Asymptotic behavior of the Boussinesq equation with nonlocal weak damping and arbitrary growth nonlinear function

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Abstract

In this paper, we consider the asymptotic behavior of the Boussinesq equation with nonlocal weak damping when the nonlinear function is arbitrary polynomial growth. We firstly prove the well-posedness of solution by means of the monotone operator theory. At the same time, we obtain the dissipative property of the dynamical system (E ,$S($ t)) associated with the problem in the space H 0 2 ( ) × L 2 ( ) and D (A 3 4) x H 0 1 ( ) , respectively. After that, the asymptotic smoothness of the dynamical system (E ,$S($ t)) and the further quasi-stability are demonstrated by the energy reconstruction method. Finally, different from [21] we show not only existence of the finite global attractor but also existence of the generalized exponential attractor.

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Abstract: In this paper, we consider the asymptotic behavior of the Boussinesq equation with nonlocal weak damping when the nonlinear function is arbitrary polynomial growth. We firstly prove the well-posedness of solution by means of the monotone operator theory. At the same time, we obtain the dissipative property of the dynamical system $(\mathbb{E}, S(t))$ associated with the problem in the space $H_0^2(\Omega) \times L^2(\Omega)$ and $D(A^{\frac{3}{4}}) \times H_0^1(\Omega)$, respectively. After that, the asymptotic smoothness of the dynamical system $(\mathbb{E}, S(t))$ and the further quasi-stability are demonstrated by the energy reconstruction method. Finally, different from [21] we show not only existence of the finite global attractor but also existence of the generalized exponential attractor.

Keywords: Nonlocal weak damping; Boussinesq equation; Asymptotic smoothness; Global attractor; Generalized exponential attractor.

1 Introduction

We are concerned with the following nonlocal weak damping Boussinesq equation

$$\begin{cases} \varepsilon u_{tt} + \Delta^2 u + \|u_t\|^r_{L^2(\Omega)} u_t - \Delta g(u) = f(x), \ x \in \Omega, \ t \geq 0, \\ u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0, \qquad t \geq 0, \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x), \qquad x \in \Omega, \end{cases} \tag{1.1}$$

where $\varepsilon \in (0,1)$, $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, ν is an unit outward vector of $\partial\Omega$, $\|u_t\|^r_{L^2(\Omega)} u_t$ $(r \geq 0)$ is nonlocal weak damping, the forcing term $f \in H^{-1}(\Omega)$. We give rise to the following conditions with respect to the nonlinear function $g(u)$,

$$g \in C^3(\mathbb{R}, \mathbb{R}), \ g(0) = 0, \tag{1.2}$$

$$|g''(s)| \leq C(1 + |s|^p), \ \forall\, p > 0, \tag{1.3}$$

$$g'(s) \geq -l, \ \forall\, s \in \mathbb{R}, \ 0 < l < \frac{\sqrt{\lambda_1}}{2}, \tag{1.4}$$

where $C > 0$ is a constant, $\lambda_1 > 0$ is the first eigenvalue of Δ^2 with boundary conditions $u|_{\partial\Omega} = \frac{\partial u}{\partial \nu}|_{\partial\Omega} = 0$.

Let we recall simply the background and development of Boussinesq equation. In 1872, Boussinesq

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([1]) established the following equation

$$u_{tt} - u_{xx} + \mu u_{xxxx} = a(u^2)_{xx}, \tag{1.5}$$

which it was the first science explanation about the solitary wave phenomena discovered and recorded by Scolt Russell in [2]; μ and a are constants depending on the depth of fluid and characteristic velocities of water waves, u is the motion of free surface of fluid. As we know that (1.5) is called a good Boussinesq equation when $\mu > 0$ and a bad Boussinesq equation when $\mu < 0$. After that, the generalized Boussinesq equations have been applied into various models, such as the model of surface waves in shallow water([3,4]) as well as the small lateral oscillation of nonlinear beam ([5,6,7]). Besides, such equations can model not only the oscillation of the nonlinear strings but also the two-dimensional irrigational flows of an inviscid liquid in a uniform rectangular channel as $\mu > 0$ ([8,9]); meanwhile, they can also be exploited to describe the propagation of ion-sound waves in a uniform isotropic plasma and nonlinear lattice waves as $\mu < 0$ ([3,4]). To the best of our knowledge, there are quite a lot of profound researches to the Boussinesq equations from various view of dynamical system, see [10-18] and references therein. For instance, in [10] the finite time blow-up of solution, existence and uniqueness of local mild solution were achieved for the cauchy problem of dissipative Boussinesq equations. Liu and Wang proved existence and scattering of a small global amplitude solution for the nonlinear Boussinesq equation in line with the estimates of dispersion along with the principle of Banach contracting mapping, see [11] for details.

As far as we know, global attractor is a key concept to study the long-time behavior of solutions for dissipative nonlinear evolution equations coming from physics and mechanics as well as atmospheric sciences and so on, please refer to [22-26] and references therein. In the matter of Boussinesq equations, study of global attractor has attracted lots of mathematicians, see [12-18]. In these literatures, Li and Yang ([12]) investigated the following Boussinesq equation with perturbation damping

$$\epsilon u_{tt} + \Delta^2 u - \Delta u_t - \Delta f(u) = g(x), \tag{1.6}$$

in which they first of all obtained the local well-posedness of weak solution for (1.6), and then proved the global well-posedness and dissipativity when the initial data could be controlled by a constant R_ϵ relying on ϵ, while R_ϵ was blow-up as $\epsilon \to 0$. In [13] the long-time behavior of solution for (1.6) was studied for $\epsilon = 1$ when nonlinear function $f(u)$ satisfied non-supercritical growth conditions. Simultaneously, existence of uniform attractor was shown for the nonautonomous Boussinesq equation with critical growth nonlinearity in [14].

To the limit of our knowledge, for study to the nonlinear evolution equations with nonlocal damping, Silve, Narciso and Vicente in [19] investigated the global well-posedness of solutions and polynomial stability as well as non-exponential decay estimates to the following nonlinear beam equations with nonlocal energy damping

$$u_{tt} - k\Delta u + \Delta^2 u - \lambda(\|\Delta u\|_{L^2}^2 + \|\Delta u_t\|_{L^2}^2)^q \Delta u_t + f(u) = 0. \tag{1.7}$$

Zhao and Zhong considered the following extensible beam model with nonlocal weak damping in [21],

$$u_{tt} + \Delta^2 u - m(\|\nabla u\|^2)\Delta u + \|u_t\|^p u_t + f(u) = h(x). \tag{1.8}$$

They above all obtained a global well-posedness of solution by virtue of the monotone operator theory; subsequently, the asymptotic smoothness of the semigroup associated with (1.8) was verified via the energy reconstruction method; ultimately, existence of a global attractor was achieved under the condition that the subcritical growth of nonlinear term. At the same time, above three authors ([20]) also focused on the following wave equation with nonlocal weak damping and nonlocal weak anti-damping

$$u_{tt} - \Delta u + k\|u_t\|^p_{L^2} u_t + f(u) = \int_\Omega k(x,y)u_t(y)dy + h(x). \tag{1.9}$$

With the aid of monotone operator theory similar to [21] and the condition (C) which was first proposed by Ma, Wang and Zhong ([28]), they proved again existence and uniqueness of a global solution as well as global attractors for (1.9) in a bounded domain. Throughout these writings all mentioned above, there has no any results on study of attractors for Boussinesq equation with nonlocal weak damping, it is just our concerned and interested.

The aim of the present paper is to solve the following questions. (i) Taking advantage with the monotone operator theory, which is similar to those in [20, 21], we obtain the global well-posedness of solution for (1.1). However, it is interesting that the condition of positive constant l in dissipative assumption (1.4) is different from that of [12], in which $g'(s) \geq -l$ and $l > \sqrt{\lambda_1}$ since their damping is $-\Delta u_t$, while $0 < l < \frac{\sqrt{\lambda_1}}{2}$ in our problem. Besides, the growth order of nonlinear term $g(u)$ only satisfies $p > 0$ and has no else restrict condition. (ii) Existence of global attractors is proved by using the energy reconstruction technique. (iii) We utilize the quasi-stable method to show the finite fractal dimension of global attractor, and from this we further achieve existence of generalized exponential attractor. Some results are extend and improvement of [21].

2 Well Posedness

Without loss of generality, denote $H = L^2(\Omega)$ equipped with the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$, and $\|\cdot\|_q$ is the norm of $L^q(\Omega)$. Let $V_1 = H^1_0(\Omega)$, $V_2 = \mathcal{D}(A^{\frac{1}{2}}) = H^2_0(\Omega)$, $V_3 = \mathcal{D}(A^{\frac{3}{4}})$, where $A = \Delta^2 : V_2 \to V_2'$, and the operator $A^s (s \in \mathbb{R})$ is strictly positive. We define a family of Hilbert spaces $V_s = D(A^{\frac{s}{4}})$ with the following inner products and norms respectively,

$$(u,v)_s = (A^{\frac{s}{4}}u, A^{\frac{s}{4}}v), \quad \|u\|_{V_s} = \|A^{\frac{s}{4}}u\|,$$

especially,

$$\|u\|_{V_1} = \|A^{\frac{1}{4}}u\| = \|\nabla u\|, \quad \|u\|_{V_2} = \|A^{\frac{1}{2}}u\| = \|\Delta u\|.$$

Besides, provided that X is a separable Banach space, and

$$W^{1,p}(a,b;X) = \{f \in C(a,b;X) : \ f' \in L_p(a,b;X)\},$$

especially,

$$W^{1,1}(a,b;V_2) = \{f \in C(a,b;V_2) : \ f' \in L_1(a,b;V_2)\},$$

where $L_p(a,b;X)$ $(1 \leq p \leq \infty)$ is the identity class spaces consisting of Bochner measure functions $f : [a,b] \mapsto X$, endowed with the norm $\|f\|_{L_p(a,b;X)} = (\int_a^b \|f(t)\|^p_X \mathrm{d}t)^{\frac{1}{p}}$, that is $\|f(\cdot)\|_X \in L_p(a,b)$. $C(a,b;X)$ denotes all of continuous functions valued on X acting on $[a,b]$.

