A new reproducing kernel method for solving the fractional differential equations

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Abstract

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Abstract. In this paper, we investigate an efficient technique for solving fractional differential equations (FDEs). The proposed technique is based upon Legendre polynomials to construct reproducing kernel spaces, the ε-approximate method is presented in space, and stability and convergence analysis are given by analyzing the condition number of the matrix of the linear system. Finally, comparison with the existing algorithm by the numerical experiments illustrates that efficiency and stability of the proposed method.

Keywords: Caputo fractional derivative; ε-approximate solution; least-squares; Fractional differential equations; Reproducing kernel space.

1 Introduction

In recent years, there have been more and more attentions on the research of fractional differential equations, the main reason is that fractional calculus operators can describe many problems in the engineering field more accurately than integer order calculus operators [1-3]. Fractional Differential Equations (FDEs)have attracted great attention from various scientific fields, such as physics, dynamics of earthquakes, signal processing, fluid mechanics [4], chaotic dynamics, biology [5], electromagnetic waves, polymer science, thermodynamics and so on [6-7]. FDEs are always with weakly singular kernel and more complicated than integer ones. Actually, in many cases, it is difficult to obtain the analytical solution, so many experts are dedicated to study the approximate solution of the equation and emerging a lot of methods [8]. Such as, finite difference method [9], the local meshless method based on Laplace transform [10], Laplace transform method [11], variational iteration method [12], spectral methods [13], shooting method [14], etc. To the best of our knowledge, the reproducing kernel space is an ideal space framework for the study of numerical analysis. In previous work, the Taylors formula or Delta function was used to construct the reproducing kernel space [15-17], which has been proved to be an effective tool to solving various kinds of differential equations [18-19].

In this paper, we are concerned with the approximation of FDEs as follows:

$$D^{\alpha}u(x)+a_1(x)u'(x)+a_0(x)u(x)=f(x), \quad x\in(0,1), \quad \alpha>0, \tag{1.1}$$

$$u(0)-\alpha_0 u'(0)=\gamma_0, \tag{1.2}$$

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$$u(1) + \alpha_1 u'(1) = \gamma_1. \tag{1.3}$$

Here,

$$D^\alpha u(x) = \frac{1}{\Gamma(k-\alpha)} \int_0^x (x-t)^{k-\alpha-1} u^{(k)}(t) dt, \tag{1.4}$$

where $k := [\alpha]$, $a_i(x)(i = 0, 1), f(x) \in L^2[0, 1]$. The existence and uniqueness of the solution $u(x)$ of problem (1.1) are established in [20].

The remaining part of this paper is organized as follows: Constructing Basis for reproducing kernel space based on Legendre polynomials in section 2. In section 3, we give an efficient technique based on ε-approximate method, theoretical analysis of the approximate solution for homogeneous equation and its unique solvability, stability and convergence analysis are given. In section 4, we do some numerical experiments.

2 Basis of reproducing kernel space based on Legendre polynomials

The well-known Legendre polynomials is defined on the interval [-1, 1] and its recurrence formula:

$$\begin{cases} L_0(z) = 1, \\ L_n(z) = \dfrac{1}{2^n n!} \dfrac{d^n}{dz^n} (z^2 - 1)^n, n = 1, 2, \cdots \end{cases} z \in [-1, 1].$$

Let $z = 2x - 1$, we can get the following formulation in the interval [0, 1]:

$$\begin{cases} L_0(x) = 1, \\ L_n(x) = \dfrac{1}{n!} \dfrac{d^n}{dx^n} (x^2 - x)^n, n = 1, 2, \cdots \end{cases} x \in [0, 1].$$

The Legendre polynomials has following properties:

$$\int_0^1 \sqrt{2n+1} L_n(x) \sqrt{2m+1} L_m(x) dx = \delta_{mn}.$$

The first seven terms of the polynomials of $(2n+1)^{\frac{1}{2}} L_n(x)$ are listed in the table.

