Higher-order multiplicative derivative iterative scheme to solve the nonlinear problems

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Abstract

Grossman and Katz (five decade ago) suggested a new definition of differential and integral calculus which utilize the multiplicative and division operator as compared to the addition and subtraction. Multiplicative Calculus is a vital part of the applied Mathematics because of its application in the area of Biology, Science and Finance, Biomedical, Economic, etc. Therefore, we used a multiplicative calculus approach to develop a new fourth-order iterative scheme for multiple roots based on the well-known King's method. In addition, we also propose the detailed convergence analysis of our scheme with the help of multiplicative calculus approach rather than the normal one. Different kinds of numerical comparisons has been suggested and obtained results are very impressive as compared to the ordinary derivative methods. Finally, the convergence of our technique is also analyzed by basin of attractions that also support the theoretical aspects.

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Abstract

Grossman and Katz (five decade ago) suggested a new definition of differential and integral calculus which utilize the multiplicative and division operator as compared to the addition and subtraction. Multiplicative Calculus is a vital part of the applied Mathematics because of its application in the area of Biology, Science and Finance, Biomedical, Economic, etc. Therefore, we used a multiplicative calculus approach to develop a new fourth-order iterative scheme for multiple roots based on the well-known King's method. In addition, we also propose the detailed convergence analysis of our scheme with the help of multiplicative calculus approach rather than the normal one. Different kinds of numerical comparisons has been suggested and obtained results are very impressive as compared to the ordinary derivative methods. Finally, the convergence of our technique is also analyzed by basin of attractions that also support the theoretical aspects.

Mathematics subject classification: 65H05,65G99.
Keywords: Multiplicative derivative, Nonlinear equations, Order of convergence.

1 Introduction

In the $70s$ of the 20^{th} century multiplicative calculus introduced by Grossman and Katz [1]. Many of scholars applied the multiplicative calculus in various branches. In 2008 Bashirov et al. [2] discussed the theoretical foundations as well as various applications of multiplicative calculus. Florack and Van Assen [3] used multiplicative calculus in Biomedical image analysis. Filip and Piatecki [4] used it to investigate economic growth. In addition, Mısırlı Gurefe [5], Riza et al. [6], and Özyapıcı and Mısırlı [7] used multiplicative calculus to develop multiplicative numerical methods. On the other hands, Bashirov et al. [8] applied it to develop multiplicative differential equations. Further, Bashirov and Riza [9] and Uzer [10] extended the multiplicative calculus to include complex valued functions of complex variables, which was previously applicable only to positive real valued functions of real variables.

From the above discussion, it is straightforward to say that multiplicative calculus approach is very important part of the applied Mathematics, Computational Engineering and Applied Sciences. In last few years, researchers used multiplicative derivatives for developing new iterative schemes for the solutions of the nonlinear equations. Özyzpici et al. [11] and Ali Özyzpici [12] adopted the multiplicative calculus approach (MCA) in order to construct

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an one-point and second order scheme. But, one-point methods have many problems regarding their order of convergence and efficiency index (See Traub [13] for more details). Obtaining the multiple root of a nonlinear equation is more complicated and challenging as compared to simple root. A few main reasons behind this are: the lengthy and complicated calculations and retaining or increasing order of convergence.

Keeping these things in our mind, we suggest a new multipoint iterative technique by adopting the MCA. Till today, we didn't have a single multipoint iterative method for multiple roots that utilize the multiplicative calculus approach. Our new scheme stands on the principles of MCA and the well-known fourth-order King's method [14]. For a fair comparison of our methods with the existing methods, we choose the six different ways that are: (i) absolute error difference between two consecutive iterations (ii) order of convergence (iii) number of iterations (iv) CPU timing (v) the graphs of absolute errors and (vi) bar graphs. On the basis of six different ways of comparisons, we conclude that our new King's scheme perform much better in comparison of the existing methods. Finally, we study the basin of attraction which also support the numerical results.

The rest content of the paper are summarized as: Section 2 discussed the definition and basics terms of multiplicative calculus. The proposed method and its analysis of convergence is represented in Section 3. Section 4 depicts the numerical results. The basins of attraction of proposed method are discussed in Section 5. Finally, conclusion is represented in Section 6.

