# A calculus for intuitionistic fuzzy values

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**Abstract**

We introduce $^{\oplus}$calculus and $^{\otimes}$calculus for intuitionistic fuzzy values and prove some basic theorems by using multiplicative calculus which has useful tools to represent the concepts of introduced calculi. Besides, we construct some isomorphic mappings to interpret the relationships between $^{\oplus}$calculus and $^{\otimes}$calculus. This paper reveals also new calculi for fuzzy sets in particular.

# A calculus for intuitionistic fuzzy values

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**Abstract:** We introduce $^{\oplus}$calculus and $^{\otimes}$calculus for intuitionistic fuzzy values and prove some basic theorems by using multiplicative calculus which has useful tools to represent the concepts of introduced calculi. Besides, we construct some isomorphic mappings to interpret the relationships between $^{\oplus}$calculus and $^{\otimes}$calculus. This paper reveals also new calculi for fuzzy sets in particular.

## 1 Introduction

Fuzzy set theory [21] is an extension of classical set theory and it provides researchers with tools to handle the elements which are not categorizable by classical sets. Fuzzy sets consider every element in the universe of discourse by assigning a membership value to each of them, while classical sets consider only the elements which are either member or nonmember of the set. In other words, classical sets exclude the partial membership while fuzzy sets include. Fuzzy sets are also extended to intuitionistic fuzzy sets(IFS) by Atanassov [4] in consideration of the partial nonmembership values. Following its introduction, IFSs are studied by many mathematicians from different aspects. In particular, many concepts of intuitionistic fuzzy calculus are introduced and applied to problems having two facets of uncertainty, namely, fuzziness and hesitancy [2, 3, 10–12, 22].

In [20], we defined the concepts of $^{\oplus}$convergence and $^{\otimes}$convergence for sequences of intuitionistic fuzzy values(IFV) and illustrated their advantage over the literature by an example [20, Example 4.3]. To be more precise, while the convergence types in the literature are either inapplicable to many sequences of IFVs or they assign multiple limits to a sequence, $^{\oplus}$convergence and $^{\otimes}$convergence are applicable to almost every sequence of IFVs and reveal a unique limit provided that the limit exists. In [20], there are also methods to recover the convergence of sequences of IFVs which do not $^{\oplus}$converge(or $^{\otimes}$converge) ordinarily. In the light of these results, now there is a need to define the concepts of $^{\oplus}$limit and $^{\otimes}$limit for intuitionistic fuzzy valued functions(IFVF) in order to extend the aforementioned advantages to intuitionistic fuzzy calculus, and a need to construct corresponding calculi. The aim of this paper is to define $^{\oplus}$limit and $^{\otimes}$limit for IFVFs and construct corresponding intuitionistic fuzzy calculi by utilizing the tools of multiplicative calculus [8, 17] which has close relation with the new calculi. The constructed calculi reveals also a new calculi for fuzzy sets in the absence of hesitancy.

Before to continue with main results, we give some preliminaries concerning IFSs and multiplicative calculus. Let $X$ be a non-empty set. Then, an Atanassov's intuitionistic fuzzy set [4] has the following form: $A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in X\}$ where $\mu : X \to [0, 1]$ is called membership function and $\nu : X \to [0, 1]$ is called non-membership function. For any $x \in X$, $0 \le \mu_A(x) + \nu_A(x) \le 1$. In special case $\mu_A(x) + \nu_A(x) = 1$, A-IFS degenerates to fuzzy set [21]. Following [6, 7, 19], we use the notation $\alpha = (\alpha_1, \alpha_2)$ for an IFV where $\alpha_1 \in [0, 1]$, $\alpha_2 \in [0, 1]$, and $0 \le \alpha_1 + \alpha_2 \le 1$. We denote the set of all IFVs by $\mathcal{L}$. Besides, by an IFVF we mean $F : I \subseteq \mathbb{R} \to \mathcal{L}$ where $F(t) = (f_1(t), f_2(t))$. In this case, $f_1, f_2 : I \to [0, 1]$ and $0 \le f_1(t) + f_2(t) \le 1$ for each $t \in I$.

**Definition 1.1.** *[7] Let $\alpha = (\alpha_1, \alpha_2)$ and $\beta = (\beta_1, \beta_2)$ be two IFVs. Then*

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*(i)* *If* $\alpha_1 \geq \beta_1$ *and* $\alpha_2 \leq \beta_2$*, then* $\alpha \geq_L \beta$

*(ii)* *If* $\alpha_1 \leq \beta_1$ *and* $\alpha_2 \geq \beta_2$*, then* $\alpha \leq_L \beta$

*(iii)* *If* $\alpha_1 = \beta_1$ *and* $\alpha_2 = \beta_2$*, then* $\alpha = \beta$

**Remark 1.2.** *Definition of strict order* $<_L$ *can also be given similar to Definition 1.1 via replacing* $\leq_L$ *and* $\leq$ *by* $<_L$ *and* $<$*, respectively.*

**Definition 1.3.** *[10, 18, 19] Let* $\alpha = (\alpha_1, \alpha_2)$ *and* $\beta = (\beta_1, \beta_2)$ *be two IFVs and* $\lambda \geq 0$*. Then,*

*(i)* $\alpha \oplus \beta = (1 - (1 - \alpha_1)(1 - \beta_1), \alpha_2\beta_2)$

*(ii)* $\alpha \otimes \beta = (\alpha_1\beta_1, 1 - (1 - \alpha_2)(1 - \beta_2))$

*(iii)* *Assuming* $\beta <_L (1, 0)$*,*

$$\alpha \ominus \beta = \begin{cases} \left(\dfrac{\alpha_1 - \beta_1}{1 - \beta_1}, \dfrac{\alpha_2}{\beta_2}\right), & \text{if } \alpha_1 \geq \beta_1,\ \alpha_2 \leq \beta_2, \text{ and} \\ & \alpha_2\pi_\beta \leq \pi_\alpha\beta_2 \\ (0, 1), & otherwise \end{cases}$$

*where* $\pi_\alpha = 1 - \alpha_1 - \alpha_2$ *and* $\pi_\beta = 1 - \beta_1 - \beta_2$

*(iv)* *Assuming* $\beta >_L (0, 1)$*,*

$$\alpha \oslash \beta = \begin{cases} \left(\dfrac{\alpha_1}{\beta_1}, \dfrac{\alpha_2 - \beta_2}{1 - \beta_2}\right), & \text{if } \alpha_1 \leq \beta_1,\ \alpha_2 \geq \beta_2, \text{ and} \\ & \alpha_1\pi_\beta \leq \pi_\alpha\beta_1 \\ (1, 0), & otherwise \end{cases}$$

*(v)* $\lambda\alpha = (1 - (1 - \alpha_1)^\lambda, \alpha_2^\lambda),\ where\ \alpha <_L (1, 0)$

*(vi)* $\alpha^\lambda = (\alpha_1^\lambda, 1 - (1 - \alpha_2)^\lambda),\ where\ \alpha >_L (0, 1)$

**Definition 1.4.** *[12] Let* $F : (a, b) \to \mathcal{L}$ *and* $t_1, t_2 \in (a, b)$*. Then,*

*(i)* $F$ *is increasing on* $I$ *if* $F(t_1) <_L F(t_2)$ *whenever* $t_1 < t_2$*,*

*(ii)* $F$ *is nondecreasing on* $I$ *if* $F(t_1) \leq_L F(t_2)$ *whenever* $t_1 < t_2$*,*

*(iii)* $F$ *is decreasing on* $I$ *if* $F(t_2) <_L F(t_1)$ *whenever* $t_1 < t_2$*,*

*(ii)* $F$ *is nonincreasing on* $I$ *if* $F(t_2) \leq_L F(t_1)$ *whenever* $t_1 < t_2$*.*

**Remark 1.5.** *For the local monotonicity, a function* $F$ *is nondecreasing at a point* $t_0 \in (a, b)$ *if there is a* $\delta > 0$ *such that* $F(u) \leq_L F(t_0) \leq_L F(v)$ *for all* $u \in (t_0 - \delta, t_0)$ *and* $v \in (t_0, t_0 + \delta)$*.* $F$ *is nondecreasing on* $(a, b)$ *if and only if* $F$ *is nondecreasing at every* $t \in (a, b)$*. The other types of local monotonicities are similar(see [9], [16, pp. 125]).*

Note that operations $\oplus, \otimes, \ominus, \oslash$ of IFVs implement multiplication and division on membership and nonmembership degrees of IFVs. Besides, many other operations on IFVs such as integrals [1, 13], intuitionistic fuzzy aggregation operators [18, 19], convergence methods [20], infinite series and products [22] include again multiplication and division of membership-nonmemberships. On the other hand, multiplication and division operations are also crucial in multiplicative calculus and the tools of multiplicative calculus are useful to represent and to handle some intuitionistic fuzzy concepts. For this reason, we here give some basic concepts of multiplicative calculus [5, 8, 17] which will be used in Sections 2–4.

