On a boundary value problem for a class of equations of mixed type

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Abstract

In this paper we study a nonlocal boundary value problem for Gellerstedt equation with singular coefficient in an unbounded domain. With the help of the method of integral equations and the principle of extremum we prove the unique solvability of the considered problem.

KEYWORDS:
Principle of extremum; uniqueness of a solution; existence of a solution; index of equation; singular coefficient; integral equations.

1 | INTRODUCTION AND FORMULATION OF A PROBLEM

The theory of non-local boundary value problems for equations of mixed typeis of great importance, it has its application in technology and nature sciences, including gas dynamics in studying the state of gas and oil basins, in aerodynamics in studying heat and mass transfer in objects with a complex structure, in hydrodynamics in groundwater filtration, in the study of fluid flow in channels surrounded by a porous medium, in electrodynamics in the construction of mathematical models of electrical oscillations in conductors, and other phenomena, as well as in a number of other areas.

The first fundamental research for a model equation of mixed type was the study conducted by F. Tricomi[1]. In the Tricomi problem, at all points of the boundary characteristic, the value of the desired function is given. L. Wolfersdorf[2] considered the Tricomi problem for the generalized Tricomi equation in an unbounded domain, the elliptic part of which is the upper half-plane, and the hyperbolic part is the characteristic triangle. For the generalized Tricomi equation, S. Gellerstedt[3] studied boundary value problems, in the formulation of which, in the hyperbolic part of the considered domain the values of the desired solution were given on two internal characteristics of the equation. The Gellerstedt problem has important applications in transonic gas dynamics. In[4], the problem with the Bitsadze–Samarskii conditions on the ellipticity boundary and on the degeneration line was studied for the Gellerstedt equation with a singular coefficient in a mixed domain, when the ellipticity boundary conditions with the segment of the y-axis and the normal curve of the equation. The study in[5] is devoted to the study of a problem with the Frankl condition type on a segment of the line of degeneracy for a mixed-type equation with a singular coefficient. In[6], for the Gellerstedt equation with a singular coefficient, we prove uniqueness and existence theorems for a solution to a problem with nonlocal conditions on parts of the boundary characteristics and the Frankl condition type on a segment of the line of degeneracy.

A problem with a shift for a mixed-type equation in an unbounded domain, the elliptic part of which is the upper half-plane, was studied in[7]. In[8] in an unbounded domain, the problem with the Bitsadze–Samarskii condition on parallel characteristics for the Gellerstedt equation with singular coefficients in an unbounded domain was studied. A nonlocal problem with integral gluing condition for a third-order loaded equation was studied in[9]. Boundary value problems for the third-order loaded equation were investigated in[10]. A nonlocal boundary value problems for a mixed type equation with singular coefficients was solved in[11],[12]. The studies in[13],[14] are devoted to study the nonlocal problems for mixed-type equation with partial Riemann-Liouville fractional derivative. In[15], the fundamental solution for the Tricomi type equation in the hyperbolic domain was found. Tricomi type equations with terms of lower order in the plane was studied in[16]. Solutions are required to satisfy conditions on one part of a characteristic in the hyperbolic region of the considered domain and on some parts of the line of parabolic degeneracy.

We consider the following equation

$$signy|y|^m u_{xx} + u_{yy} + \frac{\beta_0}{y} u_y = 0, \tag{1}$$

in domain $D = D^+ \cup D^- \cup I$ of complex plane $z = x + iy$, where D^+ is the first quadrant of the plane, D^- is the finite domain in the fourth quadrant of the plane bounded by characteristics OC and BC of equation (1) going out from points $O(0,0)$, $B(1,0)$ and intersecting at point $C(\frac{1}{2}, -\left(\frac{m+2}{4}\right)^{\frac{2}{m+2}})$ and by the segment OB of the straight line $y = 0$, $I = \{(x,y) : 0 < x < 1, y = 0\}$. In (1) m, β_0 are some real numbers satisfying conditions $m > 0$, $-\frac{m}{2} < \beta_0 < 1$.

We introduce the following notation: $I_0 = \{(x,y) : 0 < y < \infty, x = 0\}$, $I_1 = \{(x,y) : 1 < x < \infty, y = 0\}$, C_0 and C_1 are correspondingly, points of intersection of characteristics OC and BC with the characteristics going out from point $E(c,0)$, where $c \in I$ is an arbitrary fixed number.

Let $q(x) \in C^1[c,1]$ be a diffeomorphism from the set of points of segment $[c,1]$ to the segment of points of segment $[0,c]$, such that $q'(x) < 0$, $q(c) = c$, $q(1) = 0$. As an example of such a function consider the linear function $q(x) = k(1-x)$, where $k = \frac{c}{1-c}$.

Problem. In domain D find a function $u(x,y)$ with the following properties:

1)$u(x,y) \in C(\bar{D})$ where $\bar{D} = \bar{D^-} \cup D^+ \cup \bar{I}_0 \cup \bar{I}_1$;

2) $u(x,y) \in C^2(D^+)$ and satisfies equation (1) in this domain;

3) $u(x,y)$ is a generalized solution from class R_1 [17] in domain D^-;

4) the following relations hold

$$\lim_{R\to\infty} u(x,y) = 0, R^2 = x^2 + \frac{4}{(m+2)^2} y^{m+2}, x > 0, y > 0; \tag{2}$$

5) $u(x,y)$ satisfies the boundary conditions

$$u(0,y) = \varphi(y), \ \ y \geq 0, \tag{3}$$

$$u(x,0) = \tau_1(x), \ \ x \in \bar{I}_1, \tag{4}$$

$$x^\beta D_{0,x}^{1-\beta} u[\theta(x)] = \rho(x)(x-c)^\beta D_{c,x}^{1-\beta} u[\theta^\star(x)] + \psi(x), \ \ c < x < 1, \tag{5}$$

$$u(q(x),0) = \mu u(x,0) + f(x), \ \ c \leq x \leq 1, \tag{6}$$

and the transmission condition

$$\lim_{y\to+0} y^{\beta_0} \frac{\partial u}{\partial y} = \lim_{y\to-0} (-y)^{\beta_0} \frac{\partial u}{\partial y}, \quad x \in I \setminus \{c\}, \tag{7}$$

