Inverse coefficient problem for a time - fractional wave equation with initial - boundary and integral type overdetermination conditions

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November 22, 2022

Abstract

This paper considers the inverse problem of determining the time-dependent coefficient in the time-fractional diffusion-wave equation. In this case, an initial boundary value problem was set for the fractional diffusion-wave equation, and an additional condition was given for the inverse problem of determining the coefficient from this equation. First of all, it was considered the initial boundary value problem. By the Fourier method, this problem is reduced to equivalent integral equations. Then, using the Mittag-Leffler function and the generalized singular Gronwall inequality, we get apriori estimate for solution via unknown coefficient which we will need to study of the inverse problem. The inverse problem is reduced to the equivalent integral of equation of Volterra type. The principle of contracted mapping is used to solve this equation. Local existence and global uniqueness results are proved. The stability estimate is also obtained.

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Abstract - This paper considers the inverse problem of determining the time-dependent coefficient in the time-fractional diffusion-wave equation. In this case, an initial boundary value problem was set for the fractional diffusion-wave equation, and an additional condition was given for the inverse problem of determining the coefficient from this equation. First of all, it was considered the initial boundary value problem. By the Fourier method, this problem is reduced to equivalent integral equations. Then, using the Mittag-Leffler function and the generalized singular Gronwall inequality, we get apriori estimate for solution via unknown coefficient which we will need to study of the inverse problem. The inverse problem is reduced to the equivalent integral of equation of Volterra type. The principle of contracted mapping is used to solve this equation. Local existence and global uniqueness results are proved. The stability estimate is also obtained.

Keywords and phrases: Time-fractional diffusion equation, Riemann-Liouville fractional derivative, inverse problem, integral equation, Fourier series, Banach fixed point theorem.

1. Introduction

Recently, a large number of applied problems have been formulated on fractional differential equations and consequently considerable attention has been given to the solutions of those equations. Many physical and chemical processes are described by fractional differential equations [1-3]. Problems in viscoelasticity, dynamic processes in self-similar structures, system control theory, electrochemistry, diffusion processes leading to fractional order differential equations are consider in [4-7]. Fractional time derivatives are used to model diffusion wave or dispersion, a phenomenon observed in many problems. Some works providing an introduction to fractional calculus related to diffusion problems are, for instance [8,9].

The existence and uniqueness of the solution to an Cauchy type problem for the fractional differential equations were studied in many papers (see [10-12]). The idea of reducing the Cauchy problem for fractional differential equations to the Volterra integral equation was carried out by Pitcher and Sewel [13]. While is known that one can consider the initial - boundary value problems for differential equation with Riemann-Liouville fractional derivative were investigated [14].

Inverse problems for classical integro-differential wave propagation equations have been extensively studied. Nonlinear inverse coefficient problems with various types of sufficient determination conditions are often found in the literature (e.g., [15-22] and references therein). In [23-26] both the existence and uniqueness of a solution to the inverse problem are proved.

In this paper, we investigate the local existence and global uniqueness of an inverse problem of determining time-dependent coefficient in the time-fractional diffusion wave equation with initial-boundary and overdetermination conditions.

In the domain $\Omega_T := \{(x,t) : 0 < x < l,\ 0 < t \leq T\}$ consider the time-fractional diffusion wave equation

$$\left(\mathcal{D}^{\alpha}_{0+,t}u\right)(x,t) - u_{xx} + q(t)u(x,t) = f(x,t),\ (x,t) \in \Omega_T, \tag{1}$$

with initial and boundary conditions

$$\left(\mathcal{D}^{\alpha-1}_{0+,t}u(x,t)\right)_{t=0+} = \varphi(x),\ \left(\mathcal{D}^{\alpha-2}_{0+,t}u(x,t)\right)_{t=0+} = \psi(x),\ x \in [0,l], \tag{2}$$

$$u(0,t) = u(l,t) = 0,\ 0 \leq t \leq T, \tag{3}$$

where $\mathcal{D}^{\alpha}_{0+,t}$ is the Reimann-Liouville fractional derivative of order $1 < \alpha < 2$ in the time variable (see definition 1, 2 in preliminaries) and $\varphi(x),\ \psi(x),\ f(x,t)$ are given smooth functions.

We pose the inverse problem as follows: find the function $q(t), t > 0$ in (1), if the solution of the initial-boundary problem (1)-(3) satisfies condition:

$$\int\limits_0^l w(x)u(x,t)dx = g(t),\ 0 \leq t \leq T, \tag{4}$$

$g(t)$ is a given function.