Table: The first seven polynomials of $(2n+1)^{\frac{1}{2}} L_n(x)$

n	$(2n+1)^{\frac{1}{2}} L_n(x)$
0	1
1	$3^{\frac{1}{2}}(-1 + 2x)$
2	$5^{\frac{1}{2}}(1 - 6x + 6x^2)$
3	$7^{\frac{1}{2}}(-1 + 12x - 30x^2 + 20x^3)$
4	$3(1 - 20x + 90x^2 - 140x^3 + 70x^4)$
5	$11^{\frac{1}{2}}(-1 + 30x - 210x^2 + 560x^3 - 630x^4 + 252x^5)$
6	$13^{\frac{1}{2}}(1 - 42x + 420x^2 - 1680x^3 + 3150x^4 - 2772x^5 + 924x^6)$

Theorem 2.1. *Let*

$$\overline{W}_m = Span\{L_0(x), \sqrt{3}L_1(x), \sqrt{5}L_2(x), \cdots, \sqrt{2m+1}L_m(x)\},$$

the inner product $\overline{W}_m$ is given:

$$\langle u(x), v(x) \rangle = \int_0^1 u(x)v(x)dx, \quad u(x), v(x) \in \overline{W}_m.$$

Then

$$R(x,y) = R_y(x) = \sum_{i=0}^{m}(2i+1)L_i(x)L_i(y)$$

is reproducing kernel of $\overline{W}_m$.

Proof. Using [21], we can prove that $\overline{W}_m$ is a reproducing kernel Hilbert space. Next we proof $R_x(y)$ is a reproducing kernel of $\overline{W}_m$ for $\forall u(y) \in \overline{W}_m$.

Let

$$u(x) = \sum_{i=0}^{m} a_i\sqrt{2i+1}L_i(x),$$

we have

$$\begin{aligned}\langle u(x), R_y(x)\rangle &= \langle\sum_{i=0}^{m} a_i\sqrt{2i+1}L_i(x), \sum_{j=0}^{m}(2j+1)L_j(x)L_j(y)\rangle \\ &= \sum_{i=0}^{m} a_i\sqrt{2i+1}\langle L_i(x), \sum_{j=0}^{m}(2j+1)L_j(x)L_j(y)\rangle \\ &= \sum_{i=0}^{m} a_i\sqrt{2i+1}L_i(y) \\ &= u(y),\end{aligned}$$

so $R_x(y)$ is the reproducing kernel of $\overline{W}_m$. □

Using [22] and the reproducing kernel of $\overline{W}_m$, we can get the new reproducing kernel spaces W_m and reproducing kernel $R_m(x,y)$. Next, we list some reproducing kernel spaces:

• Space $W_2 = \{u(x) \mid u(x) \in \overline{W}_2, u(0) = 0\}$, W_2 has the same inner product as $\overline{W}_2$, and it is a reproducing kernel space. Its reproducing kernel:

$$R_2(x,y) = R(x,y) - \frac{R(x,0)R(0,y)}{R(0,0)} = 4xy\big(12 - 15y + 5x(-3+4y)\big).$$

• Space $W_3 = \{u(x) \mid u(x) \in \overline{W}_3, u(0) = 0, u(1) = 0\}$, W_3 has the same inner product as $\overline{W}_3$, and it is a reproducing kernel space. Its reproducing kernel:

$$R_3(x,y) = \overline{R}_3(x,y) - \frac{\overline{R}_3(x,1)\overline{R}_3(1,y)}{\overline{R}_3(1,1)} = 60(-1+x)x(-1+y)y\big(4 - 7y + 7x(-1+2y)\big),$$

where $\overline{R}_3(x,y) = R(x,y) - \frac{R(x,0)R(0,y)}{R(0,0)}$.