moreover these limits at $x = 0$, $x = 1$, $x = c$ can have singularities of order less than $1 - 2\beta$, where $\beta = \frac{m+2\beta_0}{2(m+2)}$, $f(x) \in C[c,1] \cap C^{1,\delta_1}(c,1)$, $f(1) = 0$, $f(c) = 0$,$\mu = const$, $\rho(x), \psi(x) \in C[c,1] \cap C^{1,\delta_2}(c,1)$, $\tau_1(x) \in C(\bar{I}_1)$, moreover the function $\tau_1(x)$ near point $x = 1$ is representable in the form $\tau_1(x) = (1-x)\tilde{\tau}_1(x)$, $\tilde{\tau}_1(x) \in C(\bar{I}_1)$ and for sufficiently large x satisfies the equality $|\tau_1(x)| \leq \frac{M}{x^\varepsilon}$, ε, M are positive constants, $\tau_1(x)$ is satisfies the Hölder condition on arbitrary segment $[1,N]$, $N > 1$, $\varphi(y) \in C(\bar{I}_0)$, $y^{\frac{3m+2\beta_0}{4}} \varphi(y) \in L(0,\infty)$, $\varphi(y)$ is satisfies Hölder condition on arbitrary segment $[0,N]$, $N > 0$, $\varphi(\infty) = 0$, $\varphi(0) = 0$, $D_{0,x}^{1-\beta}$, $D_{c,x}^{1-\beta}$ are fractional differentiation operators in the sense of Riemann–Liouville [17], points of intersection of characteristics $C_0C(EC_1)$ with the characteristics from the point $(x_0,0)$, $x_0 \in (c,1)$, are

$$\theta(x_0) = \left(\frac{x_0}{2}, -\left(\frac{m+2}{4}x_0\right)^{2/(m+2)}\right),$$

$$\theta^\star(x_0) = \left(\frac{c+x_0}{2}, -\left(\frac{m+2}{4}(x_0-c)\right)^{2/(m+2)}\right).$$

In [4], the problem was investigated in a bounded domain where characteristic OC was arbitrarily divided into two parts (OC_0, C_0C), and on the first part the Tricomi condition was imposed, and on the second part of the characteristic parallel to it, the Bitsadze-Samarskii condition was imposed.

This paper, devoted to the study of the problem in an unbounded domain, differs from [18] in that here the characteristic OC_0 is freed from the boundary condition (Tricomi's condition [1]), which is equivalently replaced by the non-local Frankl condition [19] on the segment of the degeneracy line.

2 | UNIQUENESS OF SOLUTION TO THE PROBLEM

Theorem 1. Let the following conditions be fulfilled $\varphi(y) \equiv 0$, $\psi(x) \equiv 0$, $f(x) \equiv 0$, $\tau_1(x) \equiv 0$, $0 < \mu < 1$, $\rho(x) \leq 0$. Then the problem has only a trivial solution.

Proof of Theorem 1. It is known that the solution of the modified Cauchy problem $u(x,0) = \tau(x)$, $x \in \bar{I}$, $\lim\limits_{y\to -0}(-y)^{\beta_0}u_y = \nu(x)$, $x \in I$, has the following form

$$u(x,y) = \gamma_1 \int_0^1 \tau\left(x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1)\right) t^{\beta-1}(1-t)^{\beta-1}dt$$

$$-\gamma_2(-y)^{1-\beta_0} \int_0^1 \nu\left(x + \frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1)\right) t^{-\beta}(1-t)^{-\beta}dt, \tag{8}$$

where $\gamma_1 = \frac{\Gamma(2\beta)}{\Gamma^2(\beta)}$ and $\gamma_2 = \frac{2}{(m+2)}\frac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}$, $\Gamma(z)$ is gamma function[17].

From the formula (8) we have

$$u[\theta(x)] = \gamma_1 \int_0^1 \tau\left(\frac{x}{2} + \frac{2}{m+2}\frac{m+2}{4}x(2t-1)\right) t^{\beta-1}(1-t)^{\beta-1}dt-$$

$$-\gamma_2\left(\frac{m+2}{4}x\right)^{\frac{2(1-\beta_0)}{m+2}} \int_0^1 \nu\left(\frac{x}{2} + \frac{2}{m+2}\frac{m+2}{4}x(2t-1)\right) t^{-\beta}(1-t)^{-\beta}dt =$$

$$= \gamma_1 \int_0^1 \tau(xt)t^{\beta-1}(1-t)^{\beta-1}dt - \gamma_2\left(\frac{m+2}{4}x\right)^{1-2\beta} \int_0^1 \nu(xt)t^{-\beta}(1-t)^{-\beta}dt.$$

Let's replace the variable integration $z = xt$. Then we obtain

$$u[\theta(x)] = \gamma_1 \int_0^x \tau(z)\left(\frac{z}{x}\right)^{\beta-1}\left(\frac{x-z}{x}\right)^{\beta-1}\frac{dz}{x}-$$

$$-\gamma_2\left(\frac{m+2}{4}\right)^{1-2\beta} x^{1-2\beta} \int_0^x \nu(z)\left(\frac{z}{x}\right)^{-\beta}\left(\frac{x-z}{x}\right)^{-\beta}\frac{dz}{x} =$$

$$= \gamma_1 x^{1-2\beta} \int_0^x \frac{\tau(z)z^{\beta-1}dz}{(x-z)^{1-\beta}} - \gamma_2\left(\frac{m+2}{4}\right)^{1-2\beta} \int_0^x \frac{\nu(z)z^{-\beta}dz}{(x-z)^{\beta}}.$$

By virtue of the fractional integration operator in the sense of Riemann-Liouville[17]

$$D_{0,x}^{\alpha}f(x) = \frac{1}{\Gamma(-\alpha)} \int_0^x \frac{f(t)dt}{(x-t)^{1+\alpha}}, \alpha < 0,$$

we obtain

$$u[\theta(x)] = \gamma_1 x^{1-2\beta}\Gamma(\beta)D_{0,x}^{-\beta}\tau(x)x^{\beta-1} - \gamma_2\left(\frac{m+2}{4}\right)^{1-2\beta}\Gamma(1-\beta)D_{0,x}^{\beta-1}\nu(x)x^{-\beta}.$$