The functions $\varphi(x),\ \varphi(x),\ f(x,t)$ and $g(t)$ satisfy the following assumptions

A1) $\{\varphi,\psi\} \in C^3[0,l],\ \{\varphi^{(4)},\psi^{(4)}\} \in L_2[0,l],\ \varphi(0) = \varphi(l) = 0,\ \psi(0) = \psi(l) = 0,\ \varphi''(0) = \varphi''(l) = 0,\ \psi''(0) = \psi''(l) = 0$ *and* $\varphi^{(4)}(0) = \varphi^{(4)}(l) = 0, \psi^{(4)}(0) = \psi^{(4)}(l) = 0$;

A2) $f(x,\cdot) \in C[0,T]$ *and for* $t \in [0,T], f(\cdot,t) \in C^3[0,l],\ f(\cdot,t)^{(4)} \in L_2[0,l],\ f(0,t) = f(l,t) = 0,\ f_{xx}(0,t) = f_{xx}(l,t) = 0$ *and* $f_{xxxx}(0,t) = f_{xxxx}(l,t) = 0$;

A3) $w(x) \in C^2[0,T]$ *and* $w(0) = w(l) = 0$ *and* $w''(0) = w''(l) = 0$;

A4) $\mathcal{D}^{\alpha}_{0+,t}g(t) \in C[0,T]$ *and* $|g(t)| \geq g_0 > 0$, g_0 *is a given number,* $\int\limits_0^l w(x)\varphi(x)dx = \left(\mathcal{D}^{\alpha-1}_{0+}g(t)\right)_{t=0+}$, $\int\limits_0^l w(x)\psi(x)dx = \left(\mathcal{D}^{\alpha-2}_{0+}g(t)\right)_{t=0+}$.

The article is organized as follows: In Section 2, we give some basic definitions and results needed in the sequel. In Section 3, the existence and uniqueness of the solution to direct problem (1)-(3) are obtained. Here also the stability estimate for this solution is given. Section 4 is devoted to the solving of inverse problem (1)-(4).

2. PRELIMINARIES

In this section, we present some useful definitions and results , which will be use in the future.

Definition 1. The Riemann-Liouville fractional integral of order $n-1 < \alpha < n$ for an integrable function $u(x,t)$ is defined by

$$I^{n-\alpha}_{0+,t}u(x,t) := \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\ t > 0.$$

Definition 2. The Riemann-Liouville fractional derivative of order $n-1 < \alpha < n$ of the integrable function $u(x,t)$ is defined by

$$\left(D^{\alpha}_{0+,t}u\right)(x,t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_0^t \frac{u(x,\tau)}{(t-\tau)^{\alpha-n+1}}d\tau,\ t > 0.$$

Two parameter Mittag-Leffler (M-L) function. The two parameter M-L function $E_{\alpha,\beta}(z)$ is defined by the following series:

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},$$

where $\alpha,\beta,z\in\mathbb{C}$ with $\Re(\alpha)>0$, $\Re(\alpha)-$denote the real part of the complex number α, $\Gamma(\cdot)$ is Euler's Gamma function. The Mittag-Leffler function has been studied by many authors who have proposed and studied various generalizations and applications. A very interesting work that meets many results about this function is due to Kilbas et al. (see [[1], pp. 42-44]).

Proposition 1. *Let $0<\alpha<2$ and $\beta\in\mathbb{R}$ be arbitrary. We suppose that κ is such that $\pi\alpha/2<\kappa<\min\{\pi,\pi\alpha\}$. Then there exists a constant $C=C(\alpha,\beta,\kappa)>0$ such that*

$$|E_{\alpha,\beta}(z)|\leq\frac{C}{1+|z|},\quad \kappa\leq|arg(z)|\leq\pi.$$

For the proof, we refer to [[21], pp. 40-45] for example.

Definition 3. We consider the weighted spaces of continuous functions [[1], pp.4-5, 162-163].

$$C_\gamma[a,b]:=\{f:(a,b]\to R:(x-a)^\gamma f(x)\in C[a,b],\ 0\leq\gamma<1,\},$$

$$C_\gamma^{2,\alpha}(\Omega)=\left\{u(x,t):\ u(\cdot,t)\in C^2(0,1);\ t\in[0,T]\text{ and }\partial_{0t}^{\alpha}u(x,\cdot)\in C_\gamma(0,T];\ x\in[0,1],\ 1<\alpha<2\right\},$$

$$C_\gamma^0[a,b]=C_\gamma[a,b],$$

with the norms

$$\|f\|_{C_\gamma}=\|(x-a)^\gamma f(x)\|_C,$$

and

$$\|f\|_{C_\gamma^n}=\sum_{k=0}^{n-1}\|f^{(k)}\|_C+\|f^{(n)}\|_{C_\gamma}.$$

Lemma 1. [[27], p.188] Suppose $b\geq 0,\ \alpha>0$ and $a(t)$ nonnegative function locally integrable on $0\leq t<T$ (some $T\leq+\infty$) and suppose $u(t)$ is nonnegative and locally integrable on $0\leq t<T$ with

$$u(t)\leq a(t)+b\int_0^t(t-s)^{\alpha-1}u(s)ds$$

on this interval; then

$$u(t)\leq a(t)+b\Gamma(\alpha)\int_0^t(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(b\Gamma(\alpha)(t-s)^\alpha\right)a(s)ds.$$