• Space $W_4 = \{u(x) \mid u(x) \in \overline{W}_4, u(0) = u'(0) = u''(0) = 0\}$, W_3 has the same inner product as $\overline{W}_4$, and it is a reproducing kernel space. Its reproducing kernel:

$$R_4(x,y) = \overline{\overline{R}}_4(x,y) - \frac{\frac{\partial^2\overline{\overline{R}}_4(x,0)}{\partial y^2}\frac{\partial^2\overline{\overline{R}}_4(0,y)}{\partial x^2}}{\frac{\partial^4\overline{\overline{R}}_4(0,0)}{\partial x^2\partial y^2}} = 8x^3y^3\big(56 - 63y + 9x(-7+8y)\big),$$

where $\overline{\overline{R}}_4(x,y) = \overline{R}_4(x,y) - \frac{\frac{\partial\overline{R}_4(x,0)}{\partial y}\frac{\partial\overline{R}_4(0,y)}{\partial x}}{\frac{\partial^2\overline{R}_4(0,0)}{\partial x\partial y}}, \overline{R}_4(x,y) = R(x,y) - \frac{R(x,0)R(0,y)}{R(0,0)}$.

Let

$$v(x) = u(x) + (-\alpha_0 u(x) - \gamma_0)(1-x) + (\alpha_1 u'(x) - \gamma_1),$$

then after homogenization, eqs.(1.1)-(1.3) are converted to the following from:

$$\begin{cases} D^{\alpha}v(x) + a_1(x)v'(x) + a_0(x)v(x) = g(x), \quad x \in (0,1), \\ v(0) = v(1) = 0, \end{cases} \tag{2.1}$$

where $g(x) = f(x) + D^{\alpha}\big((-\alpha_0 u(x) - \gamma_0)(1-x) + (\alpha_1 u'(x) - \gamma_1)\big) + a_1(x)\big((-\alpha_0 u(x) - \gamma_0)(1-x) + (\alpha_1 u'(x) - \gamma_1)\big)' + a_0(x)\big((-\alpha_0 u(x) - \gamma_0)(1-x) + (\alpha_1 u'(x) - \gamma_1)\big)$.

According to (2.1), a linear operator $\mathcal{P} : W_m[0,1] \to L^2[0,1]$ is defined by:

$$\mathcal{P}v(x) = D^{\alpha}v(x) + a_1(x)v'(x) + a_0(x)v(x). \tag{2.2}$$

Theorem 2.2. *The operator $\mathcal{P}$ defined in (2.2) is a bounded and linear operator.*

Proof. Obviously, $\mathcal{P}$ is linear. By applying Cauchy Schwartzs inequality, we have

$$\| \mathcal{P}v(x) \|_{L^2} \leq \| D^{\alpha}v(x) \|_{L^2} + \| a_1(x)v'(x) \|_{L^2} + \| a_0(x)v(x) \|_{L^2} . \tag{2.3}$$

Let $v(x) \in W_m[0,1]$, by the reproducibility property of the reproducing kernel function $R_m(x,y) \in W_m$, there exist positive constants $S_i (i = 0,1,2,\cdots)$, such that

$$|v^{(i)}(x)| = |\langle v(x), \frac{\partial^{(i)} R_m(\cdot,x)}{\partial x^{(i)}} \rangle \leq \| \frac{\partial^{(i)} R_m(\cdot,x)}{\partial x^{(i)}} \| v(x)|_{W_m} \leq S_i \| v(x)|_{W_m}.$$

Using the above formula, a direct calculation together with the use of Cauchy-Schwartzs inequality immediately yields there exist positive constants C and S, such that

$$|D^{\alpha}v(x)| \leq \frac{1}{\Gamma(m-\alpha)} \int_0^x |(x-t)^{m-\alpha-1}||v(x)^{(m)}(t)|dt \leq \frac{M\|v(x)\|_{W_m}}{\Gamma(m-\alpha)} \int_0^x |(x-t)^{m-\alpha-1}|dt \leq \frac{CS\|v(x)\|_{W_m}}{\Gamma(m-\alpha)}.$$