Similarly, it is easy to show that

$$u^{\star}[\theta(x)] = \gamma_1(x-c)^{1-2\beta}\Gamma(\beta)D_{c,x}^{-\beta}\tau(x)(x-c)^{\beta-1} - \gamma_2\left(\frac{m+2}{4}\right)^{1-2\beta}\Gamma(1-\beta)D_{c,x}^{\beta-1}\nu(x)(x-c)^{-\beta}.$$

Given these relations, from the boundary condition (5), after simple calculations, we obtain

$$\nu(x) = \gamma\omega(x)\left[D_{0,x}^{1-2\beta}\tau(x) - \rho(x)D_{c,x}^{1-2\beta}\tau(x)\right] + \Psi_1(x), x \in (c,1), \tag{9}$$

where $\Psi_1(x) = -\frac{\psi(x)}{\gamma_2\left(\frac{m+2}{4}\right)^{1-2\beta}\Gamma(1-\beta)(1-\rho(x))}$,

$\gamma = \frac{2\Gamma(1-\beta)\Gamma(2\beta)}{\Gamma(\beta)\Gamma(1-2\beta)}\left(\frac{m+2}{4}\right)^{2\beta}$, $\omega(x) = \frac{1}{1-\rho(x)}$.

The equality (9) is the first functional relation between the unknown functions $\tau(x)$ and $\nu(x)$, brought to the interval $(c, 1)$ of the axis $y = 0$ from the hyperbolic part D^- of mixed domain D.

Now let us prove that if $\varphi(y) \equiv 0$, $\psi(x) \equiv 0$, $f(x) \equiv 0$, $\tau_1(x) \equiv 0$, $0 < \mu < 1$, $\rho(x) \leq 0$ then the solution of the problem in domain of $D^+ \cup \bar{I}_0 \cup \bar{I} \cup \bar{I}_1$ by virtue of (2), is identically equal to zero.

Let D_R^+ be a finite domain cut out of domain D^+ by the arc $A_R B_R$ of the circle: $x^2 + \frac{4y^{m+2}}{(m+2)^2} = R^2, 0 \leq x \leq R, 0 \leq y \leq ((m+2)R/2)^{2/(m+2)}$, $A_R\left(0, \left(\frac{m+2}{2}R\right)^{\frac{2}{m+2}}\right)$, $B_R(R, 0)$.

Let (x_0, y_0) be a point of positive maximum of the function $u(x, y)$ in domain $\bar{D}_R^+$. In view of formula (2) for any $\varepsilon > 0$ there exists $R_0 = R_0(\varepsilon)$, such that for $R > R_0(\varepsilon)$ the inequality

$$|u(x, y)| < \varepsilon, \quad (x, y) \in A_R B_R. \tag{10}$$

By virtue of notation $u(x, 0) = \tau(x), \quad x \in \bar{I}$ the condition (6) is rewritten in the form

$$\tau(q(x)) = \mu\tau(x) + f(x), \quad x \in [c, 1]. \tag{11}$$

According to the Hopf principle[20] the function $u(x, y)$ does not attain its positive maximum and negative minimum at the inner points of domain $\bar{D}_R^+$ (Hereinafter, these points will be called extremum points functions $u(x, y)$). Let us assume that the function $u(x, y)$ reaches its positive maximum and negative minimum in domain $\bar{D}_R^+$ at the point $(x_0, 0)$ of the intervals $(0, c)$ and $(c, 1)$ of the axis $y = 0$.

Let us consider two cases separately: $x_0 \in (0, c)$ and $x_0 \in (c, 1)$. Let us assume that $x_0 \in (c, 1)$, then at this point in the case of a positive maximum (negative minimum)

$$\nu(x_0) < 0(\nu(x_0) > 0). \tag{12}$$

It is well known that at the point of the positive maximum (negative minimum) of the function $\tau(x)$ the fractional differentiation operators satisfy the inequality

$D_{c,x}^{1-2\beta}\tau(x)|_{x=x_0} > 0$ ($D_{c,x}^{1-2\beta}\tau(x)|_{x=x_0} < 0$). Then by virtue of (9) (where $\Psi(x) \equiv 0$), we have

$$\nu(x_0) = \gamma\omega(x_0)\left[D_{0,x}^{1-2\beta}\tau(x) - \rho(x)D_{c,x}^{1-2\beta}\tau(x)\right]_{x=x_0} > 0$$

$$(\nu(x_0) = \gamma\omega(x_0)\left[D_{0,x}^{1-2\beta}\tau(x) - \rho(x)D_{c,x}^{1-2\beta}\tau(x)\right]_{x=x_0} < 0). \tag{13}$$

Inequalities (12) and (13) contradict the conjugation (7), whence we deduce that $x_0 \notin (c, 1)$.

Now let us assume that $x_0 \in (0, c)$. Let $x_1 \in (c, 1)$ be the solution to the equation $q(x_1) = x_0$. In this case from (11) (where $f(x) \equiv 0$) we have

$$\tau(x_0) = \tau(q(x_1)) = \mu\tau(x_1). \tag{14}$$

Equality (14) shows, that the point x_1 is the extremum point of the function $\tau(x)$ in the interval $(c, 1)$, which contradicts the previous case. Hence $x_0 \notin (0, c)$.

We show that the point $E(c, 0)$ is also not the extremum point of the function $u(x, y)$. Indeed, from equality (11), where $f(x) \equiv 0$, we have $\tau(q(c)) = \mu\tau(c)$. Then, by virtue of the equality $q(c) = c$, it follows that $\tau(c)(1 - \mu) = 0$, i.e. $\tau(c) = 0$.

Consequently, there are no points of positive maximum (negative minimum) of the function $u(x, y)$ on the interval OB. Let $R > R_0$. From the Hopf principle and the previous reasoning, if $(x_0, y_0) \in A_R B_R$, then by virtue of (10) we have $|u(x_0, y_0)| < \varepsilon$.Therefore, $|u(x, y)| < \varepsilon$ for any $(x, y) \in \bar{D}_R^+$. Hence, by virtue of the arbitrariness of ε, with $R \to +\infty$ we conclude that $u(x, y) \equiv 0$ in domain $D^+ \cup \bar{I}_0 \cup \bar{I} \cup \bar{I}_1$. Then

$$\lim_{y\to+0} u(x, y) = 0, \quad x \in \bar{I}; \lim_{y\to+0} y^{\beta_0}u_y = 0, x \in I. \tag{15}$$

Taking into account (15), due to the continuity of the solution in domain $\bar{D}_R^+$ and the conjugation condition (7), restoring the sought function $u(x, y)$ in domain D^- as a solution of the modified Cauchy problem with homogeneous data, we obtain $u(x, y) \equiv 0$ in domain $\bar{D}^-$. Theorem 1 is proved. □

3 | EXISTENCE OF THE SOLUTION TO THE PROBLEM

Theorem 2. Let the conditions $q(x) = k(1-x)$, $0 < \mu < 1$, $\rho(x) \leq 0$, $\mu k^{\frac{1}{2}-3\alpha}(1 + 2sin(\beta\pi)\omega(c)) < 1$, $\beta_0 > -\frac{m-1}{3}$, where $\alpha = (1-2\beta)/4$, $k = c/(1-c)$, $\omega(c) = 1/(1-\rho(c))$ be fulfilled. Then the solution to the problem exists.

