Fractional analysis of biological population model via novel transform

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October 18, 2022

Abstract

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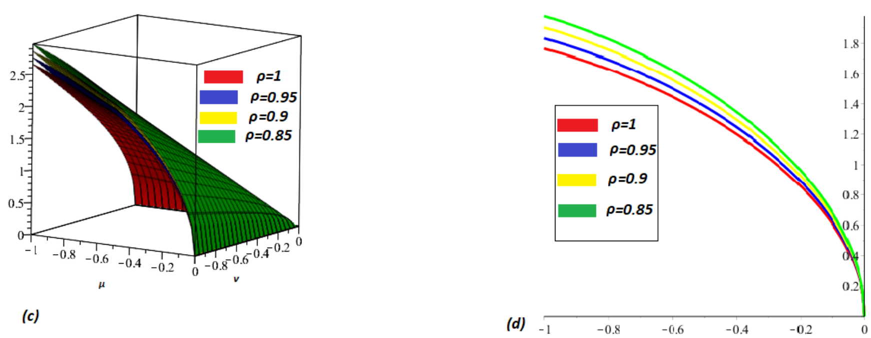

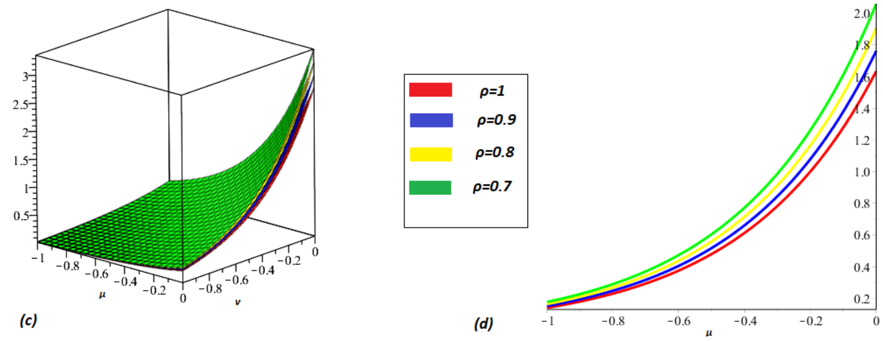