Then

$$\|D^{\alpha}v(x)\|_{L_2}^2 \leq \int_0^1 \frac{C^2 S^2 \|v(x)\|_{W_m}^2}{\Gamma^2(m-\alpha)} dx \leq M_1^2 \|v(x)\|_{W_m}^2. \tag{2.4}$$

Similarly, we have

$$\|a_1(x)v'(x)\|_{L_2} \leq M_2 \|v(x)\|_{W_m}, \quad \|a_0(x)v(x)\|_{L_2} \leq M_3 \|v\|_{W_m}. \tag{2.5}$$

Applying (2.3), (2.4), (2.5), one gets

$$\| \mathcal{P}v(x) \|_{L^2} \leq (M_1 + M_2 + M_3)\|v(x)\|_{W_m},$$

where $M_i (i = 1,2,3)$ is a constant, so the theorem holds true. □

Let $\{x_i\}_{i=0}^{\infty}$ be nodes in interval $[0,1] \times [0,1]$, so we construct the bases $\phi_i^m(x) \in W_m$, $m = 2,3,\cdots$,

$$\phi_i^m(x) = \mathcal{P}R_m(y,x)|_{y=x_i}, \quad i = 0,1,2,\cdots,\infty.$$

3 An efficient technique based on ε-approximate method

3.1 Theoretical Analysis of the Approximate Solution for (2.1) and its unique solvability

In this section, combining the least square theory and the idea of residuals in the spaces $W_m (m = 2,3,\cdots)$, we provide an efficient approach for problem (2.1) by the ε-approximate method.

Definition 3.1. For $\forall\varepsilon > 0$, if $\|\mathcal{P}v_n(x) - g\|_{L^2} < \varepsilon$, then $v_n(x)$ is called ε-approximate solution of (2.1).

According to (1.4), when $k = m - 1(m = 2, 3, \cdots)$, there is $v(x), \phi_i^m \in W_m$, let

$$v_n(x) = \sum_{i=0}^{n} c_i \phi_i^m(x)$$

is the ε-approximate solution for (2.1), and have

$$G(c_0, c_1, c_2, \cdots, c_n) = \|\mathcal{P}v_n(x) - g\|_{L^2}^2 = \|\sum_{i=0}^{n} c_i \mathcal{P}\phi_i^m(x) - g\|_{L^2}^2 < \varepsilon^2, \tag{3.1}$$

where $c_i(i = 0, 1, 2, \cdots, n)$ are constants.
Numerical Scheme: The ε-approximate method for eq(2.1) read as: seeking $c_0^*, c_1^*, c_2^*, \cdots, c_n^*$ such that

$$G(c_0^*, c_1^*, c_2^*, \cdots, c_n^*) = \|\sum_{i=0}^{n} c_i^* \mathcal{P}\phi_i^m(x) - g\|_{L^2}^2 \triangleq min\|\sum_{i=0}^{n} c_i \mathcal{P}\phi_i^m(x) - g\|_{L^2}^2. \tag{3.2}$$

Remark 3.1. The coefficients $c_i^*(i = 0, 1, 2, \cdots, n)$ are identified as the minimum value of $G(c_0, c_1, c_2, \cdots, c_n)$. Therefore, c_i^* satisfies the following:

$$\frac{\partial}{\partial c_i^*} G(c_0^*, c_1^*, c_2^*, \cdots, c_n^*) = 0,$$

which implies that

$$\sum_{i=0}^{n} c_i^* \langle \mathcal{P}\phi_i^m(x), \mathcal{P}\phi_j^m(x)\rangle_{L^2} = \langle \mathcal{P}\phi_j^m(x), g^*\rangle_{L^2}, \quad j = 0, 1, 2, \cdots, n. \tag{3.3}$$

thus, $c_0^*, c_1^*, c_2^*, \cdots, c_n^*$ can be determined by:

$$AX = d,$$

where

$$A = (\langle \mathcal{P}\phi_i^m(x), \mathcal{P}\phi_j^m(x)\rangle_{L^2})_{(n+1)\times(n+1)}, \quad X = (x_0, x_1, x_2, \cdots, x_n)^T, \quad d = (\langle \mathcal{P}\phi_j^m(x), g^*\rangle)_{(n+1)\times 1}^T.$$

