases (states) coexist. In order to understand the nature of transitions and find the dynamic phase transition points, the researchers [8] studied the thermal evolution of the dynamic order parameters for the spin-1/2 Ising system on a triangular lattice.

In the rigorous approach, the GM with no boundary conditions corresponds to the disordered phase of the Ising model on the CT [12] (see [13] for details). In [14], the authors studied the 3-state Potts-SOS model on the CT of order two in some conditions and described all the TISGMs for the Potts-SOS model. Khakimov [15] established the KSC for a phase associated with fertile hard-core models.

In Ref. [16], we proved that the (1,1/2)-MSIM on a CT of order two has three TIGMs in both the ferromagnetic and anti-ferromagnetic regions, while the Ising model having single-spin [18, 19, 20, 21, 22] does not have such GMs in the anti-ferromagnetic region. At the same time, for the Potts models [23] on the CT, the phase transition occurs only in the ferromagnetic regime. In Ref. [17], by means of the KCC we studied (2,1/2)-MSIM on CT of second-order and showed numerically that the model displays chaotic behavior in several regimes.

The phase transition issue for the Ising model can be studied with the help of the stability of the dynamic system corresponding to the model at fixed points (see [16]). In regions with repelling fixed points, it is known that there are additional fixed points as well. As a result, we will investigate the phase transition phenomena for the (1,1/2)-MSIM on the CT of arbitrary order by looking at the stability of the relevant dynamic system.

The extreme disordered phases of the models on the lattices have a very important place in the information flow theory [24, 25, 26]. Consequently, there are several publications that focus on the extreme GMs issue for a wide range of models on BL and CR (see [14, 15, 27, 28, 29, 30]). In [16], we worked on the exact detection of the TISGMs, and studied extremality of GMs by using a tree-indexed Markov chain approach on CT of order two. In the present paper, we continue to identify the TISGMs for the (1,1/2)-MSIM constructed on a CT of arbitrary order. We generalize the results obtained in [24, 25, 26] to CT having arbitrary order. We are going to construct the Markov chain with tree-index corresponding to the model and describe the criterion of extremality of a disordered phase associated the Ising model on CT of arbitrary order by means of the relevant Markov chain with tree-index. Considering the KSC [31], we elucidate the non-extreme TISGMs by means of the stochastic matrix associated with (1,1/2)-MSIM on a CT of order $k \geq 2$.

2. Preliminaries and Compatibility conditions

In this section, we recall definitions and fundamental results in the construction of GMs for mixed spin Ising model on a semi-finite CT of arbitrary order. For the definition of the semi-finite CT, we refer the reader to references [4, 22, 30, 32].

2.1. **Semi-finite Cayley tree.** For the sake of completeness, we will explain how a semi-finite Cayley tree of arbitrary order are constructed. A CT is a connected graph without any circuits [4, 22, 32, 33]. We denote a semi-infinite CT of order k ($k \geq 1$) with the root $x^{(0)}$ by $\Gamma^k = (V, L)$.

Here V is the set of vertices and L is the set of edges. The root vertex $x^{(0)}$ of the semi-finite CT of order k is connected with only k vertices only with one edge, nevertheless all other vertices of the tree are connected with $(k+1)$ vertices by only one edge.

If there is an edge linking the vertices x and y, they are referred to as *nearest neighbors* and are indicated by the symbol $l = \langle x, y\rangle$. The length of the shortest path from x to y is represented by the distance $d(x, y), x, y \in V$, on the CT. The set of vertices in the nth shell is denoted by $W_n = \{x \in V \mid d(x, x^{(0)}) = n\}$. The set of all vertices in a CT with n shells is denoted by $V_n = \bigcup_{m=1}^{n} W_m \cup \{x^{(0)}\}$. The set of all edges in the CT is denoted by $L_n = \{l =< x, y >\in L \mid x, y \in V_n\}$. The set $S(x) = \{y \in W_{n+1} : d(x, y) = 1\}$ defines the set of direct successors of $x \in W_n$ (see [4, 22, 32] for details).

In this paper, the spin variables $s \in \Psi = \{-\frac{1}{2}, +\frac{1}{2}\}$ and $\sigma \in \Phi = \{-1, 0, +1\}$ are associated with sites belonging to successive generations of CT Γ^k. The spins with Ψ elements will be inserted in the tree's vertices at odd-numbered shells, whereas the spins with Φ elements will be placed there at even-numbered shells. We take into account the standard coordinate system. Denote

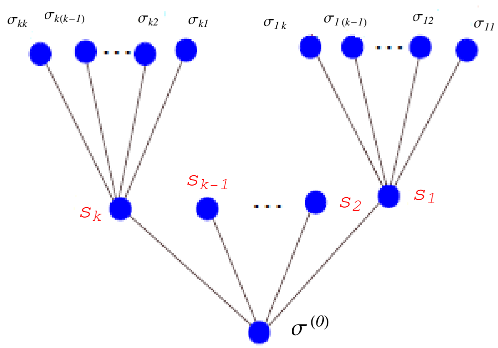

FIGURE 1. (Color online) A configuration of semi-finite CT of arbitrary order $k \geq 2$ (branching ratio is finite k) with a spin $\sigma^{(0)}$ and mixed spin having two shells, where $\sigma^{(0)}, \sigma_{ij} \in \Phi$ and $s_i \in \Psi$, $1 \leq i, j \leq k$.

$$\begin{aligned}
\Gamma_+^k &= \left\{x \in \Gamma^k : d(x^{(0)}, x) - \text{even}\right\} = W_{2m},\\
\Gamma_-^k &= \left\{x \in \Gamma^k : d(x^{(0)}, x) - \text{odd}\right\} = W_{2m+1},
\end{aligned}$$

where m is non-negative integer.

Let $\Omega_+ = \Phi^{\Gamma_+^k}$ and $\Omega_- = \Psi^{\Gamma_-^k}$ also $\Omega_{+,n} = \Phi^{\Gamma_+^k \cap V_n}$ and $\Omega_{-,n} = \Psi^{\Gamma_-^k \cap V_n}$. $\Xi = \Gamma_+ \times \Gamma_-$ represents the configuration space. Elements of Γ_+ will be denoted by $\sigma(x)$, for $x \in \Gamma_+^k$. Similarly, elements of Γ_- will be denoted by $s(x)$, for $x \in \Gamma_-^k$. Fig. 1 represents two-shell and semi-finite CT of kth order equipped with mixed spin.

2.2. **Compatibility conditions of the sequence of GMs.** In this subsection, we construct the GM and partition function for the (1,1/2)-MSIM on the semi-finite CT of arbitrary order.

Let $\xi \in \Xi$, then we have

$$\xi(x) = \begin{cases} \sigma(x); & x \in \Omega_+ \\ s(x); & x \in \Omega_-, \end{cases}$$

where $\sigma \in \Phi = \{-1, 0, +1\}$ and $s \in \Psi = \{-\frac{1}{2}, +\frac{1}{2}\}$.

