GLOBAL REGULARITY OF THE 2D STEADY COMPRESSIBLE PRANDTL EQUATIONS

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Abstract

In this paper, motivated by [Y. Wang and Z. Zhang, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 38(2021), 1989-2004], we study the global C [?] regularity of the two-dimensional steady compressible Prandtl equations in the case of the favorable pressure gradient. The proof is based on the maximum principle and interior a priori estimates.

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ABSTRACT. In this paper, motivated by [Y. Wang and Z. Zhang, Ann. Inst. H. Poincaré C Anal. Non Linéaire, 38(2021), 1989-2004], we study the global C^∞ regularity of the two-dimensional steady compressible Prandtl equations in the case of the favorable pressure gradient. The proof is based on the maximum principle and interior a priori estimates.

1. INTRODUCTION

In this paper, we study the two-dimensional steady compressible Prandtl equations in $\Omega = \{(x,y)|x>0, y>0\}$:

$$\begin{cases} u\partial_x u + v\partial_y u - \dfrac{1}{\rho}\partial_y^2 u = -\dfrac{\partial_x P(\rho)}{\rho}, \\ \partial_x(\rho u) + \partial_y(\rho v) = 0, \\ u|_{x=0} = u_0(y), \quad \lim\limits_{y\to\infty} u = U(x), \\ u|_{y=0} = v|_{y=0} = 0, \end{cases} \tag{1.1}$$

where (u,v) is velocity field, $\rho(x)$ and $U(x)$ are the traces at the boundary $\{y=0\}$ of the density and the tangential velocity of the outer Euler flow. The states ρ, U satisfies the Bernoulli law

$$U\partial_x U + \frac{\partial_x P(\rho)}{\rho} = 0. \tag{1.2}$$

The pressure $P(\rho)$ is a strictly increasing function of ρ with $0 < \rho_0 \le \rho \le \rho_1$ for some positive constants ρ_0 and ρ_1. In this paper, we assume that the pressure satisfies the favorable pressure gradient $\partial_x P \le 0$, which implies that

$$\partial_x \rho \le 0.$$

Ludwig prandtl first put forward the boundary layer theory in 1904. In [20], he obtained a degenerate parabolic equation coupled with the elliptic equation, namely the famous Prandtl equations, which was used to describe the motion of fluid in the boundary layer. Since then, the boundary layer theory has become a major tool and great achievement in fluid mechanics and many other subjects. Up to now, many scholars have developed the mathematical and physical theory of Prandtl boundary layer. See [1, 9, 16–18, 21–24, 27–36] for the relative works and the references therein.

The existence of solutions to the steady Prandtl equations has been studied by Oleinik and Samokhin in [19] by using the von Mises transformation (see Theorem 2.1.1 in [19]). As pointed out in [31], there are three natural and important problems for the steady boundary layer: (i) Boundary layer separation under the adverse pressure gradient. (ii) Under the favorable pressure gradient, whether Oleinik's global-in-x solutions are smooth up to the boundary $y=0$ for **any** $x>0$. (iii) Vanishing viscosity limit of the steady Navier-Stokes equations. In addition, similar to the incompressible boundary layer, the compressible boundary layer naturally has the above mentioned three important problems.

Separation is one of the most important problems in the boundary layer theory. Flow separation or more precisely boundary layer separation is an important phenomenon [22–25]. The mathematical theory on the separation of steady boundary layers has been developed, first by

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Caffarelli and E in an unpublished paper (see Theorem 3.2 in [5]). This result states that under some assumption on the initial data and the adverse pressure gradient, the existence time x^* of the solutions to Prandtl equations in the sense of Oleinik is finite. Moreover, the family $u_\mu(x,y) = \mu^{-\frac{1}{2}}u(x^* - \mu x, \mu^{\frac{1}{4}}y)$ is compact in $C^0(\mathbb{R}_+^2)$. More recently, Dalibard and Masmoudi [4] proved the solution behaves near the separation as $\partial_y u(x,0) \sim (x^* - x)^{\frac{1}{2}}$ for $x < x^*$. Shen, Wang and Zhang [28] also studied the local behavior of the solution near the separation. They found that when the point approaches the separation point along some typical curve, the solution near the separation point behaves like $\partial_y u(x,y) \sim (x^* - x)^{\frac{1}{4}}$ for $x < x^*$. The unsteady boundary layer separation is a more complex problem, because the appearance of the back-flow point does not necessarily lead to the boundary layer separation. There is no rigorous mathematical theory yet. As an important entry point for the study of the unsteady boundary layer separation, Wang and Zhu [30] studied the occurrence of a back-flow point of the two-dimensional unsteady boundary layer. Establishing the mathematical theory of the unsteady boundary layer separation is a challenging and important problem.

Due to degeneracy near the boundary, the problem of high regularity of solutions to the steady Prandtl equations is a difficult and meaningful problem. Recently, Guo and Iyer [11] studied the higher regularity of the solutions to the steady Prandtl equations in a local time $0 < x < x^* \ll 1$ (x is considered as time). This result shows the construction of Prandtl layer expansion up to **any** order. Wang and Zhang [31] proved that Oleinik's global-in-x solutions are smooth up to the boundary $y = 0$ for **any** $x > 0$ by using the maximum principle and the interior a priori estimates developed by Krylov in [15]. The goal of this paper is to prove the global C^∞ regularity of the two-dimensional steady compressible Prandtl equations.

For problem (iii), Gerard-Varet and Maekawa [7] studied the stability of the shear flows $(U(y/\sqrt{\mu}, 0))$ in the Sobolev space. Guo and Iyer [10] studied the stability of Blasius flow. Recently, Chen, Wu and Zhang [2] studied the stability of shear flows for the steady Navier-Stokes equations when some assumption on the linearized NS operator. Interested readers can refer to [6, 13] for more details.

Moreover, the large time behavior of Oleinik's solution is also an important problem. Serrin [21] proved the asymptotic behavior of the solution. When the initial data is a small localized perturbation of the Blasius profile, Iyer [14] proved the explicit decay of the solution. Recently. Wang and Zhang [32] found the explicit decay for general initial data with exponential decay by using the maximum principle.

The system (1.1) could be used to characterize the behavior of the solution near the boundary $y = 0$ for the two-dimensional steady compressible Navier-Stokes equations with no-slip boundary condition. In [8], Gong, Guo and Wang studied the existence of the solutions of the system (1.1) by using the von Mises transformation and the maximal principle proposed by Oleinik and Samokhin in [19]. Actually, they proved that:

Theorem 1.1. *If the initial data u_0 satisfies the following conditions:*

$$u \in C_b^{3,\alpha}([0,+\infty))(\alpha > 0), \quad u(0) = 0, \quad \partial_y u(0) > 0, \quad \partial_y u(y) \geq 0 \quad \text{for} \quad y \in [0,+\infty),$$
$$\lim_{y\to+\infty} u(y) = U(0) > 0, \quad \rho^{-1}(0)\partial_y^2 u(y) - \rho^{-1}(0)\partial_x P(0) = O(y^2),$$

and $\rho \in C^2([0,\bar{X}])$, then there exists $0 < X \leq \bar{X}$ such that the system (1.1) admits a solution $u \in C^1([0,X) \times \mathbb{R}_+)$ with the following properties:
(i) Regularity: u is bounded and continuous in $[0,X] \times \mathbb{R}_+$; $\partial_y u, \partial_y^2 u$ are bounded and continuous in $[0,X) \times \mathbb{R}_+$; $v, \partial_y v, \partial_x u$ are bounded locally in $[0,X) \times \mathbb{R}_+$.
(ii) Non-decreasing: $u(x,y) > 0$ in $[0,X) \times \mathbb{R}_+$ and for any $\bar{x} < X$, there exists $y_0, m > 0$ such that for all $(x,y) \in [0,\bar{x}] \times [0,y_0]$,$\partial_y u(x,y) \geq m$.
(iii) Global existence: if $\partial_x P \leq 0(\partial_x \rho \leq 0)$, then the solution is global-in-x.

The result shows that the local solution of problem (1.1) from Theorem 1.1 is increasing with respect to y near the boundary $y = 0$. Under the favorable pressure gradient $\partial_x P \leq 0(\partial_x \rho \leq 0)$, the solution is global-in-x. However, the adverse pressure gradient $\partial_x P > 0(\partial_x \rho > 0)$ may lead to the boundary layer separation, which is a meaningful physical phenomenon [12].