Lemma 2. [[27], p.189] Suppose $b\geq 0,\ \alpha>0,\ \gamma>0,\ \alpha+\gamma>1$ and $a(t)$ nonnegative function locally integrable on $0\leq t<T$ and suppose $t^{\gamma-1}u(t)$ is nonnegative and locally integrable on $0\leq t<T$ with

$$u(t)\leq a(t)+b\int_0^t(t-s)^{\alpha-1}s^{\gamma-1}u(s)ds$$

a.e. in $(0,T)$; then

$$u(t)\leq a(t)E_{\alpha,\gamma}\left((b\Gamma(\alpha))^{\frac{1}{\alpha+\gamma-1}}\,t\right),$$

where $E_{\alpha,\gamma}(t) = \sum\limits_{m=0}^{\infty} c_m t^{m(\alpha+\gamma-1)}$, $c_0 = 1$, $\frac{c_{m+1}}{c_m} = \frac{\Gamma(m(\alpha+\gamma-1)+\gamma)}{\Gamma(m(\alpha+\gamma-1)+\alpha+\gamma)}$ for $m \geq 0$. As $t \to +\infty$ $E_{\alpha,\gamma}(t) = O\left(t^{\frac{1}{2}\frac{\alpha+\gamma-1}{\alpha-\gamma}} \exp\left(\frac{\alpha+\gamma-1}{\alpha} t^{\frac{\alpha+\gamma-1}{\alpha}}\right)\right)$.

In the next section we will deal with finding the solution of the initial-boundary problem (1)-(3).

2. Existence and uniqueness result

By applying the Fourier method, the solution $u(x;t)$ of the problem (1)-(3) can be expanded in a uniformly convergent series in term of eigenfunctions of the form

$$u(x,t) = \sum_{n=0}^{\infty} u_n(t) X_n(x), \tag{5}$$

where

$$X_n(x) = \sqrt{\frac{2}{l}} \sin(\lambda_n x), \ \ \lambda_n = \frac{\pi k}{l}, \ \ n = 1, 2, 3, \ldots. \tag{6}$$

The coefficients $u_n(t)$ for $n \geq 1$ are to be found by making use of the orthogonality of the eigenfunctions $X(x)$. The scalar product in $L_2[0,1]$ is defined by $(f,g) = \int\limits_0^1 f(x)g(x)dx$. Let us note the expansion coefficients of $\varphi(x)$, $\psi(x)$ and $f(x,t)$ in the eigenfunctions of (6) for $n \geq 1$ are definded respectively by

$$(f(x,t), X_n(x)) = f_n(t),$$

$$(\varphi(x), X_n(x)) = \varphi_n, \ (\psi(x), X_n(x)) = \psi_n, \ \ n = 1, 2, \ldots. \tag{7}$$

We obtain in view of (1) and with $(u(x,t), X_n(x)) = \int\limits_0^l u(x,t)X_n(x)dx = u_n(t)$, and we may write

$$\mathcal{D}^{\alpha}_{0+,t} u_n(t) + \lambda_n^2 u_n + q(t)u_n(t) = f_n(t), \tag{8}$$

$$\left(\mathcal{D}^{\alpha-1}_{0+,t} u_n(t)\right)|_{t=0} = \varphi_n, \left(\mathcal{D}^{\alpha-2}_{0+,t} u_n(t)\right)|_{t=0} = \psi_n. \tag{9}$$

We suppose that $f_n(t) \in C_\gamma[0,T]$. Then, by property 3.1(a) (see [[1], p. 172]), (8), (9) is equivalent in the space $C^{\alpha}_{\gamma}[0,T]$ to the following Volterra integral equation:

$$u_n(t) = \varphi_n t^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 t^\alpha) +$$

$$+\psi_n t^{\alpha-2} E_{\alpha,\alpha-1}(-\lambda_n^2 t^\alpha) + \int\limits_0^t (t-\tau)^{\alpha-1} E_{\alpha,\alpha}(-\lambda_n^2 (t-\tau)^\alpha) \left(f_n(\tau) - q(\tau)u_n(\tau)\right) d\tau. \tag{10}$$

First we prove the following assertions:
Lemma 3. For large $n \in N$ we have the estimates

$$t^\gamma |u_n(t)| \leq \lambda_n \Big(|\varphi_n| t^{\gamma+\alpha-1} + |\psi_n| t^{\gamma+\alpha-2} + \|f_n\|_\gamma t^\alpha B(\alpha, 1-\gamma) \Big) \times$$

$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n} \|q\|_{C[0,T]} t^\gamma \Gamma(\alpha) \right)^{\frac{1}{\gamma+\alpha-1}} t \right), \ t \in [0,T],$$

$$t^{\gamma}|\mathcal{D}^{\alpha}_{0+,t}u_n(t)| \leq \|f_n\|_{\gamma}+$$

$$\lambda_n\left(\lambda_n^2+\|q\|_{C[0,T]}\right)\left(\left|\varphi_n\right|t^{\gamma+\alpha-1}+\left|\psi_n\right|t^{\gamma+\alpha-2}+\|f_n\|_{\gamma}t^{\alpha}B(\alpha,1-\gamma)\right)\times$$

$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]}t^{\gamma}\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}t\right),\ t\in[0,T],$$

where $1>\gamma>2-\alpha$.