**Definition 1.6.** *Let $f : \mathbb{R} \to \mathbb{R}^+$. The *derivative of the function $f$ is given by:*

$$f^*(t) = \lim_{h\to 0} \left( \frac{f(t+h)}{f(t)} \right)^{\frac{1}{h}}.$$

**Theorem 1.7.** *If $f : \mathbb{R} \to \mathbb{R}^+$ is differentiable at $t_0$, then it is also *differentiable at $t_0$, and*

$$f^*(t_0) = \exp\left( \frac{f'(t_0)}{f(t_0)} \right).$$

**Theorem 1.8.** *If $f : \mathbb{R} \to \mathbb{R}^+$ is *differentiable at $t_0$, and if $f^*(t_0) \neq 0$, then it is also differentiable at $t_0$, and*

$$f'(t_0) = f(t) \ln\left(f^*(t_0)\right).$$

**Theorem 1.9.** *Let $f, g : \mathbb{R} \to \mathbb{R}^+$ be *differentiable, $h : \mathbb{R} \to \mathbb{R}$ is differentiable and $\lambda > 0$. Then,*

*(i)* $(\lambda f)^*(t) = f^*(t)$

*(ii)* $(fg)^*(t) = f^*(t)g^*(t)$

*(iii)* $(f/g)^*(t) = f^*(t)/g^*(t)$

*(iv)* $(f^h)^*(t) = f^*(t)^{h(t)} \cdot f(t)^{h'(t)}$

**Theorem 1.10** (Multiplicative test for monotonicity)**.** *Let $f : (a,b) \to \mathbb{R}^+$ be *differentiable.*

*(i)* $f^*(t) > 1$ *for every $t \in (a,b)$, then $f$ is increasing*

*(ii)* $f^*(t) < 1$ *for every $t \in (a,b)$, then $f$ is decreasing*

*(iii)* $f^*(t) \geq 1$ *for every $t \in (a,b)$, then $f$ is nondecreasing*

*(iv)* $f^*(t) \leq 1$ *for every $t \in (a,b)$, then $f$ is nonincreasing*

**Definition 1.11.** *Let $f$ is a positive function. Then, *antiderivative of $f$ is given by*

$$\phi(t) = \lambda \exp\left( \int \ln\left(f(t)\right) dt \right)$$

*where $\lambda$ is a positive constant.*

**Definition 1.12** (Definite *integral)**.** *Let $f : [a,b] \to \mathbb{R}^+$. $f$ is said to be *integrable on $[a,b]$ if there exists $L$ such that for any partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and for any points $c_k \in [x_k, x_{k+1}]$, we have*

$$\lim_{\|\mathcal{P}\|\to 0} \prod_{k=0}^{n-1} f(c_k)^{\Delta x_k} = L.$$

*In that case, we write $L = \int_a^b f(t)^{dt}$.*

**Theorem 1.13.** *If $f : [a,b] \to \mathbb{R}^+$ is *integrable, then*

$$\int_a^b f(t)^{dt} = \exp\left( \int_a^b \ln\left(f(t)\right) dt \right).$$

**Theorem 1.14** (Fundamental theorem of *calculus)**.** *Let $f : [a, b] \to \mathbb{R}^+$ be continuous. Then,*

$(i)$ *The function $\phi$ defined by*

$$\phi(t) = \int_a^t f(u)^{du}$$

*is *differentiable on $[a, b]$ and $\phi^*(t) = f(t)$.*

$(ii)$ *If $\phi$ is any *antiderivative of $f$, then*

$$\int_a^b f(t)^{dt} = \frac{\phi(b)}{\phi(a)}.$$

**Theorem 1.15.** *Let $f, g : [a, b] \to \mathbb{R}^+$ be *integrable functions. Then,*

*(i)* $$\int_a^b (f(t)^\lambda)^{dt} = \left(\int_a^b f(t)^{dt}\right)^\lambda$$

*(ii)* $$\int_a^b (f(t)g(t))^{dt} = \int_a^b f(t)^{dt} \cdot \int_a^b g(t)^{dt}$$

*(iii)* $$\int_a^b \left(\frac{f(t)}{g(t)}\right)^{dt} = \frac{\int_a^b f(t)^{dt}}{\int_a^b g(t)^{dt}}$$

*(iv)* $$\int_a^b f(t)^{dt} = \int_a^c f(t)^{dt} \cdot \int_c^b f(t)^{dt}$$

*(v)* $$f \leq g \text{ on } [a, b] \implies \int_a^b f(t)^{dt} \leq \int_a^b g(t)^{dt}$$

*where $\lambda \in \mathbb{R}$ and $a \leq c \leq b$.*

**Theorem 1.16** (*Integration by parts)**.** *Let $f, g : [a, b] \to \mathbb{R}^+$ be *differentiable so the $f^g$ is *integrable. Then,*

$$\int_a^b \left(f^*(t)^{g(t)}\right)^{dt} = \frac{f(b)^{g(b)}}{f(a)^{g(a)}} \cdot \frac{1}{\int_a^b \left(f(t)^{g'(t)}\right)^{dt}}.$$

**Theorem 1.17.** *[14, 15] If $f : [a, b] \to [c, d]$ is Riemann integrable and $g$ is a continuous function on $[c, d]$, then $g \circ f$ is Riemann integrable on $[a, b]$.*

## 2 $^\oplus$Calculus for intuitionistic fuzzy sets

We define $^\oplus$limit for IFVFs as the following.

**Definition 2.1.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}$ and $c$ is a cluster point of $I$. We say that the $^\oplus$limit of $F$, as $t$ approaches $c$, is IFV $\xi$ if for any IFV $\bar{\varepsilon} = (\varepsilon, 1 - \varepsilon) >_L (0, 1)$ there exists $\delta > 0$ such that*

$$F(t) \leq_L \xi \oplus \bar{\varepsilon} \quad \text{and} \quad \xi \leq_L F(t) \oplus \bar{\varepsilon} \tag{2.1}$$

*holds whenever $t \in I$ and $0 < |t - c| < \delta$. In this case, we write ${}^\oplus\lim\limits_{t \to c} F(t) = \xi$.*

The concept of $^\oplus$limit works with any IFV $\alpha$, but in case $\alpha = (1, 0)$ many of the other concepts in $^\oplus$calculus do not work. Hence, from now on we will omit the element $(1, 0)$ in $^\oplus$calculus. We will use the set $\mathcal{L}^\oplus = \{\alpha \in \mathcal{L} : \alpha <_L (1, 0)\}$. We note that if we had used the strict order $<_L$ instead of $\leq_L$ to define $^\oplus$limit, then the element $(1, 0)$ would automatically be omitted throughout $^\oplus$calculus. See Definition 4.4. in [20].

**Theorem 2.2.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}^{\oplus}$, $F = (f_1, f_2)$ and $\xi \in \mathcal{L}^{\oplus}$. ${}^{\oplus}\lim\limits_{t\to c} F(t) = \xi$ if and only if $\lim_{t\to c} f_1(t) = \xi_1$ and $\lim\limits_{t\to c} f_2(t) = \xi_2$.*

*Proof. Necessity.* Suppose ${}^{\oplus}\lim\limits_{t\to c} F(t) = \xi$. Then, for any given $\bar{\varepsilon} = (\varepsilon, 1-\varepsilon) >_L (0, 1)$ there is $\delta > 0$ such that

$$\left.\begin{array}{l} f_1(t) \leq 1-(1-\xi_1)(1-\varepsilon) = \xi_1+\varepsilon-\varepsilon\xi_1 \leq \xi_1+\varepsilon \\ \xi_1 \leq 1-(1-f_1(t))(1-\varepsilon) = f_1(t)+\varepsilon-\varepsilon f_1(t) \leq f_1(t)+\varepsilon \end{array}\right\} \Rightarrow \xi_1-\varepsilon \leq f_1(t) \leq \xi_1+\varepsilon$$

and

$$\left.\begin{array}{l} f_2(t) \geq \xi_2(1-\varepsilon) = \xi_2-\varepsilon\xi_2 \geq \xi_2-\varepsilon \\ \xi_2 \geq f_2(t)(1-\varepsilon) = f_2(t)-\varepsilon f_2(t) \geq f_2(t)-\varepsilon \end{array}\right\} \Rightarrow \xi_2-\varepsilon \leq f_2(t) \leq \xi_2+\varepsilon$$

whenever $t \in I$ and $0 < |t-c| < \delta$. This implies $\lim_{t\to c} f_1(t) = \xi_1$ and $\lim_{t\to c} f_2(t) = \xi_2$.

*Sufficiency.* Let $\lim_{t\to c} f_1(t) = \xi_1$ and $\lim_{t\to c} f_2(t) = \xi_2$. For given $\varepsilon > 0$ followings hold:

$(i)$ There exists $\delta_1 > 0$ such that $f_1(t) - \xi_1 \leq \varepsilon(1-\xi_1)$ and $\xi_2 - f_2(t) \leq \varepsilon\xi_2$ whenever $t \in I, 0 < |t-c| < \delta_1$ and these imply $f_1(t) \leq 1-(1-\xi_1)(1-\varepsilon)$ and $\xi_2(1-\varepsilon) \leq f_2(t)$, respectively. Hence, we have $F(t) \leq_L \xi \oplus \bar{\varepsilon}$ whenever $t \in I$ and $0 < |t-c| < \delta_1$.

$(ii)$ By the assumption $\xi <_L (1, 0)$ we have $\xi_1 \neq 1$ and $\xi_2 \neq 0$ and so there exists $\delta_2 > 0$ such that $f_1(t) \leq \xi_1 + \frac{1-\xi_1}{2} = \frac{\xi_1+1}{2}$ and $f_2(t) \geq \xi_2 - \frac{\xi_2}{2} = \frac{\xi_2}{2}$ whenever $t \in I, 0 < |t-c| < \delta_2$. Besides, there is $\delta_3 > 0$ such that $\xi_1 - f_1(t) \leq \varepsilon(1 - \frac{\xi_1+1}{2})$ and $f_2(t) - \xi_2 \leq \varepsilon\frac{\xi_2}{2}$ whenever $t \in I, 0 < |t-c| < \delta_3$. These imply $\xi_1 - f_1(t) \leq \varepsilon(1-f_1(t))$ and $f_2(t) - \xi_2 \leq \varepsilon f_2(t)$ whenever $t \in I, 0 < |t-c| < \min\{\delta_2, \delta_3\}$. Hence, we have $\xi_1 \leq 1-(1-f_1(t))(1-\varepsilon)$ and $f_2(t)(1-\varepsilon) \leq \xi_2$ which implies $\xi \leq_L F(t) \oplus \bar{\varepsilon}$.