By virtue of the Poincaré inequality, there holds

$$\|A^{\frac{s}{4}}u\|^2 \geq \lambda_1^{\frac{1}{2}}\|A^{\frac{s-1}{4}}u\|^2, \quad s=1,2, \ \forall\ u \in V_2. \tag{2.1}$$

It is easy to see that (1.1) is equivalent to the following Cauchy problem:

$$\begin{cases} \varepsilon A^{-\frac{1}{2}}u_{tt} + A^{\frac{1}{2}}u + A^{-\frac{1}{2}}(\|u_t\|^r u_t) + g(u) = A^{-\frac{1}{2}}f(x), \\ u(0) = u_0, \ u_t(0) = u_1. \end{cases} \tag{2.2}$$

For convenience, write $\mathbb{E} = V_2 \times H$, $\mathbb{W} = V_3 \times V_1$, and endowed with norms respectively as follows,

$$\|(u,v)\|_{\mathbb{E}}^2 = \|A^{\frac{1}{2}}u\|^2 + \varepsilon\|v\|^2, \quad \|(u,v)\|_{\mathbb{W}}^2 = \|A^{\frac{3}{4}}u\|^2 + \varepsilon\|A^{\frac{1}{4}}v\|^2.$$

Throughout the paper, $c_i, \ c^i, \ C_i, \ C^i, \ C, \ i \in \mathbb{N}$ be the different constants for brevity.

Next we prove the well-posedness of solution for (1.1) by using the monotone operator theory. For this purpose, we define the operator $A_1 : \mathfrak{D}(A_1) \subset \mathbb{E} \to \mathbb{E}$; $B_1 : \mathbb{E} \to \mathbb{E}$ as follows:

$$A_1 = \begin{pmatrix} 0 & -I \\ \varepsilon^{-1}A & \varepsilon^{-1}D \end{pmatrix}, \quad B_1(\varphi) = \begin{pmatrix} 0 \\ \varepsilon^{-1}F(\varphi) \end{pmatrix}, \tag{2.3}$$

with domain

$$\mathfrak{D}(A_1) = \{\varphi = (u,v) \in V_2 \times V_2 : Au + Dv \in H\},$$

$\varphi = \varphi(t) = (u(t), v(t)) \in \mathbb{E}, \ v = u_t, \ F(\varphi) = f + \Delta g(u), \ D(u_t) = \|u_t\|^r u_t$. Then problem (1.1) can be written as the following form

$$\begin{cases} \frac{\mathrm{d}\varphi(t)}{\mathrm{d}t} + A_1\varphi(t) = B_1(\varphi(t)), \\ \varphi(0) = \varphi_0 = (u_0, u_1). \end{cases} \tag{2.4}$$

Based on above preliminary works, we need only to verify the following conditions in order to obtain existence and uniqueness of strong solution as well as generalized solution, this is,

(1) Operator A_1 is maximum monotone(or m-accretive);

(2) Operator $B_1 : \mathbb{E} \to \mathbb{E}$ is local Lipschitz.

Below we begin our proof from the following known results.

Lemma 2.1 [21] Let $u, v \in X$, X is a Hilbert space. Then there exists a positive constant C_r, such that

$$(\|u\|_X^{r-2}u - \|v\|_X^{r-2}v, u-v) \geq \begin{cases} C_r\|u-v\|_X^r, \ r \geq 2, \\ C_r\frac{\|u-v\|_X^2}{(\|u\|_X + \|v\|_X)^{2-r}}, \ 1 \leq r \leq 2. \end{cases} \tag{2.5}$$

In line with Lemma 2.1 we achieve the assumption 1.1 in [22] at once.

Lemma 2.2 [22] Let $D(u_t) = \|u_t\|^r u_t$, *then* D: $L^2(\Omega) \to L^2(\Omega)$ is monotone and hemicontinuous, $D(0) = 0$, i.e., for any $u_t, \ v_t \in L^2(\Omega), \ r \geq 0$, there holds

$$\big(D(u_t) - D(v_t), u_t - v_t\big) = (\|u_t\|^r u_t - \|v_t\|^r v_t, u_t - v_t) \geq C_r\|u_t - v_t\|^{r+2} \geq 0, \tag{2.6}$$

and $\lambda \mapsto (D(u_t + \lambda v_t), v_t) : \ \mathbb{R} \to \mathbb{R}$ is a continuous function. As a result, the damping operator D is strictly monotone.

Moreover, there exists a subset $W \subset H$, such that for $D(w) \in H, \ \forall w \in W$, and W is dense in H, where $W = D(A^{\frac{1}{2}}) = H_0^2(\Omega)$.

Definition 2.3[22] The function $u(t) \in C([0,T];V_2)\bigcap C^1([0,T];H)$ with $u(0)=u_0,\ u_t(0)=u_1$ is called:

(S) a strong solution to (1.1) on $[0,T]$, if the following conditions holds

(1) $u \in W^{1,1}(a,b;V_2),\ u_t \in W^{1,1}(a,b;H)$, for any $0<a<b<T$;

(2) $Au(t)+D(u_t(t)) \in H$, for almost all $t \in [0,T]$;

(3) (1.1)is satisfied in H, for almost all $t \in [0,T]$.

(G) generalized solution to (1.1) on the interval $[0,T]$, if and only if, there exists a sequence of strong solutions $\{u_n(t), u_{nt}(t)\}$ of (1.1) with initial data (u_{0n}, u_{1n}) instead of (u_0,u_1), such that

$$\lim_{n\to\infty}\max_{t\in[0,T]}\left\{|u_t(t)-u_{nt}(t)|+\left|A^{\frac{1}{2}}\big(u(t)-u_n(t)\big)\right|\right\}=0. \tag{2.7}$$

Lemma 2.4 Assume that the conditions (1.2)-(1.4) hold, then the operator $A_1:\ \mathfrak{D}(A_1)\subset\mathbb{E}\mapsto\mathbb{E}$ is maximum monotone.

Proof: We prove this Lemma by two steps.

Step I, we claim that A_1 is a monotone operator. For this aim, we take arbitrary elments $u=(u^1,u^2),\ \omega=(\omega^1,\omega^2)\in\mathfrak{D}(A_1)$, and let $\xi=(\xi_1,\xi_2)=A_1(u),\ \eta=(\eta_1,\eta_2)=A_1(\omega)$. Thus, we have

$$\xi_1=-u^2,\quad \xi_2=\varepsilon^{-1}(Au^1+D(u^2)),\quad \eta_1=-\omega^2,\quad \eta_2=\varepsilon^{-1}(A\omega^1+D(\omega^2)).$$

Since

$$(A_1(u)-A_1(\omega),u-\omega)_{\mathbb{E}}=(A^{\frac{1}{2}}(\xi_1-\eta_1),A^{\frac{1}{2}}(u^1-\omega^1))+(\varepsilon(\xi_2-\eta_2),u^2-\omega^2),$$

we get by using Lemma 2.2

$$\begin{aligned}(A_1(u)-A_1(\omega),u-\omega)_{\mathbb{E}}\ &=-(A(u^2-\omega^2),u^1-\omega^1)+\\ (A(u^1-\omega^1)+(D(u^2)-D(\omega^2)),u^2-\omega^2)\ &=(D(u^2)-D(\omega^2),u^2-\omega^2)\geq 0.\end{aligned}$$

Therefore, the operator A_1 is monotone.

Step II, we claim that the operator A_1 is maximum once we show that $R(I+A_1)=E$. In fact, given $f_0\in\mathcal{D}(A^{\frac{1}{2}})=V_2,\ f_1\in H$, such that

$$x-y=f_0,\quad Ax+Dy+\varepsilon y=\varepsilon f_1,\ as\ \ (x,y)\in D(A_1). \tag{2.8}$$

Substituting $x=y+f_0$ into above second formula, it leads to

$$Ay+Dy+\varepsilon y=\varepsilon f_1-Af_0\in V_2'. \tag{2.9}$$

Let $v=A^{\frac{1}{2}}y$, then we obtain

$$v+Sv=A^{-\frac{1}{2}}(\varepsilon f_1-Af_0)\in H, \tag{2.10}$$

where $Sv=A^{-\frac{1}{2}}D(A^{-\frac{1}{2}}v)+\varepsilon A^{-\frac{1}{2}}IA^{-\frac{1}{2}}v$. According to Lemma 2.2 we see that $A^{-\frac{1}{2}}D(A^{-\frac{1}{2}}v)$ is maximum monotone in H. Thanks to $\mathcal{D}(A^{\frac{1}{2}})\subset H$, it is clear to see that $\varepsilon A^{-\frac{1}{2}}IA^{-\frac{1}{2}}v$ is a bounded linear positive operator on H. Therefore, in line with ([26], Lemma 2.1) we deduce that the operator S is maximum monotone on H, that is, $R(I+S)=H$. As a result, there exists $v\in H$ satisfying (2.10), furthermore, $y=A^{-\frac{1}{2}}v\in D(A^{\frac{1}{2}})$ is a solution of (2.9). Thus $(x,y)\in\mathfrak{D}(A_1)$.

Below we start to verify that the operator B_1 is local Lipschitz continuous in $\mathbb{E}$, for this purpose, first of all, we need the following a priori estimates.