In [34], Xin and Zhang studied the global existence of weak solutions to the unsteady Prandtl equations under the favorable pressure gradient. For the unsteady compressible Prandtl equations, similar results are obtained in [3]. Recently, Xin, Zhang and Zhao [35] put forward a direct proof of the existence of global weak solutions to the Prandtl equations by a direct BV estimate. The key ingredients of this paper are that they studied the uniqueness and the regularity of a weak solution. This method may be applied to the compressible Prandtl equations. Moreover, for the unsteady incompressible or compressible Prandtl equations, it is an open and interesting question whether the solution also has the global C^∞ regularity up to the boundary.

To use the von Mises transformation, we set

$$\tilde{u}(x,y)=\rho(x)u(x,y),\quad \tilde{v}(x,y)=\rho(x)v(x,y),\quad \tilde{u}_0(y)=\rho(0)u_0(y).$$

Then combining with (1.1), we know that $(\tilde{u},\tilde{v})$ satisfies:

$$\begin{cases}\tilde{u}\partial_x\tilde{u}+\tilde{v}\partial_y\tilde{u}-\partial_y^2\tilde{u}-\dfrac{\partial_x\rho}{\rho}\tilde{u}^2=-\rho\partial_xP(\rho),\\ \partial_x\tilde{u}+\partial_y\tilde{v}=0,\\ \tilde{u}|_{x=0}=\tilde{u}_0(y),\quad \lim\limits_{y\to\infty}\tilde{u}=\rho(x)U(x),\\ \tilde{u}|_{y=0}=\tilde{v}|_{y=0}=0.\end{cases}\tag{1.3}$$

The following von Mises transformation is introduced:

$$x=x,\quad \psi(x,y)=\int_0^y\tilde{u}(x,z)dz,\quad w=\tilde{u}^2.\tag{1.4}$$

Combining (1.3) with (1.4), we know that $w(x,\psi)$ satisfies:

$$\partial_xw-\sqrt{w}\partial_\psi^2w-2\frac{\partial_x\rho}{\rho}w=-2\rho\partial_xP(\rho),\tag{1.5}$$

with

$$w(x,0)=0,\quad w(0,\psi)=w_0(\psi),\quad \lim_{\psi\to+\infty}w=(\rho(x)U(x))^2.\tag{1.6}$$

A direct calculation gives

$$2\partial_y\tilde{u}=\partial_\psi w,\qquad 2\partial_y^2\tilde{u}=\sqrt{w}\partial_\psi^2w.\tag{1.7}$$

The main result of this paper is as follows:

Theorem 1.2. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_xP\le 0$ are smooth. For any positive integers m,k and any positive constant X,ε with $\varepsilon<X$, there exists a positive constant C depending only on $\varepsilon,X,u_0,P(\rho),k,m$ such that for any $(x,y)\in[\varepsilon,X]\times[0,+\infty)$,*

$$|\partial_x^k\partial_y^m u(x,y)|\le C.$$

This theorem shows that the solution obtained in Theorem 1.1 are smooth up the boundary $y=0$ for **any** $x>0$. Moreover, the derivatives $|\partial_\psi^m\partial_x^kw|\le C\psi^{1-m}$ will blow up at the boundary $\psi=0$ (see Lemma 2.7). This is similar to the result of the incompressible boundary layer, despite the fluid being compressible and the degeneracy near the boundary.

Due to the degeneracy near the boundary $\psi=0$, the proof of the main result is divided into two parts, Theorem 1.3 and Theorem 1.4. First, we prove the following theorem in the domain $[\varepsilon,X]\times[0,Y_1]$ for a small Y_1. The key ingredients of proof is that we employ the maximum principle and interior a priori estimates developed by Krylov in [15].

Theorem 1.3. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_xP\le 0$ are smooth. For any positive integers m,k and any positive constant X,ε with $\varepsilon<X$, there exists a small positive constant Y_1 and a large positive constant C depending only on $\varepsilon,X,Y_1,u_0,P(\rho),k,m$ such that for any $(x,y)\in[\varepsilon,X]\times[0,Y_1]$,*

$$|\partial_x^k\partial_y^m u(x,y)|\le C.$$

Next, we prove (1.5) is a uniform parabolic equation in the domain $[\varepsilon, X] \times [Y_2, +\infty)$ for a small positive constant Y_2 in Appendix. Once we have (1.5) is a uniform parabolic equation, the global C^∞ regularity of the solution is a direct result of interior Schauder estimates and classical parabolic regularity theory. The proof can be given similarly to the steady incompressible boundary layer. More details can be found in [31] and we omit it here.

Theorem 1.4. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P \le 0$ are smooth. For any positive integers m, k and any positive constant X, ε with $\varepsilon < X$, there exists a positive constant Y_0 such that for any constant $Y_2 \in (0, Y_0)$, there exists a positive constant C depending only on $\varepsilon, X, Y_2, u_0, P(\rho), k, m$ such that for any $(x, y) \in [\varepsilon, X] \times [Y_2, +\infty)$,*

$$|\partial_x^k \partial_y^m u(x, y)| \le C.$$

Therefore, Theorem 1.2 can be directly proved by combining Theorem 1.3 with Theorem 1.4.

The organization of this paper is as follows. In Section 2, we study lower order and higher order regularity estimates. In Section 3, we prove the global C^∞ regularity of the solution by transforming back to the original coordinates (x, y). In Appendix, we prove (1.5) is a uniform parabolic equation by using the maximum principle.

2. Lower order and higher order regularity estimates

2.1. **Lower order regularity estimates.** In this subsection, we study the lower order regularity estimates via the standard interior a priori estimates developed by Krylov in [15].

Lemma 2.1. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P \le 0$ are smooth. Assume $0 < \varepsilon < X$. Then there exists some positive constants $\delta_1 > 0$, and C independent of ψ such that for any $(x, \psi) \in [\varepsilon, X] \times [0, \delta_1]$,*

$$|\partial_x w(x, \psi)| \le C\psi.$$

Proof. Due to Lemma 2.1 in [8] (or Theorem 2.1.14 in [19]), there exists $\delta_1 > 0$, for any $(x, \psi) \in [0, X] \times [0, \delta_1]$, such that for some $\alpha \in (0, \frac{1}{2})$ and positive constants m, M, (without loss of generality, we assume $\delta_1 < 1$.)

$$|\partial_x w| \le C\psi^{\frac{1}{2}+\alpha}, \quad 0 < m < \partial_\psi w < M, \quad m\psi < w < M\psi. \tag{2.1}$$

By (1.5), we obtain

$$\partial_x \partial_x w - \sqrt{w}\partial_\psi^2 \partial_x w = \frac{(\partial_x w)^2}{2w} + 2\frac{\rho \partial_x P \partial_x w}{2w} + \frac{\partial_x \rho}{\rho}\partial_x w + 2\partial_x\left(\frac{\partial_x \rho}{\rho}\right) w - 2\partial_x[\rho \partial_x P].$$

Take a smooth cut-off function $0 \le \phi(x) \le 1$ in $[0, X]$ such that

$$\phi(x) = 1, x \in [\varepsilon, X], \quad \phi(x) = 0, x \in [0, \frac{\varepsilon}{2}].$$

Then

$$\begin{aligned}
&\partial_x\left[\partial_x w \phi(x)\right] - \sqrt{w}\partial_\psi^2\left[\partial_x w \phi(x)\right] \\
&\qquad = \frac{(\partial_x w)^2}{2w}\phi(x) + 2\frac{\rho \partial_x P \partial_x w}{2w}\phi(x) + \frac{\partial_x \rho}{\rho}\partial_x w \phi(x) \\
&\qquad + 2\partial_x\left(\frac{\partial_x \rho}{\rho}\right) w\phi(x) - 2\partial_x(\rho \partial_x P)\phi(x) + \partial_x w \partial_x \phi(x) := \mathcal{W}.
\end{aligned}$$

Combining with (2.1), we know

$$|\mathcal{W}| \le C\psi^{2\alpha} + C\psi^{\alpha - \frac{1}{2}} + C\psi^{\alpha + \frac{1}{2}} + C\psi + C + C\psi^{\frac{1}{2}+\alpha} \le C\psi^{\alpha - \frac{1}{2}}. \tag{2.2}$$