Theorem 3.1. *The numerical scheme (3.2) has a unique solution provided that $\mathcal{P}$ is reversible.*

Proof. It is sufficient to prove the linear system $AX = 0$ has only zero solution. Assume that

$$\langle \sum_{i=0}^{n} \mathcal{P}\phi_i^m(x) x_i, \mathcal{P}\phi_j^m(x)\rangle_{L^2} = 0, j = 0, 1, 2, \cdots, n. \tag{3.4}$$

Multiplying both sides of (3.4) by x_j and a cumulative summation yields that

$$\langle \sum_{i=0}^{n} \mathcal{P}\phi_i^m(x) x_i, \sum_{j=0}^{n} \mathcal{P}\phi_j^m(x) x_j\rangle_{L^2} = 0,$$

which leads to

$$\mathcal{P}(\sum_{i=0}^{n} \phi_i^m(x) x_i) = 0.$$

By the linear independence of $\{\phi_i^m\}_{i=0}^n$ and reversibility of the operator $\mathcal{P}$, we immediately obtain that $x_i = 0, i = 0, 1, 2, \cdots, n$. □

3.2 Stability and convergence analysis

We will first discuss the stability of our proposed method. Here we consider the condition number of A, which is defined as follows:

$$cond(A) = |\frac{\mu_{max}}{\mu_{min}}|,$$

where μ_{max}, μ_{min} are the maximum and minimum eigenvalues of A, respectively.

Lemma 3.1. *The eigenvalue of A obtained by (3.3) is bounded by $\|\mathcal{P}\|$.*

Proof. Let μ is an eigenvalue of A $\big(A = (\langle \mathcal{P}\phi_i^m(x), \mathcal{P}\phi_j^m(x)\rangle_{L^2})_{(n+1)\times(n+1)}\big)$, then there exists a unit vector X $\big(X = (x_0, x_1, x_2, \cdots, x_n)^T\big)$, such that $AX = \mu X$. Thus,

$$\mu x_i = \sum_{j=0}^{n} a_{ij}x_j = \sum_{j=0}^{n} \langle \mathcal{P}\phi_i^m(x), \mathcal{P}\phi_j^m(x)\rangle_{L^2} x_j = \langle \mathcal{P}\phi_i^m(x), \sum_{j=0}^{n} \mathcal{P}\phi_j^m(x)x_j\rangle_{L^2},\ \forall i = 0, 1, 2, \cdots, n.$$

Summing x_i from 0 to n in above formula, we derive that

$$\mu = \mu\sum_{i=0}^{n} x_i^2 = \langle \sum_{i=0}^{n} \mathcal{P}\phi_i^m(x)x_i, \sum_{j=0}^{n} \mathcal{P}\phi_j^m(x)x_j\rangle_{L^2} = \|\mathcal{P}(\sum_{i=0}^{n} \phi_i^m(x)x_i)\|_{L^2}^2 \le \|\mathcal{P}\|^2 \|\sum_{i=0}^{n} \phi_i^m(x)x_i\|_{W_m}^2 = \|\mathcal{P}\|^2.$$

□

Lemma 3.2. *[23] If $\mathcal{P}$ is a reversible and bounded operator, then $\mathcal{P}^{-1}$ is bounded.*

Lemma 3.3. *For $v \in W_m$, if $\|v\|_{W_m} = 1$, then $\|\mathcal{P}v\|_{L^2} \ge \frac{1}{\|\mathcal{P}^{-1}\|}$.*