Proof. We write the integral equation (10) in the following form:

$$u_n(t)=\varphi_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2t^{\alpha})+$$

$$+\psi_n t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n^2t^{\alpha})+\int\limits_0^t(t-\tau)^{\alpha-1}\tau^{-\gamma}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\tau^{\gamma}f_n(\tau)d\tau-$$

$$-\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})q(\tau)u_n(\tau)d\tau.$$

This solution is bounded in $C^{\alpha}_{\gamma}[0,T]$ in view of $A1),\ A2)$. Multiplying the last equation by t^{γ}, we get

$$t^{\gamma}|u_n(t)|\leq t^{\gamma}\left|\varphi_n\right|\left|t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2t^{\alpha})\right|+$$

$$+t^{\gamma}\left|\psi_n\right|\left|t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n^2t^{\alpha})\right|+t^{\gamma}\left|\int\limits_0^t(t-\tau)^{\alpha-1}\tau^{-\gamma}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})\tau^{\gamma}f_n(\tau)d\tau\right|+$$

$$+t^{\gamma}\left|\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^{\alpha})q(\tau)u_n(\tau)d\tau\right|\leq$$

$$\leq\lambda_n\left(\left|\varphi_n\right|t^{\gamma+\alpha-1}+\left|\psi_n\right|t^{\gamma+\alpha-2}+\|f_n\|_{\gamma}t^{\alpha}B(\alpha,1-\gamma)\right)+$$

$$+\frac{1}{\lambda_n}\|q\|_{C[0,T]}t^{\gamma}\int\limits_0^t(t-\tau)^{\alpha-1}|u_n(\tau)|d\tau,$$

where $B(\alpha,1-\gamma)$ is Euler's beta function.

Next, according to Lemma 2, we have

$$t^{\gamma}|u_n(t)|\leq\lambda_n\left(\left|\varphi_n\right|t^{\gamma+\alpha-1}+\left|\psi_n\right|t^{\gamma+\alpha-2}+\|f_n\|_{\gamma}t^{\alpha}B(\alpha,1-\gamma)\right)\times$$

$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]}t^{\gamma}\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}t\right).\tag{11}$$

We get the second part of the lemma 3, from equation (8) and the first estimate of Lemma 3

$$t^{\gamma}|\mathcal{D}^{\alpha}_{0+,t}u_n(t)|\leq$$

$$\leq \|f_n\|_\gamma + \lambda_n \left(\lambda_n^2 + \|q\|_{C[0,T]}\right) \left(\left|\varphi_n\right| t^{\gamma+\alpha-1} + \left|\psi_n\right| t^{\gamma+\alpha-2} + \|f_n\|_\gamma t^\alpha B(\alpha, 1-\gamma)\right) \times$$

$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]} t^\gamma \Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}} t\right). \tag{12}$$

From the last two inequalities we immediately obtain the estimates of lemma 3 for any $t \in [0, T]$. Lemma proven.

Formally, from (5) by term-by-term differentiation we compose the series

$$\mathcal{D}_{0+,t}^\alpha u(x,t) = \sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \mathcal{D}_{0+,t}^\alpha u_n(t) \sin(\lambda_n x), \tag{13}$$

$$u_{xx}(x,t) = \sqrt{\frac{2}{l}} \sum_{n=0}^{\infty} \lambda_n^2 u_n(t) \sin(\lambda_n x). \tag{14}$$

Let us prove the uniform convergence of series (5), (13) and (14) in the domain $\overline{\Omega}$. This series for any $(x,t) \in \overline{\Omega}$ is majorized by

$$\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left(\lambda_n |\varphi_n| T^{\gamma+\alpha-1} + \lambda_n |\psi_n| T^{\gamma+\alpha-2} + \lambda_n \|f_n\|_\gamma T^\alpha B(\alpha, 1-\gamma)\right), \tag{15}$$

$$\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \Big[\left(\lambda_n^3 + \lambda_n \|q\|_{C[0,T]}\right) T^{\gamma+\alpha-1} |\varphi_n| + \left(\lambda_n^3 + \lambda_n \|q\|_{C[0,T]}\right) T^{\gamma+\alpha-2} |\psi_n| +$$

$$+ \left[1 + \left(\lambda_n^3 + \lambda_n \|q\|_{C[0,T]}\right) T^\alpha\right] \|f_n\|_\gamma B(\alpha, 1-\gamma)\Big], \tag{16}$$

$$\sqrt{\frac{2}{l}} \sum_{n=1}^{\infty} \left(\lambda_n^3 |\varphi_n| T^{\gamma+\alpha-1} + \lambda_n^3 |\psi_n| T^{\gamma+\alpha-2} + \lambda_n^3 \|f_n\|_\gamma T^\alpha B(\alpha, 1-\gamma)\right), \tag{17}$$

where $\overline{\Omega}_T := \{(x,t) : 0 \leq x \leq< l,\ 0 \leq t \leq T\}$.

We hold the following auxiliary lemma.

Lemma 4. If the conditions A1), A2) are fulfilled then there are equalities

$$\varphi_n = \frac{1}{\lambda_n^4} \varphi_n^{(4)}, \ \ \psi_n = \frac{1}{\lambda_n^4} \psi_n^{(4)}, \ \ f_n = \frac{1}{\lambda_n^4} f_n^{(4)}, \tag{18}$$

where

$$\varphi_n^{(4)} = \sqrt{\frac{2}{l}} \int_0^l \varphi^{(4)}(x) sin(\lambda_n x) dx, \ \ \psi_n^{(4)} = \sqrt{\frac{2}{l}} \int_0^l \psi^{(4)}(x) sin(\lambda_n x) dx,$$

$$f_n^{(4)} = \sqrt{\frac{2}{l}} \int_0^l f^{(4)}(x) sin(\lambda_n x) dx,$$

with the following estimates:

$$\sum_{n=1}^{\infty} \left|\varphi_n^{(4)}\right|^2 \leq \|\varphi^{(4)}\|_{L_2[0,l]}, \ \ \sum_{n=1}^{\infty} \left|\psi_n^{(4)}\right|^2 \leq \|\psi^{(4)}\|_{L_2[0,l]}, \ \ \sum_{n=1}^{\infty} \left|f_n^{(4)}\right|^2 \leq \|f^{(4)}\|_{L_2[0,T]}. \tag{19}$$

If the functions $\varphi(x)$, $\psi(x)$ and $f(x,t)$ satisfy the conditions of Lemma 4, then due to representations (18) and (19) series (5), (13) and (14) converge uniformly in the rectangle $\overline{\Omega}$, therefore, function $u(x,t)$ satisfies relations (1)–(3).