2 Basic terms of Multiplicative Calculus

Definition 2.1 *Let $g(x)$ be a real positive valued function in the open interval (a, b). Assume function $g(x)$ be changes in $x \in (a, b)$ s.t. $g(x)$ changes to $g(x + h)$. Then multiplicative forward operator [7] denoted as Δ^* defined as follows*

$$\Delta^* g(x) = \frac{g(x+h)}{g(x)} \tag{2.1}$$

By considring the operator Δ^* (2.1), multiplicative derivative can be defined as below

$$g^*(x) = \lim_{h \to 0} (\Delta^* g)^{\frac{1}{h}} \tag{2.2}$$

The function $g^*(x)$ is said to be multiplicative differentiable at x if the limit on R.H.S exists.
If g is positive function and the derivative of g at x exist, then n^{th} multiplicative derivatives of g exist and

$$g^{*(n)}(x) = exp\left\{(ln \circ g)^{(n)}(x)\right\} \tag{2.3}$$

Theorem 2.2 *(Multiplicative Taylor Theorem in one variable) [22] Let $g(x)$ be a function in open interval (a, b) s.t the functions is $n + 1$ times $*$ differentiable on (a, b). Then for any $x, x + h \in A(a, b)$, there is a number $\theta \in (a, b)$ such that*

$$g(x+h) = \prod_{m=0}^{n} \left(g^{*(m)}(x)\right)^{\dfrac{h^m}{m!}} . \left(g^{*(n+1)}(x+\theta h)\right)^{\dfrac{h^{n+1}}{(n+1)!}} \tag{2.4}$$

Theorem 2.3 *(Multiplicative Newton-Raphson theorem) [22] Consider r be a simple root of nonlinear equation $g(x) = 1$ (or $h(x) = g(x) - 1 = 0$). According to the multiplicative analysis [19], the multiplicative Newton theorem can be expressed as follows*

$$g(x) = g(x_q) \int_{x_q}^{x} g^*(z)dz = g(x_q)exp\left(\int_{x_q}^{x} (lng(z))'dz\right) \tag{2.5}$$

For definite integrals, Equation (2.5) can be written using Newton Cotes quadrature of zeroth degree as $\int_{x_q}^{x} g^*(z)dz = exp\left(\int_{x_q}^{x} (ln\ g(z))'dz\right) \equiv exp((x - x_q)(ln\ g(x_q))') = (g^*(x_q))^{x-x_q}$
Since $g(x) = 1$, the Explicit Multiplicative Newton (MN) is obtained as

$$x_{q+1} = x_q - \frac{ln\ g(x_q)}{ln\ g^*(x_q)} \tag{2.6}$$

In next section, we proposed the Multiplicative King's method scheme and its analysis of convergence.

3 The Proposed Method and Analysis of Convergence

The proposed King's iterative method in multiplicative derivative reprsented as

$$\begin{aligned} y_q &= x_q - \frac{ln\ g(x_q)}{ln\ g^*(x_q)}, \\ x_{q+1} &= y_q - \left(\frac{log\ g(x_q) + \beta log\ g(y_q)}{log\ g(x_q) + (\beta - 2)log\ g(y_q)}\right)\left(\frac{log\ g(y_q)}{log\ g^*(x_q)}\right). \end{aligned} \tag{3.1}$$

Where q is iteration step, $g^*(x)$ is multiplicative derivative, and β is a free parameter.
For convergence analysis, we have proved the following theorem.

Theorem 3.1 *For an open interval I, let $r \in I$ be a multiplicative zero of a sufficiently multiplicative differential function $g : I \subseteq \mathbb{R} \to \mathbb{R}^+$, then multiplicative King's method has fourth order of convergence with error*

$$x_{q+1} = (b_2^3 + 2\beta b_2^3 - b_2 b_3)e_q^4 + \mathcal{O}(e_q^5).$$

Proof Let r be a simple root of equation $g(x) = 1$ and $e_q = x_q - r$ be error at q^{th} iteration. Using the multiplicative Taylor expansions (2.4) for function $g(x)$, it can be written as

$$g(x_q) = g(r + e_q) = g(r)(g^*(r))^{e_q} (g^*)^2(r)^{\frac{e_q^2}{2!}} (g^*)^3(r)^{\frac{e_q^3}{3!}} \mathcal{O}^*(e_q^4). \tag{3.2}$$