From $(i)$ and $(ii)$, we conclude that

$$F(t) \leq_L \xi \oplus \bar{\varepsilon} \quad \textit{and} \quad \xi \leq_L F(t) \oplus \bar{\varepsilon}$$

whenever $t \in I$ and $0 < |t-c| < \delta = \min\{\delta_1, \delta_2, \delta_3\}$ which completes the proof. □

**Remark 2.3.** *If ${}^{\oplus}\lim\limits_{t\to c} F(t) = (1, 0)$, then there exists $\delta > 0$ such that $F(t) = (1, 0)$ for any $t \in (c-\delta, c+\delta)/\{c\}$. On the other hand, if $F(d) = (1, 0)$ for a number $d \in (c-\delta, c+\delta)/\{c\}$, then $\xi = (1, 0)$.*

**Example 2.4.** *Let $F : (0, 2) \to \mathcal{L}^{\oplus}$ be defined by $F(t) = \left(\dfrac{1}{2} - \dfrac{1}{4+t}, \dfrac{1}{3} - \dfrac{1}{4+t}\right)$. Then, ${}^{\oplus}\lim\limits_{t\to 1} F(t) = \left(\dfrac{3}{10}, \dfrac{2}{15}\right)$.*

**Definition 2.5.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}^{\oplus}$ and $\xi \in \mathcal{L}^{\oplus}$. ${}^{\oplus}\lim\limits_{t\to c^-} F(t) = \xi$ if for any IFV $\bar{\varepsilon} >_L (0, 1)$ there exists $\delta > 0$ such that (2.1) holds whenever $t \in (c-\delta, c)$. Similarly, ${}^{\oplus}\lim\limits_{t\to c^+} F(t) = \xi$ if there is $\delta > 0$ such that (2.1) holds whenever $t \in (c, c+\delta)$.*

If $I$ is a closed interval, then ${}^{\oplus}$limit, ${}^{\oplus}$continuity, ${}^{\oplus}$derivative at endpoints of $I$ are meant in the one-sided sense throughout the paper.

**Theorem 2.6.** *Let $F, G : I \subseteq \mathbb{R} \to \mathcal{L}^{\oplus}$ be two IFVFs, $\xi, \eta \in \mathcal{L}^{\oplus}$ be two IFVs; and $\lambda \geq 0$. If ${}^{\oplus}\lim\limits_{t\to c} F(t) = \xi$ and ${}^{\oplus}\lim\limits_{t\to c} G(t) = \eta$, then followings hold:*

*(i)* ${}^{\oplus}\lim\limits_{t\to c} (F(t) \oplus G(t)) = \xi \oplus \eta$

*(ii)* ${}^{\oplus}\lim\limits_{t\to c} (F(t) \ominus G(t)) = \xi \ominus \eta$ *where* $F(t) \ominus G(t) \in \mathcal{L}^{\oplus}$

*(iii)* ${}^{\oplus}\lim\limits_{t\to c} \lambda F(t) = \lambda\xi$

*Proof.* Let $F, G : I \subseteq \mathbb{R} \to \mathcal{L}^{\oplus}$ be two IFVFs such that $F = (f_1, f_2)$, $G = (g_1, g_2)$ and ${}^{\oplus}\lim\limits_{t\to c} F(t) = \xi$ and ${}^{\oplus}\lim\limits_{t\to c} G(t) = \eta$ where $\xi, \eta \in \mathcal{L}^{\oplus}$. Then, we have:

$(i)$

$$\begin{aligned}
{}^{\oplus}\lim_{t\to c}(F(t)\oplus G(t)) &= \Big(\lim_{t\to c} 1-(1-f_1(t))(1-g_1(t)), \lim_{t\to c} f_2(t)g_2(t)\Big)\\
&= \Big(\lim_{t\to c} f_1(t), \lim_{t\to c} f_2(t)\Big) \oplus \Big(\lim_{t\to c} g_1(t), \lim_{t\to c} g_2(t)\Big)\\
&= {}^{\oplus}\lim_{t\to c} F(t) \oplus {}^{\oplus}\lim_{t\to c} G(t)\\
&= \xi \oplus \eta
\end{aligned}$$

by virtue of Theorem 2.2.

$(ii)$ Suppose $F(t) \ominus G(t) = \left(1 - \dfrac{1-f_1(t)}{1-g_1(t)}, \dfrac{f_2(t)}{g_2(t)}\right) \in \mathcal{L}^{\oplus}$. Then, we have

$$f_1(t) \geq g_1(t), \quad f_2(t) \leq g_2(t), \quad \frac{f_2(t)}{g_2(t)} \leq \frac{1-f_1(t)}{1-g_1(t)}$$

and, as $t$ approaching $c$,

$$\xi_1 \geq \eta_1, \quad \xi_2 \leq \eta_2 \quad \frac{\xi_2}{\eta_2} \leq \frac{1-\xi_1}{1-\eta_1}$$

implying $\xi \ominus \eta = \left(1 - \dfrac{1-\xi_1}{1-\eta_1}, \dfrac{\xi_2}{\eta_2}\right) \in \mathcal{L}^{\oplus}$. Hence, we get

$$\begin{aligned}
{}^{\oplus}\lim_{t\to c}(F(t)\ominus G(t)) &= \left(\lim_{t\to c} 1-\frac{1-f_1(t)}{1-g_1(t)}, \lim_{t\to c}\frac{f_2(t)}{g_2(t)}\right)\\
&= \left(1-\frac{1-\lim\limits_{t\to c} f_1(t)}{1-\lim\limits_{t\to c} g_1(t)}, \frac{\lim\limits_{t\to c} f_2(t)}{\lim\limits_{t\to c} g_2(t)}\right)\\
&= \Big(\lim_{t\to c} f_1(t), \lim_{t\to c} f_2(t)\Big) \ominus \Big(\lim_{t\to c} g_1(t), \lim_{t\to c} g_2(t)\Big)\\
&= {}^{\oplus}\lim_{t\to c} F(t) \ominus {}^{\oplus}\lim_{t\to c} G(t)\\
&= \xi \ominus \eta
\end{aligned}$$

by virtue of Theorem 2.2.

$(iii)$ The proof can be done similarly by using Theorem 2.2, hence omitted. $\square$

**Definition 2.7.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}^{\oplus}$ and $t_0 \in I$. $F$ is said to be ${}^{\oplus}$continuous at $t_0$ if for any IFV $\bar{\varepsilon} = (\varepsilon, 1-\varepsilon) >_L (0,1)$ there exists $\delta > 0$ such that*

$$F(t) \leq_L F(t_0) \oplus \bar{\varepsilon} \quad \text{and} \quad F(t_0) \leq_L F(t) \oplus \bar{\varepsilon}$$

*holds whenever $t \in I$ and $|t - t_0| < \delta$.*

**Theorem 2.8.** *Let $F : (a,b) \to \mathcal{L}^{\oplus}$ and $t_0 \in (a,b)$. $F$ is ${}^{\oplus}$continuous at $t_0$ if and only if ${}^{\oplus}\lim\limits_{t\to t_0} F(t) = F(t_0)$.*

*Proof.* Since $t_0 \in (a,b)$ is a cluster point, the proof is straightforward from Definition 2.1 by taking $c = t_0$ and $\xi = F(t_0)$. $\square$

**Definition 2.9.** *Let $F : (a,b) \to \mathcal{L}^{\oplus}$ and $t_0 \in (a,b)$. $F$ is said to be right-${}^{\oplus}$continuous at $t_0$ if ${}^{\oplus}\lim\limits_{t\to t_0^+} F(t) = F(t_0)$, and said to be left-${}^{\oplus}$continuous at $t_0$ if ${}^{\oplus}\lim\limits_{t\to t_0^-} F(t) = F(t_0)$.*

**Definition 2.10.** *$F:[a,b]\to\mathcal{L}^{\oplus}$ is said to be $^{\oplus}$continuous on $[a,b]$ if $F$ is right-$^{\oplus}$continuous at $a$, left-$^{\oplus}$continuous at $b$ and $^{\oplus}$continuous at all interior points of $[a,b]$.*

**Theorem 2.11.** *Let $F:[a,b]\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is $^{\oplus}$continuous on $[a,b]$ if and only if $f_1$ and $f_2$ are continuous on $[a,b]$.*

*Proof.* In view of Theorem 2.2, the proof is straightforward. □

**Definition 2.12.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $t_0\in(a,b)$. $F$ is said to be $^{\oplus}$differentiable at $t_0$ if $F(t_0+h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0-h)$ exist in $\mathcal{L}^{\oplus}$ for sufficiently small $h$ and there is an IFV $\xi\in\mathcal{L}^{\oplus}$ such that*

$$^{\oplus}\lim_{h\to0^+}\frac{F(t_0+h)\ominus F(t_0)}{h}={}^{\oplus}\lim_{h\to0^+}\frac{F(t_0)\ominus F(t_0-h)}{h}=\xi.$$

*In this case, we write $\xi=F^{\oplus}(t_0)$.*

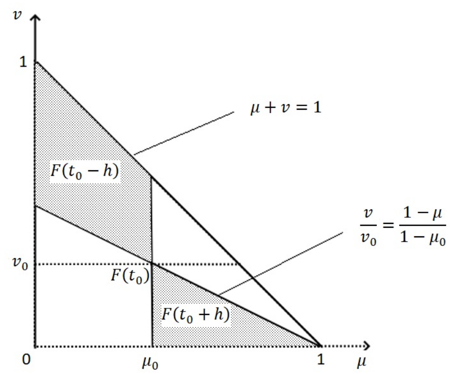


Figure 1: Regions where $F(t_0+h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0-h)$ exist in $\mathcal{L}^{\oplus}$

Figure 1 illustrates addition and subtraction regions of $F(t_0)$. For more information we refer to [10, 11].