Using the above results, we obtain the following assertion.

Theorem 1. Let $q(t)\in C[0,T]$, A1), A2) are satisfied, then there exists a unique solution of the direct problem (1)-(3) $u(x,t)\in C_{\gamma}^{2,\alpha}(\overline{\Omega})$.

Let us derive an estimate for the norm of the difference between the solution of the original integral equation (10) and the solution of this equation with perturbed functions $\widetilde{q}$, $\widetilde{\varphi}_n$, $\widetilde{\psi}_n$ and $\widetilde{f}_n$. Let $\widetilde{u}_n(t)$ be solution of the integral equation (10) corresponding to the functions $\widetilde{q}$, $\widetilde{\varphi}_n$, $\widetilde{\psi}_n$, $\widetilde{f}_n$; i.e.,

$$\widetilde{u}_n(t)=\widetilde{\varphi}_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2t^\alpha)+$$

$$+\widetilde{\psi}_n t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n^2t^\alpha)+\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)\left(\widetilde{f}_n(\tau)-\widetilde{q}(\tau)\widetilde{u}_n(\tau)\right)d\tau. \tag{20}$$

Composing the difference $u_n(t)-\widetilde{u}_n(t)$ with the help of the equations (10), (20) and introducing the notations $\widehat{u}_n(t)=u_n(t)-\widetilde{u}_n(t)$, $\widehat{q}=q(t)-\widetilde{q}(t)$, $\widehat{\varphi}_n(t)=\varphi_n(t)-\widetilde{\varphi}_n(t)$, $\widehat{\psi}_n(t)=\psi_n(t)-\widetilde{\psi}_n(t)$, $\widehat{f}_n(t)=f_n(t)-\widetilde{f}_n(t)$ we obtain the integral equation

$$\widehat{u}_n(t)=\widehat{\varphi}_n t^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2t^\alpha)+$$

$$+\widehat{\psi}_n t^{\alpha-2}E_{\alpha,\alpha-1}(-\lambda_n^2t^\alpha)+\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)\left(\widehat{f}_n(\tau)-\widehat{q}(\tau)u_n(\tau)\right)d\tau-$$

$$-\int\limits_0^t(t-\tau)^{\alpha-1}E_{\alpha,\alpha}(-\lambda_n^2(t-\tau)^\alpha)\widetilde{q}(\tau)\widehat{u}_n(\tau)d\tau, \tag{21}$$

from which, is derived the following linear integral inequality for $t^\gamma\left|\widehat{u}_n(t)\right|$

$$t^\gamma\left|\widehat{u}_n(t)\right|\le\lambda_n\left(\left|\widehat{\varphi}_n\right|t^{\gamma+\alpha-1}+\left|\widehat{\psi}_n\right|t^{\gamma+\alpha-2}+\|\widehat{f}_n\|_\gamma t^\alpha B(\alpha,1-\gamma)\right)+$$

$$+\|\widehat{q}\|_{C[0,T]}t^\gamma\left(\left|\varphi_n\right|t^{\gamma+\alpha-1}+\left|\psi_n\right|t^{\gamma+\alpha-2}+\|f_n\|_\gamma t^\alpha B(\alpha,1-\gamma)\right)\times$$

$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]}t^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}t\right)+\frac{1}{\lambda_n}\|\widetilde{q}\|_{C[0,T]}t^\gamma\int\limits_0^t(t-\tau)^{\alpha-1}\left|\widehat{u}_n(\tau)\right|d\tau.$$

Using the Lemma 2 from last inequality, we arrive at the estimate:

$$t^\gamma\left|\widehat{u}_n(t)\right|\le\left[\lambda_n\left(\left|\widehat{\varphi}_n\right|t^{\gamma+\alpha-1}+\left|\widehat{\psi}_n\right|t^{\gamma+\alpha-2}+\|\widehat{f}_n\|_\gamma t^\alpha B(\alpha,1-\gamma)\right)+\right.$$

$$+\|\widehat{q}\|_{C[0,T]}t^\gamma\left(\left|\varphi_n\right|t^{\gamma+\alpha-1}+\left|\psi_n\right|t^{\gamma+\alpha-2}+\|f_n\|_\gamma t^\alpha B(\alpha,1-\gamma)\right)\times$$

$$\left.\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]}t^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}t\right)\right]E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|\widetilde{q}\|_{C[0,T]}t^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}t\right). \tag{22}$$

Indeed, the expression (22) is stability estimate for the solution to the problem (1)-(3). The uniqueness of this solution follows from (22).