Proof. Let $\mathcal{P}v = g$, according to the condition $\|v\|_{W_m} = 1$, one gets

$$1 = \|v\|_{W_m} = \|\mathcal{P}^{-1}g\|_{W_m} \le \|\mathcal{P}^{-1}\|_{W_m}\|g\|_{L^2},$$

that is

$$\|\mathcal{P}v\|_{L^2} = \|g\|_{L^2} \ge \frac{1}{\|\mathcal{P}^{-1}\|}.$$

□

Theorem 3.2. *The numerical scheme for getting the ε-approximate solution of (3.2) is stable .*

Proof. Denote μ be the eigenvalue of A obtained by (3.2), then we have

$$\mu = \|\sum_{i=0}^{n} x_i \mathcal{P}\phi_i^m(x)\|_{L^2}^2.$$

Combining the fact that $\|\sum\limits_{i=0}^{n} x_i\phi_i^m(x)\|_{W_m} = 1$ and Lemma 3.3, yields

$$\mu = \|\mathcal{P}\sum_{i=0}^{n} x_i\phi_i^m(x)\|_{L^2}^2 \ge \frac{1}{\|\mathcal{P}^{-1}\|^2}.$$

So, the condition number

$$cond(A) = |\frac{\mu_{max}}{\mu_{min}}| \le \frac{\|\mathcal{P}\|^2}{\frac{1}{\|\mathcal{P}^{-1}\|^2}} = \|\mathcal{P}\|^2\|\mathcal{P}^{-1}\|^2.$$

It means that the condition number is bounded, so our algorithm is stable. □

Next, we will provide the convergence analysis of the presented scheme .

Theorem 3.3. *Let $v \in W_m[0,1]$ be the exact solution of (2.1), then approximate solution $v_n \in W_m[0,1]$ converges to u uniformly.*

Proof. Note that

$$|v(x) - v_n(x)| = |\langle v(x) - v_n(x), R_m(\cdot, x)\rangle_{W_m}| \leq \|v(x) - v_n(x)\|_{W_m} \|R_m(\cdot, x)\|_{W_m},$$

and for $\forall \varepsilon > 0$, have

$$\|v - v_n\|_{W_m} = \|\mathcal{P}^{-1}\mathcal{P}(v - v_n)\|_{W_m} \leq \|\mathcal{P}^{-1}\|\|\mathcal{P}(v - v_n)\|_{L^2} = \|\mathcal{P}^{-1}\|\|g - \mathcal{P}v_n\|_{L^2} < \varepsilon \to 0.$$

It means v_n converges to v uniformly on interval [0,1]. □

Similarly, we can proof each order derivative $v_n'(x), v_n''(x), \cdots$ of ε-approximate solution for (2.1) uniformly converge to $v'(x), v''(x), \cdots$ respectively.

4 Numerical result

In this section, three numerical examples are tested to show the validity of our proposed algorithm.

Example 4.1. [25] Consider the example with $\alpha = 0.5$:

$$D^\alpha u(x) - a_0(x)u(x) = f(x),\ x \in [0,1],\ u(0) = 0,$$

where

$$f(x) = \begin{cases} \dfrac{8x^{\frac{3}{2}}}{3\sqrt{\pi}}, & x \in [0, 0.5], \\ \dfrac{8x^{\frac{3}{2}}}{3\sqrt{\pi}} - 1, & x \in (0.5, 1]. \end{cases} \qquad a_0(x) = \begin{cases} 0, & x \in [0, 0.5], \\ \dfrac{1}{x^2}, & x \in (0.5, 1]. \end{cases}$$

The exact solution is $u(x) = x^2$. Here $\alpha = 0.5$, so we use the basis of W_2 to construct the ε-approximate solution $u_n(x)$. Error $e^n \triangleq \sqrt{\sum\limits_{i=0}^{n} \big(u(x_i) - u_n(x_i)\big)}$ $(x_i = ih, i = 0, 1, 2, \cdots, n, h = \frac{1}{n})$, where n is the number of basis, and also the number of points, the result is shown in Table 1 compared with ref.[25].