Let us denote $\mathbf{h} = (\mathbf{h}_{\xi(x)}(x))_{x\in\Gamma^k}$, where

$$\boldsymbol{h}_{\xi(x)}(x) = \begin{cases} \boldsymbol{h}_{\sigma(x)}, & x \in \Gamma_+; \\ \widetilde{\boldsymbol{h}}_{s(x)}, & x \in \Gamma_-. \end{cases}$$

and we denote $\boldsymbol{h}(x) = (h_{-1}(x), h_0(x), h_{+1}(x))$, $\widetilde{\boldsymbol{h}} = (\widetilde{h}_{-\frac{1}{2}}(x), \widetilde{h}_{\frac{1}{2}}(x))$.

We aim to analyze the Ising model with mixed spins $\Phi = \{-1, 0, +1\}$ and $\Psi = \{-\frac{1}{2}, +\frac{1}{2}\}$ built on the CT, having Hamiltonian

$$H(\xi) = -J \sum_{\langle x,y\rangle} \xi(x)\xi(y), \tag{2.1}$$

and partition function

$$Z_n \overset{def}{\equiv} Z_n(\beta, \boldsymbol{h}) = \sum_{\eta_n \in \Xi} \exp\left\{ \beta J \sum_{\langle x,y\rangle \subset V_n} \xi(x)\xi(y) + \sum_{x\in W_n} \boldsymbol{h}_{\xi(x)}(x) \right\}, \tag{2.2}$$

where $\xi \in \sigma \times s = \Xi$, Z_l is the partition function for a semi-finite CT having l shells, $\beta = 1/T$, J is the coupling constant and $\boldsymbol{h}_{\xi(x)}(x)$ is external field. Given that the CT is finite, we can begin to build the summation of equation (2.2) by adding the boundary spins, or the spins on the n-th shell.

The following equation summarizes the relationship between free energy and boundary conditions:

$$F(\beta, \boldsymbol{h}) = -\lim_{n\to\infty} \frac{1}{\beta |V_n|} \ln Z_n(\beta, \boldsymbol{h}). \tag{2.3}$$

The density free energy function given in (2.3) is an analytic function of β. Therefore, a spontaneous magnetization is not produced by the Ising model on the CT (see [4] for details).

We construct the GMs corresponding to the Hamiltonian (2.1). Now, for each $n \geq 1$, we define GMs $\mu_n^{\boldsymbol{h}}$ by

$$\mu_n^{\boldsymbol{h}}(\xi) = \frac{1}{Z_n} \exp\left\{ -\beta H_n(\xi) + \sum_{x\in W_n} \boldsymbol{h}_{\xi(x)}(x) \right\}, \tag{2.4}$$

where $\xi \in \Xi = \Omega_{+,n} \times \Omega_{-,n}$.

The Bethe-Peierls method [34] is represented by the eq. (2.4). The Cavity-Method in physics [35] and Belief-Propagation in computer science [36] are other names for this equation (see [35]).

For each $n \geq 1$ and $\xi_{n-1} \in \Xi_{n-1}$, we will describe conditions on $\boldsymbol{h}$ for which the sequence of the measures $\{\mu_n^{\boldsymbol{h}}\}$ are compatible, i.e.

$$\mu_{n-1}^{\boldsymbol{h}}(\xi_{n-1}) = \sum_{w \in \Xi^{w_n}} \mu_n^{\boldsymbol{h}}(\xi_{n-1} \vee w), \quad \text{for all } n \geq 1, \tag{2.5}$$

where

$$\Xi^{W_n} = \begin{cases} \Phi^{W_n}, & n\text{-even}; \\ \Psi^{W_n}, & n\text{-odd}. \end{cases}$$

where $\xi_{n-1} \vee w$ is the concatenation of ξ_{n-1} and w. Therefore, for all $\xi_n \in \Xi_n$ we have a unique measure μ on Ξ_n such that

$$\mu(\{\xi|_{V_n} = \xi_n\}) = \mu_n^{\boldsymbol{h}}(\xi_n). \tag{2.6}$$

The measure μ given in (2.6) is called the splitting GM associated with the model (2.1) (see [16] for details). The conditions on $\boldsymbol{h}_x$ and $\widetilde{h}_x$ that guarantees the compatibility of the sequence of GMs $\{\mu_n^{\boldsymbol{h}}\}$ is described in the following theorem.

Theorem 2.1. *[[16]] Probability distributions $\{\mu_n^{\boldsymbol{h}}\}$, $n = 1, 2, ...$ in (2.5) are compatible iff for any $x \in V$ the following equations hold;*

$$\exp(\boldsymbol{h}_{-1}(x) - \boldsymbol{h}_0(x)) = \prod_{y \in S(x)} \left(\frac{\theta^2 + \exp\left(\widetilde{h}_{\frac{1}{2}}(y))) - \widetilde{h}_{-\frac{1}{2}}(y)\right)}{\theta\left(1 + \exp\left(\widetilde{h}_{\frac{1}{2}}(y))) - \widetilde{h}_{-\frac{1}{2}}(y)\right)\right)} \right), \tag{2.7}$$

$$\exp(\boldsymbol{h}_{1}(x) - \boldsymbol{h}_0(x)) = \prod_{y \in S(x)} \left(\frac{1 + \theta^2 \exp\left(\widetilde{h}_{\frac{1}{2}}(y))) - \widetilde{h}_{-\frac{1}{2}}(y)\right)}{\theta\left(1 + \exp\left(\widetilde{h}_{\frac{1}{2}}(y))) - \widetilde{h}_{-\frac{1}{2}}(y)\right)\right)} \right), \tag{2.8}$$

$$\exp(\widetilde{\boldsymbol{h}}_{\frac{1}{2}}(x) - \widetilde{\boldsymbol{h}}_{-\frac{1}{2}}(x)) = \prod_{y \in S(x)} \left(\frac{\exp(\boldsymbol{h}_{-1}(y) - \boldsymbol{h}_0(y)) + \theta^2 \exp(\boldsymbol{h}_1(y) - \boldsymbol{h}_0(y)) + \theta}{\theta^2 \exp(\boldsymbol{h}_{-1}(y) - \boldsymbol{h}_0(y)) + \exp(\boldsymbol{h}_1(y) - \boldsymbol{h}_0(y)) + \theta} \right), \tag{2.9}$$

where $\theta = e^{\frac{J\beta}{2}}$ and $y \in S(x)$.

We refer the reader to ref. [16] for the proof of the theorem.

3. Construction of the TIGMs

Here, by considering the equations (2.7)-(2.9), we are going to determine whether there are any TISGMs associated with the Ising model with mixed $(1, 1/2)$ on a CT of arbitrary order.

Suppose that $U_1(x) = e^{\boldsymbol{h}_{-1}(x) - \boldsymbol{h}_0(x))}, U_2(x) = e^{\boldsymbol{h}_1(x) - \boldsymbol{h}_0(x))}$ and $V(x) = e^{\widetilde{\boldsymbol{h}}_{1/2}(x) - \widetilde{\boldsymbol{h}}_{-1/2}(x))}$, from the equations (2.7)-(2.9), we get

$$\exp(U_1(x)) = \left(\frac{\theta^2 + \exp(V(y))}{\theta(1 + \exp(V(y)))} \right)^k, \tag{3.1}$$

$$\exp(U_2(x)) = \left(\frac{1 + \theta^2 \exp(V(y))}{\theta(1 + \exp(V(y))} \right)^k. \tag{3.2}$$

Similarly, one gets

$$\exp(V(x)) = \left(\frac{\theta + \theta^2 \exp(U_2(y)) + \exp(U_1(y))}{\theta + \theta^2 \exp(U_1(y)) + \exp(U_2(y))} \right)^k, \tag{3.3}$$

where $\theta = \exp\left(\frac{J\beta}{2}\right)$, $x \in V$ and $y \in S(x)$.