We take $\varphi(\psi) = \mu_1 \psi - \mu_2 \psi^{1+\beta}$ with constants μ_1, μ_2. Then by (2.1) and (2.2), we get

$$\begin{aligned}
\partial_x\left[\partial_x w \phi(x) - \varphi\right] - \sqrt{w}\partial_\psi^2\left[\partial_x w \phi(x) - \varphi\right] &\le |\mathcal{W}| - \mu_2\sqrt{w}\beta(1+\beta)\psi^{\beta-1} \\
&\le C\psi^{\alpha - \frac{1}{2}} - \mu_2\sqrt{m}\beta(1+\beta)\psi^{\beta - \frac{1}{2}}.
\end{aligned}$$

By taking μ_2 sufficiently large and $\alpha = \beta$, then for $(x, \psi) \in (0, X] \times (0, \delta_1)$, we have

$$\partial_x \left[\partial_x w \phi(x) - \varphi\right] - \sqrt{w}\partial_\psi^2 \left[\partial_x w \phi(x) - \varphi\right] < 0.$$

For any $\psi \in [0, \delta_1]$, let $\mu_1 \geq \mu_2$, we have

$$\left(\partial_x w \phi - \varphi\right)(0, \psi) \leq 0,$$

and take μ_1 large enough depending on M, δ_1, μ_2 such that

$$\left(\partial_x w \phi - \varphi\right)(x, \delta_1) \leq M\delta_1^{\frac{1}{2}+\alpha} - \mu_1\delta_1 + \mu_2\delta_1^{1+\beta} \leq 0.$$

Since $w(x, 0) = 0$, we know that for any $x \in [0, X]$,

$$\left(\partial_x w \phi - \varphi\right)(x, 0) = 0.$$

By the maximum principle, it holds in $[0, X] \times [0, \delta_1]$ that

$$\left(\partial_x w \phi - \varphi\right)(x, \psi) \leq 0.$$

Let δ_1 is chosen suitably small, for $(x, \psi) \in [\varepsilon, X] \times [0, \delta_1]$, we obtain

$$\partial_x w(x, \psi) \leq \mu_1\psi - \mu_2\psi^{1+\beta} \leq \frac{\mu_1}{2}\psi.$$

Considering $-\partial_x w\phi - \varphi$, the result $-\partial_x w \leq \frac{\mu_1}{2}\psi$ in $[\varepsilon, X] \times [0, \delta_1]$ can be proved similarly. This completes the proof of the lemma. □

Lemma 2.2. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P \leq 0$ are smooth. Assume $0 < \varepsilon < X$. Then there exists some positive constants $\delta_2 > 0$, and C independent of ψ such that for any $(x, \psi) \in [\varepsilon, X] \times [0, \delta_2]$,*

$$|\partial_\psi \partial_x w(x, \psi)| \leq C, \quad |\partial_x^2 w(x, \psi)| \leq C\psi^{-\frac{1}{2}}, \quad |\partial_\psi^2 \partial_x w(x, \psi)| \leq C\psi^{-1}.$$

Proof. From Lemma 2.1, there exists $\delta_1 > 0$ such that for any $(x, \psi) \in [\frac{\varepsilon}{2}, X] \times [0, \delta_1]$,

$$|\partial_x w(x, \psi)| \leq C\psi.$$

Set $\Psi_0 = \min\{\frac{2}{3}\delta_1, \frac{\varepsilon}{2}\}$, for any $(x_0, \psi_0) \in [\varepsilon, X] \times (0, \Psi_0]$ we denote

$$\Omega = \left\{(x, \psi) | x_0 - \psi_0^{\frac{3}{2}} \leq x \leq x_0, \frac{1}{2}\psi_0 \leq \psi \leq \frac{3}{2}\psi_0\right\}.$$

By the definition of Ψ_0, we know $\Omega \subseteq [\frac{\varepsilon}{2}, X] \times [0, \delta_1]$. Then it holds in Ω that

$$|\partial_x w| \leq C\psi. \tag{2.3}$$

The following transformation f are defined:

$$\Omega \to \widetilde{\Omega} := [-1, 0]_{\tilde{x}} \times [-\frac{1}{2}, \frac{1}{2}]_{\tilde{\psi}}, \quad (x, \psi) \mapsto (\tilde{x}, \tilde{\psi}),$$

where $x - x_0 = \psi_0^{\frac{3}{2}}\tilde{x}, \psi - \psi_0 = \psi_0\tilde{\psi}$. Since $\partial_{\tilde{x}} = \psi_0^{\frac{3}{2}}\partial_x, \partial_{\tilde{\psi}} = \psi_0\partial_\psi$, it holds in Ω that

$$\partial_{\tilde{x}}\left(\psi_0^{-1}w\right) - \psi_0^{-\frac{1}{2}}\sqrt{w}\partial_{\tilde{\psi}}^2\left(\psi_0^{-1}w\right) - 2\frac{\partial_{\tilde{x}}\rho}{\rho}(\psi_0^{-1}w) = -2\rho\partial_{\tilde{x}}P\psi_0^{-1}.$$

Combining with (2.1), we get $0 < c \leq \psi_0^{-\frac{1}{2}}\sqrt{w} \leq C, |\psi_0^{-1}w| \leq C$ and for any $\widetilde{z}_1, \widetilde{z}_2 \in \widetilde{\Omega}$,

$$|\psi_0^{-\frac{1}{2}}\sqrt{w}(\tilde{z}_1) - \psi_0^{-\frac{1}{2}}\sqrt{w}(\tilde{z}_2)| = \psi_0^{-\frac{1}{2}}\frac{|w(\tilde{z}_1) - w(\tilde{z}_2)|}{\sqrt{w}(\tilde{z}_1) + \sqrt{w}(\tilde{z}_2)} \leq C\frac{\psi_0|\tilde{z}_1 - \tilde{z}_2|}{\psi_0} = C|\tilde{z}_1 - \tilde{z}_2|.$$

This means that for any $\alpha \in (0, 1)$, we have

$$|\psi_0^{-\frac{1}{2}}\sqrt{w}|_{\mathcal{C}^\alpha(\widetilde{\Omega})} \leq C.$$

Since P and ρ are smooth, we have

$$|\rho^{-1}\partial_{\tilde{x}}\rho|_{\mathcal{C}^{0,1}([-1,0]_{\tilde{x}})} + |\rho\partial_{\tilde{x}}P\psi_0^{-1}|_{\mathcal{C}^{0,1}([-1,0]_{\tilde{x}})} \leq C.$$

By standard interior a priori estimates (see Theorem 8.11.1 in [15] or Proposition 2.3 in [31]), we have

$$|w\psi_0^{-1}|_{\mathcal{C}^\alpha([-\frac{1}{2},0]_{\tilde{x}}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}})}+|\partial_{\tilde{\psi}}^2 w\psi_0^{-1}|_{\mathcal{C}^\alpha([-\frac{1}{2},0]_{\tilde{x}}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}})}\leq C. \tag{2.4}$$

We denote $f:=\partial_x w\psi_0^{-1}$, by (1.5), we get

$$\partial_{\tilde{x}}f-\frac{\sqrt{w}}{\psi_0^{\frac{1}{2}}}\partial_{\tilde{\psi}}^2 f-\frac{\partial_{\tilde{\psi}}^2 w}{2\sqrt{w}\psi_0^{\frac{1}{2}}}f-2\frac{\partial_{\tilde{x}}\rho}{\rho}f=-2\partial_x[\rho\partial_{\tilde{x}}P]\psi_0^{-1}+2\partial_x\left(\frac{\partial_{\tilde{x}}\rho}{\rho}\right)(\psi_0^{-1}w).$$

By (2.3), we have $|f|\leq C$ in $\widetilde{\Omega}$. Due to

$$\left|\psi_0^{\frac{1}{2}}w^{-\frac{1}{2}}(\tilde{z}_1)-\psi_0^{\frac{1}{2}}w^{-\frac{1}{2}}(\tilde{z}_2)\right|=\psi_0^{\frac{1}{2}}\frac{\left|\frac{w(\tilde{z}_1)-w(\tilde{z}_2)}{w(\tilde{z}_1)w(\tilde{z}_2)}\right|}{w^{-\frac{1}{2}}(\tilde{z}_1)+w^{-\frac{1}{2}}(\tilde{z}_2)}\leq C\left|\tilde{z}_1-\tilde{z}_2\right|,$$

we have

$$\left|\psi_0^{\frac{1}{2}}w^{-\frac{1}{2}}\right|_{\mathcal{C}^\alpha(\widetilde{\Omega})}\leq C. \tag{2.5}$$