Proof of Theorem 2. The solution of the Dirichlet problem in domain D^+ satisfying the conditions (2)-(4) and the condition $u(x,0) = \tau(x)$, $x \in \bar{I}$, can be represented in the following form

$$u(x,y) = k_2 y^{1-\beta_0} \int_0^1 \tau(t)\times$$

$$\times\left(\left((t-x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1} - \left((t+x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1}\right)dt+$$

$$+k_2 y^{1-\beta_0} \int_1^\infty \tau_1(t)\times$$

$$\times\left(\left((t-x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1} - \left((t+x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1}\right)dt+$$

$$+\frac{2}{m+2}y^{\frac{1-\beta_0}{2}} \int_0^\infty t^{\frac{2m+1+\beta_0}{2}}\varphi(t)dt\times$$

$$\times\int_0^\infty se^{-sx} J_{\frac{1-2\beta}{2}}\left(\frac{2st^{\frac{m+2}{2}}}{m+2}\right) J_{\frac{1-2\beta}{2}}\left(\frac{2sy^{\frac{m+2}{2}}}{m+2}\right)ds, \tag{16}$$

where $k_2 = \frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(1-\beta)(1-\beta_0)}{\Gamma(2-2\beta)}$, $\beta = \frac{2\beta_0+m}{2(m+2)}$, $J_\nu(z)$ is the Bessel function of the first kind. Differentiating equality (16) in y we get

$$\frac{\partial u}{\partial y} = k_2 \int_0^1 \tau(t)\frac{\partial}{\partial y}y^{1-\beta_0}\times$$

$$\times\left(\left((t-x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1} - \left((t+x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1}\right)dt + \frac{\partial}{\partial y}F_1(x,y) + \frac{\partial}{\partial y}F_2(x,y), \tag{17}$$

where

$$F_1(x,y) = k_2 \int_1^\infty \tau_1(t)y^{1-\beta_0}\times$$

$$\times\left(\left((t-x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1} - \left((t+x)^2 + \frac{4}{(m+2)^2}y^{m+2}\right)^{\beta-1}\right)dt,$$

$$F_2(x,y) = \frac{2}{m+2}y^{\frac{1-\beta_0}{2}} \int_0^\infty t^{\frac{2m+1+\beta_0}{2}}\varphi(t)dt \int_0^\infty se^{-sx} J_{\frac{1-2\beta}{2}}\left(\frac{2st^{\frac{m+2}{2}}}{m+2}\right) J_{\frac{1-2\beta}{2}}\left(\frac{2sy^{\frac{m+2}{2}}}{m+2}\right)ds.$$

By virtue of equality

$$\frac{\partial}{\partial y}y^{1-\beta_0}\left(\left[(x-t)^2 + \frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1} - \left[(x+t)^2 + \frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right) =$$

$$= \frac{m+2}{2}y^{-\beta_0}\frac{\partial}{\partial t}\times$$

$$\times\left((x-t)\left[(x-t)^2 + \frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1} + (x+t)\left[(x+t)^2 + \frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)$$

from (17) we have

$$\frac{\partial u}{\partial y}=k_2\frac{m+2}{2}y^{-\beta_0}\int\limits_0^1\tau(t)\times$$

$$\times\frac{\partial}{\partial t}\left((x-t)\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}+(x+t)\left[(x+t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)dt+$$

$$+\frac{\partial}{\partial y}F_1(x,y)+\frac{\partial}{\partial y}F_2(x,y). \tag{18}$$

In the integral of the right side of equality (18), having performed the integration operation in parts, taking into account $\tau(0)=0$, $\tau(1)=0$, after simple calculations we have

$$\frac{\partial u}{\partial y}=-k_2\frac{m+2}{2}y^{-\beta_0}\int\limits_0^1\tau'(t)\times$$

$$\times\left((x-t)\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}+(x+t)\left[(x+t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)dt+$$

$$+\frac{\partial}{\partial y}F_1(x,y)+\frac{\partial}{\partial y}F_2(x,y). \tag{19}$$

Multiplying both parts of the equality (19) by y^{β_0}, then moving to the limit at $y\to+0$, we get

$$\nu(x)=-k_2\frac{m+2}{2}\int\limits_0^1\tau'(t)\left[(x-t)|x-t|^{2\beta-2}+(t+x)^{2\beta-1}\right]dt+\Phi_0(x),\, x\in(0,1), \tag{20}$$

where

$$\Phi_0(x)=\lim_{y\to+0}y^{\beta_0}\frac{\partial}{\partial y}(F_1(x,y)+F_2(x,y))=k_2(1-\beta_0)\int\limits_0^\infty\tau_1(t)\left[(t-x)^{2\beta-2}-(t+x)^{2\beta-2}\right]dt+$$

$$+\frac{2}{(m+2)^{\frac{1-2\beta}{2}}\Gamma(\frac{1}{2}-\beta)}\int\limits_0^\infty\varphi(t)t^{\frac{2m+1+\beta_0}{2}}dt\int\limits_0^\infty s^{\frac{3-2\beta}{2}}e^{-sx}J_{\frac{1-2\beta}{2}}\left(\frac{2s^{\frac{m+2}{2}}}{m+2}\right)ds.$$