**Theorem 2.13.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is $^{\oplus}$differentiable at $t_0$ if and only if $f_1'(t_0), f_2'(t_0)$ exist, $F$ is nondecreasing at $t_0$ and $\frac{f_2}{1-f_1}$ is nonincreasing at $t_0$. Furthermore,*

$$F^{\oplus}(t_0)=\left(1-\exp\left(\frac{(1-f_1)'(t_0)}{(1-f_1)(t_0)}\right),\exp\left(\frac{f_2'(t_0)}{f_2(t_0)}\right)\right).$$

*Proof. Necessity.* Let $F:(a,b)\to\mathcal{L}^{\oplus}$ be $^{\oplus}$differentiable at $t_0$. Then, in view of the facts that

$$\begin{aligned}F^{\oplus}(t_0^+)={}^{\oplus}\lim_{h\to0^+}\frac{F(t_0+h)\ominus F(t_0)}{h}&=\left(\lim_{h\to0^+}\left[1-\left(\frac{1-f_1(t_0+h)}{1-f_1(t_0)}\right)^{1/h}\right],\lim_{h\to0^+}\left(\frac{f_2(t_0+h)}{f_2(t_0)}\right)^{1/h}\right)\\&=\left(1-\exp\left(\frac{(1-f_1)'(t_0^+)}{(1-f_1)(t_0)}\right),\exp\left(\frac{f_2'(t_0^+)}{f_2(t_0)}\right)\right),\end{aligned}$$

$$F^{\oplus}(t_0^-)=\left(1-\exp\left(\frac{(1-f_1)'(t_0^-)}{(1-f_1)(t_0)}\right),\exp\left(\frac{f_2'(t_0^-)}{f_2(t_0)}\right)\right)$$

we conclude $f_1'(t_0), f_2'(t_0)$ exist. Besides, since $F(t_0+h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0-h)$ exist, we have

$$f_1(t_0-h)\le f_1(t_0)\le f_1(t_0+h)\quad\text{and}\quad f_2(t_0-h)\ge f_2(t_0)\ge f_2(t_0+h)$$

by the property of subtraction operation and this implies $F$ is nondecreasing at $t_0$.

On the other hand, since $F(t_0+h)\ominus F(t_0)$ exists we have $f_2(t_0+h)\pi_{F(t_0)}\le f_2(t_0)\pi_{F(t_0+h)}$ by the property of subtraction operation implying

$$f_2(t_0+h)[1-f_1(t_0)-f_2(t_0)]\le f_2(t_0)[1-f_1(t_0+h)-f_2(t_0+h)].$$

So, we have

$$\begin{aligned} 0 &\geq (1-f_1(t_0))[f_2(t_0+h)-f_2(t_0)] - f_2(t_0)[1-f_1(t_0+h)-(1-f_1(t_0))] \\ &\geq \frac{(1-f_1(t_0))[f_2(t_0+h)-f_2(t_0)] - f_2(t_0)[1-f_1(t_0+h)-(1-f_1(t_0))]}{[1-f_1(t_0)][1-f_1(t_0+h)]} \\ &= \Delta_h\left(\frac{f_2(t_0)}{1-f_1(t_0)}\right) \end{aligned}$$

where $\Delta_h$ is the forward difference operator with step $h$. Similary, since $F(t_0) \ominus F(t_0-h)$ exists we have $f_2(t_0)\pi_{F(t_0-h)} \leq f_2(t_0-h)\pi_{F(t_0)}$ which reveals

$$\nabla_h\left(\frac{f_2(t_0)}{1-f_1(t_0)}\right) \leq 0$$

where $\nabla_h$ is the backward difference operator with step $h$. These imply $\frac{f_2}{1-f_1}$ is nonincreasing at $t_0$.

*Sufficiency.* Let $f_1'(t_0), f_2'(t_0)$ exist, $F$ be nondecreasing at $t_0$ and $\frac{f_2}{1-f_1}$ be nonincreasing at $t_0$. Since, $F$ is nondecreasing at $t_0$ and $\frac{f_2}{1-f_1}$ is nonincreasing at $t_0$ we guarantee, by following above calculation steps reversely, the existence of $F(t_0+h)\ominus F(t_0)$ and $F(t_0)\ominus F(t_0-h)$ for sufficiently small $h$. Besides, existence of $f_1'(t_0), f_2'(t_0)$ guarantee the existence of $F^{\oplus}(t_0)$. ☐

**Definition 2.14.** *$F:(a,b)\to\mathcal{L}^{\oplus}$ is said to be $^{\oplus}$differentiable on $(a,b)$ if $F$ is $^{\oplus}$differentiable for each $t_0\in(a,b)$.*

**Theorem 2.15.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is $^{\oplus}$differentiable on $(a,b)$ if and only if $F$ is nondecreasing on $(a,b)$, $f_1, f_2$ are differentiable on $(a,b)$ and $\left(\frac{f_2}{1-f_1}\right)' \leq 0$. Furthermore,*

$$F^{\oplus} = \left(1-\exp\left(\frac{(1-f_1)'}{(1-f_1)}\right), \exp\left(\frac{f_2'}{f_2}\right)\right). \tag{2.2}$$

*Proof.* In view of Theorem 2.13 and Remark 1.5, the proof is straightforward. ☐

**Example 2.16.** *Let $F:(3,4)\to\mathcal{L}^{\oplus}$ be defined by*

$$F(t) = \left(1-\frac{1}{t}, \exp\left(-t^2\right)\right).$$

*Then,*

$$F^{\oplus}(t) = \left(1-\exp\left(-\frac{1}{t}\right), \exp\left(-2t\right)\right)$$

*which is also an IFVF.*

Here, the tools of multiplicative calculus [8, 17] may be useful to represent (2.2). Besides, we have

$$\left(\frac{f_2}{1-f_1}\right)' \leq 0 \Longleftrightarrow \frac{f_2'}{f_2} \leq \frac{(1-f_1)'}{1-f_1} \not\Longleftrightarrow f_2' \leq (1-f_1)'$$

which means that the condition $\left(\frac{f_2}{1-f_1}\right)' \leq 0$ in Theorem 2.15 is related directly to *relative rate of changes* of $(1-f_1)$ and $f_2$ rather than the rate of changes of $(1-f_1)$ and $f_2$. At this point multiplicative *derivative, which has a close relation with *relative rate of changes*, may also be useful. In fact, we have

$$\left(\frac{f_2}{1-f_1}\right)' \leq 0 \Longleftrightarrow \frac{f_2'}{f_2} \leq \frac{(1-f_1)'}{1-f_1} \Longleftrightarrow f_2^* \leq (1-f_1)^*.$$

We give following two theorems as the representation of Theorem 2.13 and Theorem 2.15 by means of the concept of *derivative. The results are straightforward in view of Theorem 1.7 and Theorem 1.10, and hence the proofs are omitted.

**Theorem 2.17.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is $^{\oplus}$differentiable at $t_0$ if and only if $(1-f_1)^*(t_0)$, $f_2^*(t_0)$ exists, $F$ is nondecreasing at $t_0$ and $\frac{f_2}{1-f_1}$ nonincreasing at $t_0$. Furthermore,*

$$F^{\oplus}(t_0)=\left(1-(1-f_1)^*(t_0),f_2^*(t_0)\right).$$

**Theorem 2.18.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is $^{\oplus}$differentiable on $(a,b)$ if and only if $F$ is nondecreasing on $(a,b)$, $(1-f_1),f_2$ are *differentiable on $(a,b)$ and $\left(\frac{f_2}{1-f_1}\right)^*\leq 1$. Furthermore,*

$$F^{\oplus}=\left(1-(1-f_1)^*,f_2^*\right). \tag{2.3}$$

**Theorem 2.19.** *Let $F,G:(a,b)\to\mathcal{L}^{\oplus}$ be $^{\oplus}$differentiable IFVFs, $h:(a,b)\to\mathbb{R}^+\cup\{0\}$ be differentiable and nondecreasing real valued function and $\lambda\geq 0$. Then,*

*(i)* $(F\oplus G)^{\oplus}(t)=F^{\oplus}(t)\oplus G^{\oplus}(t)$

*(ii)* $(\lambda F)^{\oplus}(t)=\lambda F^{\oplus}(t)$

*(iii)* $(hF)^{\oplus}(t)=\left(hF^{\oplus}(t)\right)\oplus\left(h'F(t)\right)$

*Moreover, if $(F\ominus G)^{\oplus}(t)$ exists then*

*(iv)* $(F\ominus G)^{\oplus}(t)=F^{\oplus}(t)\ominus G^{\oplus}(t)$.

*Proof.* Let $F,G:(a,b)\to\mathcal{L}^{\oplus}$ be $^{\oplus}$differentiable IFVFs such that $F=(f_1,f_2)$, $G=(g_1,g_2)$ and $\lambda\geq 0$. Then, by Theorem 2.18 we have $f_1,g_1$ are nondecreasing, $f_2,g_2$ are nonincreasing, $(1-f_1)^*,f_2^*,(1-g_1)^*,g_2^*$ exist and $\left(\frac{f_2}{1-f_1}\right)^*\leq 1$, $\left(\frac{g_2}{1-g_1}\right)^*\leq 1$ hold.