If we take the natural logarithm on both sides, we get

$$
\begin{aligned}
\ln g(x_q) &= \ln g(r) + \ln g^*(r)e_q + \ln(g^*)^2(r)\frac{e_q^2}{2!} + \ln(g^*)^3(r)\frac{e_q^3}{3!}\mathcal{O}(e_q^4), \\
&= \ln g^*(r)\left(e_q + \frac{1}{2!}\frac{\ln(g^*)^2(r)}{\ln g^*(r)}e_q^2 + \frac{1}{3!}\frac{\ln(g^*)^3(r)}{\ln g^*(r)}e_q^3 + \mathcal{O}(e_q^4)\right), \\
&= \ln g^*(r)\left(e_q + b_2e_q^2 + b_3e_q^3 + \mathcal{O}(e_q^4)\right),
\end{aligned} \tag{3.3}
$$

where $b_j = \dfrac{1}{j!}\dfrac{\ln(g^*)^j(r)}{\ln g^*(r)}$.
On the other hand, we have

$$
\begin{aligned}
\ln g^*(x_q) &= \ln g^*(r) + \ln(g^*)^2(r)e_q + \ln(g^*)^3(r)\frac{e_q^2}{2!} + \mathcal{O}(e_q^3), \\
&= \ln g^*(r)\left(1 + \frac{1}{2!}\frac{\ln(g^*)^2(r)}{\ln g^*(r)}e_q + \frac{1}{3!}\frac{\ln(g^*)^3(r)}{\ln g^*(r)}e_q^2 + \mathcal{O}(e_q^3)\right), \\
&= \ln g^*(r)\left(1 + 2b_2e_q + 3b_3e_q^2 + \mathcal{O}(e_q^3)\right).
\end{aligned} \tag{3.4}
$$

On dividing equation (3.3) by (3.4), we have

$$
\frac{\ln g(x_q)}{\ln g^*(x_q)} = \frac{(e_q + b_2e_q^2 + b_3e_q^3 + \mathcal{O}(e_q^4))}{(1 + 2b_2e_q + 3b_3e_q^2 + \mathcal{O}(e_q^3))}. \tag{3.5}
$$

Now the first step of scheme (3.1) is obtained by using equation (3.5)

$$
y_q = r + b_2e_q^2 + 2(b_3 - b_2^2)e_q^3 + \mathcal{O}(e_q^4). \tag{3.6}
$$

By using the multiplicative Taylor expansion upon $g(y_q)$ about r, we obtain

$$
g(y_q) = g(r)(g^*(r))^{e_q}((g^*)^2(r))^{\frac{e_q^2}{2!}}((g^*)^3(r))^{\frac{e_q^3}{3!}}\mathcal{O}(e_q^4). \tag{3.7}
$$

As a result of taking the natural logarithm from both sides, we get

$$
\ln g(y_q) = \ln g^*(r)(e_q + b_2e_q^2 + b_3e_q^3 + \mathcal{O}(e_q^4)). \tag{3.8}
$$

By using equation (3.3),(3.4),(3.6) and (3.8) , we obtained the final error of scheme

$$
\begin{aligned}
x_{q+1} &= y_q - \frac{log\ g(x_q) + \beta log\ g(y_q)}{log\ g(x_q) + (\beta - 2)log\ g(y_q)}.\frac{log\ g(y_q)}{log\ g^*(x_q)}, \\
&= (b_2^3 + 2\beta b_2^3 - b_2b_3)e_q^4 + \mathcal{O}(e_q^5).
\end{aligned} \tag{3.9}
$$

Hence, the method (3.1) has fourth order of convergence.

4 Numerical Examples

In this section, we solve the nonlinear equation $g(x) = 0$ using ordinary King's method [14] denoted as (KM$_1$ $for\ \beta = 3$, KM$_2$ $for\ \beta = \frac{1}{2}$,KM$_3$ $for\ \beta = -1$ respectively), Chun method [23] denoted as (CM), Jnawali method [24] denoted as (JM) and the proposed multiplicative King's method denoted as (MKM$_1$ $for\ \beta = 3$, MKM$_2$ $for\ \beta = \frac{1}{2}$, MKM$_3$ $for\ \beta = -1$ respectively). The results obtained using these methods are presented in Tables 1–5. All computations have done in Mathematica version 11.1.1 software and the stopping criteria $|x_{q+1} - x_q| < \epsilon$ and $\epsilon = 10^{-200}$ is used. Moreover, the Approximated computational order of convergence (ACOC) is computed by using the following.

$$\rho \cong \frac{ln|\left|\dfrac{x_{q+1} - r}{x_q - r}\right|}{ln\left|\dfrac{x_q - r}{x_{q-1} - r}\right|}. \tag{4.1}$$

Numerical results indicate in the Tables 1–5 that the proposed method execute less number of iterations and reduce the computational time.