Since $\frac{\partial_{\tilde{\psi}}^2 w}{2\sqrt{w}\psi_0^{\frac{1}{2}}}=\partial_{\tilde{\psi}}^2 w\psi_0^{-1}\frac{\psi_0^{\frac{1}{2}}}{2\sqrt{w}}$, which along with (2.4) and (2.5) gives

$$\left|\frac{\partial_{\tilde{\psi}}^2 w}{2\sqrt{w}\psi_0^{\frac{1}{2}}}\right|_{\mathcal{C}^\alpha([-\frac{1}{2},0]_{\tilde{x}}\times[-\frac{1}{4},\frac{1}{4}]_{\tilde{\psi}})}\leq C.$$

As before, by (2.4) and the density ρ and P are smooth, via the standard interior a priori estimates yield that

$$|\partial_{\tilde{x}}f|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})}+|\partial_{\tilde{\psi}}f|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})}+|\partial_{\tilde{\psi}}^2 f|_{L^\infty([-\frac{1}{4},0]_{\tilde{x}}\times[-\frac{1}{8},\frac{1}{8}]_{\tilde{\psi}})}\leq C.$$

Therefore, we obtain

$$|\partial_x^2 w\,(x_0,\psi_0)\,|\leq C\psi_0^{-\frac{1}{2}},\quad |\partial_\psi\partial_x w\,(x_0,\psi_0)\,|\leq C,\quad |\partial_\psi^2\partial_x w\,(x_0,\psi_0)\,|\leq C\psi_0^{-1}.$$

This completes the proof of the lemma. □

2.2. **Higher order regularity estimates.** In this subsection, we study the higher order regularity estimates via the maximum principle. The two main results of this subsection are Lemma 2.3 and Lemma 2.7.

Lemma 2.3. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P\leq 0$ are smooth. Assume $0<\varepsilon<X$ and $k\geq 2$. Then there exists some positive constants $\delta>0$, and C independent of ψ such that for any $(x,\psi)\in[\varepsilon,X]\times[0,\delta]$,*

$$|\partial_x^k w|\leq C\psi,\quad |\partial_\psi\partial_x^k w|\leq C,\quad |\partial_\psi^2\partial_x^k w|\leq C\psi^{-1}.$$

Proof. By Lemma 2.1 and Lemma 2.2, we may inductively assume that for $0\leq j\leq k-1$, there holds that in $\left[\frac{\varepsilon}{2},X\right]\times[0,\delta_3]$, (without loss of generality, assume $\delta_3\ll 1$.)

$$|\partial_\psi\partial_x^j w|\leq C,\quad |\partial_\psi^2\partial_x^j w|\leq C\psi^{-1},\quad |\partial_x^j w|\leq C\psi,\quad |\partial_x^j\sqrt{w}|\leq C\psi^{\frac{1}{2}},\quad |\partial_x^k w|\leq C\psi^{-\frac{1}{2}}. \tag{2.6}$$

We will prove that there exists $\delta_4<\delta_3$ so that in $[\varepsilon,X]\times[0,\delta_4]$,

$$|\partial_\psi\partial_x^k w|\leq C,\quad |\partial_\psi^2\partial_x^k w|\leq C\psi^{-1},\quad |\partial_x^k w|\leq C\psi,\quad |\partial_x^k\sqrt{w}|\leq C\psi^{\frac{1}{2}},\quad |\partial_x^{k+1}w|\leq C\psi^{-\frac{1}{2}}. \tag{2.7}$$

The above results are deduced from the following Lemma 2.4, Lemma 2.5, and Lemma 2.6. □

Lemma 2.4. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P\leq 0$ are smooth. Assume that* (2.6) *holds, then there is a positive constant M_1 for any $(x,\psi)\in\left[\frac{7\varepsilon}{8},X\right]\times[0,\delta_3]$ and $0<\beta\ll 1$,*

$$|\partial_x^k w|<M_1\psi^{1-\beta},\quad |\partial_x^k\sqrt{w}|\leq M_1\psi^{\frac{1}{2}-\beta}.$$

Proof. Take a smooth cut-off function $0 \leq \phi(x) \leq 1$ in $[0, X]$ such that

$$\phi(x) = 1, x \in [\frac{7\varepsilon}{8}, X], \quad \phi(x) = 0, x \in [0, \frac{5\varepsilon}{8}].$$

As in [31], fix any $h < \frac{\varepsilon}{8}$. Set

$$\Omega = \{(x,\psi)|0 < x \leq X, 0 < \psi < \delta_3\},$$

and let

(i) $f = \frac{\partial_x^{k-1} w(x-h,\psi) - \partial_x^{k-1} w(x,\psi)}{-h}\phi + M\psi \ln\psi, \qquad (x,\psi) \in [\frac{5\varepsilon}{8}, X] \times [0, +\infty),$

(ii) $f = M\psi \ln\psi, \qquad (x,\psi) \in [0, \frac{5\varepsilon}{8}) \times [\psi, +\infty),$

so we get $f(x, 0) = 0$, $f(0, \psi) \leq 0$. By taking M large enough, we have

$$f(x, \delta_3) \leq C(\delta_3)^{-\frac{1}{2}} + M\delta_3 \ln\delta_3 \leq 0.$$

Then we prove that the positive maximum of f can not be achieved in the interior by choosing appropriate M. Finally, the lemma can be proved by the arbitrariness of h. Assume that there exists a point $p_0 = (x_0, \psi_0) \in \Omega$ such that $f(p_0) = \max_{\bar{\Omega}} f > 0$. It is easy to know that $x_0 > \frac{5\varepsilon}{8}$ and $\partial_x^{k-1} w(x_0 - h, \psi_0) < \partial_x^{k-1} w(x_0, \psi_0)$. By (2.1), denote $\xi = \sqrt{m}$, we have

$$-\sqrt{w}\partial_\psi^2(M\psi \ln\psi) = -M\sqrt{w}\psi^{-1} \leq -\xi M\psi^{-\frac{1}{2}}. \tag{2.8}$$

By (1.5), a direct calculation gives

$$\begin{aligned}
\partial_x\partial_x^{k-1}w - &\sqrt{w}\partial_\psi^2\partial_x^{k-1}w \\
&= -2\partial_x^{k-1}(\rho\partial_x P) + \sum_{m=1}^{k-2} C_{k-1}^m \left(\partial_x^{k-1-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w + \left(\partial_x^{k-1}\sqrt{w}\right)\partial_\psi^2 w \\
&\quad + 2\sum_{m=0}^{k-1} C_{k-1}^m \partial_x^{k-1-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w \\
&= -2\partial_x^{k-1}(\rho\partial_x P) + \sum_{m=1}^{k-2} C_{k-1}^m \left(\partial_x^{k-1-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w + \frac{\partial_x^{k-1}w}{2\sqrt{w}}\frac{\partial_x w}{\sqrt{w}} \\
&\quad + \left(\frac{\partial_x^{k-1}w}{2\sqrt{w}}\right)\frac{2\rho\partial_x P}{\sqrt{w}} - \left(\frac{\partial_x^{k-1}w}{2\sqrt{w}}\right)\frac{2\frac{\partial_x\rho}{\rho}w}{\sqrt{w}} + 2\sum_{m=0}^{k-1} C_{k-1}^m \partial_x^{k-1-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w \\
&\quad + \sum_{m=0}^{k-3} C_{k-2}^m \partial_\psi^2 w \partial_x^{m+1} w \partial_x^{k-2-m}\frac{1}{2\sqrt{w}} := \sum_{i=1}^{4} I_i,
\end{aligned}$$

and

$$\begin{aligned}
I_1 &= -2\partial_x^{k-1}(\rho\partial_x P) + \sum_{m=1}^{k-2} C_{k-1}^m \left(\partial_x^{k-1-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w + \frac{\partial_x^{k-1}w}{2\sqrt{w}}\frac{\partial_x w}{\sqrt{w}}, \\
I_2 &= \frac{\rho\partial_x P}{w}\partial_x^{k-1}w, \\
I_3 &= -\frac{\partial_x\rho}{\rho}\partial_x^{k-1}w + 2\sum_{m=0}^{k-1} C_{k-1}^m \partial_x^{k-1-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w, \\
I_4 &= \sum_{m=0}^{k-3} C_{k-2}^m \partial_\psi^2 w \partial_x^{m+1} w \partial_x^{k-2-m}\frac{1}{2\sqrt{w}}.
\end{aligned}$$

For $x \geq \frac{5\varepsilon}{8}$, we consider the following equality

$$\partial_x f_1 - \sqrt{w(p_1)}\partial_\psi^2 f_1 = \frac{\sqrt{w(p_1)} - \sqrt{w(p)}}{-h}\partial_\psi^2\partial_x^{k-1}w(p) + \sum_{i=1}^{4}\frac{1}{-h}\left(I_i(p_1) - I_i(p)\right), \tag{2.9}$$

where $f_1 = \frac{1}{-h}(\partial_x^{k-1}w(p_1) - \partial_x^{k-1}w(p))$, with $p_1 = (x-h,\psi), p = (x,\psi)$.