$(i)$ $F\oplus G=(1-(1-f_1)(1-g_1),f_2g_2)$. We apply Theorem 2.18. $1-(1-f_1)(1-g_1)$ is nondecreasing and $f_2g_2$ is nonincreasing. Hence, $F\oplus G$ is nondecreasing. Besides, we know the existence of $((1-f_1)(1-g_1))^*$, $(f_2g_2)^*$ and $\left(\frac{f_2g_2}{(1-f_1)(1-g_1)}\right)^*\leq 1$. This implies $(F\oplus G)^{\oplus}$ exist and

$$\begin{aligned}(F\oplus G)^{\oplus}&=\left(1-((1-f_1)(1-g_1))^*,(f_2g_2)^*\right)\\&=\left(1-(1-f_1)^*(1-g_1)^*,f_2^*g_2^*\right)\\&=\left(1-(1-f_1)^*,f_2^*\right)\oplus\left(1-(1-g_1)^*,g_2^*\right)\\&=F^{\oplus}\oplus G^{\oplus}.\end{aligned}$$

$(ii)$ $\lambda F=\left(1-(1-f_1)^{\lambda},f_2^{\lambda}\right)$. We apply Theorem 2.18. $1-(1-f_1)^{\lambda}$ is nondecreasing and $f_2^{\lambda}$ is nonincreasing. Hence, $\lambda F$ is nondecreasing. Besides, we know that $(1-f_1)^{\lambda}$, $f_2^{\lambda}$ are *differentiable and $\left(\left(\frac{f_2}{1-f_1}\right)^{\lambda}\right)^*\leq 1$. This implies $(\lambda F)^{\oplus}$ exists and

$$(\lambda F)^{\oplus}=\left(1-\left((1-f_1)^{\lambda}\right)^*,(f_2^{\lambda})^*\right)=\left(1-((1-f_1)^*)^{\lambda},(f_2^*)^{\lambda}\right)=\lambda\left(1-(1-f_1)^*,f_2^*\right)=\lambda F^{\oplus}.$$

$(iii)$ Let $h:(a,b)\to\mathbb{R}^+\cup\{0\}$ be differentiable and nondecreasing on $(a,b)$. Hence, $hF=\left(1-(1-f_1)^h,f_2^h\right)$ is nondecreasing. Besides, we have

$$\left(\frac{f_2^h}{(1-f_1)^h}\right)^*=\left(\left(\frac{f_2}{1-f_1}\right)^h\right)^*=\left(\left(\frac{f_2}{1-f_1}\right)^*\right)^h\left(\frac{f_2}{1-f_1}\right)^{h'}\leq 1$$

in view of the facts $\left(\frac{f_2}{1-f_1}\right)^*\leq 1$, $\frac{f_2}{1-f_1}\leq 1$ and $h'\geq 0$. Hence, $hF$ is $^{\oplus}$differentiable by Theorem 2.18 and

$$(hF)^{\oplus}\;=\;\left(1-\left((1-f_1)^h\right)^*,\left(f_2^h\right)^*\right)$$

$$
\begin{aligned}
&= \left(1-((1-f_1)^*)^h\,(1-f_1)^{h'},(f_2^*)^h\,f_2^{h'}\right)\\
&= \left(1-((1-f_1)^*)^h,(f_2^*)^h\right)\oplus\left(1-(1-f_1)^{h'},f_2^{h'}\right)\\
&= \left(hF^{\oplus}\right)\oplus\left(h'F\right).
\end{aligned}
$$

$(iv)$ Let $(F\ominus G)^{\oplus}$ exist. Then,

$$F\ominus G=\left(\frac{f_1-g_1}{1-g_1},\frac{f_2}{g_2}\right)=\left(1-\frac{1-f_1}{1-g_1},\frac{f_2}{g_2}\right)$$

exists and

$$\left(\frac{1-f_1}{1-g_1}\right)^*\leq 1,\quad \left(\frac{f_2}{g_2}\right)^*\leq 1,\quad \left(\frac{f_2(1-g_1)}{g_2(1-f_1)}\right)^*\leq 1$$

hold by Theorem 2.18 and Theorem 1.10. Hence, we have

$$(1-f_1)^*\leq(1-g_1)^*,\quad f_2^*\leq g_2^*,\quad 1-\left(\frac{1-f_1}{1-g_1}\right)^*+\left(\frac{f_2}{g_2}\right)^*\leq 1$$

which implies the existence of $F^{\oplus}\ominus G^{\oplus}$ by the property of subtraction operation. Then, we conclude

$$
\begin{aligned}
(F\ominus G)^{\oplus} &= \left(1-\left(\frac{1-f_1}{1-g_1}\right)^*,\left(\frac{f_2}{g_2}\right)^*\right)\\
&= \left(1-\frac{(1-f_1)^*}{(1-g_1)^*},\frac{f_2^*}{g_2^*}\right)\\
&= (1-(1-f_1)^*,f_2^*)\ominus(1-(1-g_1)^*,g_2^*)\\
&= F^{\oplus}\ominus G^{\oplus}.
\end{aligned}
$$

□

**Definition 2.20.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. The ${}^{\oplus}$antiderivative $\Phi$ of $F$ is defined by*

$$\Phi(t)=\left(1-\lambda_1\exp\left(\int\ln(1-f_1)dt\right),\lambda_2\exp\left(\int\ln(f_2)dt\right)\right)$$

*where $\lambda_1,\lambda_2>0$ are arbitrary constants such that $\Phi$ is an IFV.*

In view of the definition above and the concept of *integral, we give following theorem.

**Theorem 2.21.** *Let $F:(a,b)\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. If $\Phi$ is ${}^{\oplus}$antiderivative of $F$, then*

$$\Phi(t)=\left(1-\lambda_1\int(1-f_1)^{dt},\lambda_2\int(f_2)^{dt}\right)\tag{2.4}$$

*where $\lambda_1,\lambda_2>0$ are arbitrary constants such that $\Phi$ is an IFV.*

We note that ${}^{\oplus}$antiderivative $\Phi(t)$ of $F=(f_1,f_2)$ is an IFV if and only if

$$0<\lambda_1\leq\frac{1}{\int(1-f_1)^{dt}},\quad 0<\lambda_2\leq\frac{1}{\int(f_2)^{dt}},\quad \int\left(\frac{f_2}{1-f_1}\right)^{dt}\leq\frac{\lambda_1}{\lambda_2}.$$

**Theorem 2.22.** *If $F:(a,b)\to\mathcal{L}^{\oplus}$ is ${}^{\oplus}$continuous, then ${}^{\oplus}$antiderivative $\Phi(t)$ exists and $\Phi^{\oplus}(t)=F(t)$.*

*Proof.* Let $F:(a,b)\to\mathcal{L}^{\oplus}$, $F(t)=(f_1(t),f_2(t))$ be ${}^{\oplus}$continuous. Then, $f_1$ and $f_2$ are continuous which implies the existence of *antiderivatives

$$\lambda_1\int(1-f_1(t))^{dt}\quad\text{and}\quad\lambda_2\int f_2(t)^{dt}$$

where

$$0<\lambda_1\le\frac{1}{\int(1-f_1)^{dt}},\quad 0<\lambda_2\le\frac{1}{\int(f_2)^{dt}},\quad \int\left(\frac{f_2}{1-f_1}\right)^{dt}\le\frac{\lambda_1}{\lambda_2}.$$

Hence, ${}^{\oplus}$antiderivative $\Phi(t)$ in (2.4) exists.

Now, we check the conditions of Theorem 2.18 for ${}^{\oplus}$differentiability of $\Phi(t)$. $\Phi(t)$ is nondecreasing in view of the facts that

$$\left(\lambda_1\int(1-f_1)^{dt}\right)^{*}=1-f_1\le1\quad\text{and}\quad\left(\lambda_2\int f_2(t)^{dt}\right)^{*}=f_2\le1$$

and in view of Theorem 1.10. Besides,

$$\left(\frac{\lambda_2\int f_2(t)^{dt}}{\lambda_1\int(1-f_1)^{dt}}\right)^{*}=\frac{f_2(t)}{1-f_1(t)}\le1.$$

by virtue of the properties of *derivative and *integral. Hence, $\Phi(t)$ satisfies Theorem 2.18 which means that $\Phi(t)$ is ${}^{\oplus}$differentiable. Furthermore, we have

$$\begin{aligned}\Phi^{\oplus}(t)&=\left(1-\left(\lambda_1\int(1-f_1)^{dt}\right)^{*},\left(\lambda_2\int f_2(t)^{dt}\right)^{*}\right)\\&=(f_1(t),f_2(t))\\&=F(t).\end{aligned}$$

which completes the proof. ☐

**Definition 2.23** (Definite ${}^{\oplus}$integral)**.** *$F:[a,b]\to\mathcal{L}^{\oplus}$ is said to be ${}^{\oplus}$integrable on $[a,b]$ if there exists an IFV $\xi\in\mathcal{L}^{\oplus}$ such that for any partition $\mathcal{P}=\{x_0,x_1,\cdots,x_n\}$ of $[a,b]$ and for any points $c_k\in[x_k,x_{k+1}]$, we have*

$$^{\oplus}\lim_{\|\mathcal{P}\|\to0}\bigoplus_{k=0}^{n-1}F(c_k)\Delta x_k=\xi.$$

*In that case, we write $\xi={}^{\oplus}\!\!\int_a^b F(t)dt$.*

**Theorem 2.24.** *Let $F:[a,b]\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is ${}^{\oplus}$integrable on $[a,b]$ if and only if $f_1$ and $f_2$ are integrable on $[a,b]$. Furthermore,*

$$^{\oplus}\!\!\int_a^b F(t)dt=\left(1-\exp\left(\int_a^b\ln(1-f_1)dt\right),\exp\left(\int_a^b\ln(f_2)dt\right)\right).\tag{2.5}$$

*Proof.* Let $F:[a,b]\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. $F$ is ${}^{\oplus}$integrable on $[a,b]$ if and only if $f_1$ and $f_2$ are integrable on $[a,b]$ in view of the fact

$$\begin{aligned}{}^{\oplus}\!\!\int_a^b F(t)dt&={}^{\oplus}\lim_{\|\mathcal{P}\|\to0}\bigoplus_{k=0}^{n-1}F(c_k)\Delta x_k\\&=\left(1-\lim_{\|\mathcal{P}\|\to0}\prod_{k=0}^{n-1}(1-f_1(c_k))^{\Delta x_k},\prod_{k=0}^{n-1}(f_2(c_k))^{\Delta x_k}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(1-\exp\left(\lim_{\|\mathcal{P}\|\to 0}\sum_{k=0}^{n-1}\Delta x_k \ln\left(1-f_1(c_k)\right)\right), \exp\left(\lim_{\|\mathcal{P}\|\to 0}\sum_{k=0}^{n-1}\Delta x_k \ln\left(f_2(c_k)\right)\right)\right)\\
&= \left(1-\exp\left(\int_a^b \ln(1-f_1)dt\right), \exp\left(\int_a^b \ln(f_2)dt\right)\right)
\end{aligned}
$$

by Theorem 2.2 and Theorem 1.17. □

**Theorem 2.25.** *Let $F:[a,b]\to\mathcal{L}^{\oplus}$ and $F=(f_1,f_2)$. If $F$ is ${}^{\oplus}$integrable on $[a,b]$, then*

$$
{}^{\oplus}\!\!\int_a^b F(t)dt = \left(1-\int_a^b (1-f_1)^{dt}, \int_a^b (f_2)^{dt}\right). \tag{2.6}
$$

*Proof.* In view of (2.5) and the concept of *integral, the proof is straightforward. □

**Theorem 2.26** (Fundamental theorem of ${}^{\oplus}$calculus)**.** *Let $F:[a,b]\to\mathcal{L}^{\oplus}$ be a continuous. Then, following statements hold:*

$(i)$ *The function $\psi$ defined by*

$$
\psi(t) = {}^{\oplus}\!\!\int_a^t F(u)du
$$

*is ${}^{\oplus}$differentiable on $[a,b]$ and $\psi^{\oplus}(t)=F(t)$.*

$(ii)$ *If $\Phi$ is any ${}^{\oplus}$antiderivative of $F$, then*

$$
{}^{\oplus}\!\!\int_a^b F(t)dt = \Phi(b)\ominus\Phi(a).
$$

*Proof.* Let $F:[a,b]\to\mathcal{L}^{\oplus}$ be continuous.