(i) **A priori estimates on $\mathbb{E}$**

Taking the inner product of (2.2) with $A^{\frac{1}{2}}v = A^{\frac{1}{2}}u_t + \alpha A^{\frac{1}{2}}u$ in $L^2(\Omega)$, we find

$$\frac{d}{dt}Q(t) + Q_1(t) = 0, \tag{2.11}$$

where

$$\begin{aligned} Q(t) &= P(t) + \varepsilon\alpha(u_t, u), \\ P(t) =& E_0(t) + \frac{1}{2}(g'(u), |A^{\frac{1}{4}}u|^2) - (f, u), \\ E_0(t) &= \frac{\varepsilon}{2}\|u_t\|^2 + \frac{1}{2}\|A^{\frac{1}{2}}u\|^2; \\ Q_1(t) &= \alpha\|A^{\frac{1}{2}}u\|^2 - \varepsilon\alpha\|u_t\|^2 + (\|u_t\|^r u_t, u_t + \alpha u) + \\ &\alpha(g'(u),|A^{\frac{1}{4}}u|^2) - (f, \alpha u) - \frac{1}{2}(g''(u)|A^{\frac{1}{4}}u|^2, u_t). \end{aligned}$$

By virtue of (1.3),(1.4),(2.1), and Young as well as interpolation inequalities, we arrive at

$$(g'(u), |A^{\frac{1}{4}}u|^2) \geq -l\|A^{\frac{1}{4}}u\|^2 \geq -\frac{l}{\sqrt{\lambda_1}}\|A^{\frac{1}{2}}u\|^2, \tag{2.12}$$

taking advantage with (1.3), (2.23) and Sobolev imbedding $W^{2,2}(\Omega) \hookrightarrow C(\overline{\Omega})$ $(n = 3)$, we conclude

$$\begin{aligned} \|g'(u)\|_{L^\infty} &\leq C(1 + \|u\|_{L^\infty}^{p+1}) \leq C(1 + \|A^{\frac{1}{2}}u\|^{p+1}), \\ \|g''(u)\|_{L^\infty} &\leq C(1 + \|u\|_{L^\infty}^{p}) \leq C(1 + \|A^{\frac{1}{2}}u\|^{p}). \end{aligned} \tag{2.13}$$

So

$$\begin{aligned} |(g''(u)|A^{\frac{1}{4}}u|^2, u_t)| &\leq \|g''(u)\|_{L^\infty}\|A^{\frac{1}{4}}u\|_{L^4}^2\|u_t\| \\ &\leq \alpha\|u_t\|^2 + C_\alpha(1 + \|u\|_{L^\infty}^{2p})\|A^{\frac{1}{4}}u\|_{L^4}^4 \\ &\leq \alpha\|u_t\|^2 + C_\alpha(1 + \|A^{\frac{1}{2}}u\|^{2p})\|A^{\frac{1}{2}}u\|^4 \\ &\leq \alpha\|u_t\|^2 + C_\alpha E_0(t)^{p+2}. \end{aligned} \tag{2.14}$$

Together with Hölder, Young inequalities and (2.1), yields

$$|(f, u)| \leq \frac{l}{4\sqrt{\lambda_1}}\|A^{\frac{1}{2}}u\|^2 + \frac{1}{l}\|f\|_{H^{-1}}^2. \tag{2.15}$$

Therefore, from (2.12)-(2.15) we deduce

$$P(t) \geq c_1 E_0(t) - C_1, \tag{2.16}$$

as $0 < l < \frac{\sqrt{\lambda_1}}{2}$, and $c_1 = \min\{1, 1 - \frac{l}{\sqrt{\lambda_1}}\} > 0,\ C_1 = \frac{1}{l}\|f\|_{H^{-1}}^2$.

By means of Hölder and Young inequalities again, we conclude

$$\varepsilon\alpha|(u_t, u)| \leq \varepsilon\alpha\|u_t\|\|u\| \leq \frac{c_1\varepsilon}{4}\|u_t\|^2 + \frac{\varepsilon\alpha^2}{c_1\lambda_1}\|A^{\frac{1}{2}}u\|^2, \tag{2.17}$$

thus, choosing $\alpha > 0$ small enough, such that

$$Q(t) \geq c_2 E_0(t) - C_2, \tag{2.18}$$

where $c_2=\min\{\frac{c_1}{2},c_1-\frac{2\varepsilon\alpha^2}{c_1\lambda_1}\}>0,\ C_2=\frac{1}{l}\|f\|^2_{H^{-1}}$. As a result, (2.11) can be rewritten as follows

$$\frac{d}{dt}Q(t)+\alpha Q(t)+\Upsilon=0, \tag{2.19}$$

where

$$\begin{aligned}\Upsilon=&(\|u_t\|^r u_t,u_t+\alpha u)-\frac{3\varepsilon\alpha}{2}\|u_t\|^2+\frac{\alpha}{2}\|A^{\frac12}u\|^2+\\ &\frac{\alpha}{2}(g'(u),|A^{\frac14}u|^2)-\frac12(g''(u)|A^{\frac14}u|^2,u_t)-\varepsilon\alpha^2(u_t,u).\end{aligned}$$

With the aid of Young inequality, there exists positive constants c_3,c_4, such that

$$(u_t,u_t)=\|u_t\|^2\le c_3+c_4\|u_t\|^{r+2}. \tag{2.20}$$

Combining with Cauchy and Young inequalities as well as (2.1), it follows that

$$\begin{aligned}|(\|u_t\|^r u_t,\alpha u)|\le&\frac{\alpha}{2}\|u_t\|^{r+2}+\frac{\alpha}{2}\|u_t\|^r\|u\|^2\\ \le&\frac{\alpha}{2}\|u_t\|^{r+2}+\frac{\alpha}{2\lambda_1}(C_\delta\|u_t\|^{r+2}+\delta)\|A^{\frac12}u\|^2\\ \le&\frac{\alpha}{2}\|u_t\|^{r+2}+\frac{\alpha C_\delta}{2\lambda_1}E_0(t)\cdot\|u_t\|^{r+2}+\frac{\alpha\delta}{2\lambda_1}E_0(t).\end{aligned} \tag{2.21}$$

Moreover, in line with (2.11), (2.14) and (2.16), there holds

$$\begin{aligned}\frac{d}{dt}P(t)+\alpha P(t)\le\frac{d}{dt}P(t)+\alpha P(t)+\|u_t\|^{r+2}=&\frac12(g''(u)|A^{\frac14}u|^2,u_t)+\alpha P(t)\\ \le\frac{\alpha}{2}\|u_t\|^2+\frac{C_\alpha}{2}E_0(t)^{p+2}+\alpha P(t)\le& C_3\alpha^{2(p+2)}P(t)^{p+2}+C_4.\end{aligned}$$

We infer from Lemma 4.1 in [27] that $P(t)\le C_B,\ t\ge t_0$, so

$$E_0(t)\le C_B,\ t\ge t_0. \tag{2.22}$$

Integer with (2.14) and (2.17)-(2.22), we obtain

$$\begin{aligned}\Upsilon\ge&\Big(1-\frac{\alpha}{2}-\frac{\alpha C_\delta C_B}{2\lambda_1}\Big)\|u_t\|^{r+2}-\frac{\alpha\delta C_B}{2\lambda_1}-\frac{3\varepsilon\alpha}{2}\|u_t\|^2+\frac{\alpha}{2}\|A^{\frac12}u\|^2-\\ &\frac{\alpha l}{4\sqrt{\lambda_1}}\|A^{\frac12}u\|^2-\frac{\alpha}{2}\|u_t\|^2-\frac{C_\alpha}{2}E_0(t)^{p+2}-\frac{\varepsilon\alpha c_1}{4}\|u_t\|^2-\frac{\varepsilon\alpha^3}{c_1\lambda_1}\|A^{\frac12}u\|^2\\ \ge&\Big(\frac{1}{c_4}\Big(1-\frac{\alpha}{2}-\frac{\alpha C_\delta C_B}{2\lambda_1}\Big)-\frac{3\varepsilon\alpha}{2}-\frac{\alpha}{2}-\frac{\varepsilon\alpha c_1}{4}\Big)\|u_t\|^2+\Big(\frac{\alpha}{2}-\frac{\alpha l}{4\sqrt{\lambda_1}}-\frac{\varepsilon\alpha^3}{c_1\lambda_1}\Big)\|A^{\frac12}u\|^2\\ &-\frac{c_3}{c_4}\Big(1-\frac{\alpha}{2}-\frac{\alpha C_\delta C_B}{2\lambda_1}\Big)-\frac{\alpha\delta C_B}{2\lambda_1}-C_5Q(t)^{p+2}-2^{p+1}C_\alpha\big(\frac{C_2}{c_2}\big)^{p+2}.\end{aligned} \tag{2.23}$$

Choosing small enough α, such that

$$\begin{aligned}&\frac{1}{c_4}\Big(1-\frac{\alpha}{2}-\frac{\alpha C_\delta C_B}{2\lambda_1}\Big)-\frac{3\varepsilon\alpha}{2}-\frac{\alpha}{2}-\frac{\varepsilon\alpha c_1}{4}>0,\\ &\frac{\alpha}{2}-\frac{\alpha l}{4\sqrt{\lambda_1}}-\frac{\varepsilon\alpha^3}{c_1\lambda_1}>0,\ 1-\frac{\alpha}{2}-\frac{\alpha C_\delta C_B}{2\lambda_1}>0,\end{aligned}$$

for $0<l<\frac{\sqrt{\lambda_1}}{2}$. Thus $\Upsilon\ge-\alpha C_6-C_5Q(t)^{p+2}$, furthermore, $\frac{d}{dt}Q(t)+\alpha Q(t)\le\alpha C_6+C_5Q(t)^{p+2}$. By virtue of Lemma 4.1 in [27], we know that $Q(t)\le C_7,\ \forall\ t\ge t_0$, and then together with (2.17), we claim that

$$\|(u,u_t)\|^2_{\mathbb{E}}\le\frac{2C_7+2C_2}{c_2}=R^2. \tag{2.24}$$

(ii) **A priori estimates on $\mathbb{W}$**

Taking the inner product of (2.2) with $Au_t + \alpha Au$ in $L^2(\Omega)$, we have

$$\begin{aligned} &\frac{d}{dt}\mathcal{E}(t) + \alpha\|A^{\frac{3}{4}}u\|^2 - \varepsilon\alpha\|A^{\frac{1}{4}}u_t\|^2 + (\|u_t\|^r u_t, A^{\frac{1}{2}}u_t + \alpha A^{\frac{1}{2}}u) \\ &\quad + \alpha(g(u), Au) - (f, \alpha A^{\frac{1}{2}}u) = (g'(u)u_t, Au), \end{aligned} \tag{2.25}$$

where

$$\mathcal{E}(t) = E_1(t) + (g(u), Au) - (f, A^{\frac{1}{2}}u) + \varepsilon\alpha(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u),$$
$$E_1(t) = \frac{\varepsilon}{2}\|A^{\frac{1}{4}}u_t\|^2 + \frac{1}{2}\|A^{\frac{3}{4}}u\|^2.$$