For any $x \geq \frac{5\varepsilon}{8}$, by (2.6), it is easy to conclude that

$$\begin{aligned}
&\left|\frac{1}{-h}(\sqrt{w}(p_1)-\sqrt{w}(p))\partial_\psi^2\partial_x^{k-1}w(p)\right| \leq C\psi^{-\frac{1}{2}},\\
&\left|\frac{1}{-h}(I_1(p_1)-I_1(p))\right| \leq C\psi^{-\frac{1}{2}},\\
&\left|\sum_{i=3}^{4}\frac{1}{-h}(I_i(p_1)-I_i(p))\right| \leq C\psi^{-\frac{1}{2}}.
\end{aligned} \tag{2.10}$$

Since

$$\frac{1}{-h}(I_2(p_1)-I_2(p)) = f_1\cdot\big[\frac{\rho\partial_x P}{w}(p_1)\big] + \partial_x^{k-1}w(p)\frac{1}{-h}\big[\frac{\rho\partial_x P}{w}(p_1)-\frac{\rho\partial_x P}{w}(p)\big].$$

Combining with (2.6), $f_1(p_0)>0$ and $\partial_x P \leq 0$, it holds at $p=p_0$ that

$$\frac{1}{-h}(I_2(p_1)-I_2(p_0)) \leq C. \tag{2.11}$$

Summing up (2.10) and (2.11), we conclude that at $p=p_0$,

$$\partial_x f_1 - \sqrt{w}\partial_\psi^2 f_1 \leq C_0\psi^{-\frac{1}{2}}.$$

This along with (2.8) shows that for $x \geq \frac{5\varepsilon}{8}$, it holds at $p=p_0$ that

$$\partial_x f - \sqrt{w}\partial_\psi^2 f \leq C\psi^{-\frac{1}{2}} - \xi M\psi^{-\frac{1}{2}}. \tag{2.12}$$

By taking M large enough, we have $\partial_x f(p_0) - \sqrt{w}\partial_\psi^2 f(p_0) < 0$. By the definition of p_0, we obtain

$$\partial_x f(p_0) - \sqrt{w}\partial_\psi^2 f(p_0) \geq 0,$$

which leads to a contradiction. Therefore, we have $\max_{\overline{\Omega}} f \leq 0$.

We can similarly prove that $\min_{\overline{\Omega}} f \geq 0$ by replacing $M\psi\ln\psi$ in f with $-M\psi\ln\psi$. By the arbitrariness of h, for any $(x,\psi)\in(\frac{7\varepsilon}{8},X]\times(0,\delta_3]$ we have

$$|\partial_x^k w| \leq -M\psi\ln\psi.$$

Due to

$$2\sqrt{w}\partial_x^k\sqrt{w} + \sum_{m=1}^{k-1}C_k^m(\partial_x^m\sqrt{w}\partial_x^{k-m}\sqrt{w}) = \partial_x^k(\sqrt{w}\sqrt{w}) = \partial_x^k w, \tag{2.13}$$

which along with (2.6) shows that in $(\frac{7}{8}\varepsilon,X]\times(0,\delta_3]$,

$$|\sqrt{w}\partial_x^k\sqrt{w}| \leq -C\psi\ln\psi.$$

This completes the proof of the lemma. □

Lemma 2.5. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P \leq 0$ are smooth. Assume that* (2.6) *holds. Then for any $(x,\psi)\in[\frac{15}{16}\varepsilon,X]\times[0,\delta_3]$,*

$$|\partial_x^k w| \leq C\psi, \quad |\partial_x^k\sqrt{w}| \leq C\psi^{\frac{1}{2}}.$$

Proof. Take a smooth cut-off function $\phi(x)$ so that

$$\phi(x)=1, x\in[\frac{15\varepsilon}{16},X], \quad \phi(x)=0, x\in[0,\frac{7\varepsilon}{8}].$$

Set $f=\partial_x^k w\phi - \mu_1\psi + \mu_2\psi^{\frac{3}{2}-\beta}$ with constants μ_1,μ_2. Let β be small enough in Lemma 2.4. Then it holds in $[\frac{7\varepsilon}{8},X]\times[0,\delta_3]$ that

$$|\partial_x^k w| \leq C\psi^{1-\beta}, \quad |\partial_x^k\sqrt{w}| \leq C\psi^{\frac{1}{2}-\beta}. \tag{2.14}$$

We denote $\Omega=\{(x,\psi)|0<x\leq X, 0<\psi<\delta_3\}$. As in [31], we have $f(x,0)=0$, $f(0,\psi)\leq 0$ and $f(x,\delta_3)\leq 0$ by taking μ_1 large depending on μ_2.

We claim that the maximum of f can not be achieved in the interior. By (1.5), we have

$$\partial_x\partial_x^k w-\sqrt{w}\partial_\psi^2\partial_x^k w=-2\partial_x^k(\rho\partial_x P)+\sum_{m=0}^{k-1}C_k^m(\partial_x^{k-m}\sqrt{w})\partial_\psi^2\partial_x^m w+2\sum_{m=0}^{k}C_k^m\partial_x^{k-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w,$$

and

$$\partial_\psi^2\partial_x^m w=\partial_x^m\partial_\psi^2 w=\partial_x^m\left(\frac{\partial_x w}{\sqrt{w}}+\frac{2\rho\partial_x P}{\sqrt{w}}-\frac{2\partial_x\rho}{\rho}\sqrt{w}\right).$$

For any $x\geq\frac{7\varepsilon}{8}$, $0\leq j\leq k-1$ and $0\leq m\leq k-1$, from (2.6) and (2.14), we get

$$|\partial_x^j w|\leq C\psi,\quad |\partial_x^k w|\leq C\psi^{1-\beta},\quad |\partial_x^{k-m}\sqrt{w}|\leq C\psi^{\frac12-\beta}.$$

Then let $\beta\ll\frac12$, for $0\leq m\leq k-1$ and $x\geq\frac{7\varepsilon}{8}$, we obtain

$$|\partial_\psi^2\partial_x^m w|\leq C\psi^{\frac12-\beta}+C\psi^{-\frac12}+C\psi^{\frac12-\beta}\leq C\psi^{-\frac12}.$$

Therefore, we conclude that for $x\geq\frac{7\varepsilon}{8}$,

$$\partial_x\partial_x^k w-\sqrt{w}\partial_\psi^2\partial_x^k w\leq C+C\psi^{-\beta}+C\psi^{1-\beta}\leq C\psi^{-\beta}.$$