Equality (20) is a functional relation between the unknown functions $\tau(x)$ and $\nu(x)$, brought to I from the elliptic part D^+ of the mixed domain D. Note that the relation (20) is valid for the entire interval I. Next, the integration interval $(0,1)$ is divided into the intervals $(0,c)$ and $(c,1)$, and then in integrals with the limit $(0,c)$ by replacing the variable integration $t=q(s)=k(1-s)$, given the equality (10), the relation (20) is reduced to the form

$$\nu(x)=-k_2\frac{m+2}{2}\left(\int\limits_c^x\tau'(t)(x-t)^{2\beta-1}dt-\int\limits_x^c\tau'(t)(t-x)^{2\beta-1}dt\right)-$$

$$-k_2\frac{m+2}{2}\left(\int\limits_c^1\tau'(t)(t+x)^{2\beta-1}dt-\mu\int\limits_c^1\tau'(s)\left[(x-q(s))^{2\beta-1}+(x+q(s))^{2\beta-1}\right]ds\right)+\Phi_1(x),\;\; x\in(c,1), \tag{21}$$

where

$$\Phi_1(x)=k_2\frac{m+2}{2}\int\limits_c^1 f'(s)\left[(x-q(s))^{2\beta-1}(x+q(s))^{2\beta-1}\right]ds+\Phi_0(x).$$

By virtue of (7), excluding the function $\nu(x)$ from (9) and (21) we get

$$-\frac{2\gamma\omega(x)}{(m+2)k_2}\left[D_{0,x}^{1-2\beta}\tau(x)-\rho(x)D_{c,x}^{1-2\beta}\tau(x)\right]+\Psi_2(x)=\int\limits_c^x\tau'(t)(x-t)^{2\beta-1}dt-\int\limits_x^1\tau'(t)(t-x)^{2\beta-1}dt+$$

$$+\int\limits_c^1\tau'(t)(t+x)^{2\beta-1}dt-\mu\int\limits_c^1\tau'(s)\left[(x-q(s))^{2\beta-1}+(x+q(s))^{2\beta-1}\right]ds,\;\; x\in(c,1), \tag{22}$$

where $\Psi_2(x) = -\frac{2}{(m+2)k_2}\left(\Psi_1(x) - \Phi_1(x)\right)$. By applying the operator $\Gamma(1-2\beta)D_{c,x}^{2\beta-1}$ to both parts of equality (22) we have

$$-\frac{2\gamma}{(m+2)k_2}\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\omega(x)\left[D_{0,x}^{1-2\beta}\tau(x) - \rho(x)D_{c,x}^{1-2\beta}\tau(x)\right] + \Gamma(1-2\beta)D_{c,x}^{2\beta-1}\Psi_2(x) =$$

$$= \Gamma(1-2\beta)D_{c,x}^{2\beta-1}\left(\int_c^x \tau'(t)(x-t)^{2\beta-1}dt - \int_x^1 \tau'(t)(t-x)^{2\beta-1}dt\right) +$$

$$+\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\left(\int_c^1 \tau'(t)(t+x)^{2\beta-1}dt - \mu\int_c^1 \tau'(s)\left[(x-q(s))^{2\beta-1} + (x+q(s))^{2\beta-1}\right]ds\right),\ x \in (c,1), \tag{23}$$

It is not difficult to make sure that

$$\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\int_c^x \tau'(t)(x-t)^{2\beta-1}dt = \Gamma(2\beta)\Gamma(1-2\beta)\tau(x), \tag{24}$$

$$\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\int_x^1 \tau'(t)(t-x)^{2\beta-1}dt = \pi ctg(2\beta\pi)\tau(x) + \int_c^1 \left(\frac{x-c}{t-c}\right)^{1-2\beta}\frac{\tau(t)dt}{t-x}, \tag{25}$$

$$\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\int_c^1 \tau'(t)(t+x)^{2\beta-1}dt = \int_c^1 \left(\frac{x-c}{c+t}\right)^{1-2\beta}\frac{\tau(t)dt}{t+x}, \tag{26}$$

$$\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\int_c^1 \tau'(s)(x-q(s))^{2\beta-1}ds = -\int_c^1 \left(\frac{x-c}{c-q(s)}\right)^{1-2\beta}\frac{\tau(s)q'(s)ds}{x-q(s)}, \tag{27}$$

$$\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\int_c^1 \tau'(s)(x+q(s))^{2\beta-1}ds = \int_c^1 \left(\frac{x-c}{c+q(s)}\right)^{1-2\beta}\frac{\tau(s)q'(s)ds}{x+q(s)}. \tag{28}$$

$$\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\omega(x)\left[D_{0,x}^{1-2\beta}\tau(x) - \rho(x)D_{c,x}^{1-2\beta}\tau(x)\right] =$$

$$= \frac{\mu\omega(x)}{\Gamma(2\beta)}\int_c^1 \left(\frac{x-c}{c-q(z)}\right)^{1-\alpha-\beta}\frac{\tau(z)q'(z)dz}{x-q(z)} + \frac{\omega(x)}{\Gamma(2\beta)}\int_c^1 \left(\frac{x-c}{c-q(z)}\right)^{1-\alpha-\beta}\frac{f(z)q'(z)dz}{x-q(z)} +$$

$$+\frac{\mu(\omega(c)-\omega(x))}{\Gamma(2\beta)(x-c)^{2\beta}}\int_c^1 \frac{\tau(z)q'(z)dz}{(c-q(z))^{1-2\beta}} + \frac{\omega(c)-\omega(x)}{\Gamma(2\beta)(x-c)^{2\beta}}\int_c^1 \frac{f(z)q'(z)dz}{(c-q(z))^{1-2\beta}} +$$

$$+\frac{\mu}{\Gamma(2\beta)}\int_c^1 \tau(z)q'(z)dz\int_c^x \left[\frac{\omega'(s)}{(x-s)^{2\beta}} + \frac{2\beta(\omega(s)-\omega(x))}{(x-s)^{1+2\beta}}\right]\frac{ds}{(s-q(z))^{1-2\beta}} +$$

$$+\frac{\mu}{\Gamma(2\beta)}\int_c^1 f(z)q'(z)dz\int_c^x \left[\frac{\omega'(s)}{(x-s)^{2\beta}} + \frac{2\beta(\omega(s)-\omega(x))}{(x-s)^{1+2\beta}}\right]\frac{ds}{(s-q(z))^{1-2\beta}} + \Gamma(1-2\beta)\tau(x). \tag{29}$$