By means of Hölder, Young and Gagliardo-Nirenberg inequalities, and together with (2.13), (2.24), there holds

$$\begin{aligned} &\|A^{\frac{1}{2}}g(u)\| = \|g'(u)A^{\frac{1}{2}}u + g''(u)(A^{\frac{1}{4}}u)^2\| \\ \le&\|g'(u)\|_{L^\infty}\|A^{\frac{1}{2}}u\| + \|g''(u)\|_{L^\infty}\|A^{\frac{1}{4}}u\|^2_{L^4} \le C(R), \end{aligned} \tag{2.26}$$

so from (2.26) yields

$$\begin{aligned} &|(g(u), Au)| \le \|A^{\frac{1}{2}}g(u)\|\|A^{\frac{1}{2}}u\| \\ \le&C(R)\|A^{\frac{1}{2}}u\| \le \frac{3C(R)}{\sqrt{\lambda_1}} + \frac{1}{12}\|A^{\frac{3}{4}}u\|^2, \end{aligned} \tag{2.27}$$

$$\begin{aligned} &|(g'(u)u_t, Au)| = |(g''(u)A^{\frac{1}{4}}u\cdot u_t + g'(u)A^{\frac{1}{4}}u_t, A^{\frac{3}{4}}u)| \\ \le&\|g''(u)\|_{L^\infty}\|A^{\frac{1}{4}}u\|_{L^4}\|u_t\|_{L^4}\|A^{\frac{3}{4}}u\| + \|g'(u)\|_{L^\infty}\|A^{\frac{1}{4}}u_t\|\|A^{\frac{3}{4}}u\| \\ \le&\frac{\alpha}{2}\|A^{\frac{1}{4}}u_t\|^2 + C(1 + \|A^{\frac{1}{2}}u\|^{2p} + \|A^{\frac{1}{2}}u\|^{2(p+1)})\|A^{\frac{3}{4}}u\|^2, \end{aligned} \tag{2.28}$$

$$|(f, A^{\frac{1}{2}}u)| \le 3\|f\|^2_{H^{-1}} + \frac{1}{12}\|A^{\frac{3}{4}}u\|^2, \tag{2.29}$$

$$|\varepsilon\alpha(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u)| \le \frac{3\varepsilon^2\alpha^2}{\lambda_1}\|A^{\frac{1}{4}}u_t\|^2 + \frac{1}{12}\|A^{\frac{3}{4}}u\|^2. \tag{2.30}$$

Collecting all above estimates, it follows that

$$\mathcal{E}(t) \ge c_6E_1(t) - C_8, \tag{2.31}$$

where $c_6 = \min\{1 - \frac{6\varepsilon\alpha^2}{\lambda_1}, \frac{1}{2}\}$, and $0 < c_6 < 1$, $c_8 = \frac{3}{\sqrt{\lambda_1}}C(R) + 3\|f\|^2_{H^{-1}}$. Thus (2.25) is transformed into the following equality

$$\frac{d}{dt}\mathcal{E}(t) + \alpha\mathcal{E}(t) + \Upsilon_1 = (g'(u)u_t, Au), \tag{2.32}$$

where

$$\Upsilon_1 = (\|u_t\|^r u_t, A^{\frac{1}{2}}u_t + \alpha A^{\frac{1}{2}}u) - \frac{3\varepsilon\alpha}{2}\|A^{\frac{1}{4}}u_t\|^2 + \frac{\alpha}{2}\|A^{\frac{3}{4}}u\|^2 - \varepsilon\alpha^2(A^{\frac{1}{4}}u_t, A^{\frac{1}{4}}u).$$

In line with Hölder inequality and (2.23) yields

$$|(\|u_t\|^r u_t, \alpha A^{\frac{1}{2}}u)| \le \alpha\|u_t\|^{r+1}\|A^{\frac{1}{2}}u\| \le \alpha R^{r+1}\|A^{\frac{1}{2}}u\| \le \alpha R^{r+2}. \tag{2.33}$$

Therefore, collecting all (2.20), (2.30) and (2.33), we get

$$
\begin{aligned}
\Upsilon_1 &\geq \Big(\|u_t\|^r u_t, A^{\frac{1}{2}} u_t\Big) - \alpha R^{r+2} - \frac{3\varepsilon\alpha}{2}\|A^{\frac{1}{4}} u_t\|^2 + \frac{\alpha}{2}\|A^{\frac{3}{4}} u\|^2 - \frac{3\varepsilon^2\alpha^3}{\lambda_1}\|A^{\frac{1}{4}} u_t\|^2 - \frac{\alpha}{12}\|A^{\frac{3}{4}} u\|^2 \\
&= \|u_t\|^r\|A^{\frac{1}{4}} u_t\|^2 + \frac{5\alpha}{12}\|A^{\frac{3}{4}} u\|^2 - \Big(\frac{3\varepsilon\alpha}{2} + \frac{3\varepsilon^2\alpha^3}{\lambda_1}\Big)\|A^{\frac{1}{4}} u_t\|^2 - \alpha R^{r+2} \\
&\geq -\Big(\frac{R^r}{\varepsilon} + \frac{3\alpha}{2} + \frac{3\varepsilon\alpha^3}{\lambda_1}\Big)\varepsilon\|A^{\frac{1}{4}} u_t\|^2 + \frac{\alpha}{4}\|A^{\frac{3}{4}} u\|^2 - \alpha R^{r+2} \\
&\geq -\alpha R^{r+2} - \varepsilon C_8\|A^{\frac{1}{4}} u_t\|^2,
\end{aligned}
$$

where $C_8 = \frac{R^r}{\varepsilon} + \frac{3\alpha}{2} + \frac{3\varepsilon\alpha^3}{\lambda_1}$. Combining with (2.28) and (2.32) we arrive at

$$
\begin{aligned}
\frac{d}{dt}\mathcal{E}(t) + \alpha\mathcal{E}(t) &\leq \Big(\varepsilon C_8 + \frac{\alpha}{2}\Big)\|A^{\frac{1}{4}} u_t\|^2 + C(1 + R^p + R^{p+1})\|A^{\frac{3}{4}} u\|^2 + \alpha R^{r+2} \\
&\leq C_9 E_1(t) + \alpha R^{r+2},
\end{aligned}
$$

where $C_9 = \min\{2C_8 + \frac{\alpha}{\varepsilon}, 2C(1 + R^p + R^{p+1})\}$. Using (2.31) we achieve

$$
\frac{d}{dt}\mathcal{E}(t) + \frac{\alpha}{2}\mathcal{E}(t) \leq C_{10}\mathcal{E}(t) + C_{11},
$$

where $C_{10} = \frac{C_9}{c_6}$, $C_{11} = \frac{C_8 C_9}{c_6} + \alpha R^{r+2}$. According to Gronwall Lemmas we deduce that

$$
\mathcal{E}(t) \leq C(R)e^{-\frac{\alpha}{4}t}\mathcal{E}(0) + C_{12},
$$

hence, there exists $t_0 = t(\bar{B}) = \frac{4}{\alpha}\ln\frac{C(R)\mathcal{E}(0)}{C_{12}}$, such that $\mathcal{E}(t) \leq 2C_{12}$, $\forall\ t \geq t_0$. Together with (2.31) it follows that

$$
\|(u, u_t)\|^2_{\mathbb{W}} \leq \frac{4C_{12} + 2C_8}{c_6} = \bar{R}^2. \tag{2.34}
$$

Thus, in line with above two estimates, we infer the following results.

Lemma 2.5 The Operator B_1 is local Lipschitz continuous in $\mathbb{E}$.

Proof: By means of (1.2)-(1.3), (2.24) and embedding $W^{2,2}(\Omega) \hookrightarrow C(\overline{\Omega})$ $(n = 3)$, similar to the estimate (2.13) we conclude

$$
\|g'''(v + \theta z)\|_{L^\infty} \leq C(R),\ \|g''(v + \theta z)\|_{L^\infty} \leq C(R),\ \|g'(v + \theta z)\|_{L^\infty} \leq C(R),\ \text{for } 0 \leq \theta \leq 1.
$$

Then we infer from above inequality

$$
\begin{aligned}
&\|g(u) - g(v)\|_{V_2} = \Big(\int_\Omega \big|A^{\frac{1}{2}}\big(g'(v + \theta(u - v))(u - v)\big)\big|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
&\leq \Big(\int_\Omega |g'''(v + \theta z)[A^{\frac{1}{4}}(u + \theta z)]v^2 z|^2 \mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\int_\Omega |g''(v + \theta z)A^{\frac{1}{2}}(u + \theta z)z|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
&\quad + 2\Big(\int_\Omega |g''(v + \theta z)A^{\frac{1}{4}}(v + \theta z)A^{\frac{1}{4}} z|^2 \mathrm{d}x\Big)^{\frac{1}{2}} + \Big(\int_\Omega |g'(v + \theta z)A^{\frac{1}{2}} z|^2 \mathrm{d}x\Big)^{\frac{1}{2}} \\
&\leq \|g'''(v + \theta z)\|_{L^\infty}\|A^{\frac{1}{4}}(v + \theta z)\|^2_{L^4}\|z\|_{L^\infty} + \|g''(v + \theta z)\|_{L^\infty}\|A^{\frac{1}{2}}(v + \theta z)\|\|z\|_{L^\infty} \qquad (2.35) \\
&\quad + 2\|g''(v + \theta z)\|_{L^\infty}\|A^{\frac{1}{4}}(v + \theta z)\|_{L^4}\|A^{\frac{1}{4}} z\|_{L^4} + \|g'(v + \theta z)\|_{L^\infty}\|A^{\frac{1}{2}} z\| \\
&\leq C(R)(\|A^{\frac{1}{4}} u\|^2_{L^4} + \|A^{\frac{1}{4}} v\|^2_{L^4})\|A^{\frac{1}{2}} z\| + C(R)(\|A^{\frac{1}{2}} u\| + \|A^{\frac{1}{2}} v\|)\|A^{\frac{1}{2}} z\| \\
&\quad + 2C(R)(\|A^{\frac{1}{4}} u\|_{L^4} + \|A^{\frac{1}{4}} v\|_{L^4})\|A^{\frac{1}{2}} z\| + C(R)\|A^{\frac{1}{2}} z\| \\
&\leq C(R)\|A^{\frac{1}{2}} z\|,
\end{aligned}
$$

where $z = u - v$. Therefore,

$$\|\varepsilon^{-1}(\Delta g(u) + f(x)) - \varepsilon^{-1}(\Delta g(v) + f(x))\| = \varepsilon^{-1}\|g(u) - g(v)\|_{V_2} \leq C(R, \varepsilon)\|\Delta z\|,$$

that is, the operator $B_1(\varphi) = \begin{pmatrix} 0 \\ \varepsilon^{-1}F(\varphi) \end{pmatrix}$ is local Lipschitz in $\mathbb{E}$.