INVESTIGATION OF THE INVERSE PROBLEM (1)-(4)

Let us now we in westigate to inverse problem (1)-(4)

Applying $\mathcal{D}^{\alpha}_{0+,t}$ to the over-determination condition (4), we obtain the following equation

$$\int\limits_0^l w(x)\Big\{ \left(\mathcal{D}^{\alpha}_{0+,t}u\right)(x,t) - u_{xx} + q(t)u(x,t)\Big\}dx = \int\limits_0^l w(x)f(x,t)dx,$$

we form

$$\left(\mathcal{D}^{\alpha}_{0+,t}g\right)(t) + q(t)g(t) - \int\limits_0^l w''(x)u(x,t)dx = \int\limits_0^l w(x)f(x,t)dx,$$

which yields

$$q(t) = \frac{1}{g(t)}\left(\int\limits_0^l w(x)f(x,t)dx - \left(\mathcal{D}^{\alpha}_{0+}g\right)(t) + \int\limits_0^l w''(x)u(x,t)dx\right).$$

The functions $u_n(t)$ depend on $q(t)$, i.e. $u_n(t;q)$. After simple converting, we get the following integral equation for determining $q(t)$:

$$q(t) = q_0(t) + \frac{1}{g(t)}\sum_{n=1}^{\infty} w_n u_n(t;q), \tag{23}$$

where

$$w_n = \sqrt{\frac{2}{l}}\int\limits_0^l w''(x)\sin(\lambda_n x)dx,\;\; q_0(t) = \frac{1}{g(t)}\left(\int\limits_0^l w(x)f(x,t)dx - \left(\mathcal{D}^{\alpha}_{0+}g\right)(t)\right).$$

We introduce an operator $\mathcal{F}$ defining it by the right hand side of (23):

$$\mathcal{F}[q](t) = q_0(t) + \frac{1}{g(t)}\sum_{n=1}^{\infty} w_n u_n(t;q). \tag{24}$$

Then, the equation (24) is written in more convenient form as

$$\mathcal{F}[q](t) = q(t). \tag{25}$$

Let

$$q_{00} := \max_{t\in[0;T]}|q_0(t)| = \left\|\frac{1}{g(t)}\left(\int\limits_0^l w(x)f(x,t)dx - \left(\mathcal{D}^{\alpha}_{0+}g\right)(t)\right)\right\|_{C[0,T]}.$$

Fix a number $r > 0$ and consider the ball

$$B(q_0, r) := \{q(t) : q(t) \in C[0,T],\; \|q - q_0\| \le r\}.$$

Theorem 2. Let A1)-A3) are satisfied. Then there exists a number $T^* \in (0;T)$, such that there exists a unique solution $q(t) \in C[0,T^*]$ of the inverse problem (1)-(4).

Proof. Let us first prove that for an enough small $T>0$ the operator $\mathcal{F}$ maps the ball $B(q_0,r)$ implies that $\mathcal{F}[q](t)\in B(q_0,r)$. Indeed, for any continuous function $q(t)$, the function $\mathcal{F}[q](t)$ calculated using formula (25) will be continuous. Moreover, estimating the norm of the differences, we find that

$$\|\mathcal{F}[q](t)-q_0(t)\|\leq\frac{w_0}{g_0}\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_n|T^{\gamma+\alpha-1}+|\psi_n|T^{\gamma+\alpha-2}+\|f_n\|_\gamma T^\alpha B(\alpha,1-\gamma)\Big)\times$$
$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]}T^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right).$$

Here we have used the estimate (11). In view of Lemmas 3 and 4 last series is convergent series. Note that the function occurring on the right-hand side in this inequality is monotone increasing with T, and the fact that the function $q(t)$ belongs to the ball $B(q_0,r)$ implies the inequality $\|q\|\leq\|q_0\|+r$. Therefore, we only strengthen the inequality if we replace $\|q\|$ in this inequality with the expression $\|q_0\|+r$. Performing these replacements, we obtain the estimate

$$\|\mathcal{F}[q](t)-q_0(t)\|\leq\frac{w_0}{g_0}\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_n|T^{\gamma+\alpha-1}+|\psi_n|T^{\gamma+\alpha-2}+\|f_n\|_\gamma T^\alpha B(\alpha,1-\gamma)\Big)\times$$
$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\left(\|q_0\|+r\right)T^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right).$$

Let T_1 be a positive root of the equation

Therefore if by T_1 we denote the positive root of the equation (for T)

$$\frac{w_0}{g_0}\sum_{n=1}^{\infty}\lambda_n\Big(|\varphi_n|T^{\gamma+\alpha-1}+|\psi_n|T^{\gamma+\alpha-2}+\|f_n\|_\gamma T^\alpha B(\alpha,1-\gamma)\Big)\times$$
$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\left(\|q_0\|+r\right)T^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right)=r,$$

then $\|\mathcal{F}[q](t)-q_0(t)\|\leq r$ for $T\leq T_1$; i.e. $\mathcal{F}[q](t)\in B(q_0,r)$.