Remark: The meaning of expression $m(\pm n)$ is $m \times 10^{\pm n}$ and d represents that scheme is divergent in all the tables.

Example 4.1 *Firstly, we consider the population growth model that formulate the following nonlinear equation*

$$g(x) = \frac{1000}{1564}e^x + \frac{435}{1564}(e^x - 1) - 1.$$

In this model we evaluate the birth rate denoted as x, if in a specific local area has 1000 thousand people at first and 435 thousand move into the local area in the first year. Likewise, assume 1564 thousand individuals toward the finish of one year. The computed results towards the root $x_r = 0.1009979\ldots$ are displayed in Table 1. Clearly, the proposed methods MKM_1, MKM_2, MKM_3 shows better results in terms of consecutive error and number of iteration in comparison of existing ones.

Example 4.2 *Next, we apply the proposed method on some of the following academic problems.*

(a) $g(x) = (x + 2)e^x - 1$ *having approximate root* $x_r = -0.4428544\ldots.$

(b) $g(x) = (x - 1)^6 - 1$ *having exact root* $x_r = 2.$

(c) $g(x) = e^{x^3+7x-30} - 1$ *with an approximate root* $x_r = 2.3741\ldots.$

(d) $g(x) = xe^{x^2} - sin^2x + 3cos\ x - 4$ *having approximate root* $x_r = 1.0651\ldots.$

In Table 2, 3 and 5 it is clearly seen that the proposed method shows more effective results as compared to others in terms of absolute error and consecutive error. In Table 4 the proposed method converges and giving the results while all other methods fails to converge.

Method	*q*	$\lvert x_q - x_{q-1}\rvert$	$\lvert g(x_q)\rvert$	ρ	*No. of iteration*	*C.P.U time*
KM_1	2 3 4	4.4(−5) 2.4(−18) 2.1(−71)	3.7(−5) 2.0(−18) 1.8(−71)	4.000	5	0.32
MKM_1	2 3 4	2.7(−18) 4.8(−74) 4.6(−297)	1.000	4.000	4	0.32
KM_2	2 3 4	4.1(−7) 3.8(−27) 3.0(−107)	3.5(−7) 3.3(−27) 2.5(−107)	4.000	5	0.39
MKM_2	2 3 4	2.7(−19) 2.4(−78) 1.5(−314)	1.000	4.000	4	0.26
KM_3	2 3 4	1.8(−5) 1.8(−20) 1.9(−80)	1.5(−5) 1.6(−20) 1.6(−80)	4.000	5	0.39
MKM_3	2 3 4	2.2(−20) 5.5(−83) 2.0(−333)	1.000	4.000	4	0.29
CM	2 3 4	1.8(−5) 4.8(−20) 2.5(−78)	1.5(-5) 4.1(−20) 2.1(−78)	4.000	5	0.37
JM	2 3 4	4.4(−6) 8.8(−23) 1.5(−89)	3.7(−6) 7.5(−23) 1.3(−89)	4.000	5	0.34

Table 1: Results of population growth model with initial guess $x_0 = 1$

Method	*q*	$\lvert x_q - x_{q-1}\rvert$	$\lvert g(x_q)\rvert$	ρ	*No. of Iteration*	*C.P.U time*
KM_1	2 3 4	3.4(−1) 1.2(−2) 4.0(−8)	7.4(−1) 2.0(−2) 6.6(−8)	3.996	7	0.25
MKM_1	2 3 4	4.2(−10) 2.3(−40) 2.0(−161)	1.000	4.000	5	0.25
KM_2	2 3 4	1.2(−1) 7.5(−5) 1.5(−17)	2.1(−1) 1.2(−4) 2.4(−17)	4.000	6	0.29
MKM_2	2 3 4	1.7(−10) 2.3(−42) 8.0(−170)	1.000	4.000	5	0.28
KM_3	2 3 4	*d*	*d*	*d*	*d*	*d*
MKM_3	2 3 4	4.7(−9) 4.2(−36) 2.8(−144)	1.000	4.000	5	0.28
CM	2 3 4	2.9(−1) 5.7(−3) 1.6(−9)	6.1(−1) 9.4(−3) 2.5(−9)	4.000	5	0.39
JM	2 3 4	2.3(−1) 1.6(−3) 5.7(−12)	4.5(−1) 2.7(−3) 9.3(−12)	4.000	6	0.26