From above discussions we obtain our first main result in this paper.

Theorem 2.6(Well-posedness) Assume that the conditions (1.2)-(1.4) hold, then for any $T > 0$, we have

(1) For every $(u_0, u_1) \in V_2 \times V_2$, such that $Au_0 + D(u_1) \in H$, there exists a unique strong solution $u(t)$ of (1.1), satisfying

$$\begin{aligned} &(u_t, u_{tt}) \in L^\infty(0, T; V_2 \times H), \ u_t \in C_r([0, T]; V_2), \\ &u_{tt} \in C_r([0, T]; H), \ Au(t) + D(u_t(t)) \in C_r([0, T]; H), \end{aligned} \tag{2.36}$$

where C_r denotes the right continuous function space, and

$$P(t) + \int_0^t \|u_t(\tau)\|^{r+2}\mathrm{d}\tau = P(0) + \frac{1}{2}\int_0^t (g''(u)|A^{\frac{1}{4}}u|^2, u_t)\mathrm{d}\tau. \tag{2.37}$$

(2) For every $(u_0, u_1) \in V_2 \times H$, there exists a unique generalized solution $u(t)$ of (1.1), satisfying

$$(u, u_t) \in C([0, T]; V_2 \times H). \tag{2.38}$$

Especially, this generalized solution is also weak solution.

Proof: On the basis of Lemmas 2.4, 2.5 and applying the same argument as in the proof of Theorem 7.2 of [25], it is easy to know that for any $\varphi_0 = (u_0, u_1) \in \mathfrak{D}(A_1)$, there exists $t_{max} \leq \infty$, such that (2.4) has a unique strong solution $\varphi = (u, u_t)$ on the interval $[0, t_{max})$. In addition, we infer from Lemma 2.2 that $\overline{\mathfrak{D}(A_1)} = \mathbb{E}$ because the set $\mathfrak{D}(A_1) \times H_0^2(\Omega) \subset \mathfrak{D}(A_1)$ and W is dense in H. Meanwhile, if $\varphi_0 \in \overline{\mathfrak{D}(A_1)}$, then there exists a unique generalized solution $\varphi \in C([0, t_{max}); \mathbb{E})$ of (2.4). Besides, under such two cases we have

$$\lim_{t \to t_{max}} \|\varphi(t)\|_{\mathbb{E}} = \infty, \text{ provided } t_{max} < \infty. \tag{2.39}$$

In accordance with (2.7) we know that the generalized solution is hold to a priori estimates on $\mathbb{E}$. Therefore, thanks to above a priori estimates, we obtain the global existence(uniqueness) of strong and generalized solution.

On the other hand, due to $D(u_t) = \|u_t\|^r u_t$ we show that $D : H \to H$ satisfies

$$\|D(u_t)\| = \|u_t\|^{p+1} \leq C_\rho, \quad \forall \ u_t \in H, \ \|u_t\| \leq \rho, \ \rho > 0,$$

then the operator D is bounded on bounded sets, along with Lemma 2.2 we show that a generalized solution is also a weak solution.

As a result, making use of Theorem 2.6, we define a dynamical system $(\mathbb{E}, S(t))$ associated with problem (1.1) in $\mathbb{E} = V_2 \times H$, i.e.,

$$S(t)\varphi(0) = \varphi(t) = (u(t), u_t(t)),$$

and $u(t)$ is a weak solution of (1.1) with initial data $\varphi(0) = (u_0, u_1)$.

3 Some Abstract Results

In order to obtain our next two main results, that is, existence of global attractors and generalized exponential attractors of (1.1), some abstract results are necessary.

Definition 3.1[22] A dynamical system $(X, S(t))$ is said to be asymptotically smooth, if for any positive invariant bounded subset $M \subset X$ (i.e., $\forall\ t > 0,\ \ S(t)M \subset M$), there exists a compact subset $K \subset \bar{M}$, such that $\lim\limits_{t\to+\infty} d_X\{S(t)M|K\} = 0$, where $d_X\{\mathcal{A}|B\} = \sup\limits_{x\in\mathcal{A}} \mathrm{dist}_X(x, B)$ is Hausdorff semi-distance.

Definition 3.2[22] A bounded closed set $\mathcal{A} \subset X$ is said to be a global attractor of $(X, S(t))$, if the following conditions hold,

(1) $\mathcal{A}$ is invariant, namely, $S(t)\mathcal{A} = \mathcal{A}$, $\forall\ t \geq 0$;

(2) $\mathcal{A}$ is uniformly attracting, i.e., for all bounded set $M \subset X$, $\lim\limits_{t\to+\infty} d_X\{S(t)M|\mathcal{A}\} = 0$.

Theorem 3.3[22] Let $(X, S(t))$ be a dynamical system on a complete metric space X endowed with a metric d. Assume that for any bounded positive invariant set $B \subset X$, there exists $T > 0$, and a continuous non-decreasing function $h:\ \mathbb{R}^+ \to \mathbb{R}^+$ as well as speudometric ϱ_B^T on $C(0, T; X)$, such that

(1) $h(0) = 0$, and $h(s) < s,\ \forall\ s > 0$;

(2) Speudometric ϱ_B^T is precompact, w.r.t. X in the following sense: for any sequence $\{x_n\} \subset B$, there exists subsequence $\{x_{nk}\} \subset \{x_n\}$, such that $y_k(\tau) = S(\tau)x_{nk}$ and $\{y_k\} \subset C(0, T; X)$ is Cauchy sequence w.r.t. ϱ_B^T;

(3) For any y_1, $y_2 \in B$, there holds

$$d(S(T)y_1, S(T)y_2) \leq h\Big(d(y_1, y_2) + \varrho_B^T\big(\{S(\tau)y_1\}, \{S(\tau)y_2\}\big)\Big), \tag{3.1}$$

where $y_i(\tau) = S(\tau)y_i$. Then dynamical system $(X, S(t))$ is asymptotically smooth.

Remark 3.4[22] Instead of (3) we can also assume that

$$d(S(T)y_1, S(T)y_2) \leq h\Big(d(y_1, y_2)\Big) + \varrho_B^T\big(\{S(\tau)y_1\}, \{S(\tau)y_2\}\big).$$

Theorem 3.5[22] Assume that dynamical system $(X, S(t))$ is dissipative, then $(X, S(t))$ possesses a compact global attractor $\mathcal{A}$ if and only if $(X, S(t))$ is asymptotically smooth.

Theorem 3.6[23] Let the dynamical system $(X, S(t))$ possesses a compact global attractor $\mathcal{A}$ and it is quasi-stable on $\mathcal{A}$, i.e., there exists a compact seminorm n_X *on* X and non-negative scalar functions $a(t), b(t), c(t)$ *on* $\mathbb{R}^+$ such that

(1) $a(t), c(t)$ are locally bounded on $[0, \infty)$; (2) $b(t) \in L_1(\mathbb{R}^+)$ possesses the property $\lim\limits_{t\to\infty} b(t) = 0$;
(3) for every $y_1, y_2 \in B$ and $t > 0$ we have

$$\|S(t)y_1 - S(t)y_2\|_X^2 \leq a(t)\|y_1 - y_2\|_X^2, \tag{3.2}$$

$$\|S(t)y_1 - S(t)y_2\|_X^2 \leq b(t)\|y_1 - y_2\|_X^2 + c(t) \sup_{s\in[0,t]} [n_X z(s)]^2 \tag{3.3}$$

where $S(t)y_i = y_i(t), i = 1, 2, z(t) = u(t) - v(t)$. Then the attractor $\mathcal{A}$ has a finite fractal dimension.

Theorem 3.7[23] Let the dynamical system $(X, S(t))$ have a positively invariant absorbing set $\mathbb{B}$ and it is quasi-stable. We also assume that there exists a space $\tilde{X} \supseteq X$ such that for any $T > 0$, we have

$$\|S(t_1)y - S(t_2)y\|_{\tilde{X}} \leq C_{\mathbb{B}}|t_1 - t_2|^\gamma, \ t_1, \ t_2 \in [0, T], y \in \mathbb{B}, \tag{3.4}$$

where $C_{\mathbb{B}} > 0, \ \gamma \in (0, 1]$. Then the dynamical system $(X, S(t))$ possesses a gengralized exponential attractor $\mathcal{A}_{\exp} \subset X$ whose dimension is finite in the space $\tilde{X}$.

4 Global Attractors

In this section, first of all, we show the following two dissipative results from a priori estimates in section 2.

Theorem 4.1 Assume that the conditions (1.2)-(1.4) hold, then the dynamical system $(\mathbb{E}, S(t))$ generated by (1.1) is dissipative, namely, there exists a constant $R > 0$, for any bounded set $B \subset \mathbb{E}$, there is a $t_0 = t(B) > 0$ such that $\|S(t)y\|_{\mathbb{E}} = \|\big(u(t), u_t(t)\big)\|_{\mathbb{E}} \leq R$, for any $y \in B$ and $t \geq t_0$. Especially, the set

$$\mathbb{B}_0 = \{(u, u_t) \in \mathbb{E}; \ \|(u, u_t)\|_{\mathbb{E}} \leq R\}$$

is a bounded absorbing set in $\mathbb{E}$ for dynamical system $(\mathbb{E}, S(t))$.