Definition 3.1. *Suppose $x \in V$, $i \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $j \in \{-1, 0, 1\}$, if $\widetilde{\boldsymbol{h}}_i(x) = \widetilde{\boldsymbol{h}}_i(y) = \widetilde{\boldsymbol{h}}_i$ and $\boldsymbol{h}_j(x) = \boldsymbol{h}_j(y) = \boldsymbol{h}_j$ for all $y \in S(x)$, the vector valued functions $\widetilde{\boldsymbol{h}} = \{\widetilde{\boldsymbol{h}}_{-\frac{1}{2}}(x), \widetilde{\boldsymbol{h}}_{\frac{1}{2}}(x)\}$ and $\boldsymbol{h} = \{\boldsymbol{h}_{-1}(x), \boldsymbol{h}_0(x), \boldsymbol{h}_1(x)\}$ are called translation-invariant. The measures corresponding to the functions $\widetilde{\boldsymbol{h}}$ and $\boldsymbol{h}$ is called TISGMs.*

In the Ref. [16], we determined TISGMs for the CT of order two (i.e. $k = 2$). We shall take the following abbreviations into consideration for completeness: for all $x \in \Gamma^2_+$ $\boldsymbol{h}_j := \boldsymbol{h}_j(x)$ and $x \in \Gamma^2_-$, $i \in \Psi$, $j \in \Phi$, $\widetilde{\boldsymbol{h}}_i := \widetilde{\boldsymbol{h}}_i(x)$, respectively.

Let $X=\exp(U_1)$, $Y=\exp(U_2)$ and $Z=\exp(V)$, from the equations (3.1)-(3.3), we have

$$X = \left(\frac{\theta^2+Z}{\theta(1+Z)}\right)^k, \tag{3.4}$$

$$Y = \left(\frac{1+\theta^2 Z}{\theta(1+Z)}\right)^k, \tag{3.5}$$

$$\begin{aligned} Z &= \left(\frac{X+\theta^2 Y+\theta}{\theta^2 X+Y+\theta}\right)^k \\ &= \left(\frac{\theta+\left(\frac{Z+\theta^2}{(1+Z)\theta}\right)^k+\theta^2\left(\frac{1+Z\theta^2}{(1+Z)\theta}\right)^k}{\theta+\theta^2\left(\frac{Z+\theta^2}{(1+Z)\theta}\right)^k+\left(\frac{1+Z\theta^2}{(1+Z)\theta}\right)^k}\right)^k := F(Z). \end{aligned} \tag{3.6}$$

We notice that one of the solutions of the equations (3.4)-(3.6) is

$$\ell := \left(X_0=\left(\frac{1+\theta^2}{2\theta}\right)^k, Y_0=\left(\frac{1+\theta^2}{2\theta}\right)^k, Z_0=1\right). \tag{3.7}$$

Throughout the article we will analyze the Gibbs measure corresponding to the fixed point given in (3.7). The TISGM $\mu_0(k)$ corresponding to roots $Z_0=1$ and $X_0=Y_0=\left(\frac{1+\theta^2}{2\theta}\right)^k$ is called *disordered phase* of the model (see [16] for details). Also, one can show that the set of solutions of the equations (3.4)-(3.6) is uncountable according to θ.

4. The stability analysis of the dynamical system (3.4)-(3.6)

This section will examine the stability issue for the fixed point $\left(\left(\frac{1+\theta^2}{2\theta}\right)^k, \left(\frac{1+\theta^2}{2\theta}\right)^k, 1\right)$ of the dynamical system given in (3.4)-(3.6) to determine whether there is a phase transition for (1,1/2)-MSIM on the CT of arbitrary order.

From (3.6), if we choose initial condition as $Z_0=1$, then we obtain $X_0=Y_0=\left(\frac{\theta^2+1}{2\theta}\right)^k$. Therefore, the Jacobian matrix of the dynamical system (3.4)-(3.6) at the fixed point $(\left(\frac{\theta^2+1}{2\theta}\right)^k, \left(\frac{\theta^2+1}{2\theta}\right)^k, 1)$ is calculated as

$$J_F(\theta,k) = \begin{pmatrix} 0 & 0 & -\frac{k(\theta^2-1)\left(1+\theta^2\right)^{k-1}}{2^{k+1}\theta^k} \\ 0 & 0 & \frac{k(\theta^2-1)\left(1+\theta^2\right)^{k-1}}{2^{k+1}\theta^k} \\ -\frac{2^k k(\theta^2-1)\theta^k}{2^k\theta^{k+1}+(1+\theta^2)^{k+1}} & \frac{2^k k(\theta^2-1)\theta^k}{2^k\theta^{k+1}+(1+\theta^2)^{k+1}} & 0 \end{pmatrix}. \tag{4.1}$$

The eigenvalues of $J_F(\theta,k)$ given in (4.1) are obtained as:

$$\begin{cases} \lambda_1(\theta,k)=0, \\ \lambda_2(\theta,k) = -\frac{k(\theta^2-1)\left(1+\theta^2\right)^{\frac{k-1}{2}}}{\sqrt{2^k\theta^{k+1}+(1+\theta^2)^{k+1}}}, \\ \lambda_3(\theta,k) = \frac{k(\theta^2-1)\left(1+\theta^2\right)^{\frac{k-1}{2}}}{\sqrt{2^k\theta^{k+1}+(1+\theta^2)^{k+1}}}. \end{cases}$$

Characterizing behavior of the dynamical system (3.4)-(3.6) is necessary to check the eigenvalues of the matrix $J_F(\theta,k)$ given in (4.1). We will consider the equation:

$$|\lambda_2(\theta,k)| = |\lambda_3(\theta,k)| := \left|\frac{k(\theta^2-1)\left(1+\theta^2\right)^{\frac{k-1}{2}}}{\sqrt{2^k\theta^{k+1}+(1+\theta^2)^{k+1}}}\right|. \tag{4.2}$$

Let $\theta_1^{crt}(k)$ and $\theta_1^{crt}(k)$ be the positive roots of equation $|\lambda_3(\theta,k)|-1=0$ such that $\theta_1^{crt}(k)<\theta_2^{crt}(k)$. It is well-known that for $\theta\in(\theta_1^{crt}(k),\theta_2^{crt}(k))$, we get $|\lambda_3(\theta,k)|<1$ and for $\theta\in(0,\theta_1^{crt}(k))\cup(\theta_2^{crt}(k),\infty)$, we have $|\lambda_3(\theta,k)|>1$ (see Fig. 2).