Table 2: *Example* 4.2(*a*) *at initial point* $x_0 = 2$

Method	q	$\lvert x_q - x_{q-1} \rvert$	$\lvert g(x_q) \rvert$	ρ	*No. of iteration*	*C.P.U time*
KM_1	2 3 4	1.4(−2) 3.5(−6) 1.5(−20)	8.9(−2) 2.1(−5) 9.2(−20)	4.000	6	0.28
MKM_1	2 3 4	1.8(−5) 7.3(−20) 2.0(−77)	1.000	4.000	4	0.29
KM_2	2 3 4	2.1(−3) 4.8(−10) 1.2(−36)	1.3(−2) 2.9(−9) 7.3(−36)	4.000	6	0.37
MKM_2	2 3 4	3.7(−11) 1.5(−43) 4.5(−173)	1.000	4.000	4	0.39
KM_3	2 3 4	d	d	d	d	d
MKM_3	2 3 4	4.3(−9) 9.7(−35) 2.6(−137)	1.000	4.000	4	0.31
CM	2 3 4	1.0(−2) 7.1(−7) 1.8(−23)	6.4(−2) 4.3(−6) 1.1(−22)	3.609	4	0.31
JM	2 3 4	5.9(−3) 4.3(−8) 1.2(−28)	3.6(−2) 2.9(−7) 7.4(−28)	4.000	5	0.26

Table 3: *Example* 4.2(b) *with initial guess* $x_0 = 2.5$

Method	q	$\lvert x_q - x_{q-1} \rvert$	$\lvert g(x_q) \rvert$	ρ	*No. of iteration*	*C.P.U time*
KM_1	2 3 4	d	d	d	d	d
MKM_1	2 3 4	7.9(−4) 6.7(−14) 3.4(−54)	1.000	3.922	4	0.23
KM_2	2 3 4	d	d	d	d	d
MKM_2	2 3 4	2.6(−5) 1.9(−20) 4.8(−81)	1.000	4.000	4	0.26
KM_3	2 3 4	d	d	d	d	d
MKM_3	2 3 4	6.3(−5) 6.1(−19) 5.3(−75)	1.000	3.231	4	0.32
CM	2 3 4	d	d	d	d	d
JM	2 3 4	d	d	d	d	d

Table 4: *Example* 4.2(c) *at initial point* $x_0 = 4.5$

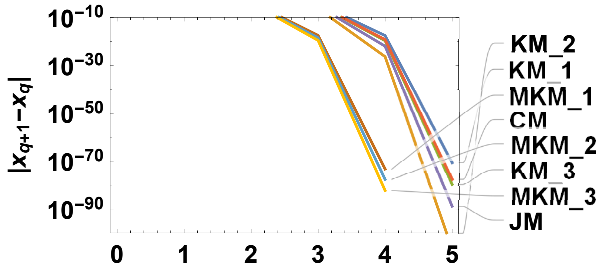
Method	q	$\lvert x_{q+1}-x_q \rvert$	$\lvert g(x_q) \rvert$	ρ	*No. of iteration*	*C.P.U time*
KM_1	2 3 4	2.2(−1) 5.7(−2) 6.0(−4)	3.9 4.5(−1) 4.0(−3)	3.853	5	0.20
MKM_1	2 3 4	1.2(−2) 4.8(−8) 1.2(−29)	1.000	4.000	3	0.32
KM_2	2 3 4	1.4(−1) 4.8(−3) 1.3(−8)	1.5 3.3(−2) 9.0(−8)	3.997	5	0.20
MKM_2	2 3 4	1.5(−2) 1.0(−9) 1.3(−38)	1.000	4.000	3	0.14
KM_3	2 3 4	d	d	d	d	d
MKM_3	2 3 4	1.6(−2) 7.9(−8) 5.4(−29)	1.000	4.000	3	0.25
CM	2 3 4	3.3(−1) 2.2(−1) 4.2(−2)	3.4 3.2(−1) 1.0(−3)	3.922	5	0.34
JM	2 3 4	2.0(−1) 2.5(−2) 1.3(−5)	2.8 1.8(−1) 8.9(−5)	3.979	5	0.39