Theorem 4.2 Assume that the conditions (1.2)-(1.4) hold, then there exists $\bar{R} > 0$, for any bounded set $\bar{B} \subset \mathbb{W}$, there is a $t_0 = t(\bar{B}) > 0$, such that $\|S(t)y\|_{\mathbb{W}} = \|\big(u(t), u_t(t)\big)\|_{\mathbb{W}} \leq R$, for any $y \in \bar{B}$ and $t \geq t_0$. Especially,

$$\bar{\mathbb{B}}_0 = \{(u, u_t) \in \mathbb{W}; \ \|(u, u_t)\|_{\mathbb{W}} \leq \bar{R}\}$$

is a bounded absorbing set for dynamical system $(\mathbb{E}, S(t))$ in $\mathbb{W}$.

Now we start to restructure the energy for dynamical system $(\mathbb{E}, S(t))$.

Lemma 4.3 Assume that the conditions (1.2)-(1.4) hold, $(u_0, u_1), \ (v_0, v_1) \in V_2 \times V_2$, then there exists $T_0 > 0$, and $C > 0$ independent of T, such that for any two strong solutions $u, \ v$ of (1.1),

$$\begin{aligned} TE_z(T) + \int_0^T E_z(t)\mathrm{d}t \leq C\Big\{ \int_0^T \|z_t\|^2\mathrm{d}t + \int_0^T |\big(D(t, z_t), z\big)|\mathrm{d}t + \\ \int_0^T \big(D(t, z_t), z_t\big)\mathrm{d}t + \Big|\int_0^T (g(u) - g(v), A^{\frac{1}{2}}z)\mathrm{d}t\Big| + \\ \Big|\int_0^T (g(u) - g(v), A^{\frac{1}{2}}z_t)\mathrm{d}t\Big| + \Big|\int_0^T \mathrm{d}t \int_t^T (g(u) - g(v), A^{\frac{1}{2}}z_t)\mathrm{d}\tau\Big|\Big\}, \end{aligned} \tag{4.1}$$

for $T \geq T_0$, where $z(t) = u(t) - v(t)$, and

$$E_z(t) = \frac{1}{2}(\varepsilon\|z_t\|^2 + \|A^{\frac{1}{2}}z\|^2), \quad D(t, z_t) = \|u_t\|^r u_t - \|v_t\|^r v_t.$$

Proof: Let $z(t) = u(t) - v(t)$, it satisfies

$$\varepsilon A^{-\frac{1}{2}}z_{tt} + A^{\frac{1}{2}}z + A^{-\frac{1}{2}}(\|u_t\|^r u_t) - A^{-\frac{1}{2}}(\|v_t\|^r v_t) + g(u) - g(v) = 0. \tag{4.2}$$

Taking the inner product of (4.2) with $A^{\frac{1}{2}}z_t(t)$ in $L^2(\Omega)$, we get

$$\frac{1}{2}\frac{d}{dt}\big(\varepsilon\|z_t\|^2 + \|A^{\frac{1}{2}}z\|^2\big) + \big(D(t, z_t), z_t\big) = -\big(g(u) - g(v), A^{\frac{1}{2}}z_t\big), \tag{4.3}$$

where $D(t,z_t)=\|u_t\|^r u_t-\|v_t\|^r v_t$. Denote

$$E_z(t)=\frac{1}{2}(\varepsilon\|z_t\|^2+\|A^{\frac{1}{2}}z\|^2),$$

and integrating (4.3) over $[t,T]$, it follows that

$$E_z(T)+\int_t^T \big(D(t,z_t),z_t\big)\mathrm{d}\tau=E_z(t)-\int_t^T \big(g(u)-g(v),A^{\frac{1}{2}}z_t\big)\mathrm{d}\tau. \tag{4.4}$$

Taking the inner product of (4.2) with $A^{\frac{1}{2}}z(t)$ in $L^2(\Omega)$, we arrive at

$$\varepsilon\frac{d}{dt}(z_t,z)-\varepsilon\|z_t\|^2+\|A^{\frac{1}{2}}z\|^2+\big(D(t,z_t),z\big)=-\big(g(u)-g(v),A^{\frac{1}{2}}z\big).$$

Integrating above inequality over $[0,T]$, it yields

$$\begin{aligned}2\int_0^T E_z(t)\mathrm{d}t-2\varepsilon\int_0^T\|z_t\|^2\mathrm{d}t+\int_0^T\big(D(t,z_t),z\big)\mathrm{d}t+\varepsilon(z_t,z)\big|_0^T\\ =-\int_0^T\big(g(u)-g(v),A^{\frac{1}{2}}z\big)\mathrm{d}t.\end{aligned}$$

Exploiting (2.1) we deduce

$$|(z_t,z)|\le\frac{1}{2}(\|z_t\|^2+\frac{1}{\lambda_1}\|A^{\frac{1}{2}}z\|^2)\le\frac{C}{2}E_z(t).$$

Therefore,

$$\begin{aligned}2\int_0^T E_z(t)\mathrm{d}t\le C\big(E_z(0)-E_z(T)\big)+2\varepsilon\int_0^T\|z_t\|^2\mathrm{d}t-\\ \int_0^T\big(D(t,z_t),z\big)\mathrm{d}t-\int_0^T\big(g(u)-g(v),A^{\frac{1}{2}}z\big)\mathrm{d}t.\end{aligned} \tag{4.5}$$

For (4.4), let $t=0$, we have

$$E_z(0)=E_z(T)+\int_0^T\big(D(t,z_t),z_t\big)\mathrm{d}t+\int_0^T\big(g(u)-g(v),A^{\frac{1}{2}}z_t\big)\mathrm{d}t. \tag{4.6}$$

In line with the monotone property of D, integrating (4.4) over $[0,T]$, it leads to

$$TE_z(T)\le\int_0^T E_z(t)\mathrm{d}t-\int_0^T\mathrm{d}t\int_t^T\big(g(u)-g(v),A^{\frac{1}{2}}z_t\big)\mathrm{d}\tau. \tag{4.7}$$

Thus we can infer (4.1) from (4.5)-(4.7).

In order to obtain the asymptotical smoothness of dynamical system$(\mathbb{E},S(t))$, we need to verify the three conditions of Theorem 3.3.

Proposition 4.4(Energy reconstruction) Assume that the conditions (1.2)-(1.4) hold, $z(t)=u(t)-v(t),\ 0<\tilde{\beta}<\frac{1}{4}$, then there exists $T_0>0,\ C>0$, such that for any solution u,v of (1.1),

$$E_z(T)\le C_{\mathbb{B},T}(K+I)\Big(\int_0^T\big(D(t,z_t),z_t\big)\mathrm{d}t\Big)+C_{\mathbb{B},T}\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\|+C_{\mathbb{B},T}\sup_{t\in[0,T]}\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\| \tag{4.8}$$

as $T\ge T_0$.

Proof: Let $\mathbb{B}=\overline{\bigcup_{t\ge t_0}S(t)\mathbb{B}_0}$. It is easy to know that from Theorem 3.1 $\mathbb{B}$ is a bounded closed forward invariant set. Then for any bounded set B, there is a $t(B)\ge0$, such that $S(t)B\subset\mathbb{B}_0$ for any $t\ge t(B)$. Besides, $\mathbb{B}_0$ is also a bounded absorbing set, so there exists $t_0\ge0$, such that $S(t)\mathbb{B}_0\subset\mathbb{B}_0$ for any $t\ge t_0$.

Therefore, when $t \geq t_0 + t(B)$, we show that $S(t)B \subset \mathbb{B}$. From above discussions it follows that $\mathbb{B}$ is a absorbing set. Assume that $u(t), v(t)$ are two weak solutions, and

$$(u(t), u_t(t)) \equiv S(t)y_0,\ (v(t), v_t(t)) \equiv S(t)y_1,\ \forall\ y_0, y_1 \in \mathbb{B}. \tag{4.9}$$

Let $T > 0$, by virtue of a priori estimates in $\mathbb{E}$, along with (2.24), we conclude

$$\int_0^T (D(u_t), u_t)\mathrm{d}t + \int_0^T (D(v_t), v_t)\mathrm{d}t \leq C_{\mathbb{B}}. \tag{4.10}$$

Rewrite (4.1) as follows

$$\begin{aligned} TE_z(T) + \int_0^T E_z(t)\mathrm{d}t \leq & C\Big\{\int_0^T \|z_t\|^2\mathrm{d}t + \int_0^T |(D(t, z_t), z)|\mathrm{d}t \\ & + \int_0^T (D(t, z_t), z_t)\mathrm{d}t + \Gamma_T(u, v)\Big\}, \end{aligned} \tag{4.11}$$

where

$$\begin{aligned} \Gamma_T(u, v) = & \Big|\int_0^T (g(u) - g(v), A^{\frac{1}{2}}z)\mathrm{d}t\Big| + \Big|\int_0^T (g(u) - g(v), A^{\frac{1}{2}}z_t)\mathrm{d}t\Big| \\ & + \Big|\int_0^T \mathrm{d}t\int_t^T (g(u) - g(v), A^{\frac{1}{2}}z_t)\mathrm{d}\tau\Big| \\ \leq & C_T\Big\{\int_0^T \Big|(g(u) - g(v), A^{\frac{1}{2}}z)\Big|\mathrm{d}t + \int_0^T \Big|(g(u) - g(v), A^{\frac{1}{2}}z_t)\Big|\mathrm{d}t\Big\}. \end{aligned} \tag{4.12}$$

Taking advantage with (2.1), (2.35) and Hölder inequality, there exists a constant $0 < \alpha < \frac{1}{4}$, it leads to

$$\begin{aligned} \big|\big(g(u) - g(v), A^{\frac{1}{2}}z\big)\big| = & \big|\big(A^{\frac{1}{2}}(g(u) - g(v)), z\big)\big| \leq \|g(u) - g(v)\|_{V_2}\|z\| \\ \leq & C(R)\|A^{\frac{1}{2}}z\|\|z\| \leq C_{\mathbb{B},T}\|A^{\frac{1}{2}}z\| \leq C_{\mathbb{B},T}\|A^{\frac{3}{4}-\alpha}z\|, \end{aligned} \tag{4.13}$$