Now let us take any functions $q(t)$, $\widetilde{q}(t)\in B(q_0,r)$ and estimate the distance between their images $\mathcal{F}[q](t)$ and $\mathcal{F}[\widetilde{q}](t)$ in the space $C[0,T]$. The function $\widetilde{u}_n(t)$ corresponding to $\widetilde{q}(t)$ satisfies the integral equation (20) with $\varphi_n=\varphi_n$, $\psi=\psi_n$ and $f_n=\widetilde{f_n}$. Composing the difference $\mathcal{F}[q](t)-\mathcal{F}[\widetilde{q}](t)$ with the help of equations (10), (20) and then estimating its norm, we obtain

$$\|\mathcal{F}[q](t)-\mathcal{F}[\widetilde{q}](t)\|\leq\frac{w_0}{g_0}\sum_{n=1}^{\infty}\|u_n(t,q)-\widetilde{u}_n(t,\widetilde{q})\|\leq$$
$$\leq\frac{w_0}{g_0}\sum_{n=1}^{\infty}T^\gamma\Big(|\varphi_n|T^{\gamma+\alpha-1}+|\psi_n|T^{\gamma+\alpha-2}+\|f_n\|_\gamma T^\alpha B(\alpha,1-\gamma)\Big)\times$$
$$\times E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|q\|_{C[0,T]}T^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right)\Bigg)E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\|\widetilde{q}\|_{C[0,T]}T^\gamma\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right)\|\widehat{q}\|_{C[0,T]}. \tag{26}$$

The functions $q(t)$ and $\widetilde{q}(t)$ belong to the ball $B(q_0,r)$, and hence for each of these functions one has inequality $\|q\|\leq\|q_0\|+r$. Note that the function on the right-hand side in inequality (26) at the factor $\|q\| - \|\widetilde{q}\|$ is monotone increasing with $\|q\|$, $\|\widetilde{q}\|$ and T. Consequently, replacing $\|q\|$ and $\|\widetilde{q}\|$ in inequality (26) with $\|q\|+r$ will only strengthen the inequality. This, we have

$$\|\mathcal{F}[q](t) - \mathcal{F}[\widetilde{q}](t)\| \leq \frac{w_0}{g_0}\sum_{n=1}^{\infty}\|u_n(t,q) - \widetilde{u}_n(t,\widetilde{q})\|$$

$$\leq \frac{w_0}{g_0}\sum_{n=1}^{\infty}T^{\gamma}\Big(|\varphi_n|T^{\gamma+\alpha-1} + |\psi_n|T^{\gamma+\alpha-2} + \|f_n\|_{\gamma}T^{\alpha}B(\alpha,1-\gamma)\Big)\times$$

$$\times\left(E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\left(\|q\|+r\right)T^{\gamma}\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right)\right)^2\|\widehat{q}\|_{C[0,T]}.$$

Therefore, if T_2 is the positive root of the equation (for T)

$$\frac{w_0}{g_0}\sum_{n=1}^{\infty}T^{\gamma}\Big(|\varphi_n|T^{\gamma+\alpha-1} + |\psi_n|T^{\gamma+\alpha-2} + \|f_n\|_{\gamma}T^{\alpha}B(\alpha,1-\gamma)\Big)\times$$

$$\times\left(E_{\alpha,\gamma}\left(\left(\frac{1}{\lambda_n}\left(\|q\|+r\right)T^{\gamma}\Gamma(\alpha)\right)^{\frac{1}{\gamma+\alpha-1}}T\right)\right)^2 = 1$$

then for $T \in [0,T_2)$ the operator $\mathcal{F}$ contracts the distance between the elements $q(t)$, $\widetilde{q}(t) \in B(q_0,r)$. Consequently, if we choose $T^* < \min(T_1,T_2)$ then the operator $\mathcal{F}$ is a contraction in the ball $B(q_0,r)$. However, in accordance with the Banach theorem (see [[28], p.p. 87–97]), the operator $\mathcal{F}$ has unique fixed point in the ball $B(q_0,r)$ i.e., there exists a unique solution of equation (25). Theorem 2 is proven.

Let T,l be positive fixed numbers. Consider the set D_{ν_0} of the given functions $(\varphi,\ \psi,\ g,\ f)$ for which all conditions from A1)-A4) are fulfilled and

$$\max\{\|\varphi\|_{C^4[0,l]},\ \|\psi\|_{C^4[0,l]},\ \|g\|_{C^{\alpha}[0,T]},\ \|f\|_{C^4(\overline{\Omega})}\} \leq \nu_0.$$

We denote by Q_{ν_1} the set of function $q(t)$ that for some $T>0,\ l>0$ satisfy the following condition $\|Q\|_{C[0,T]} \leq \nu_1,\ \nu_1>0$.

Theorem 3. Let $(\varphi,\ \psi,\ g,\ f) \in D_{\nu_0}$, $(\widetilde{\varphi},\ \hat{\psi},\ \widetilde{g},\ \widetilde{f}) \in D_{\nu_0}$ and $q \in Q_{\nu_1}$, $\widetilde{q} \in Q_{\nu_1}$. Then, for solution of the inverse problem (1)-(4) the following stability estimate holds:

$$\|q-\widetilde{q}\|_{C[0,T]} \leq \rho\Big[\|\varphi-\widetilde{\varphi}\|_{C[0,l]} + \|\psi-\widetilde{\psi}\|_{C[0,l]} + \|g-\widetilde{g}\|_{C^{\alpha}[0,T]} + \|f-\widetilde{f}\|_{C(\overline{\Omega})}\Big], \tag{27}$$

where the constant ρ depends only on $\nu_0,\ \nu_1,\ T,\ l,\ \alpha$, and $\Gamma(\alpha),\ B(\alpha,1-\gamma)$.