Substituting (24)-(29) in (23),after some calculations we have

$$\left[\frac{2\gamma\Gamma(1-2\beta)}{(m+2)k_2} + \Gamma(2\beta)\Gamma(1-2\beta) - \pi ctg(2\beta\pi)\right]\tau(x) - \int_c^1 \left(\frac{x-c}{t-c}\right)^{1-2\beta}\frac{\tau(t)dt}{t-x} =$$

$$= -\mu\int_c^1 \left(\frac{x-c}{c-q(t)}\right)^{1-2\beta}\frac{\tau(t)q'(t)dt}{x-q(t)} - \frac{2\gamma}{(m+2)k_2}\frac{\mu\omega(x)}{\Gamma(2\beta)}\int_c^1 \left(\frac{x-c}{c-q(t)}\right)^{1-2\beta}\frac{\tau(t)q'(t)dt}{x-q(t)} + R[\tau] + F_1(x), \tag{30}$$

where

$$R[\tau]=-\int_c^1\left(\frac{x-c}{c+t}\right)^{1-2\beta}\frac{\tau(t)dt}{t+x}+\mu\int_c^1\left(\frac{x-c}{c+q(t)}\right)^{1-2\beta}\frac{\tau(t)q'(t)dt}{x+q(t)}+$$
$$+\frac{\mu}{\Gamma(2\beta)}\frac{\omega(c)-\omega(x)}{(x-c)^{1-2\beta}}\int_c^1\frac{\tau(t)q'(t)dt}{(c-q(t))^{1-2\beta}}+\frac{\mu}{\Gamma(2\beta)}\int_c^1\tau(t)q'(t)dt\int_c^x\left[\frac{\omega'(s)}{(x-s)^{2\beta}}+\frac{2\beta(\omega(s)-\omega(x))}{(x-s)^{1+2\beta}}\right]\times$$
$$\times\frac{ds}{(s-q(t))^{1-2\beta}}$$

is the regular operator,

$$F_1(x)=-\frac{2\gamma}{(m+2)k_2}\left[\frac{\omega(x)}{\Gamma(2\beta)}\int_0^1\left(\frac{x-c}{c-q(t)}\right)^{1-2\beta}\frac{f(t)q'(t)dt}{x-q(t)}+\frac{1}{\Gamma(2\beta)}\frac{\omega(c)-\omega(x)}{(x-c)^{2\beta}}\int_c^1\frac{f(t)q'(t)dt}{(c-q(t))^{1-2\beta}}\right]-$$
$$-\frac{2\gamma}{(m+2)k_2}\frac{1}{\Gamma(2\beta)}\int_c^1 f(t)q'(t)dt\int_c^x\left[\frac{\omega'(s)}{(x-s)^{2\beta}}+\frac{2\beta(\omega(s)-\omega(x))}{(x-s)^{1+2\beta}}\right]\times$$
$$\times\frac{ds}{(s-q(t))^{1-2\beta}}+\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\Psi_2(x).$$

Equality (30) is written in the following form

$$\tau(x)-\lambda\int_c^1\left(\frac{x-c}{t-c}\right)^{1-2\beta}\frac{\tau(t)dt}{t-x}=g(x),x\in(c,1),\tag{31}$$

where

$$g(x)=\mu k\lambda(1+2sin(\beta\pi)\omega(x))\int_c^1\left(\frac{x-c}{c-q(t)}\right)^{1-2\beta}\frac{\tau(t)dt}{x-q(t)}+R[\tau]+F_1(x),\tag{32}$$

$$R[\tau]=-\lambda\int_c^1\left(\frac{x-c}{c+t}\right)^{1-2\beta}\frac{\tau(t)dt}{x+t}-\mu k\lambda\int_c^1\left(\frac{x-c}{c+q(t)}\right)^{1-2\beta}\frac{\tau(t)dt}{x+q(t)}$$

is the regular operator, $F_1(x)=\lambda\Gamma(1-2\beta)D_{c,x}^{2\beta-1}\Psi_2(x)$, $\lambda=\frac{cos(\beta\pi)}{\pi(1+sin(\beta\pi))}$.

The first integral operator in $g(x)$ is not regular, since the integrand for $x=c$, $t=c$ has an isolated first-order singularity. Therefore, this term in (31) is highlighted separately.

In (31) assuming $(x-c)^{2\beta-1}\tau(x)=\rho(x)$, $(x-c)^{2\beta-1}g(x)=g_1(x)$ we obtain

$$\rho(x)-\lambda\int_c^1\frac{\rho(t)dt}{t-x}=g_1(x),\ x\in(c,1).\tag{33}$$

We will seek the solution to the equation (33) in the class of functions satisfying the Hölder condition on $(c,1)$ and bounded at $x=1$, and with $x=c$, which can turn to infinity of the order of less than $1-2\beta$. In this class, the index of the equation (33) is zero. Applying the Carleman-Vecua[17] method to the equation (33), we obtain its solution

$$\rho(x)=\frac{1+sin(\beta\pi)}{2}g_1(x)+\frac{cos(\beta\pi)}{2\pi}\left(\frac{1-x}{x-c}\right)^{\frac{1}{4}(1-2\beta)}\times$$
$$\times\int_c^1\frac{g_1(t)dt}{\left(\frac{1-t}{t-c}\right)^{\frac{1}{4}(1-2\beta)}(t-x)}.\tag{34}$$

In (34) returning to the previous functions we get

$$\tau(x)=cos^2(\pi\alpha)g(x)+\frac{sin(2\pi\alpha)}{2\pi}\int_c^1\left(\frac{(1-x)(x-c)^3}{(1-t)(t-c)^3}\right)^{\alpha}\frac{g(t)dt}{t-x},\tag{35}$$

where $\alpha=(1-2\beta)/4$.