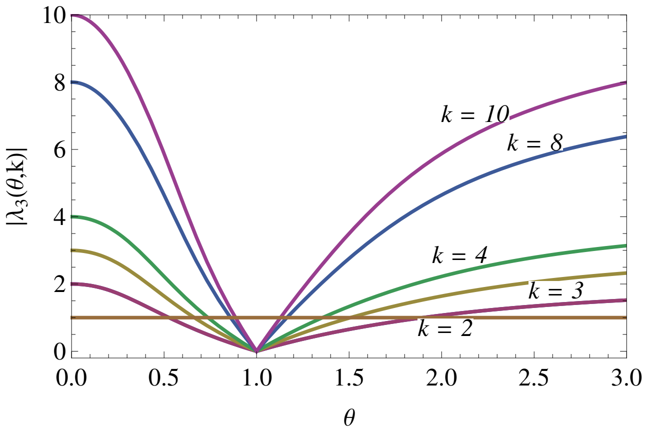

FIGURE 2. (Color online) Plots of the function $|\lambda_3(\theta,k)|$ in (4.2) for $k=2,3,4,8,10$.

One can show that $\theta_1^{crt}(k)<1$ and $\theta_2^{crt}(k)>1$. When performing in both ferromagnetic and anti-ferromagnetic regimes, since $|\lambda_3(\theta,k)|>1$, there are repelling fixed points. The repelling fixed points indicate that there exists more than one GM and phase transition happens, while the single-spin Ising model does not have the TIGMs in the anti-ferromagnetic regime [18, 20, 21, 33, 37, 38].

Remark 4.1. If $\theta\in(0,\theta_1^{crt}(k))\cup(\theta_2^{crt}(k),\infty)$, one can show that the system of equations (3.4)-(3.6) contains additional fixed points since the fixed point $\left(\left(\frac{1+\theta^2}{2\theta}\right)^k,\left(\frac{1+\theta^2}{2\theta}\right)^k,1\right)$ is repelling. Therefore, it indicates that the phase transition occurs.

Remark 4.2. As seen in the figure 2, as the order of the CT increases for the given model, the region where the fixed point is repelling increases depending on the relevant eigenvalues. Therefore, the region of phase transitions is expanding.

4.1. **Illustrative examples for the phase transition phenomena of model.** In order to emphasize θ and k, let us denote the rational function given in (3.6) by $F_{\theta,k}(Z)$. In Fig 3, we plot some graphs of the rational function $F_{\theta,k}(Z)$. The effect of both θ and k on the phase transition of the model is evident. For example, let us assume $\theta=1.12$, while no phase transition occurs for $k=8$, phase transition occurs for $k=12$ (see Fig 3). Another important finding is that as the values of θ, including $\theta\in(0,1)$, decreases, the phase transition phenomena tends to occur depending on the values of k. Assume $\theta>1$, as the value of θ increases, the tendency of the phase transition to occur also increases. Let us take $k=8$ for example. In this case, while the phase transition does not occur for $\theta=1.12$, the phase transition occurs for $\theta=1.2$ (see Fig 3).

5. THE MARKOV CHAINS WITH TREE-INDEX FOR TISGMS

In this section, we will construct a tree-indexed Markov chain that correlates to (1,1/2)-MSIM on a CT of order arbitrary. Considering the orders (shells) of the tree, we shall combine spins to design the CT.

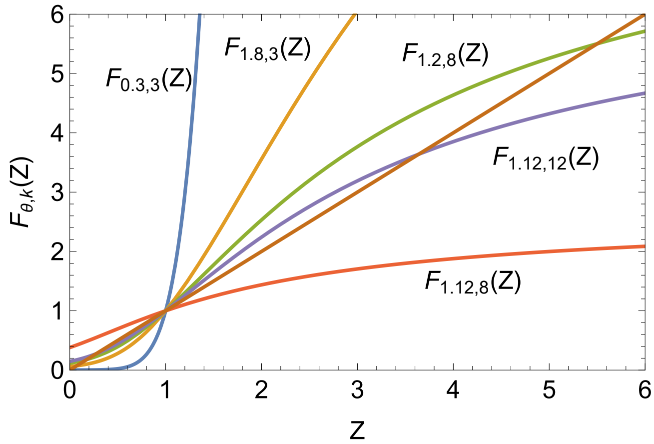

FIGURE 3. (Color online) Plots of the rational function $F_{\theta,k}(Z)$ given in (3.6) for some relevant θ and k.

5.1. **Extremality of the disordered phase.** To study a TISGM's extremality, we employ the Dobrushin coefficient from the reconstruction theory of trees [27]. Therefore, we determine the regions of the extremality of the disordered phase corresponding to the (1,1/2)-MSIM on a CT of arbitrary order. By applying arguments of a reconstruction on trees, we are going to derive the stochastic matrix (see [1, 16, 28, 39] for details). Georgii [37] introduced a suitable notion of a Markov chain associated with Ising model on a CT of order k.

Definition 5.1. [37, Definition 12.2] *A probability measure μ on the measurable space $(\Omega, \mathcal{F})$ is called a Markov chain if*

$$\mu\left(\sigma_j = y | \mathcal{F}_{]-\infty,ij[}\right) = \mu\left(\sigma_j = y|\mathcal{F}_{\{i\}}\right) \quad \mu - a.s.$$

for all $ij \in \overrightarrow{B}$ and $y \in E$, where the symbol $\overrightarrow{B}$ stands for the set of all oriented bonds. Let E be a state space. For all $y \in E$, the stochastic matrix $\mathbb{P}_{ij}$ on E such that

$$\mathbb{P}_{ij}(\sigma_i, y) = \mu\left(\sigma_j = y|\mathcal{F}_{\{i\}}\right) \quad \mu - a.s.$$

is called a transition matrix from i to j for the measure μ.

Now, let us introduce a tree-indexed Markov chain for (1,1/2)-MSIM on a CA of order $k \geq 2$. Let us consider the fixed point $l = (X_0, Y_0, Z_0) = (\left(\frac{1+\theta^2}{2\theta}\right)^k, \left(\frac{1+\theta^2}{2\theta}\right)^k, 1)$ of the equations (3.4)-(3.6), the TISGM for a vector $l \in \mathbb{R}^3$ is a Markov chain [27].

One defines the entries of the stochastic matrix $\mathbb{P} = (P_{ij})$ as

$$P_{ij} = \frac{e^{(ij\beta J + \widetilde{\boldsymbol{h}}_j)}}{\sum\limits_{v\in\{-\frac{1}{2},\frac{1}{2}\}} e^{(iv\beta J + \widetilde{\boldsymbol{h}}_v)}},$$

where $i \in \{-1, 0, 1\}$ and $j \in \{-\frac{1}{2}, \frac{1}{2}\}$ (see [16]).