$(i)$

$$
\psi(t) = \left(1-\int_a^t (1-f_1)^{du}, \int_a^t (f_2)^{du}\right).
$$

In view of $0<1-f_1\le 1$, $0<f_2\le 1$ and Theorem 1.10 we have $\psi$ is nondecreasing. Besides, since $f_1+f_2\le 1$ we have

$$
\left(\frac{\int_a^t (f_2)^{du}}{\int_a^t (1-f_1)^{du}}\right)^{*} = \frac{f_2}{1-f_1}\le 1
$$

by Theorem 1.14 and we conclude $\psi$ is ${}^{\oplus}$differentiable on $[a,b]$ in view of Theorem 2.18. Besides,

$$
\psi^{\oplus}(t) = (f_1(t), f_2(t)) = F(t)
$$

in view of (2.3) and Theorem 1.14.

$(ii)$ Let $\Phi$ be an ${}^{\oplus}$antiderivative of $F$ and $\widetilde{(1-f_1)}(t)=\lambda_1\int(1-f_1)^{dt}$ and $\tilde{f}_2(t)=\lambda_2\int(f_2)^{dt}$ are *antiderivatives of $(1-f_1)$ and $f_2$, respectively. Then, we have

$$
\begin{aligned}
{}^{\oplus}\!\!\int_a^b F(t)dt &= \left(1-\int_a^b (1-f_1)^{dt}, \int_a^b (f_2)^{dt}\right)\\
&= \left(1-\frac{\widetilde{(1-f_1)}(b)}{\widetilde{(1-f_1)}(a)}, \frac{\tilde{f}_2(b)}{\tilde{f}_2(a)}\right)\\
&= \left(1-\widetilde{(1-f_1)}(b), \tilde{f}_2(b)\right)\ominus\left(1-\widetilde{(1-f_1)}(a), \tilde{f}_2(a)\right)\\
&= \Phi(b)\ominus\Phi(a)
\end{aligned}
$$

in view of (2.4), (2.6) and Theorem 1.14. □

**Theorem 2.27.** *Let $F, G : [a,b] \to \mathcal{L}^{\oplus}$ are $^{\oplus}$integrable on $[a,b]$ and $\lambda \geq 0$. Then,*

*(i)* $$\int_a^{\oplus b} \lambda F(t)dt = \lambda \int_a^{\oplus b} F(t)dt$$

*(ii)* $$\int_a^{\oplus b} (F(t) \oplus G(t))\, dt = \left(\int_a^{\oplus b} F(t)dt\right) \oplus \left(\int_a^{\oplus b} G(t)dt\right)$$

*(iii)* $$\int_a^{\oplus b} F(t)dt = \int_a^{\oplus c} F(t)dt \oplus \int_c^{\oplus b} F(t)dt, \quad a \leq c \leq b.$$

*Moreover, if $F \ominus G$ exists then*

*(iv)* $$\int_a^{\oplus b} (F(t) \ominus G(t))\, dt = \left(\int_a^{\oplus b} F(t)dt\right) \ominus \left(\int_a^{\oplus b} G(t)dt\right).$$

*Proof.* The proofs of $(i)$, $(ii)$ and $(iii)$ are straightforward from (2.6) and Theorem 1.15.

$(iv)$ Let $F = (f_1, f_2)$ and $G = (g_1, g_2)$ are $^{\oplus}$integrable IFVFs on $[a,b]$ and $F \ominus G$ exists. So, we have

$$f_1 \geq g_1, \quad f_2 \leq g_2, \quad \frac{f_2}{g_2} \leq \frac{1-f_1}{1-g_1}$$

which implies

$$\int_a^b (1-f_1)^{dt} \leq \int_a^b (1-g_1)^{dt}, \quad \int_a^b (f_2)^{dt} \leq \int_a^b (g_2)^{dt}, \quad \int_a^b \left(\frac{f_2}{g_2}\right)^{dt} \leq \int_a^b \left(\frac{1-f_1}{1-g_1}\right)^{dt}.$$

Hence, $\left(\int_a^{\oplus b} Fdt\right) \ominus \left(\int_a^{\oplus b} Gdt\right)$ exists and

$$\begin{aligned}
\int_a^{\oplus b} (F \ominus G)\, dt &= \left(1 - \int_a^b \left(\frac{1-f_1}{1-g_1}\right)^{dt}, \int_a^b \left(\frac{f_2}{g_2}\right)^{dt}\right) \\
&= \left(1 - \frac{\int_a^b (1-f_1)^{dt}}{\int_a^b (1-g_1)^{dt}}, \frac{\int_a^b (f_2)^{dt}}{\int_a^b (g_2)^{dt}}\right) \\
&= \left(1 - \int_a^b (1-f_1)^{dt}, \int_a^b (f_2)^{dt}\right) \ominus \left(1 - \int_a^b (1-g_1)^{dt}, \int_a^b (g_2)^{dt}\right) \\
&= \left(\int_a^{\oplus b} Fdt\right) \ominus \left(\int_a^{\oplus b} Gdt\right).
\end{aligned}$$

in view of (2.6) and Theorem 1.15. □

**Theorem 2.28** ($^{\oplus}$Integration by parts)**.** *Let $F : [a,b] \to \mathcal{L}^{\oplus}$ be $^{\oplus}$differentiable and let $h : [a,b] \to \mathbb{R}^+$ be differentiable and nondecreasing. Then,*

$$\int_a^{\oplus b} h(t)F^{\oplus}(t)dt = (h(b)F(b) \ominus h(a)F(a)) \ominus \int_a^{\oplus b} F(t)h'(t)dt.$$

*Proof.* The proof is straightforward from Theorem 1.16. □

# 3 $^{\otimes}$Calculus for intuitionistic fuzzy sets

We define $^{\otimes}$limit for IFVFs as the following.

**Definition 3.1.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}$ and $c$ is a cluster point of $I$. We say that the $^{\otimes}$limit of $F$, as $t$ approaches $c$, is IFV $\xi$ if for any IFV $\bar{\varepsilon} = (1-\varepsilon, \varepsilon) <_L (1,0)$ there exists $\delta > 0$ such that*

$$\xi \geq_L F(t) \otimes \bar{\varepsilon} \quad \textit{and} \quad F(t) \geq_L \xi \otimes \bar{\varepsilon} \tag{3.1}$$

*holds whenever $0 < |t-c| < \delta$, $t \in I$. In this case, we write $^{\otimes}\lim\limits_{t\to c} F(t) = \xi$.*

$^{\otimes}$Limit works with any IFV, but we will omit the element $(0,1)$ in $^{\otimes}$calculus since the other concepts of $^{\otimes}$calculus do not work properly with $(0,1)$. We will use the set $\mathcal{L}^{\otimes} = \{\alpha \in \mathcal{L} : \alpha >_L (0,1)\}$.

**Theorem 3.2.** *Let $F : I \subseteq\to \mathcal{L}^{\otimes}$, $F = (f_1, f_2)$ and $\xi \in \mathcal{L}^{\otimes}$. $^{\otimes}\lim\limits_{t\to c} F(t) = \xi$ if and only if $\lim_{t\to c} f_1(t) = \xi_1$ and $\lim\limits_{t\to c} f_2(t) = \xi_2$.*

*Proof.* *Necessity.* Suppose $^{\otimes}\lim\limits_{t\to c} F(t) = \xi$. Then, for any given $\bar{\varepsilon} = (1-\varepsilon, \varepsilon) <_L (1,0)$ there is $\delta > 0$ such that

$$\left.\begin{array}{l} \xi_1 \geq f_1(t)(1-\varepsilon) = f_1(t) - \varepsilon f_1(t) \geq f_1(t) - \varepsilon \\ f_1(t) \geq \xi_1(1-\varepsilon) = \xi_1 - \varepsilon\xi_1 \geq \xi_1 - \varepsilon \end{array}\right\} \Rightarrow \xi_1 - \varepsilon \leq f_1(t) \leq \xi_1 + \varepsilon$$

and

$$\left.\begin{array}{l} \xi_2 \leq 1 - (1-f_2(t))(1-\varepsilon) = f_2(t) + \varepsilon - \varepsilon f_2(t) \leq f_2(t) + \varepsilon \\ f_2(t) \leq 1 - (1-\xi_2)(1-\varepsilon) = \xi_2 + \varepsilon - \varepsilon\xi_2 \leq \xi_2 + \varepsilon \end{array}\right\} \Rightarrow \xi_2 - \varepsilon \leq f_2(t) \leq \xi_2 + \varepsilon$$

whenever $t \in I$ and $0 < |t-c| < \delta$. This implies $\lim_{t\to c} f_1(t) = \xi_1$ and $\lim_{t\to c} f_2(t) = \xi_2$.