Now, taking into account the expression for $g(x)$ from (32), we convert the solution (35) to the form

$$\tau(x) = \lambda\mu k(1+2sin(\pi\beta\omega(x))cos^2(\pi\alpha)\int\limits_c^1 \left(\frac{x-c}{c-q(t)}\right)^{4\alpha}\frac{\tau(t)dt}{x-q(t)}+ \\ +\lambda\mu k\frac{sin(2\pi\alpha)}{2\pi}(1-x)^{\alpha}(x-c)^{3\alpha}\int\limits_c^1 \frac{\tau(s)ds}{(c-q(s))^{4\alpha}}\times \\ \times\int\limits_c^1 \left(\frac{t-c}{1-t}\right)^{\alpha}\frac{(1+2sin(\beta\pi)\omega(t))dt}{(t-q(s))(t-x)}+R_1[\tau]+F_2(x). \tag{36}$$

where

$$R_1[\tau] = cos^2(\pi\alpha)R[\tau]+\frac{sin(2\pi\alpha)}{2\pi}\int\limits_c^1 \left(\frac{(1-x)(x-c)^3}{(1-t)(t-c)^3}\right)^{\alpha}\frac{R[\tau]dt}{t-x}$$

is the regular operator,

$$F_2[x] = cos^2(\pi\alpha)F_1(x)+\frac{sin(2\pi\alpha)}{2\pi}\int\limits_c^1 \left(\frac{(1-x)(x-c)^3}{(1-t)(t-c)^3}\right)^{\alpha}\frac{F_1(t)dt}{t-x}.$$

Equation (36) is transformed to the form

$$\tau(x) = \lambda\mu k(1+2sin(\pi\beta\omega(x))cos^2(\pi\alpha)\int\limits_c^1 \left(\frac{x-c}{c-q(t)}\right)^{4\alpha}\frac{\tau(t)dt}{x-q(t)}+ \\ +\lambda\mu k\frac{sin(2\pi\alpha)}{2\pi}(1+2sin(\beta\pi)\omega(x))(1-x)^{\alpha}(x-c)^{3\alpha}\int\limits_c^1 \frac{\tau(s)ds}{(c-q(s))^{4\alpha}}\times \\ \times\int\limits_c^1 \left(\frac{t-c}{1-t}\right)^{\alpha}\frac{dt}{(t-q(s))(t-x)}+R_2[\tau]+F_2(x), \tag{37}$$

where

$$R_2[\tau] = R_1[\tau]-\frac{\lambda\mu k sin(2\pi\alpha)sin(\beta\pi)(1-x)^{\alpha}(x-c)^{3\alpha}}{\pi}\int\limits_c^1 \frac{\tau(s)ds}{(c-q(s))^{4\alpha}}\int\limits_c^1 \left(\frac{t-c}{1-t}\right)^{\alpha}\frac{(\omega(t)-\omega(x))dt}{(t-x)(t-q(s))}$$

is the regular operator.

By virtue of $q(x) = k(1-x), k = \frac{c}{1-c}$, the equation (37) is written as

$$\tau(x) = \lambda\mu k^{1-4\alpha}(1+2sin(\beta\pi)\omega(x))cos^2(\pi\alpha)\int\limits_c^1 \left(\frac{x-c}{t-c}\right)^{4\alpha}\frac{\tau(t)dt}{x-k(1-t)}+ \\ +\lambda\mu k^{1-4\alpha}\frac{sin(2\pi\alpha)}{2\pi}(1+2sin(\beta\pi)\omega(x))(1-x)^{\alpha}(x-c)^{3\alpha}\int\limits_c^1 \frac{\tau(s)ds}{(s-c)^{4\alpha}}\times \\ \times\int\limits_c^1 \left(\frac{t-c}{1-t}\right)^{\alpha}\frac{dt}{(t+ks-k)(t-x)}+R_2[\tau]+F_2(x). \tag{38}$$

Further, it is not difficult to make sure that the value of the internal integral in (38) has the form

$$A(x,s) = \int\limits_c^1 \left(\frac{t-c}{1-t}\right)^{\alpha}\frac{dt}{(t-x)(t-k+ks)} = \frac{1}{x+ks-k}\times \\ \times\left(-\pi ctg(\pi\alpha)\frac{(x-c)^{\alpha}}{(1-x)^{\alpha}}-\Gamma(-\alpha)\Gamma(1+\alpha)-\frac{1-c}{1+ks-k}\Gamma(1+\alpha)\Gamma(1-\alpha)F(1-\alpha,1,2;\frac{1-c}{1+ks-k})\right). \tag{39}$$

Now, substituting (39) into (38), we get

$$\tau(x)=\lambda\mu k^{1-4\alpha}(1+2sin(\beta\pi)\omega(x))cos^2(\pi\alpha)\int\limits_c^1\left(\frac{x-c}{t-c}\right)^{4\alpha}\frac{\tau(t)dt}{x-k(1-t)}+$$
$$+\lambda\mu k^{1-4\alpha}\frac{sin(2\pi\alpha)}{2\pi}(1+2sin(\beta\pi)\omega(x))(1-x)^{\alpha}(x-c)^{3\alpha}\int\limits_c^1\frac{\tau(s)ds}{(s-c)^{4\alpha}}\frac{1}{x+ks-k}\times \tag{40}$$
$$\times\left(-\pi ctg(\pi\alpha)\frac{(x-c)^{\alpha}}{(1-x)^{\alpha}}-\Gamma(-\alpha)\Gamma(1+\alpha)-\frac{1-c}{1+ks-k}\Gamma(1+\alpha)\Gamma(1-\alpha)F(1-\alpha,1,2;\frac{1-c}{1+ks-k})\right)+$$
$$+R_2[\tau]+F_2(x).$$

After simple calculations, the equation (40) is written as

$$\tau(x)=\lambda\int\limits_c^1\frac{K(x,s)\tau(s)ds}{x+ks-k}+R_2[\tau]+F_2(x),x\in(c,1), \tag{41}$$

where

$$K(x,s)=-\mu k^{1-4\alpha}(1+2sin(\beta\pi)\omega(x))\frac{sin(2\pi\alpha)}{2\pi}\frac{(1-x)^{\alpha}(x-c)^{3\alpha}}{(s-c)^{4\alpha}}\times$$
$$\times\left(\Gamma(-\alpha)\Gamma(1+\alpha)+\frac{1-c}{1+ks-k}\Gamma(1+\alpha)\Gamma(1-\alpha)F(1-\alpha,1,2;\frac{1-c}{1+ks-k})\right).$$

Applying the Boltz formula for the hypergeometric function, after some calculations, the kernel $K(x,s)$ takes the following form

$$K(x,s)=\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(x))cos(\pi\alpha)\left(\frac{1-x}{1-q(s)}\right)^{\alpha}\left(\frac{x-c}{s-c}\right)^{3\alpha}.$$