Table 5: *Example* 4.2(d) *with initial guess* $x_0 = 2$

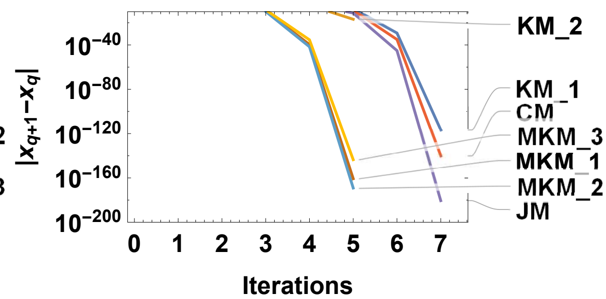
Remark: The Figure 1 represents the error analysis of numerical 4.1 to 4.2(d). It is clear from all sub figures of Figure 1 that proposed method error reduction is more faster than existing methods. Since, in the example 4.2(a) ,(b) *and* (d), the methods KM_3, and in the example 4.2(c) the methods KM_1, KM_2, KM_3, CM, JM divergence so these are not shown in sub figures (a), (b), (c), *and* (d). In similar way, iteration comparisons of different existing methods with proposed methods is depicted in the Figure 2. Clearly, the proposed method converges to root in less number of iterations as compared with other schemes. Further, the Examples 4.2(a), (b), (c), (d) by the methods KM_3, and KM_1, KM_2, KM_3, CM, JM are not approaching to desired root so these are not shown in the Figure 2.

5 Basin of Attraction

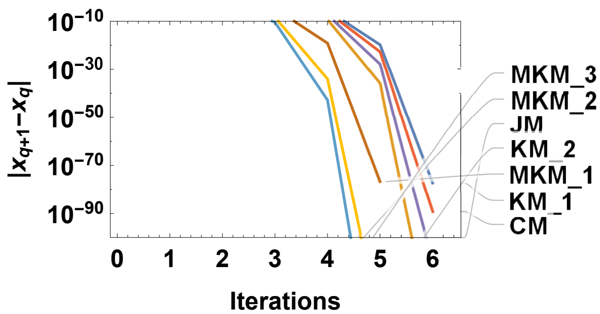
The concept of basin of attraction confirms the convergence of all the possible roots of the nonlinear equation within specified rectangular region. So, here we presents the convergence of ordinary King's methods(KM_1, KM_2, KM_3), multiplicative King's methods(MKM_1, MKM_2, MKM_3), Chun method(CM), and Jnawali method(JM) on different initial values in the rectangular region $[-2,2] \times [-2,2]$ by dynamical planes explained in [25]. The basin of attractions are shown in Figure 3 for the scaler equation $z^3 - 1$ and each image is plotted by an initial guess as an ordered pair of 256 complex points of abscissa and coordinate axis. If an initial point does not converge to the root then it is plotted with black color otherwise different colors are used to represent different roots $1, -i, i$ with a tolerance of 10^{-3}. We observe that very less divergence area is depicted by proposed schemes MKM_1, MKM_2, MKM_3.



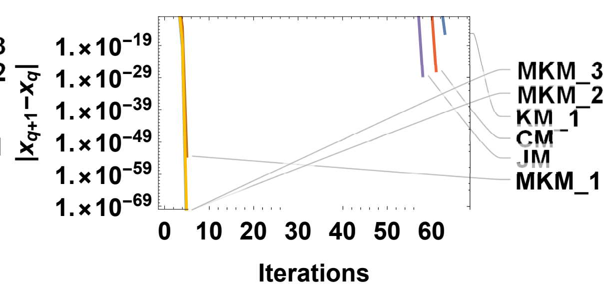
(a) Example 4.1



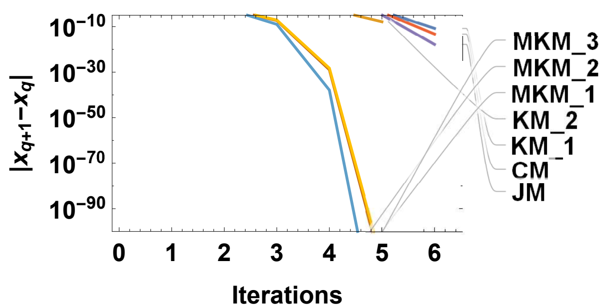
(b) Example 4.2(a)