*Sufficiency.* This part can be done by replacing $\oplus$ with $\otimes$ and changing the roles of $f_1$, $f_2$ in the sufficiency part of the proof of Theorem 2.2. □

**Remark 3.3.** *If $^{\otimes}\lim\limits_{t\to c} F(t) = (0,1)$, then there exists $\delta > 0$ such that $F(t) = (0,1)$ for any $t \in (c-\delta, c+\delta)/\{c\}$. On the other hand, if $F(d) = (0,1)$ for a number $d \in (c-\delta, c+\delta)/\{c\}$, then $\xi = (0,1)$.*

The proofs of the other theorems in this section can be done in a similar way to those of Section 2 by replacing $\oplus$ with $\otimes$ and changing the roles of $f_1$, $f_2$. Hence, the proofs are omitted.

**Definition 3.4.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}^{\otimes}$ and $\xi \in \mathcal{L}^{\otimes}$. $^{\otimes}\lim\limits_{t\to c^-} F(t) = \xi$ if for any IFV $\bar{\varepsilon} <_L (1,0)$ there exists $\delta > 0$ such that* (3.1) *holds whenever $t \in (c-\delta, c)$. Similarly, $^{\otimes}\lim\limits_{t\to c^+} F(t) = \xi$ if there is $\delta > 0$ such that* (3.1) *holds whenever $t \in (c, c+\delta)$.*

If $I$ is a closed interval, then $^{\otimes}$limit, $^{\otimes}$continuity, $^{\otimes}$derivative at endpoints of $I$ are meant in the one-sided sense throughout the paper.

**Theorem 3.5.** *Let $F, G : I \subseteq \mathbb{R} \to \mathcal{L}^{\otimes}$ be two IFVFs; $\xi, \eta \in \mathcal{L}^{\otimes}$ be two IFVs; and $\lambda \geq 0$. If $^{\otimes}\lim\limits_{t\to c} F(t) = \xi$ and $^{\otimes}\lim\limits_{t\to c} G(t) = \eta$, then followings hold:*

*(i)* $^{\otimes}\lim\limits_{t\to c} (F(t) \otimes G(t)) = \xi \otimes \eta$

*(ii)* $^{\otimes}\lim\limits_{t\to c} (F(t) \oslash G(t)) = \xi \oslash \eta$ *where* $F(t) \oslash G(t) \in \mathcal{L}^{\otimes}$

*(iii)* $^{\otimes}\lim\limits_{t\to c} (F(t))^{\lambda} = \xi^{\lambda}$

**Definition 3.6.** *Let $F : I \subseteq \mathbb{R} \to \mathcal{L}^{\otimes}$ and $t_0 \in I$. $F$ is said to be ${}^{\otimes}$continuous at $t_0$ if for any IFV $\bar{\varepsilon} = (1-\varepsilon, \varepsilon) <_L (1, 0)$ there exists $\delta > 0$ such that*

$$F(t_0) \geq_L F(t) \otimes \bar{\varepsilon} \quad \text{and} \quad F(t) \geq_L F(t_0) \otimes \bar{\varepsilon}$$

*holds whenever $t \in I$ and $|t - t_0| < \delta$.*

**Theorem 3.7.** *Let $F : (a, b) \to \mathcal{L}^{\otimes}$ and $t_0 \in (a, b)$. $F$ is ${}^{\otimes}$continuous at $t_0$ if and only if ${}^{\otimes}\lim\limits_{t \to t_0} F(t) = F(t_0)$.*

**Definition 3.8.** *Let $F : (a, b) \to \mathcal{L}^{\otimes}$ and $t_0 \in (a, b)$. $F$ is right-${}^{\otimes}$continuous at $t_0$ if ${}^{\otimes}\lim\limits_{t \to t_0^+} F(t) = F(t_0)$, and left-${}^{\otimes}$continuous at $t_0$ if ${}^{\otimes}\lim\limits_{t \to t_0^-} F(t) = F(t_0)$.*

**Definition 3.9.** *$F : [a, b] \to \mathcal{L}^{\otimes}$ is said to be ${}^{\otimes}$continuous on $[a, b]$ if $F$ is right-${}^{\otimes}$continuous at a, left-${}^{\otimes}$continuous at $b$ and ${}^{\otimes}$continuous at all interior points of $[a, b]$.*

**Theorem 3.10.** *Let $F : [a, b] \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. $F$ is ${}^{\otimes}$continuous on $[a, b]$ if and only if $f_1$ and $f_2$ are continuous on $[a, b]$.*

**Definition 3.11.** *Let $F : (a, b) \to \mathcal{L}^{\otimes}$ and $t_0 \in (a, b)$. $F$ is said to be ${}^{\otimes}$differentiable at $t_0$ if $F(t_0 + h) \oslash F(t_0)$ and $F(t_0) \oslash F(t_0 - h)$ exist in in $\mathcal{L}^{\otimes}$ for sufficiently small $h$ and there is an IFV $\xi \in \mathcal{L}^{\otimes}$ such that*

$${}^{\otimes}\lim_{h \to 0^+} (F(t_0 + h) \oslash F(t_0))^{1/h} = {}^{\otimes}\lim_{h \to 0^+} (F(t_0) \oslash F(t_0 - h))^{1/h} = \xi.$$

*In this case, we write $\xi = F^{\otimes}(t_0)$.*

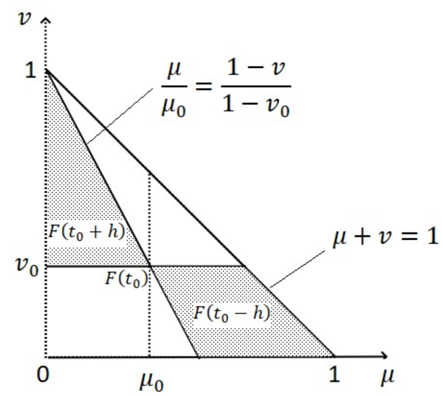


Figure 2: Regions where $F(t_0 + h) \oslash F(t_0)$ and $F(t_0) \oslash F(t_0 - h)$ exist in $\mathcal{L}^{\otimes}$

Figure 2 illustrates multiplication and division regions of $F(t_0)$. For more information we refer to [10, 11].

**Theorem 3.12.** *Let $F : (a, b) \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. $F$ is ${}^{\otimes}$differentiable at $t_0$ if and only if $f_1'(t_0), f_2'(t_0)$ exists, $F$ and $\frac{f_1}{1-f_2}$ are nonincreasing at $t_0$. Furthermore,*

$$F^{\otimes}(t_0) = \left( \exp\left( \frac{f_1'(t_0)}{f_1(t_0)} \right), 1 - \exp\left( \frac{(1 - f_2)'(t_0)}{(1 - f_2)(t_0)} \right) \right).$$

**Definition 3.13.** *$F : (a, b) \to \mathcal{L}^{\otimes}$ is said to be ${}^{\otimes}$differentiable on $(a, b)$ if $F$ is ${}^{\otimes}$differentiable for each $t_0 \in (a, b)$.*

**Theorem 3.14.** *Let $F : (a, b) \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. $F$ is ${}^{\otimes}$differentiable on $(a, b)$ if and only if $F$ is nonincreasing on $(a, b)$, $f_1, f_2$ are differentiable on $(a, b)$ and $\left(\frac{f_1}{1-f_2}\right)' \leq 0$. Furthermore,*

$$F^{\otimes} = \left( \exp\left( \frac{f_1'}{f_1} \right), 1 - \exp\left( \frac{(1 - f_2)'}{(1 - f_2)} \right) \right).$$

**Theorem 3.15.** *Let $F : (a,b) \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. $F$ is ${}^{\otimes}$differentiable at $t_0$ if and only if $f_1^*(t_0), (1-f_2)^*(t_0)$ exists, $F$ and $\frac{f_1}{1-f_2}$ are nonincreasing at $t_0$. Furthermore,*

$$F^{\otimes}(t_0) = \left(f_1^*(t_0), 1 - (1-f_2)^*(t_0)\right).$$

**Theorem 3.16.** *Let $F : (a,b) \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. $F$ is ${}^{\otimes}$differentiable on $(a,b)$ if and only if $F$ is nonincreasing on $(a,b)$, $f_1, (1-f_2)$ are *differentiable on $(a,b)$ and $\left(\frac{f_1}{1-f_2}\right)^* \leq 1$. Furthermore,*

$$F^{\otimes} = \left(f_1^*, 1 - (1-f_2)^*\right).$$

**Theorem 3.17.** *Let $F, G : (a,b) \to \mathcal{L}^{\otimes}$ be ${}^{\otimes}$differentiable IFVFs, $h : (a,b) \to \mathbb{R}^+ \cup \{0\}$ be differentiable and nondecreasing real valued function and $\lambda \geq 0$. Then,*

*(i)* $(F \otimes G)^{\otimes}(t) = F^{\otimes}(t) \otimes G^{\otimes}(t)$

*(ii)* $\left(F^{\lambda}\right)^{\otimes}(t) = \left(F^{\otimes}(t)\right)^{\lambda}$

*(iii)* $\left(F^{h}\right)^{\otimes}(t) = \left(F^{\otimes}(t)\right)^{h(t)} \otimes \left(F(t)^{h'(t)}\right)$

*Moreover, if $(F \oslash G)^{\otimes}(t)$ exists then*

*(iv)* $(F \oslash G)^{\otimes}(t) = F^{\otimes}(t) \oslash G^{\otimes}(t)$.