Proof. To prove this theorem, using (23) we write down the equations for $\widetilde{q}(t)$ and compose the difference $\widehat{q} = q(t) - \widetilde{q}(t)$. Then after evaluating this expression and using estimates $u_n(t)$,$\widehat{u}_n(t)$, we obtain following estimates

$$\|q-\widetilde{q}\|_{C[0,T]} = \max_{0\leq t\leq T}\left|\frac{1}{g(t)}\left(\int_0^l w(x)f(x,t)dx - \left(\mathcal{D}^{\alpha}_{0+}g\right)(t)\right) + \frac{1}{g(t)}\int_0^l w''(x)u(x,t)dx - \right.$$

$$\left.\frac{1}{\widetilde{g}(t)}\left(\int_0^l w(x)\widetilde{f}(x,t)dx - \left(\mathcal{D}^{\alpha}_{0+}\widetilde{g}\right)(t)\right) + \frac{1}{\widetilde{g}(t)}\int_0^l w''(x)\widetilde{u}(x,t)dx\right| \leq$$

$$\le \max_{0\le t\le T}\left\{\frac{w_0}{g_0^2}\left|\int_0^l \left[\widetilde{g}(t)\left(f(x,t)-\widetilde{f}(x,t)\right)+\widetilde{f}(x,t)\left(g(t)-\widetilde{g}(t)\right)\right]dx+\right.\right.$$

$$\left.\left.+\int_0^l \left[\widetilde{g}(t)\left(\left(\mathcal{D}_{0+}^{\alpha}g\right)(t)-\left(\mathcal{D}_{0+}^{\alpha}\widetilde{g}\right)(t)\right)+\left(\mathcal{D}_{0+}^{\alpha}\widetilde{g}\right)(t)\left(g(t)-\widetilde{g}(t)\right)\right]dx\right|\right\}+$$

$$+\max_{0\le t\le T}\left\{\frac{w_0}{g_0^2}\left|\int_0^l \left[\widetilde{g}(t)\left(u(x,t)-\widetilde{u}(x,t)\right)+\widetilde{u}(x,t)\left(g(t)-\widetilde{g}(t)\right)\right]dx\right|\right\}\le$$

$$\le \rho_0\left(\|\varphi-\widetilde{\varphi}\|+\|\psi-\widetilde{\psi}\|+\|f-\widetilde{f}\|+\|g-\widetilde{g}\|+\|\left(\mathcal{D}_{0+}^{\alpha}g\right)-\left(\mathcal{D}_{0+}^{\alpha}\widetilde{g}\right)\|\right)+$$

$$+\rho_1\int_0^t (t-\tau)^{\alpha-1}\|q(\tau)-\widetilde{q}(\tau)\|_{C[0,T]}d\tau,\ \ t\in[0,T], \tag{28}$$

where $\rho_0,\ \rho_1$ depends only on $\nu_0,\ \nu_1,\ T,\ l,\ \alpha$, and $\Gamma(\alpha),\ B(\alpha,1-\gamma)$. From (28) using lemma 1, we get the estimate

$$\|q-\widetilde{q}\|_{C[0,T]}\le \rho_0\left(\|\varphi-\widetilde{\varphi}\|+\|\psi-\widetilde{\psi}\|+\|f-\widetilde{f}\|+\|g-\widetilde{g}\|+\|\left(\mathcal{D}_{0+}^{\alpha}g\right)-\left(\mathcal{D}_{0+}^{\alpha}\widetilde{g}\right)\|\right)\times$$

$$\times E_{\alpha,1}\left(\rho_1\Gamma(\alpha)t^{\alpha}\right),\ t\in[0,T]. \tag{29}$$

This inequality implies the estimate (27), if we set $\rho=\rho_0E_{\alpha,1}\left(\rho_1\Gamma(\alpha)t^{\alpha}\right)$.

From *Theorem 3* follows also the next assertion on uniqueness in whole for solution to the inverse problem.

Theorem 4. Let the functions $\varphi,\ \psi,\ g,\ f$ and $\widetilde{\varphi},\ \hat{\psi},\ \widetilde{g},\ \widetilde{f}$ have the same meaning as in Theorem 3 and conditions A1)-A4). Moreover, if $\varphi=\widetilde{\varphi}$, $\psi=\widetilde{\psi}$, $g=\widetilde{g}$, $f=\widetilde{f}$, for $t\in[0,T]$ then $q(t)=\widetilde{q}(t)$ $t\in[0,T]$.

Conclusion

In this work, the solvability of a nonlinear inverse problem for a time-fractional diffusion equation with initial-boundary conditions was studied. Firsty we investigated solvability the initial-boundary conditional problem(1)-(3). The problem replaced by an equivalent of integral equation. Existence and uniqueness of direct problem solution were proven. The inverse problem was considered for determining the coefficient $q(t)$ included in the equation (1) with additional condition (4) of the solution of equation (1) with the initial and boundary conditions (2), (3). Conditions for given functions are obtained, under which the inverse problem has unique solution for a sufficiently small interval.

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