Table1: Results of e^n compared with Ref.[25] for Example 1.

n	e^n	$e^n(ref.[25])$
32	1.75×10^{-12}	7.56×10^{-4}
64	5.76×10^{-12}	1.30×10^{-4}
128	5.57×10^{-12}	2.28×10^{-5}
256	2.56×10^{-12}	4.08×10^{-6}
512	2.70×10^{-11}	7.54×10^{-7}

Example 4.2. Consider the example 2 in ref.[25] with $\alpha = 1.9$:

$$\begin{cases} D^\alpha u(x) - (2x+6)u(x) = f(x), & x \in (0,1), \\ u(0) - \dfrac{1}{\alpha - 1}u'(0) = \dfrac{\alpha - 4}{\alpha - 1}, & u(1) + u'(1) = 3\alpha + 8, \end{cases}$$

where the exact solution $u(x) = x^\alpha + x^{2\alpha - 1} + 1 + 3x - 7x^2 + 4x^3 + x^4$, $f(x) = -1.82736 + 14.7159x^{0.1} - 4.88079x^{0.9} - 22.9339x^{1.1} - 10.9209x^{2.1} - 2(3+x)(1 + 3x + x^{1.9} - 7x^2 + x^{2.8} + 4x^3 + x^4)$. For $\alpha = 1.9$, we use the basis of W_3 to get the ε-approximate solution $u_n(x)$, the error estimates are all discussed in ref.[25] and our paper. Let $e^N \triangleq \max\limits_{1 \leq i \leq N} |\big(u(x_i) - u_N(x_i)\big)|$ $(x_i = \frac{i}{N}, i = 0, 1, 2, \cdots, N)$, where N is the number of basis and also the number of points. Table 2 show our results and the results of ref.[25].

Table2: Results of e^N compared with Ref.[25] for Example 2.

n	e^N	$e^N(Ref.[25])$
18	5.09×10^{-4}	7.56×10^{-2}
36	5.08×10^{-4}	1.30×10^{-3}
72	5.09×10^{-4}	2.28×10^{-3}
144	5.09×10^{-4}	4.08×10^{-4}

Example 4.3. Consider the following example with $2 \leq \alpha \leq 3$:

$$\begin{cases} D^\alpha u(x) + u(x) = \dfrac{6!}{\Gamma(4.5)x^{3.5}} + x^6, \quad x \in (0,1), \\ u(0) = 0, u'(0) = 0, u''(0) = 0, \end{cases}$$

with the solution $u(x) = x^3$.

In the Table 3, we give errors of each derivative of different n when $\alpha = 2.3$, where $e_k^n \triangleq \sqrt{\sum_{i=0}^{n} \left(u^{(k)}(x_i) - u_n^{(k)}(x_i)\right)}, k = 0, 1, 2.$

Table3: Results of errors of each derivative for Example 3.

n	e_0^n	e_1^n	e_2^n
10	2.85×10^{-16}	2.27×10^{-15}	1.62×10^{-14}
20	9.81×10^{-16}	1.43×10^{-15}	4.92×10^{-15}
40	7.22×10^{-15}	3.91×10^{-14}	4.69×10^{-14}

Figure 1 and Figure 2 show that the absolute-errors corresponds to different the number of basis n and different values of fractional parameter α.

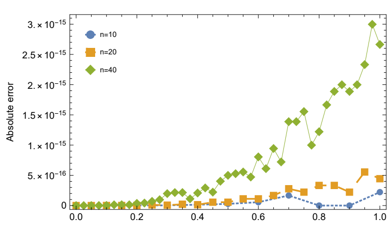

Fig1. the Absolute-errors with $\alpha = 2.3$ for different the number of basis n.

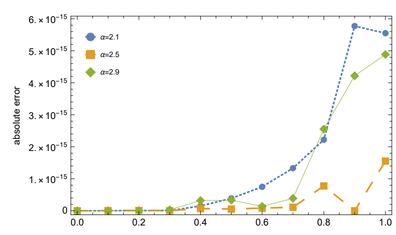

Fig2. the Absolute-errors with $n = 10$ for different values of fractional parameter α.

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