**Definition 3.18.** *Let $F : (a,b) \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. The ${}^{\otimes}$antiderivative $\Phi$ of $F$ is defined by*

$$\Phi(t) = \left(\lambda_1 \exp\left(\int \ln(f_1) dt\right), 1 - \lambda_2 \exp\left(\int \ln(1-f_2) dt\right)\right)$$

*where $\lambda_1, \lambda_2 > 0$ are arbitrary constants such that $\Phi$ is an IFV.*

**Theorem 3.19.** *Let $F : (a,b) \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. If $\Phi$ is ${}^{\otimes}$antiderivative of $F$, then*

$$\Phi(t) = \left(\lambda_1 \int (f_1)^{dt}, 1 - \lambda_2 \int (1-f_2)^{dt}\right)$$

*where $\lambda_1, \lambda_2 > 0$ are arbitrary constants such that $\Phi$ is an IFV.*

We note that ${}^{\otimes}$antiderivative $\Phi(t)$ of $F = (f_1, f_2)$ is an IFV if and only if

$$0 < \lambda_1 \leq \frac{1}{\int (f_1)^{dt}}, \quad 0 < \lambda_2 \leq \frac{1}{\int (1-f_2)^{dt}}, \quad \int \left(\frac{f_1}{1-f_2}\right)^{dt} \leq \frac{\lambda_2}{\lambda_1}.$$

**Theorem 3.20.** *If $F : (a,b) \to \mathcal{L}^{\otimes}$ is ${}^{\otimes}$continuous, then ${}^{\otimes}$antiderivative $\Phi(t)$ exists and $\Phi^{\otimes}(t) = F(t)$*

**Definition 3.21** (Definite ${}^{\otimes}$integral)**.** *$F : [a,b] \to \mathcal{L}^{\otimes}$ is said to be ${}^{\otimes}$integrable on $[a,b]$ if there exists an IFV $\xi \in \mathcal{L}^{\otimes}$ such that for any partition $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ of $[a,b]$ and for any points $c_k \in [x_k, x_{k+1}]$, we have*

$$\sideset{^{\otimes}}{}\lim_{\|\mathcal{P}\| \to 0} \bigotimes_{k=0}^{n-1} F(c_k)^{\Delta x_k} = \xi.$$

*In that case, we write $\xi = \sideset{^{\otimes}}{}\int_a^b F(t)^{dt}$.*

**Theorem 3.22.** *Let $F : [a,b] \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. $F$ is ${}^{\otimes}$integrable on $[a,b]$ if and only if $f_1$ and $f_2$ are integrable on $[a,b]$. Furthermore,*

$$\sideset{^{\otimes}}{}\int_a^b F(t)^{dt} = \left(\exp\left(\int_a^b \ln(f_1) dt\right), 1 - \exp\left(\int_a^b \ln(1-f_2) dt\right)\right).$$

**Theorem 3.23.** *Let $F : [a,b] \to \mathcal{L}^{\otimes}$ and $F = (f_1, f_2)$. If $F$ is ${}^{\otimes}$integrable on $[a,b]$, then*

$$\overset{\otimes}{\int_a^b} F(t)^{dt} = \left( \int_a^b (f_1)^{dt}, 1 - \int_a^b (1 - f_2)^{dt} \right).$$

**Theorem 3.24** (Fundamental theorem of ${}^{\otimes}$calculus)**.** *Let $F : [a,b] \to \mathcal{L}^{\otimes}$ be continuous. Then, following statements hold:*

$(i)$ *The function $\psi$ defined by*

$$\psi(t) = \overset{\otimes}{\int_a^t} F(u)^{du}$$

*is ${}^{\otimes}$differentiable on $[a,b]$ and $\psi^{\otimes}(t) = F(t)$.*

$(ii)$ *If $\Phi$ is any ${}^{\otimes}$antiderivative of $F$, then*

$$\overset{\otimes}{\int_a^b} F(t)^{dt} = \Phi(b) \oslash \Phi(a).$$

**Theorem 3.25.** *Let $F, G : [a,b] \to \mathcal{L}^{\otimes}$ are ${}^{\otimes}$integrable on $[a,b]$ and $\lambda \geq 0$. Then,*

*(i)* $$\overset{\otimes}{\int_a^b} F^{\lambda}(t)^{dt} = \left( \overset{\otimes}{\int_a^b} F(t)^{dt} \right)^{\lambda}$$

*(ii)* $$\overset{\otimes}{\int_a^b} (F(t) \otimes G(t))^{dt} = \left( \overset{\otimes}{\int_a^b} F(t)^{dt} \right) \otimes \left( \overset{\otimes}{\int_a^b} G(t)^{dt} \right)$$

*(iii)* $$\overset{\otimes}{\int_a^b} F(t)^{dt} = \left( \overset{\otimes}{\int_a^c} F(t)^{dt} \right) \otimes \left( \overset{\otimes}{\int_c^b} F(t)^{dt} \right), \quad a \leq c \leq b.$$

*Moreover, if $F \oslash G$ exists then*

*(iv)* $$\overset{\otimes}{\int_a^b} (F(t) \oslash G(t))^{dt} = \left( \overset{\otimes}{\int_a^b} F(t)^{dt} \right) \oslash \left( \overset{\otimes}{\int_a^b} G(t)^{dt} \right).$$

**Theorem 3.26** (${}^{\otimes}$Integration by parts)**.** *Let $F : [a,b] \to \mathcal{L}^{\otimes}$ be ${}^{\otimes}$differentiable and let $h : [a,b] \to \mathbb{R}^{+}$ be differentiable and nondecreasing. Then,*

$$\overset{\otimes}{\int_a^b} \left( F^{\otimes}(t)^{h(t)} \right)^{dt} = \left( F(b)^{h(b)} \oslash F(a)^{h(a)} \right) \oslash \overset{\otimes}{\int_a^b} \left( F(t)^{h'(t)} \right)^{dt}.$$

# 4 Isomorphisms with respect to some basic operations

As seen in Section 2 and Section 3, there are many parallel properties between ${}^{\oplus}$calculus and ${}^{\otimes}$calculus which can be explained by the structural analogy of $(\mathcal{L}^{\oplus}, \oplus)$, $(\mathcal{L}^{\otimes}, \otimes)$ and of $(\mathcal{L}^{\oplus}, \star)$, $(\mathcal{L}^{\otimes}, \odot)$ where $\lambda \star \alpha = \lambda\alpha$ and $\lambda \odot \alpha = \alpha^{\lambda}$. In the existing literature of theory of intuitionistic fuzzy calculus, Ai and Xu [1] are the first to account for the above phenomenon from the knowledge of abstract algebra. They showed that $(\mathcal{L}, \oplus) \cong (\mathcal{L}, \otimes)$ and $(\mathcal{L}, \star) \cong (\mathcal{L}, \odot)$ by using the isomorphism $\varphi : \mathcal{L} \to \mathcal{L}$, $\varphi(\alpha) = \overline{\alpha}$ where $\overline{\alpha} = \overline{(\alpha_1, \alpha_2)} = (\alpha_2, \alpha_1)$ is the complement of IFV $\alpha$. They also showed that $(A_1, \oplus) \cong (A_2, \oplus)$ and $(A_1, \star) \cong (A_2, \odot)$ where $A_1$ is the set of intuitionistic fuzzy multiple definite integrals(IFMDI) and $A_2$ is the set of multiplicative IFMDIs. Following [1], one can also show that $(\mathcal{L}^{\oplus}, \oplus) \cong (\mathcal{L}^{\otimes}, \otimes)$ and $(\mathcal{L}^{\oplus}, \star) \cong (\mathcal{L}^{\otimes}, \odot)$ by using the isomorphism $\varphi : \mathcal{L}^{\oplus} \to \mathcal{L}^{\otimes}$, $\varphi(\alpha) = \overline{\alpha}$. Furthermore, let

$$\mathcal{S}_1 = \left\{ F^{\oplus} \mid F : (a,b) \to \mathcal{L}^{\oplus} \text{ is } {}^{\oplus}\text{differentiable} \right\}, \quad \mathcal{S}_2 = \left\{ F^{\otimes} \mid F : (a,b) \to \mathcal{L}^{\otimes} \text{ is } {}^{\otimes}\text{differentiable} \right\}$$

$$\mathcal{S}_3 = \left\{ \overset{\oplus}{\int_a^b} F(t)dt \,\middle|\, F : [a,b] \to \mathcal{L}^{\oplus} \text{ is } {}^{\oplus}\text{integrable} \right\}, \quad \mathcal{S}_4 = \left\{ \overset{\otimes}{\int_a^b} F(t)^{dt} \,\middle|\, F : [a,b] \to \mathcal{L}^{\otimes} \text{ is } {}^{\otimes}\text{integrable} \right\}$$

and let $\varphi_1 : \mathcal{S}_1 \to \mathcal{S}_2$ be defined again by $\varphi_1(F) = \overline{F}$. Then, we have

$$\begin{aligned}
\varphi_1\left(F^{\oplus} \oplus G^{\oplus}\right) &= \overline{F^{\oplus} \oplus G^{\oplus}} \\
&= \overline{\left(1-(1-f_1)^*(1-g_1)^*, f_2^* g_2^*, \right)} \\
&= \left(f_2^* g_2^*, 1-(1-f_1)^*(1-g_1)^*\right) \\
&= \left(f_2^*, 1-(1-f_1)^*\right) \otimes \left(g_2^*, 1-(1-g_1)^*\right) \\
&= \overline{\left(1-(1-f_1)^*, f_2^*\right)} \otimes \overline{\left(1-(1-g_1)^*, g_2^*\right)} \\
&= \overline{F^{\oplus}} \otimes \overline{G^{\oplus}} \\
&= \varphi_1\left(F^{\oplus}\right) \otimes \varphi_1\left(G^{\oplus}\right)
\end{aligned}$$

and

$$\begin{aligned}
\varphi_1\left(\lambda \star F^{\oplus}\right) &= \overline{\left(1-((1-f_1)^*)^{\lambda}, (f_2^*)^{\lambda}\right)} \\
&= \left((f_2^*)^{\lambda}, 1-((1-f_1)^*)^{\lambda}\right) \\
&= \lambda \odot \left(f_2^*, 1-(1-f_1)^*\right) \\
&= \lambda \odot \overline{\left(1-(1-f_1)^*, f_2^*\right)} \\
&= \lambda \odot \overline{F^{\oplus}} \\
&= \lambda \odot \varphi_1\left(F^{\oplus}\right)
\end{aligned}$$

which imply $(\mathcal{S}_1, \oplus) \cong (\mathcal{S}_2, \otimes)$ and $(\mathcal{S}_1, \star) \cong (\mathcal{S}_2, \odot)$. In a similar way, $(\mathcal{S}_3, \oplus) \cong (\mathcal{S}_4, \otimes)$ and $(\mathcal{S}_3, \star) \cong (\mathcal{S}_4, \odot)$ can also be obtained.

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