Assume that $Z = \exp(\widetilde{\boldsymbol{h}}_{\frac{1}{2}} - \widetilde{\boldsymbol{h}}_{-\frac{1}{2}})$, we have the stochastic matrix $\mathbb{P} = (P_{ij})$ as

$$\mathbb{P} = \begin{pmatrix} P_{(-1,\frac{1}{2})} & P_{(-1,-\frac{1}{2})} \\ P_{(0,\frac{1}{2})} & P_{(0,\frac{1}{2})} \\ P_{(1,\frac{1}{2})} & P_{(1,\frac{1}{2})} \end{pmatrix} = \begin{pmatrix} \frac{\theta^2 Z}{1+\theta^2 Z} & \frac{1}{1+\theta^2 Z} \\ \frac{Z}{1+Z} & \frac{1}{1+Z} \\ \frac{Z}{\theta^2+Z} & \frac{\theta^2}{\theta^2+Z} \end{pmatrix}. \tag{5.1}$$

Similarly, we can define the stochastic matrix $\mathbb{Q} = (Q_{ij})$ by

$$Q_{ij} = \frac{e^{(ij\beta J + \boldsymbol{h}_j)}}{\sum\limits_{u\in\{-1,0,1\}} e^{(iu\beta J + \boldsymbol{h}_u)}},$$

where $i \in \{-\frac{1}{2}, \frac{1}{2}\}$ and $j \in \{-1, 0, 1\}$.

Assume that $X = e^{\boldsymbol{h}_{-1}-\boldsymbol{h}_0}$ and $Y = e^{\boldsymbol{h}_1-\boldsymbol{h}_0}$, we get the transition probability matrix $\mathbb{Q} = (Q_{ij})$ as

$$\mathbb{Q} = \begin{pmatrix} Q_{(\frac{1}{2},-1)} & Q_{(\frac{1}{2},0)} & Q_{(\frac{1}{2},1)} \\ Q_{(-\frac{1}{2},-1)} & Q_{(-\frac{1}{2},0)} & Q_{(-\frac{1}{2},1)} \end{pmatrix} = \begin{pmatrix} \frac{\theta^2 X}{\theta^2 X+\theta+Y} & \frac{\theta}{\theta^2 X+\theta+Y} & \frac{Y}{\theta^2 X+\theta+Y} \\ \frac{X}{X+\theta+\theta^2 Y} & \frac{\theta}{X+\theta+\theta^2 Y} & \frac{\theta^2 Y}{X+\theta+\theta^2 Y} \end{pmatrix}. \tag{5.2}$$

If we substitute $Z = 1$ and $X = Y = \left(\frac{1+\theta^2}{2\theta}\right)^k$, from (5.1) and (5.2), we get

$$\mathbb{P} = \begin{pmatrix} \frac{\theta^2}{1+\theta^2} & \frac{1}{1+\theta^2} \\ \frac{1}{2} & \frac{1}{2} \\ \frac{1}{\theta^2+1} & \frac{\theta^2}{\theta^2+1} \end{pmatrix}, \tag{5.3}$$

$$\mathbb{Q} = \frac{\left(1+\theta^2\right)^k}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}} \begin{pmatrix} \theta^2 & \theta\left(\frac{2\theta}{1+\theta^2}\right)^k & 1 \\ 1 & \theta\left(\frac{2\theta}{1+\theta^2}\right)^k & \theta^2 \end{pmatrix}. \tag{5.4}$$

By multiplying two $n\times n$-dimensional stochastic matrices, we can obtain a new stochastic matrix. As a result, if we product the matrix $\mathbb{P}$ by the $\mathbb{Q}$, we have a new stochastic matrix as follows;

$$\mathbb{H} = \mathbb{P}\mathbb{Q} = \frac{\left(1+\theta^2\right)^k}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}} \begin{pmatrix} \frac{(1+\theta^4)}{(1+\theta^2)} & 2^k\theta\left(\frac{\theta}{1+\theta^2}\right)^k & \frac{2\theta^2}{(1+\theta^2)} \\ \frac{(1+\theta^2)}{2} & 2^k\theta\left(\frac{\theta}{1+\theta^2}\right)^k & \frac{(1+\theta^2)}{2} \\ \frac{2\theta^2}{(1+\theta^2)} & 2^k\theta\left(\frac{\theta}{1+\theta^2}\right)^k & \frac{(1+\theta^4)}{(1+\theta^2)} \end{pmatrix}. \tag{5.5}$$

Now, the parameters κ and γ defined in the Ref. [1] will be presented in order to assess the conditions for extremality of a disordered phase. Let μ_1 and μ_1 be two measures on Ω, the variation distance between the projections of μ_1 and μ_1 onto the spin at x is defined by

$$||\mu_1 - \mu_2||_x = \frac{1}{2}\sum_{i\in\Phi} |\mu_1(\sigma(x) = i) - \mu_2(\sigma(x) = i)|.$$

We denote the configuration η with the spin at x set to s by $\eta^{x,s}$. Let $\mu^s_{\mathcal{T}_x}$ be the GM in which the parent of x has its spin fixed to s and the configuration on the bottom boundary of $\mathcal{T}_x$ (i.e., on $\partial\mathcal{T}_x \setminus \{\text{parent of } x\}$) is specified by $\mathcal{T}$ (see [1, 2, 14, 25, 39]).

Definition 5.2. [1] *For a set of Gibbs distributions $\{\mu^s_{\mathcal{T}_x}\}$, the quantities $\kappa \equiv \kappa(\{\mu^s_{\mathcal{T}_x}\})$ and $\gamma \equiv \gamma(\{\mu^s_{\mathcal{T}_x}\})$ are defined by*

(1) $\kappa = \sup\limits_{z\in\Gamma^k} \max\limits_{z,s,s'} \|\mu^s_{T_z} - \mu^{s'}_{T_z}\|_z$

(2) $\gamma = \sup\limits_{A\subset\Gamma^k} \max\limits_{z,s,s'} \|\mu^{\eta^{y,s}}_A - \mu^{\eta^{y,s'}}_A\|_z$, *where the maximum is taken over all boundary conditions η, all sites $y \in \partial A$, all neighbors $x \in A$ of y, and all spins $s, s' \in \{-1, 0, 1\}$.*

The following inequality is a necessary condition for extremality of the disordered phase (TISGM) (see [1]);

$$k\kappa\gamma < 1. \tag{5.6}$$

For $\mathbb{P}$ and $\mathbb{Q}$, the quantity κ has the form $\kappa = \sqrt{\tau_{\mathbb{P}}\tau_{\mathbb{Q}}}$, where for an arbitrary stochastic matrix $\mathbb{B} = (B_{ij})$, $\tau_{\mathbb{B}}$ is computed by

$$\tau_{\mathbb{B}} = \frac{1}{2}\max_{i,j}\left\{\sum_{\ell=1}^{3}|B_{i,\ell} - B_{j,\ell}|\right\}.$$

Taking into account the stochastic matrix given in (5.3) and (5.4), we obtain

$$\tau_{\mathbb{P}} = \frac{|\theta^2 - 1|}{\theta^2 + 1}, \quad \tau_{\mathbb{Q}} = \frac{|\theta^2 - 1|\left(1+\theta^2\right)^k}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}}. \tag{5.7}$$

Hence, from (5.7) and the definition, we get

$$\kappa^2 = \tau_{\mathbb{P}}\tau_{\mathbb{Q}} = \frac{\left(1+\theta^2\right)^{k-1}\left(\theta^2 - 1\right)^2}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}}. \tag{5.8}$$

From [14, 16, 28], the criterion of extremality of a disordered phase for the model on the CT of order $k \geq 2$ was determined by the inequality

$$k\kappa\gamma < 1. \tag{5.9}$$

From Definition 5.2 and following the demonstration given in references [14, 29], we can get $\gamma = \kappa$. Therefore, the extremality condition (5.9) is reduced to

$$k\kappa^2 - 1 = \frac{k\left(1+\theta^2\right)^{k-1}\left(\theta^2 - 1\right)^2}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}} - 1 < 0. \tag{5.10}$$

In this subsection, we are going to study the inequality (5.10).