(c) Example 4.2(b)



(d) Example 4.2(c)

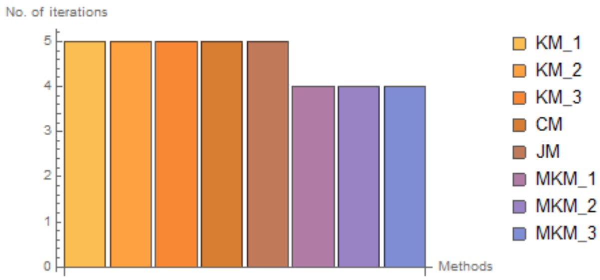


(e) Example 4.2(d)

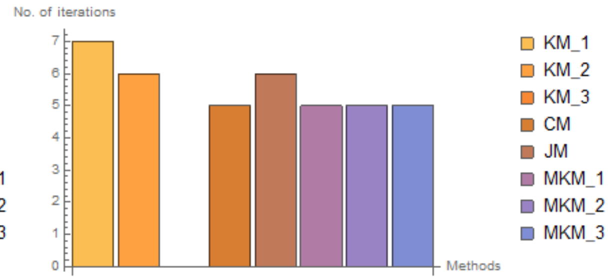
Figure 1: Graphical Error Analysis

6 Conclusion

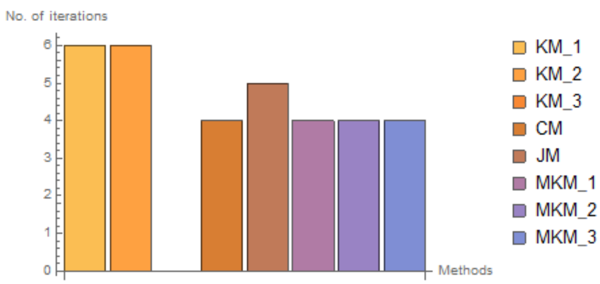
By adopting multiplicative calculus approach, we suggested a new fourth-order multipoint iterative technique for the multiple roots, when the multiplicity m is known in advance. A well-known King's method and the MCA are



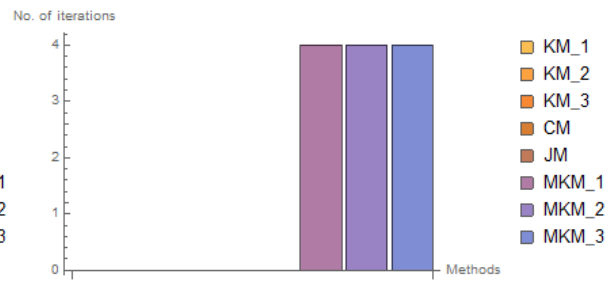
(a) Example 4.1



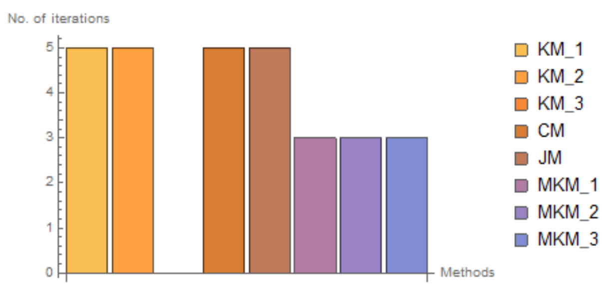
(b) Example 4.2(a)



(c) Example 4.2(b)