By the above inequality and (2.1), it holds at $p=p_0$ that

$$\partial_x f-\sqrt{w}\partial_\psi^2 f=\partial_x\partial_x^k w-\sqrt{w}\partial_\psi^2\partial_x^k w+\partial_x^k w\partial_x\phi-\sqrt{w}\partial_\psi^2(-\mu_1\psi+\mu_2\psi^{\frac32-\beta})\leq C_2\psi^{-\beta}-\xi\mu_2\psi^{-\beta},$$

where $\xi=\left(\frac32-\beta\right)\left(\frac12-\beta\right)\sqrt{m}>0$. Then we have $\partial_x f-\sqrt{w}\partial_\psi^2 f<0$ in Ω by taking μ_2 large depending on C_2. This means that the maximum of f can not be achieved in the interior. Therefore, we have $\max_{\bar\Omega} f\leq 0$. Similarly, we can prove that $\max_{\bar D}-\partial_x^k w\phi-\mu_1\psi+\mu_2\psi^{\frac32-\beta}\leq 0$. So, for any $(x,\psi)\in[\frac{15}{16}\varepsilon,X]\times[0,\delta_3]$, we have

$$|\partial_x^k w|\leq\mu_1\psi-\mu_2\psi^{\frac32-\beta}\leq\mu_1\psi.$$

Combining with (2.6) and (2.13), it holds in $[\frac{15}{16}\varepsilon,X]\times[0,\delta_3]$ that

$$|\partial_x^k\sqrt{w}|\leq C\psi^{\frac12}.$$

This completes the proof of the lemma. $\square$

Lemma 2.6. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P\leq 0$ are smooth. Assume that* (2.6) *holds. Then for any $(x,\psi)\in[\varepsilon,X]\times[0,\delta_4]$,*

$$|\partial_\psi\partial_x^k w|\leq C,\quad |\partial_\psi^2\partial_x^k w|\leq C\psi^{-1},\quad |\partial_x^{k+1}w|\leq C\psi^{-\frac12}.$$

Proof. By Lemma 2.5 and (2.6), for any $(x,\psi)\in[\frac{15}{16}\varepsilon,X]\times[0,\delta_3]$,

$$|\partial_x^j w|\leq C\psi,\quad |\partial_x^j\sqrt{w}|\leq C\psi^{\frac12},\quad 0\leq j\leq k. \tag{2.15}$$

Set $\Psi_0=\min\{\frac23\delta_3,\frac{\varepsilon}{16}\}$, for $(x_0,\psi_0)\in[\varepsilon,X]\times(0,\Psi_0]$, we denote

$$\Omega=\big\{(x,\psi)|x_0-\psi_0^{\frac32}\leq x\leq x_0,\frac12\psi_0\leq\psi\leq\frac32\psi_0\big\}.$$

A direct calculation gives

$$\begin{aligned}\partial_x\partial_x^k w-\sqrt{w}\partial_\psi^2\partial_x^k w=&-2\partial_x^k(\rho\partial_x P)+\partial_x^k\sqrt{w}\partial_\psi^2 w+\sum_{m=1}^{k-2}C_k^m\left(\partial_x^{k-m}\sqrt{w}\right)\partial_\psi^2\partial_x^m w\\&+C_k^{k-1}\frac{\partial_x w}{2\sqrt{w}}\partial_\psi^2\partial_x^{k-1}w+2\sum_{m=0}^{k}C_k^m\partial_x^{k-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w.\end{aligned}$$

By (1.5), we obtain

$$\begin{aligned}\partial_\psi^2\partial_x^m w &= \partial_x^m\partial_\psi^2 w\\ &= \partial_x^m\left(\frac{\partial_x w}{\sqrt{w}}+\frac{2\rho\partial_x P}{\sqrt{w}}-2\frac{\partial_x\rho}{\rho}\sqrt{w}\right)\\ &= \frac{\partial_x^{m+1}w}{\sqrt{w}}+\sum_{l=1}^{m}C_m^l\partial_x^{m-l+1}w\partial_x^l\frac{1}{\sqrt{w}}+\partial_x^m\left(\frac{2\rho\partial_x P}{\sqrt{w}}\right)-\partial_x^m\left(2\frac{\partial_x\rho}{\rho}\sqrt{w}\right),\end{aligned}$$

and

$$\partial_x^k\sqrt{w}=\partial_x^{k-1}\frac{\partial_x w}{2\sqrt{w}}=\frac{\partial_x^k w}{2\sqrt{w}}+\sum_{l=1}^{k-1}C_{k-1}^l\partial_x^{k-1-l+1}w\partial_x^l\frac{1}{2\sqrt{w}}.$$

Then

$$\begin{aligned}\partial_x\partial_x^k w-&\sqrt{w}\partial_\psi^2\partial_x^k w\\ =&-2\partial_x^k(\rho\partial_x P)+\sum_{m=1}^{k-2}C_k^m(\partial_x^{k-m}\sqrt{w})\partial_\psi^2\partial_x^m w\\ &+\frac{\partial_x^k w}{2\sqrt{w}}\partial_\psi^2 w+\sum_{l=1}^{k-1}C_{k-1}^l\partial_x^{k-1-l+1}w\partial_x^l\left(\frac{1}{2\sqrt{w}}\right)\partial_\psi^2 w\\ &+C_k^{k-1}\frac{\partial_x w\partial_x^k w}{2w}+2\sum_{m=0}^{k}C_k^m\partial_x^{k-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w\\ &+C_k^{k-1}\frac{\partial_x w}{2\sqrt{w}}\left[\sum_{l=1}^{k-1}C_{k-1}^l\partial_x^{k-l}w\partial_x^l\frac{1}{\sqrt{w}}+\partial_x^{k-1}\left(\frac{2\rho\partial_x P}{\sqrt{w}}\right)-\partial_x^{k-1}\left(2\frac{\partial_x\rho}{\rho}\sqrt{w}\right)\right].\end{aligned}$$

The following transformation f are defined:

$$\Omega\to\widetilde{\Omega}:=[-1,0]_{\tilde{x}}\times[-\frac{1}{2},\frac{1}{2}]_{\tilde{\psi}},\quad (x,\psi)\mapsto(\tilde{x},\tilde{\psi}),$$

where $x-x_0=\psi_0^{\frac{3}{2}}\tilde{x},\psi-\psi_0=\psi_0\tilde{\psi}$. Let $f=\partial_x^k w\psi_0^{-1}$, we get

$$\begin{aligned}\partial_{\tilde{x}}f-&\frac{\sqrt{w}}{\psi_0^{\frac{1}{2}}}\partial_{\tilde{\psi}}^2 f-\frac{1}{2\sqrt{w}}\partial_\psi^2 w\psi_0^{\frac{3}{2}}f-\frac{\partial_x w}{2w}\psi_0^{\frac{3}{2}}f\\ &=-2\psi_0^{\frac{1}{2}}\partial_x^k(\rho\partial_x P)+\psi_0^{\frac{1}{2}}\sum_{m=1}^{k-2}C_k^m(\partial_x^{k-m}\sqrt{w})\partial_\psi^2\partial_x^m w\\ &\quad+\psi_0^{\frac{1}{2}}\sum_{l=1}^{k-1}C_{k-1}^l\partial_x^{k-l}w\left(\partial_x^l\frac{1}{2\sqrt{w}}\right)\partial_\psi^2 w\\ &\quad+2\psi_0^{\frac{1}{2}}\sum_{m=0}^{k}C_k^m\partial_x^{k-m}\left(\frac{\partial_x\rho}{\rho}\right)\partial_x^m w\\ &\quad+\psi_0^{\frac{1}{2}}\frac{\partial_x w}{2\sqrt{w}}\left[\sum_{l=1}^{k-1}C_{k-1}^l\partial_x^{k-l}w\partial_x^l\frac{1}{\sqrt{w}}+\partial_x^{k-1}\left(\frac{2\rho\partial_x P}{\sqrt{w}}\right)-\partial_x^{k-1}\left(2\frac{\partial_x\rho}{\rho}\sqrt{w}\right)\right]\\ &:=F.\end{aligned}$$

From the proof of Lemma 2.2 and Lemma 2.6, we know that in $\widetilde{\Omega}$ for $\alpha\in(0,1)$,

$$|f|\le C,\quad 0<c\le\psi_0^{-\frac{1}{2}}\sqrt{w}\le C,\quad |\psi_0^{-\frac{1}{2}}\sqrt{w}|_{\mathcal{C}^\alpha(\widetilde{\Omega})}\le C.$$

By (2.6), (2.15) and the equality

$$\partial_\psi\left(\partial_\psi^2\partial_x^m w\right)=\frac{\partial_\psi\partial_x^{m+1}w}{\sqrt{w}}-\frac{\partial_\psi w\partial_x^{m+1}w}{2(\sqrt{w})^3}+\sum_{l=1}^{m}C_m^l\partial_x^{m-l+1}\partial_\psi w\partial_x^l\frac{1}{\sqrt{w}}$$
$$+\sum_{l=1}^{m}C_m^l\partial_x^{m-l+1}w\partial_x^l\frac{\partial_\psi w}{-2(\sqrt{w})^3}+\partial_x^m\left(\frac{\rho\partial_x P\partial_\psi w}{-(\sqrt{w})^3}\right)-\partial_x^m\left(\frac{\partial_x\rho}{\rho}\frac{\partial_\psi w}{\sqrt{w}}\right),$$

we can conclude that for $j\le k-1$ and $m\le k-2$,

$$\left|\nabla_{\tilde{x},\tilde{\psi}}\partial_x^j\sqrt{w}\right|\le C\psi_0^{\frac12},\quad \left|\nabla_{\tilde{x},\tilde{\psi}}\partial_x^j\left(\frac{1}{\sqrt{w}}\right)\right|\le C\psi_0^{-\frac12},\quad \left|\nabla_{\tilde{x},\tilde{\psi}}\partial_\psi^2\partial_x^m w\right|\le C\psi_0^{-\frac12}.$$