5.1.1. *The extremality conditions of the disordered phase for $k = 3$.*

Theorem 5.3. *For $k = 3$, if $\theta \in (0.477977, 2.0921)$, then the disordered phase $\mu_0(3)$ associated to the fixed point $(\left(\frac{1+\theta^2}{2\theta}\right)^3, \left(\frac{1+\theta^2}{2\theta}\right)^3, 1)$ is extreme.*

Proof. From (5.10), for $k = 3$, we get

$$g(\theta) := 3\kappa^2 - 1 \quad = \quad \frac{3\left(-1+\theta^2\right)^2\left(1+\theta^2\right)^2}{8\theta^4 + \left(1+\theta^2\right)^4} - 1 < 0. \tag{5.11}$$

From (5.11) and after some algebraic operations, we get the following equality

$$0 \quad = \quad 2\left(1 - 2\theta^2 - 10\theta^4 - 2\theta^6 + \theta^8\right). \tag{5.12}$$

From (5.12), we get

$$\theta^4\left(\left(\theta^4 + \frac{1}{\theta^4}\right) - 2\left(\theta^2 + \frac{1}{\theta^2}\right) - 10\right) = 0. \tag{5.13}$$

Assume that $\varsigma = \theta^2$ and $\varsigma + \frac{1}{\varsigma} = u$, then we get

$$u^2 - 2u - 12 = 0. \tag{5.14}$$

We obtain the root of the equation (5.14) as

$$\varsigma + \frac{1}{\varsigma} = u = 1 + \sqrt{13}.$$

Therefore, we have the following quadratic equation;

$$\varsigma^2 - (1 + \sqrt{13})\varsigma + 1 = 0.$$

From the last quadratic equation, we have

$$\theta_c' = \sqrt{\frac{1}{2}\left(1+\sqrt{13}-\sqrt{2(5+\sqrt{13})}\right)} \approx 0.477977,$$

$$\theta_c'' = \sqrt{\frac{1}{2}\left(1+\sqrt{13}+\sqrt{2(5+\sqrt{13})}\right)} \approx 2.0921.$$

Therefore, for $\theta \in (0.477977, 2.0921)$, the disordered phase $\mu_0(3)$ related to $\left(\left(\frac{1+\theta^2}{2\theta}\right)^3, \left(\frac{1+\theta^2}{2\theta}\right)^3, 1\right)$ is extreme.

This completes the proof of the theorem. $\square$

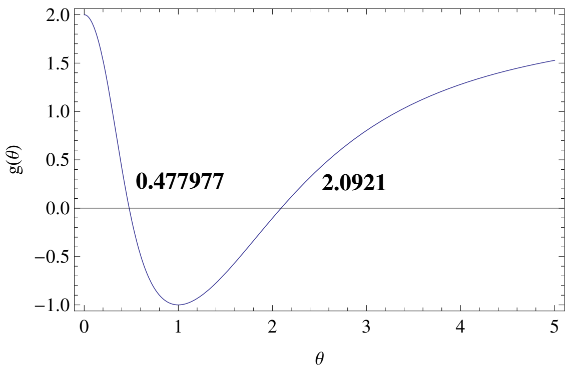

FIGURE 4. The graph of $g(\theta)$ in (5.11) for $k = 3$.

Figure 4 represents the graph in the function $g(\theta)$ given in (5.11). This function takes positive and negative numbers in different areas.

Example 5.1. For $k = 3$, and $\theta = 0.64$, the set of positive fixed points of the function F in (3.6) is obtained as $\mathrm{Fix}(F) = \{0.33077, 1, 3.02325\}$ (see Figure 5). Now let us consider the fixed point as

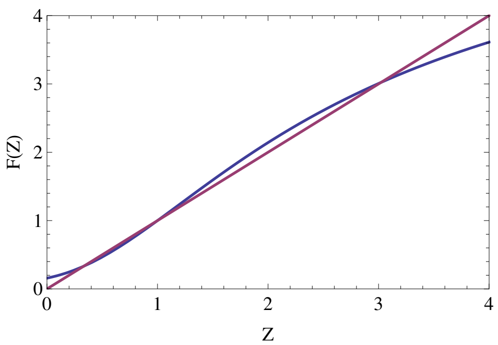

FIGURE 5. The graph of $F(Z)$ given in (3.6) for $k = 3$ and $\theta = 0.64$.

3.02325. After suitable substations, for the fixed point of dynamical sysyem given in (3.4)-(3.6) we get

$$(X_0, Y_0, Z_0) = (0.9706292766099369, 0.26906514853846664, 3.02325),$$

we get the stochastic matrices as follows;

$$\mathbb{P} = \begin{pmatrix} 0.553237 & 0.446763 \\ 0.751445 & 0.248555 \\ 0.880682 & 0.119318 \end{pmatrix},$$

$$\mathbb{Q} = \begin{pmatrix} 0.30427 & 0.489808 & 0.205922 \\ 0.564044 & 0.371912 & 0.0640438 \end{pmatrix}.$$

After some computations, we get

$$\tau_{\mathbb{P}} = \frac{1}{2}\max\{0.65489, 0.65489\} = 0.327445,$$
$$\tau_{\mathbb{Q}} = \frac{1}{2}\max\{0.25977, 0.117896, 0.1418782\} = 0.129885$$

$$\kappa^2 = (0.327445) * (0.129885) = 0.042530193825.$$

Due to $3\kappa^2 - 1 = 3(0.042530193825) - 1 \approx -0.872409418525 < 0$, the TISGM corresponding to the fixed point $(X_0, Y_0, Z_0) = (0.9706292766099369, 0.26906514853846664, 3.02325)$ is an extreme GM.

5.1.2. *The extremality conditions of the disordered phase for $k = 4$.* For $k = 4$, from (5.10) we get

$$h(\theta) := \frac{4\left(-1+\theta^2\right)^2\left(1+\theta^2\right)^3}{16\theta^5 + \left(1+\theta^2\right)^5} - 1 < 0. \tag{5.15}$$

From the inequality (5.15), we have

$$0 = \theta^{-5}\left(3 - \theta^2 - 18\theta^4 - 16\theta^5 - 18\theta^6 - \theta^8 + 3\theta^{10}\right). \tag{5.16}$$

Using Mathematica [40], we obtain the positive roots of the equation (5.16) as $\theta_c^{'} \approx 0.539833, \theta_c^{''} \approx 1.85242$ (see Fig. 6). So, we have the following theorem.