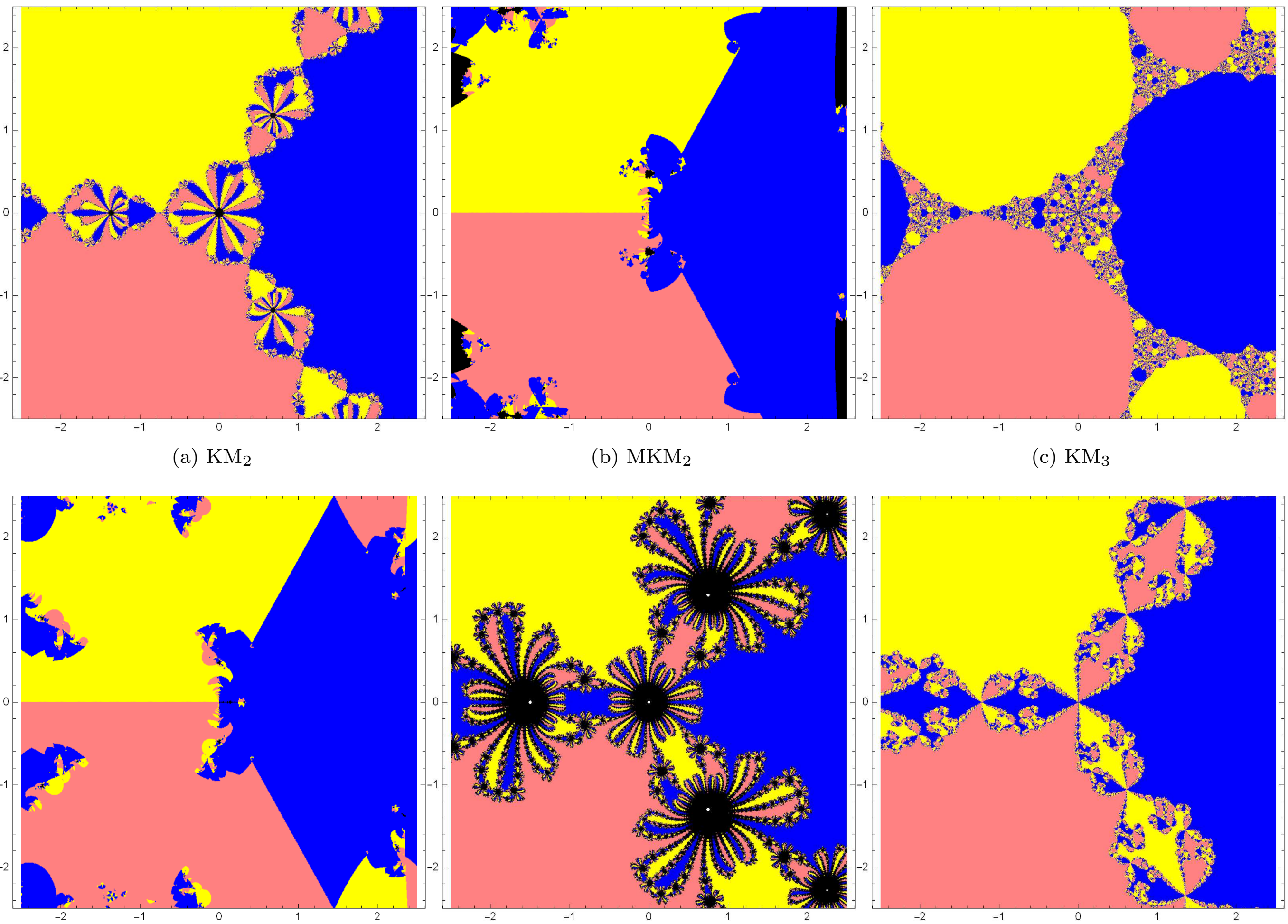
(d) Example 4.2(c)



(e) Example 4.2(d)

Figure 2: Iterations comparison

the two main pillars for the construction of new scheme. With the help of free disposable parameter β, we can obtain many new variants of fourth-order. In addition, we studied the convergence analysis of newly constructed scheme. We compare our methods with the existing techniques on the basis of absolute error difference between two consecutive iterations, order of convergence, number of iterations, CPU timing, the graphs of absolute errors and bar graphs. We found that our methods provide better approximations, which can be achieved with less computational time and complexity. Further, we also study the basin of attraction which also support the numerical results. In the future work, we will try to extend this idea for the system of nonlinear equations. In this way, this new approach of multiplicative calculus will open a new era of numerical techniques.



(a) KM_2 (b) MKM_2 (c) KM_3

(d) MKM_3 (e) CM (f) JM

Figure 3: Dynamical planes of new and existing methods for function $z^3 - 1 = 0$

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