Combining (2.4) with (2.5), we can obtain

$$\left|\frac{1}{2\sqrt{w}}\partial_\psi^2 w\psi_0^{\frac32}+\frac{\partial_x w}{2w}\psi_0^{\frac32}\right|_{\mathcal{C}^\alpha(\tilde\Omega)}+|F|_{\mathcal{C}^\alpha(\tilde\Omega)}\le C.$$

By standard interior a priori estimates, we obtain

$$|\partial_{\tilde{x}}f|_{L^\infty([-\frac14,0]_{\tilde{x}}\times[-\frac18,\frac18]_{\tilde\psi})}+|\partial_{\tilde\psi}f|_{L^\infty([-\frac14,0]_{\tilde{x}}\times[-\frac18,\frac18]_{\tilde\psi})}+|\partial_{\tilde\psi}^2f|_{L^\infty([-\frac14,0]_{\tilde{x}}\times[-\frac18,\frac18]_{\tilde\psi})}\le C.$$

Therefore, this means that

$$|\partial_x^{k+1}w\,(x_0,\psi_0)\,|\le C\psi_0^{-\frac12},\quad |\partial_\psi\partial_x^k w\,(x_0,\psi_0)\,|\le C,\quad |\partial_\psi^2\partial_x^k w\,(x_0,\psi_0)\,|\le C\psi_0^{-1}.$$

Since (x_0,ψ_0) is arbitrary, this completes the proof of the lemma. □

Lemma 2.7. *If u is a global solution of the system* (1.1) *in Theorem 1.1. Assume that the initial data u_0 satisfies the condition given in Theorem 1.1, the density ρ and $\partial_x P\le 0$ are smooth. Assume $0<\varepsilon<X$ and integer $m,k\ge 0$. Then there exists a positive constant $\delta>0$ such that for any $(x,\psi)\in[\varepsilon,X]\times[0,\delta]$,*

$$|\partial_\psi^m\partial_x^k w|\le C\psi^{1-m}. \tag{2.16}$$

Proof. From Lemma 2.1, (2.1), Lemma 2.2 and Lemma 2.3, a direct calculation can prove that

$$\left|\partial_x^k\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac12},\quad \left|\partial_x^k\partial_\psi\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac32},\quad \left|\partial_x^k\partial_\psi^2\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac52},$$

and (2.16) holds for $m=0,1,2$. Then for $0\le m\le j$ with $j\ge 1$, we inductively assume that

$$|\partial_\psi^m\partial_x^k w|\le C\psi^{1-m},\quad \left|\partial_x^k\partial_\psi^m\frac{1}{\sqrt{w}}\right|\le C\psi^{-\frac12-m}. \tag{2.17}$$

In the next part, we will prove that (2.17) still holds for $m=j+1$. By (1.5), we obtain

$$\partial_\psi^{j+1}\partial_x^k w=\partial_\psi^{j-1}\partial_x^k\partial_\psi^2 w=\partial_x^k\partial_\psi^{j-1}\left(\frac{\partial_x w}{\sqrt{w}}+\frac{2\rho\partial_x P}{\sqrt{w}}-2\frac{\partial_x\rho}{\rho}\frac{w}{\sqrt{w}}\right)$$
$$=\partial_x^k\left(\sum_{i=0}^{j-1}C_{j-1}^i\partial_\psi^{j-1-i}\partial_x w\partial_\psi^i\frac{1}{\sqrt{w}}+2\rho\partial_x P\partial_\psi^{j-1}\frac{1}{\sqrt{w}}-2\frac{\partial_x\rho}{\rho}\sum_{i=0}^{j-1}C_{j-1}^i\partial_\psi^{j-1-i}w\partial_\psi^i\frac{1}{\sqrt{w}}\right).$$

Combining with (2.17), we get

$$|\partial_\psi^{j+1}\partial_x^k w|\le C\psi^{\frac32-j}+C\psi^{\frac12-j}+C\psi^{\frac32-j}\le C\psi^{\frac12-j}. \tag{2.18}$$

By straight calculations, we get

$$0=\partial_x^k\partial_\psi^{j+1}\left(\frac{1}{\sqrt{w}}\frac{1}{\sqrt{w}}w\right)$$
$$=\partial_x^k\Bigg[2\sqrt{w}\partial_\psi^{j+1}\frac{1}{\sqrt{w}}+\sum_{i=1}^{j}\sum_{l=0}^{j+1-i}C_{j+1}^iC_{j+1-i}^l\left(\partial_\psi^i\frac{1}{\sqrt{w}}\right)\left(\partial_\psi^l\frac{1}{\sqrt{w}}\right)\partial_\psi^{j+1-l-i}w$$
$$+\sum_{l=0}^{j}C_{j+1}^l\frac{1}{\sqrt{w}}\left(\partial_\psi^l\frac{1}{\sqrt{w}}\right)\partial_\psi^{j+1-l}w\Bigg].$$

Combining the above equality withn (2.17), we can conclude that

$$|\partial_x^k \partial_\psi^{j+1} \frac{1}{\sqrt{w}}| \leq C\psi^{-\frac{3}{2}-j}.$$

This completes the proof of the lemma. $\square$

Remark 2.8. *This lemma shows that, when $m > 1$, the derivatives $|\partial_\psi^m \partial_x^k w| \leq C\psi^{1-m}$ will blow up at the boundary $\psi = 0$. Actually, the exponent $1 - m$ is* ***optimal.***

3. Global C^∞ regularity

In this section, we prove our main Theorem 1.2. First, we prove the regularity of the solution u in the domain $\{(x,\psi)|\varepsilon \leq x \leq X, 0 \leq y \leq Y_1\}$ for some small positive constant Y_1.

Proof of Theorem 1.3:

Proof. For the convenience of proof, we denote the Von Mises transformation as follows

$$(\tilde{x}, \psi) = \left(x, \int_0^y \tilde{u} dy\right).$$

A direct calculation gives (see P13 in [31])

$$\partial_y = \sqrt{w}\partial_\psi, \quad \partial_x = \partial_{\tilde{x}} + \partial_x\psi(x,y)\partial_\psi, \quad \partial_x\psi = \frac{1}{2}\sqrt{w}\int_0^\psi w^{-\frac{3}{2}}\partial_{\tilde{x}} w d\psi.$$

By (2.1) and Lemma 2.3, we have $|\partial_x\psi| \leq C\psi$. Due to $\partial_y = \sqrt{w}\partial_\psi$, we obtain

$$\begin{aligned}
\partial_x^k 2\partial_y \tilde{u} &= \left(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi\right)^k \partial_\psi w, \\
\partial_x^k 2\partial_y^2 \tilde{u} &= \left(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi\right)^k \left(\partial_{\tilde{x}} w + 2\rho\partial_x P - 2\frac{\partial_x\rho}{\rho} w\right) \\
&= \left(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi\right)^k (\partial_{\tilde{x}} w) + 2\partial_x^k(\rho\partial_x P) - 2\left(\frac{\partial_x\rho}{\rho}\right)(\partial_{\tilde{x}} + \partial_x\psi\partial_\psi)^k w - 2\partial_x^k\left(\frac{\partial_x\rho}{\rho}\right) w.
\end{aligned}$$

By $|\partial_x\psi| \leq C\psi$ and Lemma 2.7, we obtain that Theorem 1.3 holds for $m = 0, 1, 2$,

$$|\partial_x^k \partial_y \tilde{u}| + |\partial_x^k \partial_y^2 \tilde{u}| \leq C. \tag{3.1}$$