Analogously,

$$\big|\big(g(u) - g(v), A^{\frac{1}{2}}z_t\big)\big| \quad \leq \|g(u) - g(v)\|_{V_2}\|z_t\| \leq C(R)\|A^{\frac{1}{2}}z\|\|z_t\| \leq C_{\mathbb{B},T}\|A^{\frac{3}{4}-\alpha}z\|. \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.12) yields

$$\Gamma_T(u, v) \leq 2C_{\mathbb{B},T}\int_0^T \|A^{\frac{3}{4}-\alpha}z\|\mathrm{d}t. \tag{4.15}$$

According to Lemma 2.1, let $K_0(s) = C_p^{-\frac{2}{p+2}}s^{\frac{2}{p+2}}(\ p \geq 0)$, it is strictly increasing concave function, and $K_0 \in C(\mathbb{R}^+)$, $K_0(0) = 0$, hence

$$K_0\big((\|u + v\|^p(u + v) - \|u\|^p u), v\big) \geq K_0(C_p\|v\|^{p+2} = \|v\|^2,\ \forall\ u,\ v \in V_2,$$

utilizing Jensen's inequality, we achieve

$$\begin{aligned} \int_0^T \|z_t\|^2\mathrm{d}t \leq & \int_0^T K_0\big(D(t, z_t), z_t\big)\mathrm{d}t \\ & \leq TK_0\Big(\frac{1}{T}\int_0^T (D(t, z_t), z_t)\mathrm{d}t\Big) \\ & = K\Big(\int_0^T (D(t, z_t), z_t)\mathrm{d}t\Big), \end{aligned} \tag{4.16}$$

where $K(s) = TK_0(\frac{s}{T})$. In addition, there exists $0 < \beta < \frac{1}{2}$, such that

$$
\begin{aligned}
|(D(t,z_t),z)| &\le \|z\|(\int_\Omega (\|u_t\|^r u_t - \|v_t\|^r v_t)^2 dx)^{\frac{1}{2}} \\
&\le C\|z\|(\|u_t\|^{2r}\|u_t\|^2 + \|v_t\|^{2r}\|v_t\|^2)^{\frac{1}{2}} \\
&\le C_{\mathbb{B}}\|z\| \le C_{\mathbb{B}}\|A^{\frac{1}{2}-\beta}z\|,
\end{aligned}
\tag{4.17}
$$

thus, we infer from (4.11)-(4.17)

$$
TE_z(T) + \frac{1}{2}\int_0^T E_z(t)\mathrm{d}t \le C_{\mathbb{B}}\Big\{(K+I)\Big(\int_0^T (D(t,z_t),z_t)\mathrm{d}t\Big) + \int_0^T \|A^{\frac{1}{2}-\beta}z\|\mathrm{d}t + C_{\mathbb{B},T}\int_0^T \|A^{\frac{3}{4}-\alpha}z\|\mathrm{d}t\Big\}.
$$

Therefore, let $\tilde{\beta} = \min\{\alpha,\beta\}$, we obtain

$$
\begin{aligned}
E_z(T) &\le C_{\mathbb{B},T}(K+I)\Big(\int_0^T \big(D(t,z_t),z_t\big)\mathrm{d}t\Big) + C_{\mathbb{B},T}\int_0^T \|A^{\frac{1}{2}-\tilde{\beta}}z\|\mathrm{d}t + C_{\mathbb{B},T}\int_0^T \|A^{\frac{3}{4}-\tilde{\beta}}z\|\mathrm{d}t \\
&\le C_{\mathbb{B},T}(K+I)\Big(\int_0^T \big(D(t,z_t),z_t\big)\mathrm{d}t\Big) + C_{\mathbb{B},T}\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\| + C_{\mathbb{B},T}\sup_{t\in[0,T]}\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\|.
\end{aligned}
$$

Proposition 4.5 Assume that the conditions (1.2)-(1.4) hold, then the dynamical system $(\mathbb{E}, S(t))$ generated by (1.1)is asymptotically smooth in $\mathbb{E}$.

Proof: In line with of Proposition 4.4, we need only to deal with the damping term of (4.8). For our aim, let $M_0(s) = (K+I)^{-1}\big(\frac{s}{2C_{\mathbb{B},T}}\big)$, then $M_0(s)$ is a strictly concave function, so

$$
(K+I)^{-1}(s) \le s, \ \forall s \ge 0,
$$

due to (4.8) we obtain

$$
\begin{aligned}
&M_0(E_z(T)) = (K+I)^{-1}\Big(\frac{E_z(T)}{2C_{\mathbb{B},T}}\Big) \\
\le&(K+I)^{-1}\Big\{\frac{1}{2}(K+I)\Big(\int_0^T \big(D(t,z_t),z_t\big)\mathrm{d}t\Big) + \frac{1}{2}\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\| + \frac{1}{2}\sup_{t\in[0,T]}\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\|\Big\} \\
\le&\frac{1}{2}\int_0^T \big(D(t,z_t),z_t\big)\mathrm{d}t + \frac{1}{2}(K+I)^{-1}\big(\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\| + \frac{1}{2}\sup_{t\in[0,T]}\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\|\big) \\
\le&\frac{1}{2}\int_0^T \big(D(t,z_t),z_t\big)\mathrm{d}t + \frac{1}{2}\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\| + \frac{1}{2}\sup_{t\in[0,T]}\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\|.
\end{aligned}
\tag{4.18}
$$

By virtue of Hölder inequality, combining with (1.3), (2.24) as well as Sobolev imbedding inequality $W^{2,2}(\Omega) \hookrightarrow C(\Omega)$ $(n=3)$, we deduce that

$$
\begin{aligned}
&\big|\big(g(u)-g(v), A^{\frac{1}{2}}z_t\big)\big| = \big|\big(A^{\frac{1}{4}}(g(u)-g(v)), A^{\frac{1}{4}}z_t\big)\big| \\
\le&\Big|\Big(\int_0^1 g''(\theta u + (1-\theta)v)(\theta A^{\frac{1}{4}}u + (1-\theta)A^{\frac{1}{4}}v)z\mathrm{d}\theta + \int_0^1 g'(\theta u + (1-\theta)v)A^{\frac{1}{4}}z\mathrm{d}\theta, A^{\frac{1}{4}}z_t\Big)\Big| \\
\le&C\int_\Omega (1+|u|^p+|v|^p)(|A^{\frac{1}{4}}u| + |A^{\frac{1}{4}}v|)|z||A^{\frac{1}{4}}z_t|\mathrm{d}x + C\int_\Omega (1+|u|^{p+1}+|v|^{p+1})|A^{\frac{1}{4}}z||A^{\frac{1}{4}}z_t|\mathrm{d}x \\
\le&C(R)(\|A^{\frac{1}{4}}u\|_{L^4} + \|A^{\frac{1}{4}}v\|_{L^4})\|z\|_{L^4}\|A^{\frac{1}{4}}z_t\| + C(R)\|A^{\frac{1}{4}}z\|\|A^{\frac{1}{4}}z_t\| \\
\le&C(R)(\|A^{\frac{1}{2}}u\| + \|A^{\frac{1}{2}}v\|)\|A^{\frac{1}{4}}z\|\|A^{\frac{1}{4}}z_t\| + C(R)\|A^{\frac{1}{4}}z\|\|A^{\frac{1}{4}}z_t\| \\
\le&C(R)\|A^{\frac{1}{4}}z\|\|A^{\frac{1}{4}}z_t\|,
\end{aligned}
\tag{4.19}
$$

where $g(u)-g(v)=\int_0^1 g'(\theta u+(1-\theta)v)z\mathrm{d}\theta$. Taking $t=0$ in (4.4), together with (2.34), (4.19) and compact imbedding theorem, we achieve

$$\begin{aligned}\int_0^T (D(t,z_t),z_t)\mathrm{d}t &= E_z(0)-E_z(T)-\int_0^T \big(g(u)-g(v),A^{\frac{1}{2}}z_t\big)\mathrm{d}t\\ &\le E_z(0)-E_z(T)+C(R)\int_0^T \|A^{\frac{1}{4}}z\|\|A^{\frac{1}{4}}z_t\|\mathrm{d}t\\ &\le E_z(0)-E_z(T)+C_{\mathbb{B},T}\|A^{\frac{1}{4}}z\|\\ &\le E_z(0)-E_z(T)+C_{\mathbb{B},T}\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\|. \end{aligned}\tag{4.20}$$

Therefore,

$$E_z(T)+2M_0(E_z(T))\le E_z(0)+C_{\mathbb{B},T}\big(\sup_{t\in[0,T]}\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\|+\sup_{t\in[0,T]}\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\|\big).$$

Since $z(t)\in V_3$ is uniformly bounded, and $\mathcal{D}(A^{\frac{1}{2}})\hookrightarrow\hookrightarrow\mathcal{D}(A^{\frac{1}{2}-\tilde{\beta}})\hookrightarrow\hookrightarrow H$ and $\mathcal{D}(A^{\frac{3}{4}})\hookrightarrow\hookrightarrow\mathcal{D}(A^{\frac{3}{4}-\tilde{\beta}})\hookrightarrow\hookrightarrow H$, with the aid of interpolation theorem, it follows that

$$\|A^{\frac{1}{2}-\tilde{\beta}}z(t)\mid\le\|A^{\frac{1}{2}}z\|^{\theta_1}\|z\|^{1-\theta_1}\le C_R\|z(t)\|^{1-\theta_1},\ \theta_1\in(0,1),$$

and

$$\|A^{\frac{3}{4}-\tilde{\beta}}z(t)\mid\le\|A^{\frac{3}{4}}z\|^{\theta_1}\|z\|^{1-\theta_1}\le C_R\|z(t)\|^{1-\theta_1},\ \theta_1\in(0,1).$$

Hence

$$E_z(T)+2M_0(E_z(T))\le E_z(0)+C_{\mathbb{B},T}\sup_{t\in[0,T]}\|z(t)\|^{\theta_2},\ \theta_2=1-\theta_1\in(0,1].$$