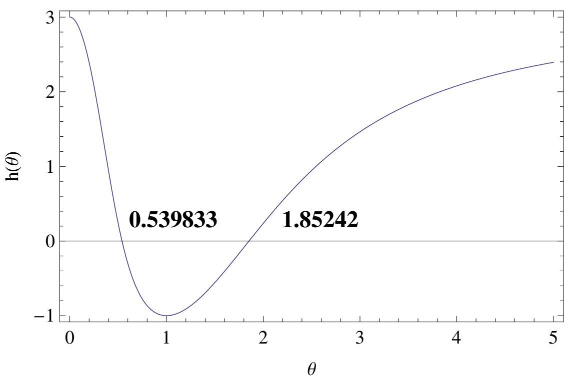

FIGURE 6. The graph of the function $h(\theta)$ in the equation (5.15) for $k = 4$.

Theorem 5.4. *If $\theta \in (0.539833, 1.85242)$, for $k = 4$ the disordered phase $\mu_0(4)$ for the fixed point $\left(\left(\frac{1+\theta^2}{2\theta}\right)^4, \left(\frac{1+\theta^2}{2\theta}\right)^4, 1\right)$ is extreme.*

Conjecture 1. Let $\theta_k^{'}(cr)$ and $\theta_k^{''}(cr)$ be the positive roots of the equation

$$f(\theta, k) := k\kappa^2 - 1 = \frac{k\left(1+\theta^2\right)^{k-1}\left(\theta^2-1\right)^2}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}} - 1 = 0 \tag{5.17}$$

such that $\theta_k'(cr) < \theta_k''(cr)$. Then for $\theta \in (\theta_k'(cr), \theta_k''(cr))$, the corresponding disordered phases are extreme and $(\theta_k'(cr), \theta_k''(cr)) \subset (\theta_{k-1}'(cr), \theta_{k-1}''(cr))$ for all positive integer $k > 1$.

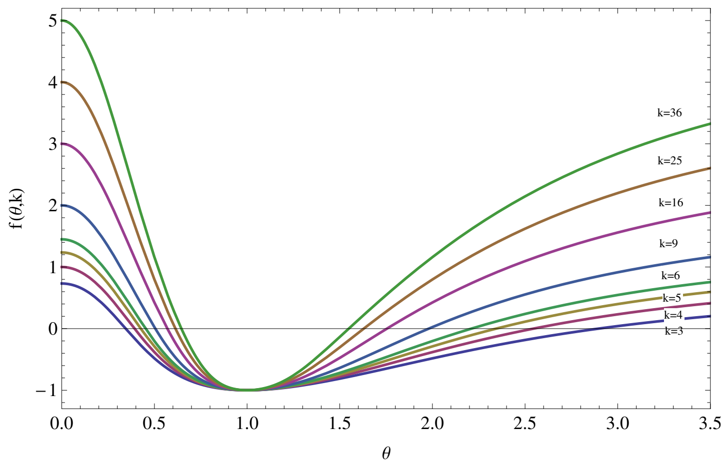

FIGURE 7. The graphs of $f(\theta, k)$ in eq. (5.17) for $k = 3, 4, 5, 6, 9, 16, 25, 36$.

In Fig. 7, we plot the graphs of $f(\theta, k)$ in equation (5.17) for $k = 3, 4, 5, 6, 9, 16, 25, 36$ by means of Mathematica [40]. As the value of k increases, the extreme regime of the disordered phase narrows.

5.2. **The non-extremality of the disordered phases.** Here, we will determine the regimes of the parameter θ for the non-extreme disordered phases in the set of all GMs constructed by the (1,1/2)-MSIM. So, in order to establish the non-extremality of the disordered phase, we will satisfy the KSC.

Let $|\lambda_{\max}|$ be the second largest eigenvalue of the stochastic matrix given in (5.5). It is well known that condition $k|\lambda_{\max}^2| - 1 > 0$ is a necessary condition (so-called Kesten-Stigum condition [31]) for non-extremality of a GM μ associated with to the stochastic matrix (see [14, 28] for details).

From the basic computations, we can obtain the set of eigenvalues of matrix $\mathbb{H}$ given in (5.5) as

$$\left\{ 0, 1, \frac{\left(\theta^2-1\right)^2 \left(1+\theta^2\right)^{k-1}}{\left(2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}\right)} \right\}.$$

The second largest of these eigenvalues is as follows

$$\lambda_{\max} = \frac{\left(\theta^2-1\right)^2 \left(1+\theta^2\right)^{k-1}}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}}. \tag{5.18}$$

It is well known that if $k\left|\lambda_{\max}\right|^2 - 1 > 0$, the equivalent disordered phase $\mu_0(k)$ is non-extreme [29]. Therefore, we get

$$k\left|\lambda_{\max}\right|^2 - 1 = k\left(\frac{\left(\theta^2-1\right)^2 \left(1+\theta^2\right)^{k-1}}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}}\right)^2 - 1 > 0. \tag{5.19}$$

It is obvious that (5.19) is equivalent to

$$\left(\frac{\sqrt{k}\left(1+\theta^2\right)^{k-1}\left(\theta^2-1\right)^2}{2^k\theta^{k+1} + \left(1+\theta^2\right)^{k+1}}\right) - 1 > 0. \tag{5.20}$$

In order to examine the non-extrematity of $\mu_0(k)$ corresponding the fixed point $(X_0, Y_0, Z_0) = \left(\left(\frac{\theta^2+1}{2\theta}\right)^k, \left(\frac{\theta^2+1}{2\theta}\right)^k, 1\right)$, we should analyze the inequality given in (5.20).

5.2.1. *The non-extremality of $\mu_0(3)$.* For $k = 3$, from (5.18) we have

$$\lambda_{\max} = \frac{\left(\theta^4 - 1\right)^2}{1 + 4\theta^2 + 14\theta^4 + 4\theta^6 + \theta^8}.$$

Therefore, we get

$$3\left|\lambda_{\max}\right|^2 - 1 > 3\left(\frac{\left(\theta^4 - 1\right)^2}{1 + 4\theta^2 + 14\theta^4 + 4\theta^6 + \theta^8}\right)^2 - 1 > 0. \tag{5.21}$$

Solving the following inequality will reveal the region where the inequality (5.21) has a solution:

$$\left(\frac{\sqrt{3}\left(\theta^4 - 1\right)^2}{1 + 4\theta^2 + 14\theta^4 + 4\theta^6 + \theta^8}\right) - 1 > 0. \tag{5.22}$$

From (5.22), we get

$$\left(\sqrt{3} - 1\right) - 4\theta^2 - \left(14 + 2\sqrt{3}\right)\theta^4 - 4\theta^6 + \left(\sqrt{3} - 1\right)\theta^8 > 0.$$

Similarly to the algebraic operations provided in (5.13) and (5.14), we obtain the critical values of θ as

$$\theta_c^{'} = \sqrt{\frac{1}{2}\left(1 + \sqrt{3} + \sqrt{2\left(8 + 5\sqrt{3}\right)} - \sqrt{-4 + \left(1 + \sqrt{3} + \sqrt{2\left(8 + 5\sqrt{3}\right)}\right)^2}\right)} \approx 0.3453,$$

$$\theta_c^{''} = \sqrt{\frac{1}{2}\left(1 + \sqrt{3} + \sqrt{2\left(8 + 5\sqrt{3}\right)} + \sqrt{-4 + \left(1 + \sqrt{3} + \sqrt{2\left(8 + 5\sqrt{3}\right)}\right)^2}\right)} \approx 2.8957.$$

As a result, the following theorem has been proven.