Then we rewrite the equations (41) in the form

$$\tau(x)=\lambda\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(x))cos(\pi\alpha)\int\limits_c^1\left(\frac{1-x}{1-q(s)}\right)^{\alpha}\left(\frac{x-c}{s-c}\right)^{3\alpha}\frac{\tau(s)ds}{x+ks-k}+R_2[\tau]+F_2(x),x\in(c,1), \tag{42}$$

In (42), highlighting the characteristic part, we get

$$\tau(x)=\lambda\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(c))cos(\pi\alpha)\int\limits_c^1\left(\frac{x-c}{s-c}\right)^{3\alpha}\frac{\tau(s)ds}{x-k(1-s)}+R_3[\tau]+F_2(x),x\in(c,1), \tag{43}$$

where

$$R_3[\tau]=\lambda\mu k^{1-3\alpha}cos(\pi\alpha)\int\limits_c^1\left(\frac{x-c}{s-c}\right)^{3\alpha}\frac{\tau(s)ds}{x-k(1-s)}\times$$
$$\times\left[\left(\frac{1-x}{1-q(s)}\right)^{\alpha}(1+2sin(\beta\pi)\omega(x))-(1+2sin(\beta\pi)\omega(c))\right]+R_2[\tau]$$

is the regular operator. The equation (43) is rewritten as

$$\tau(x)=\lambda\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(c))cos(\pi\alpha)\int\limits_c^1\left(\frac{x-c}{s-c}\right)^{3\alpha}\frac{\tau(s)ds}{(s-c)\left(k+\frac{x-c}{s-c}\right)}+R_3[\tau]+F_2(x),x\in(c,1). \tag{44}$$

In the equation (44), by replacing the variables $x=c+(1-c)e^{-\xi}$, $s=c+(1-c)e^{-t}$, and denoting $\rho(\xi)=\tau(c+(1-c)e^{-\xi})e^{(3\alpha-\frac{1}{2})\xi}$, we get

$$\rho(\xi)=\lambda\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(c))cos(\pi\alpha)\int\limits_0^{\infty}\frac{\rho(t)dt}{ke^{\frac{\xi-t}{2}}+e^{-\frac{\xi-t}{2}}}+R_4[\rho(\xi)]+F_3(\xi),\ \xi\in(0,\infty), \tag{45}$$

where $R_4[\rho(\xi)]=R_3[\tau(c+(1-c)e^{-\xi})]e^{(3\alpha-\frac{1}{2})\xi}$, $F_3(\xi)=F_2[(c+(1-c)e^{-\xi})]e^{(3\alpha-\frac{1}{2})\xi}$. By virtue of the condition $\beta_0>-\frac{m-1}{3}$, there is an inequality $3\alpha-\frac{1}{2}<0$.

Let us introduce the notation
$$N(\xi)=\frac{\lambda\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(c))cos(\alpha\pi)}{ke^{\frac{\xi}{2}}+e^{-\frac{\xi}{2}}}.$$
Then the equation (45) is written as
$$\rho(\xi)=\int\limits_0^\infty N(\xi-t)\rho(t)dt+R_4[\rho(\xi)]+F_3(\xi),\ \xi\in(0,\infty). \tag{46}$$
The equation (46) is the Wiener–Hopf integral equation[21] and using the Fourier transform it is reduced to the Riemann boundary value problem, i.e. solved in quadratures. Functions $N(\xi)$, $F_3(\xi)$ have exponential decreasing order at infinity, with $N(\xi)\in C(0,\infty)$, $F_3(\xi)\in H_{\alpha_1}(0,\infty)$. Therefore, $N(\xi), F_3(\xi)\in L_2\cap H_{\alpha_1}$. Fredholm's theorems for integral equations of the convolution type are valid only in one particular case when the index of these equations is zero. The index of the equation (46) is the index of the expression $1-N\hat{}(\xi)$ with the opposite sign, where
$$N\hat{}(\xi)=\int\limits_{-\infty}^{\infty}e^{i\xi t}N(t)dt=\lambda\mu k^{1-3\alpha}(1+2sin(\beta\pi)\omega(c))cos(\alpha\pi)\int\limits_{-\infty}^{\infty}\frac{e^{i\xi t}dt}{ke^{\frac{t}{2}}+e^{-\frac{t}{2}}}. \tag{47}$$
Having calculated the Fourier integral, using the residue theory[8] we find
$$\int\limits_{-\infty}^{\infty}\frac{e^{i\xi t}dt}{ke^{\frac{t}{2}}+e^{-\frac{t}{2}}}=\frac{\pi e^{-i\xi lnk}}{\sqrt{k}ch(\pi\xi)}. \tag{48}$$
Substituting (48) in (47), considering $\lambda=\frac{cos(\beta\pi)}{\pi(1+sin(\beta\pi))}$, $\alpha=(1-2\beta)/4$, we have
$$N\hat{}(\xi)=\mu k^{\frac{1}{2}-3\alpha}(1+2sin(\beta\pi)\omega(c))sin(\alpha\pi)\frac{e^{-i\xi lnk}}{ch(\pi\xi)}.$$
Since $\mu k^{\frac{1}{2}-3\alpha}(1+2sin(\beta\pi)\omega(c))<1$ and since
$$Re(N\hat{}(\xi))=Re\left(\mu k^{\frac{1}{2}-3\alpha}(1+2sin(\beta\pi)\omega(c))sin(\alpha\pi)\frac{e^{-i\xi lnk}}{ch(\pi\xi)}\right)=$$
$$=\mu k^{\frac{1}{2}-3\alpha}sin(\pi\alpha)(1+2sin(\beta\pi)\omega(c))\frac{cos(\xi lnk)}{ch(\pi\xi)}<\mu k^{\frac{1}{2}-3\alpha}(1+2sin(\beta\pi)\omega(c))<1$$
then $Re(1-N\hat{}(\xi))>0$. Hence, the index of the equation (46) $\chi=-Jnd(1-N\hat{}(\xi)=0$, i.e. changing the argument of the expression $1-N\hat{}(\xi))$ on the real axis, expressed in full revolutions, is zero[21]. Consequently, the equation (46) is uniquely reduced to the Fredholm integral equation of the second kind, the unambiguous solvability of which follows from the uniqueness of the solution to the problem GF. Theorem 2 is proved. □

4 | CONCLUSION

In this work, we study a new nonlocal boundary value problem for an elliptic-hyperbolic equation. Main results are new. Using these results, we can explore various boundary value problems for mixed-type equations of the second and higher orders.

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