Thus, integer with

$$E_z(T)=\frac{1}{2}\big(\varepsilon\|z_t\|^2+\|A^{\frac{1}{2}}z\|^2\big)=\frac{1}{2}\|S(T)y_1-S(T)y_2\|_{\mathbb{E}}^2,$$

we have

$$\begin{aligned}&\|S(T)y_1-S(T)y_2\|_{\mathbb{E}}=(2E_z(T))^{\frac{1}{2}}\\ \le&\sqrt{2}\Big[(I+2M_0)^{-1}\Big\{\frac{1}{2}\|y_1-y_2\|_E^2+C_{\mathbb{B},T}\sup_{t\in[0,T]}\|z(t)\|^{\theta_2}\Big\}\Big]^{\frac{1}{2}}\\ \le&\sqrt{2}\Big[(I+2M_0)^{-1}\Big\{\frac{1}{2}\Big(\|y_1-y_2\|_E+(C_{\mathbb{B},T}\sup_{t\in[0,T]}\|z(t)\|^{\theta_2})^{\frac{1}{2}}\Big)^2\Big\}\Big]^{\frac{1}{2}}\\ \le&\sqrt{2}\Big[(I+2M_0)^{-1}\Big\{\frac{1}{2}\Big(\|y_1-y_2\|_E+C_{\mathbb{B},T}\sup_{t\in[0,T]}\|z(t)\|^{\theta_3}\Big)^2\Big\}\Big]^{\frac{1}{2}},\end{aligned}\tag{4.21}$$

that is

$$\|S(T)y_1-S(T)y_2\|_{\mathbb{E}}\le h\Big(\|y_1-y_2\|_{\mathbb{E}}+\varrho_{\mathbb{B}}^T\big(\{S(\tau)y_1\},\{S(\tau)y_2\}\big)\Big),\tag{4.22}$$

where

$$h(s)=\sqrt{2}\Big((I+2M_0)^{-1}(\frac{s^2}{2})\Big)^{\frac{1}{2}},$$

$$\varrho_{\mathbb{B}}^T\big(\{S(\tau)y_1\},\{S(\tau)y_2\}\big)=C_{\mathbb{B},T}\sup_{t\in[0,T]}\|u(t)-v(t)\|^{\theta_3},\ \theta_3\in(0,\frac{1}{2}].$$

It is clear that the function h satisfies the conditions of Theorem 3.3. Besides, using the similar technique of [21], we conclude that pseudomeasure $\sup_{t\in[0,T]}\|u(t)-v(t)\|^{\theta_3}$ is precompact on the set $\mathcal{L}_{\mathbb{B},T}$, which all solutions of (1.1) on [0,T] with initial data in $\mathbb{B}$. Thus in line with Theorem 3.3, we show that the dynamical system $(\mathbb{E}, S(t))$ is asymptotically smooth.

Now taking advantage with Theorem 4.1 and Proposition 4.5, from Theorem 3.4 we claim that the following result is hold.

Theorem 4.6 The dynamical system $(\mathbb{E}, S(t))$ associated with (1.1) possesses a compact global attractor $\mathcal{A}$.

5 Fractal Dimension and Generalized Exponential Attractor

From abstract Theorem 3.5 we know that the following Lemma need only to be proved.

Lemma 5.1 The dynamical system $(\mathbb{E}, S(t))$ is quasi-stable in a bounded positive invariant set $\mathbb{B} \subset \mathbb{E}$.

Proof: In line with the definition of quasi-stability in Theorem 3.5, we only need to verify (3.2) and (3.3). For that purpose, first of all, taking the inner product of (4.2) with $A^{\frac{1}{2}}Z_t$, we achieve

$$\frac{1}{2}\frac{d}{dt}\left(\varepsilon\|z_t\|^2+\left\|A^{\frac{1}{2}}z\right\|^2\right)+\left(\|u_t\|^r u_t-\|v_t\|^r v_t, z_t\right)=-\left(g(u)-g(v), A^{\frac{1}{2}}z_t\right). \tag{5.1}$$

Lemma 2.2 implies that

$$\left(\|u_t\|^r u_t-\|v_t\|^r v_t,\ z_t\right) \geqslant C_r\|z_t\|^{r+2} \geqslant 0, \tag{5.2}$$

and due to (4.19), it leads to

$$\begin{aligned}
&\left|-\left(g(u)-g(v), A^{\frac{1}{2}}z_t\right)\right| \leq \|g(u)-g(v)\|_{V_2}\cdot\|z_t\| \\
&\leq C(R)\|z_t\|\cdot\left\|A^{\frac{1}{2}}z\right\| \\
&\leq C(R,\varepsilon)\left(\varepsilon\|z_t\|^2+\left\|A^{\frac{1}{2}}z\right\|^2\right).
\end{aligned} \tag{5.3}$$

Therefore, from (5.1) – (5.3) we deduce

$$\frac{d}{dt}\left(\varepsilon\|z_t\|^2+\left\|A^{\frac{1}{2}}z\right\|^2\right) \leq c(R,\varepsilon)\left(\varepsilon\|z_t\|^2+\left\|A^{\frac{1}{2}}z\right\|^2\right). \tag{5.4}$$

By virtue of Gronwall Lemma, it follows that

$$\varepsilon\|z_t(t)\|^2+\left\|A^{\frac{1}{2}}z(t)\right\|^2 \leqslant e^{c(R,\varepsilon)t}\left(\varepsilon\|z_t(0)\|^2+\left\|A^{\frac{1}{2}}z(0)\right\|^2\right),$$

i.e. ,

$$\|s(t)y_1-s(t)y_2\|_{\mathbb{E}}^2 \leq a(t)\|y_1-y_2\|_{\mathbb{E}}^2, \tag{5.5}$$

where $a(t)=e^{c(R,\varepsilon)t}$ is local bounded on $[0,\infty]$.

Next we prove (3.3) is true. For the sake of this aim, we presume that

$$\bar{V}=closure\{\nu\in V_2 : \|\nu\|_{\bar{V}} \equiv \mu_{V_2}(\nu)+\|\nu\|_H<\infty\}.$$

Then $V_2 \hookrightarrow\hookrightarrow \bar{V}$. Hence from Theorem 1.1.8 in [23] we claim $W^1_{\infty,2}(0,T;V_2,H) \hookrightarrow\hookrightarrow C(0,T;\bar{V})$, where $W^1_{\infty,2}(0,T;V_2,H)=\{u\in L^\infty(0,T;V_2):\ u_t\in L^2(0,T;H)\}$. So the speudometric $\varrho^T_{\mathbb{B}} \in C(0,T;\mathbb{E})$, and

$$\varrho^T_{\mathbb{B}}\big(\{S(\tau)y_1\},\{S(\tau)y_2\}\big)=c(t)\sup_{s\in[0,t]}\mu_{V_2}\big(u(s)-v(s)\big),$$

and the semi-norm is defined as follows

$$\mu_{V_2}(u(t)-v(t)) = \|u(t)-v(t)\|^{\theta_3},\ \theta_3 \in (0, \frac{1}{2}].$$

Thanks to $H^2(\Omega)\bigcap H_0^1(\Omega) \hookrightarrow\hookrightarrow L^2(\Omega)$, it is clear that $\mu_{V_2}(\cdot)$ is compact semi-norm of V_2. Now, choosing $K_{\mathbb{B}}(t)s = (I+2M_0(t))^{-\frac{1}{2}}s$ in (4.21) and combining with (4.22), we get

$$\|S(t)y_1 - S(t)y_2\|_{\mathbb{E}}^2 \leq b(t)\|y_1-y_2\|_{\mathbb{E}}^2 + c(t)\sup_{s\in[0,t]}[\mu_{V_2}(u(s)-v(s))]^2,$$

where $b(t) = (K_{\mathbb{B}}(t))^2 = (I+2M_0(t))^{-1},\ c(t) = C_{\mathbb{B},T}$. Apparently $b(t) \in L^1(\mathbb{R}^+), and\ \lim\limits_{t\to\infty} b(t) = 0$, $c(t)$ is local bounded in $[0,\infty)$. Therefore, Theorem 2.14 implies dynamical system $(\mathbb{E}, S(t))$ is quasi-stable.

Thus we conclude the following result at once.

Theorem 5.2 The fractal dimension of compact global attractor $\mathcal{A}$ of (1.1) is finite.

Theorem 5.3 Assume that the conditions (1.2)-(1.4) hold, then the dynamical system $(\mathbb{E}, S(t))$ possesses a generalized exponential attractor $\mathcal{A}_{exp} \subset \mathbb{E}$, and it has finite fractal dimension in $\tilde{\mathbb{E}} = L^2(\Omega)\times H^{-2}(\Omega) \supseteq \mathbb{E}$.

Proof: Based on Lemma 5.1, the dynamical system $(\mathbb{E}, S(t))$ is quasi-stable in a bounded positive invariant subset $\mathbb{B}\subset\mathbb{E}$, the reminder is only to verify the mapping $t\mapsto S(t)y$ is Hölder continuous in $\tilde{\mathbb{E}}$. In fact, if for any $y = \varphi(0) = (u_0, u_1) \in \mathbb{B}$, there have $S(t)y = (u(t), u_t(t)) = \varphi(t)$. Utilizing (2.36), there is $R > 0$, such that $\|u_t\|_{V_2}^2 + \|u_{tt}\|^2 \leq R^2$, hence, $\|\varphi_t(t)\|_{\tilde{\mathbb{E}}}^2 = \|u_t\|^2 + \|u_{tt}\|_{V_{-2}}^2 \leq \|u_t\|_{V_2}^2 + \|u_{tt}\|^2 \leq C_{\mathbb{B}}$. As a result, for any $0\leq t_1\leq t_2\leq T$, there holds

$$\|S(t_1)y - S(t_2)y\|_{\tilde{\mathbb{E}}} \leq \int_{t_1}^{t_2}\|\varphi_t(s)\|_{\tilde{E}}ds \leq C_{\mathbb{B}}|t_1-t_2|.$$

Thus, in accordance with Theorem 3.6 with $r = 1$, we show that the dynamical system $(\mathbb{E}, S(t))$ possesses a generalized exponential attractor $\mathcal{A}_{exp}$ in $\mathbb{E}$, and its fractal dimension is finite in $\tilde{\mathbb{E}}$.

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