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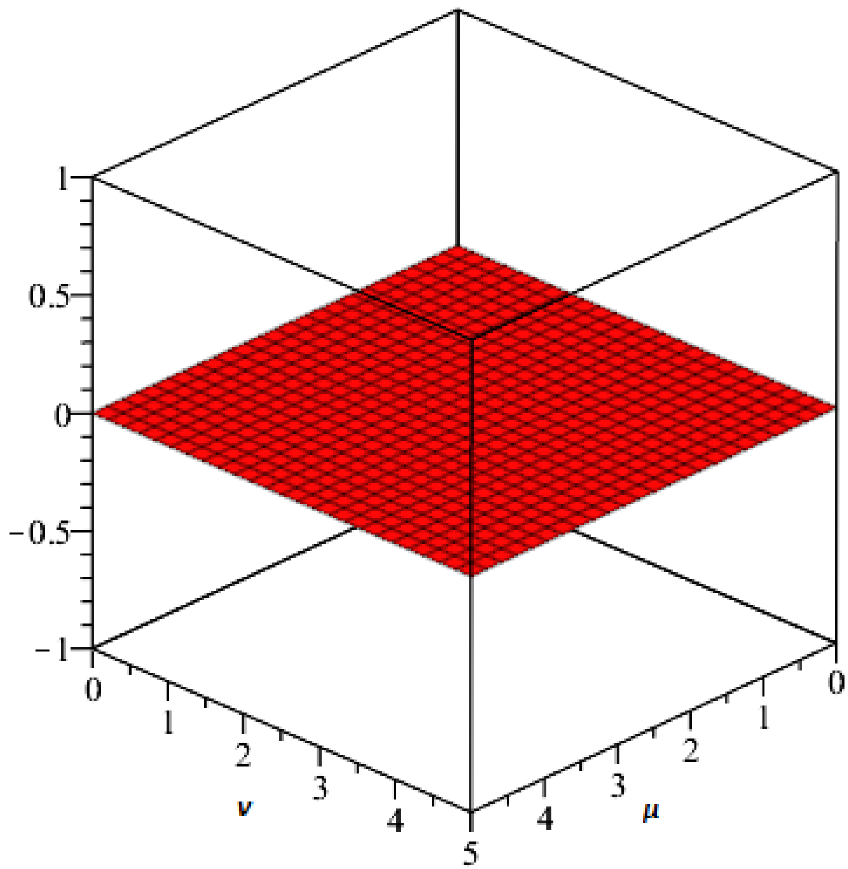
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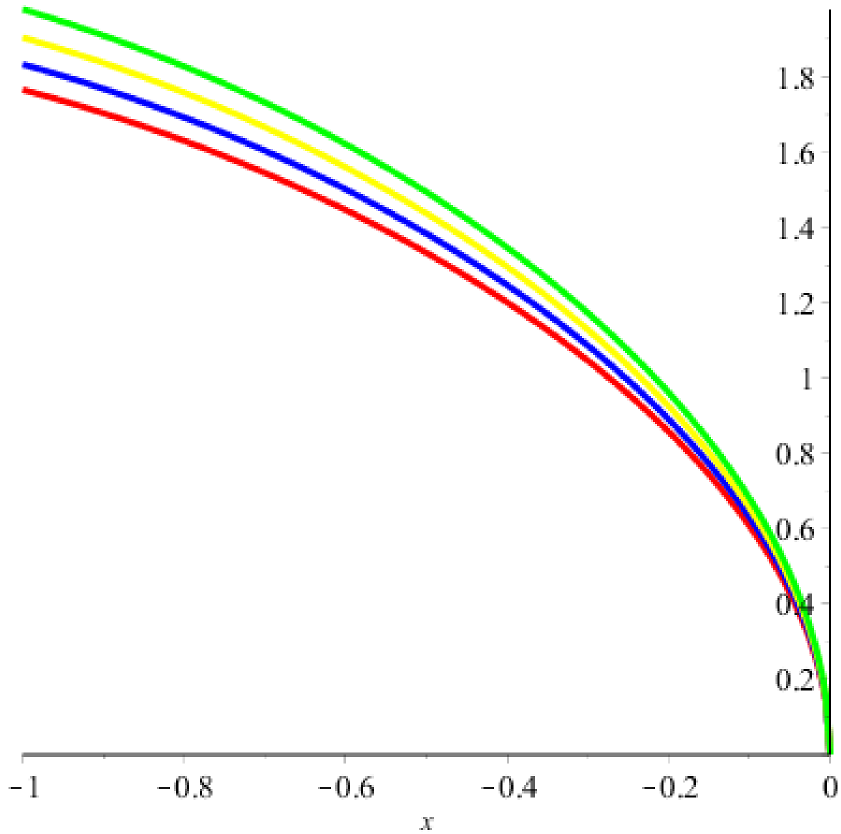
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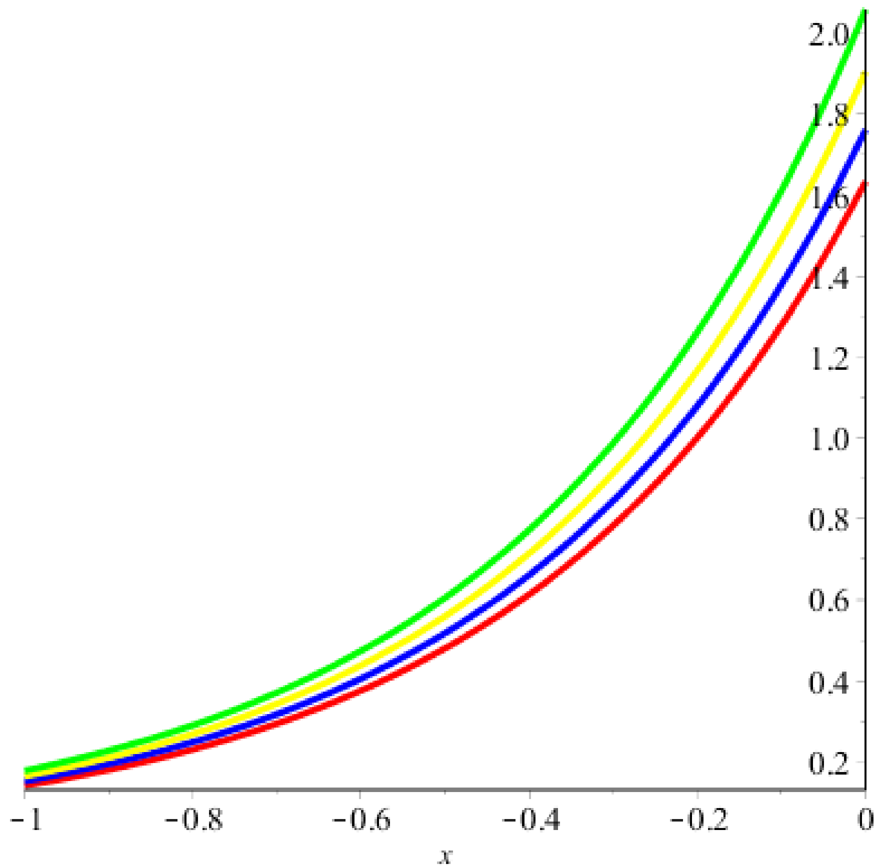