We inductively assume that for any integer k and $m \geq 1$,

$$|\partial_x^k \partial_y^j \tilde{u}| \leq C, \quad j \leq m. \tag{3.2}$$

A direct calculation gives

$$\begin{aligned}
&\partial_x^k \partial_y^{m+1} \tilde{u} \\
&= \partial_x^k \partial_y^{m-1} \partial_y^2 \tilde{u} \\
&= \partial_x^k \partial_y^{m-1} \left(\tilde{u}\partial_x\tilde{u} - \partial_y\tilde{u}\int_0^y \partial_x\tilde{u} dy - \frac{\partial_x\rho}{\rho}\tilde{u}^2\right) \\
&= \partial_x^k \Bigg(\sum_{i=0}^{m-1} C_{m-1}^i \partial_y^{m-1-i}\tilde{u}\partial_y^i\partial_x\tilde{u} - \sum_{i=0}^{m-2} C_{m-1}^{i+1}\partial_y^{m-1-i}\tilde{u}\partial_y^i\partial_x\tilde{u} - \partial_y^m\tilde{u}\int_0^y \partial_x\tilde{u} dy - \frac{\partial_x\rho}{\rho}\partial_y^{m-1}\tilde{u}^2\Bigg),
\end{aligned}$$

we can deduce from (3.1) and (3.2) that

$$|\partial_x^k \partial_y^j \tilde{u}| \leq C, \quad j \leq m+1. \quad \Rightarrow \quad |\partial_x^k \partial_y^j u| \leq C, \quad j \leq m+1.$$

This completes the proof of the theorem. $\square$

Finally, Theorem 1.2 follows directly by combining Theorem 1.3 and Theorem 1.4.

4. Appendix

In this Appendix, we prove (1.5) is a uniform parabolic equation. The proof is based on the classical parabolic maximum principle. Here we give proof for the reader's convenience.

Proof. By (1.2) and $\partial_x P \leq 0$, we obtain

$$C \geq U^2(x) = U^2(0) - 2\int_0^x \frac{\partial_x P(\rho)}{\rho} dx \geq U^2(0).$$

By (1.6) and w is increasing in ψ (see below), we know that there exist some positive constants Ψ and C_0 such that for any $(x,\psi) \in [0,X] \times [\Psi, +\infty)$

$$w \geq C_0 U^2(0). \tag{4.1}$$

From Theorem 1.1, we know that there exist positive constants y_0, M, m such that for any $(x,\psi) \in [0,X] \times [0,y_0]$, (without loss of generality, we can take y_0 to be small enough.)

$$M \geq \partial_y \tilde{u}(x,y) \geq m. \tag{4.2}$$

The fact that $\psi \sim y^2$ near the boundary $y=0$ (see Remark 4.1 in [31]). Then for some small positive constant $\kappa < 1$, we have

$$\frac{\kappa}{2} y_0^2 \leq \psi \leq \kappa y_0^2 \Rightarrow \sigma y_0 \leq y \leq \frac{y_0}{2}, \tag{4.3}$$

for some positive constant σ depending only on κ, m, M. We denote

$$\Omega = \big\{(x,\psi) | 0 \leq x \leq X, \frac{\kappa}{2} y_0^2 \leq \psi \leq +\infty\big\},$$

where $\frac{\kappa}{2} y_0^2 < \Psi$. By (4.2) and (4.3), we get $\tilde{u}(x, \sigma y_0) \geq m\sigma y_0$. Then for any $x \in [0,X]$, we have

$$w(x, \frac{\kappa}{2} y_0^2) \geq m^2 \sigma^2 y_0^2. \tag{4.4}$$

Since the initial data u_0 satisfies the condition given in Theorem 1.1 and $w = \tilde{u}^2$, we know $w(0,\psi) > 0$ for $\psi > 0$ and then we have for some positive constant ζ, such that for $\psi \in [\frac{\kappa}{2} y_0^2, \Psi]$,

$$w(0,\psi) > \zeta. \tag{4.5}$$

Then we only consider in

$$\Omega_1 = \big\{(x,\psi) | 0 \leq x \leq X, \frac{\kappa}{2} y_0^2 \leq \psi \leq \Psi\big\}.$$

We denote $G(x,\psi) := e^{-\lambda x} \partial_\psi w(x,\psi)$, which satisfies the following system in the region $\Omega_0 = \{(x,\psi) | 0 \leq x < X, 0 < \psi < +\infty\}$:

$$\begin{cases} \partial_x G - \dfrac{\partial_\psi w}{2\sqrt{w}} \partial_\psi G - \sqrt{w} \partial_\psi^2 G + (\lambda - 2\dfrac{\partial_x \rho}{\rho}) G = 0, \\ G|_{x=0} = \partial_\psi w_0(\psi), \quad G|_{\psi=0} = 2e^{-\lambda x} \partial_y \tilde{u}|_{y=0}, \quad G|_{\psi=+\infty} = 0. \end{cases} \tag{4.6}$$

To apply the maximum principle for the problem (4.6), we choose λ properly large such that $\lambda - 2\frac{\partial_x \rho}{\rho} \geq 0$. Duo to

$$G|_{x=0} = \partial_\psi w_0(\psi) \geq 0, \quad G|_{\psi=0} = 2e^{-\lambda x} \partial_y \tilde{u}|_{y=0} > 0, \quad G|_{\psi=+\infty} = 0,$$

by the maximum principle, it follows that

$$G(x,\psi) = e^{-\lambda x} F(x,\psi) = e^{-\lambda x} \partial_\psi w \geq 0, \quad (x,\psi) \in [0, X^*) \times \mathbb{R}_+,$$

which means $\partial_\psi w \geq 0$ in $[0,X) \times \mathbb{R}_+$. Hence, w is increasing in ψ. Therefore, we know that there exists a positive constant $\lambda \geq m^2 \sigma^2 y_0^2$ such that for any $x \in [0,X]$,

$$w(x,\Psi) \geq \lambda. \tag{4.7}$$

By (1.5), for any $\varepsilon > 0$, we know $W := w + \varepsilon x$ satisfies the following system in Ω_1 :

$$\begin{cases} \partial_x W - \sqrt{w}\partial_\psi^2 W - 2\frac{\partial_x \rho}{\rho} W = \mathcal{F}, \\ W|_{x=0} = W_0 > \zeta, \quad W|_{\psi=\frac{\kappa}{2}y_0^2} = W_1 \geq m^2\sigma^2 y_0^2, \quad W|_{\psi=\Psi} = W_2 \geq \lambda, \end{cases}$$

where

$$\mathcal{F} = -2\rho\partial_x P + \varepsilon - 2\varepsilon x \frac{\partial_x \rho}{\rho}.$$

Since $\partial_x P \leq 0$, we know the diffusive term $\mathcal{F} > 0$. Therefore, the minimum cannot be attained at an interior point of Ω_1. Set

$$\eta_0 = \min\{W_0, W_1, W_2\}.$$

Then by the maximum principle, we obtain $W = w + \varepsilon x \geq \eta_0$. Let $\varepsilon \to 0$, we have $w \geq \eta_0$ in Ω_1. Then we denote

$$\eta = \min\left\{\eta_0, C_0 U^2(0)\right\} > 0,$$

combining with (4.1), we have $w \geq \eta$ in Ω. Therefore, there exists some positive constant c such that $c \leq w$ in Ω. From Theorem 1.1, we have $w \leq C$ in Ω. In sum, there exist positive constants c, C such that $c \leq w \leq C$ in Ω. This further means that

$$0 < \sqrt{c} \leq \sqrt{w} \leq \sqrt{C}, \tag{4.8}$$

where C depending on X. Therefore, we prove (1.5) is a uniform parabolic equation. Furthermore, by Theorem 1.1, we know $\partial_y \tilde{u}, \partial_y^2 \tilde{u}$ are bounded and continuous in $[0, X) \times \mathbb{R}_+$. Combining ρ, $\partial_x P$ are smooth, (4.8) with

$$2\partial_y \tilde{u} = \partial_\psi w, \quad 2\partial_y^2 \tilde{u} = \sqrt{w}\partial_\psi^2 w = \partial_x w - 2\frac{\partial_x \rho}{\rho} w + 2\rho\partial_x P(\rho),$$

we obtain

$$\|\sqrt{w}\|_{\mathcal{C}^\alpha(\Omega)} \leq C.$$

Once we have the above conclusion, the proof of Theorem 1.4 can be given in a similar fashion to [31] and we omit it here. □

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