Theorem 5.5. *For $\theta \in (0, 0.3453) \cup (2.8957, \infty)$ and $k = 3$, the disordered phase $\mu_0(3)$ for $(\left(\frac{1+\theta^2}{2\theta}\right)^3, \left(\frac{1+\theta^2}{2\theta}\right)^3, 1)$ is non-extreme.*

Therefore, we obtain the family of the disordered phases that are non-extremal.

5.2.2. *The non-extremality of $\mu_0(4)$.*

Theorem 5.6. *For $\theta \in (0, 0.39294) \cup (2.54492, \infty)$ and $k = 4$, the disordered phase $\mu_0(4)$ for $(\left(\frac{1+\theta^2}{2\theta}\right)^4, \left(\frac{1+\theta^2}{2\theta}\right)^4, 1)$ is non-extreme.*

Proof. For $k = 4$, from the equation (5.18) we get

$$\lambda_{\max} = \frac{(\theta^2 - 1)^2\left(1 + \theta^2\right)^3}{1 + 5\theta^2 + 10\theta^4 + 16\theta^5 + 10\theta^6 + 5\theta^8 + \theta^{10}}.$$

From the necessary condition for non-extremality (5.19), one obtains

$$4\left|\lambda_{\max}\right|^2 - 1 = 4\left(\frac{(\theta^2 - 1)^2\left(1 + \theta^2\right)^3}{1 + 5\theta^2 + 10\theta^4 + 16\theta^5 + 10\theta^6 + 5\theta^8 + \theta^{10}}\right)^2 - 1 > 0. \tag{5.23}$$

To solve the inequality (5.23), it is sufficient to solve the following inequality

$$\left(\frac{2(\theta^2-1)^2\left(1+\theta^2\right)^3}{1+5\theta^2+10\theta^4+16\theta^5+10\theta^6+5\theta^8+\theta^{10}}\right)-1>0. \tag{5.24}$$

From (5.24), we get

$$1-3\theta^2-14\theta^4-16\theta^5-14\theta^6-3\theta^8+\theta^{10}>0.$$

Similarly the algebraic operations provided in (5.16), we obtain the critical values of θ as

$$\theta_c^{'}\approx 0.39294, \theta_c^{''}\approx 2.54492.$$

Thus, if $\theta\in(0,0.39294)\cup(2.54492,\infty)$, then for $k=4$ the KSC (5.19) is satisfied. This is sufficient to prove the theorem. $\square$

Conjecture 2. Let $\theta_k^{'}(cr)$ and $\theta_k^{''}(cr)$ be the positive roots of the equation

$$g(\theta,k):=\sqrt{k\,|\lambda_{\max}|}-1=\frac{\sqrt{k}\left(\theta^2-1\right)^2\left(1+\theta^2\right)^{k-1}}{2^k\theta^{1+k}+\left(1+\theta^2\right)^{1+k}}-1=0 \tag{5.25}$$

such that $\theta_k^{'}(cr)<\theta_k^{''}(cr)$. Then for $\theta\in(0,\theta_k^{'}(cr))\cup(\theta_k^{''}(cr),\infty)$, the corresponding TISGMs are non-extreme and $(\theta_{k+1}^{'}(cr),\theta_{k+1}^{''}(cr))\subset(\theta_k^{'}(cr),\theta_k^{''}(cr))$ for all positive integer $k>1$.

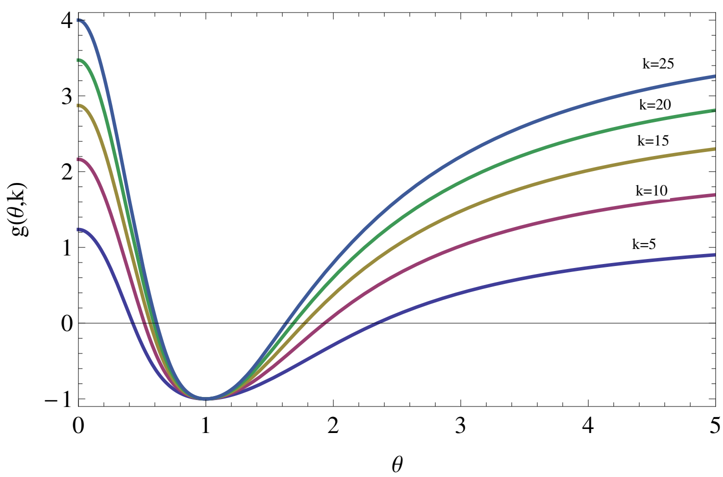

FIGURE 8. The graphs of $g(\theta,k)$ in eq. (5.25) for $k=5,10,15,20,25$.

In Fig. 8, we plot the graphs of $g(\theta,k)$ in eq. (5.25) for $k=5,10,15,20,25$ by means of Mathematica [40]. As the value of k increases, the non-extreme regime of the disordered phase $\mu(k)$ expands.

6. CONCLUSIONS

In this paper, using the approach of Akın and Mukhamedov [16], we have established the TISGMs for the (1,1/2)-MSIM on a CT of arbitrary order. We demonstrated that this model has phase transition circumstances in both the ferromagnetic and antiferromagnetic regimes, in contrast to the single-spin Ising model, which is devoid of such GMs in the antiferromagnetic regime [18, 20, 22].

For the Potts model with q-state on a CT, in [23] we overcome the phase transitions phenomena problem and demonstrated that phase transition occurred in the peaks at critical temperature

$$T_c = \frac{J_p}{\ln\left(\frac{1}{2}\left(2-q+\sqrt{q^2+32(q-1)}\right)\right)}$$

if $T < T_c$, where J_p is prolonged next-nearest-neighbor potential.

Using a tree-indexed Markov chain on the CT with any order, we have evaluated the extremality of the disordered phases connected to the proposed model. Taking the KSC [31] into account, we have determined the regimes with non-extremality conditions for a TISGM by means of the stochastic matrix associated with(1,1/2)-MSIM on a CT of order $k \geq 2$.

We know that for the single-spin Ising model on the CT, the density free energy can be written as an analytic function of the inverse temperature, therefore a spontaneous magnetization is not produced by the Ising model on the CT [3, 4]. We do not yet know whether the spontaneous magnetization feature is provided for the model given here. So, we will explore this issue in our future work. Using the Fourier transform, Seino [41] calculated the free energy associated with the random Ising model on the BL as a single integral. The free energy per site has not been explored in this study. Furthermore, using the methods given in the Refs [38, 42], we will endeavor to evaluate the free energy and entropy quantities of TISGMs corresponding to the given model. We will also analyze the phase transition problem for larger k values. It is well-known that as a CT is very sensitive to the boundary conditions [4], depending on the given boundary conditions, the free energy formulas differ (see [30, 38, 42, 43] for details). In our next studies, we will provide precise formulas for the free energy function related to the (1,1/2)-MSIM for a few boundary conditions.

Acknowledgements

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