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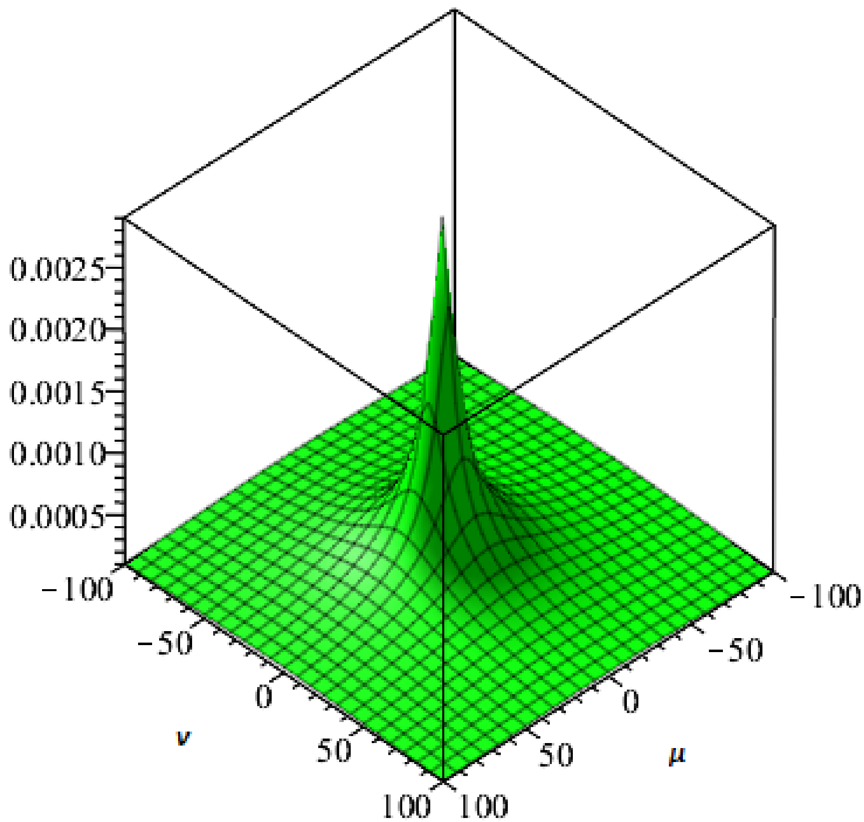

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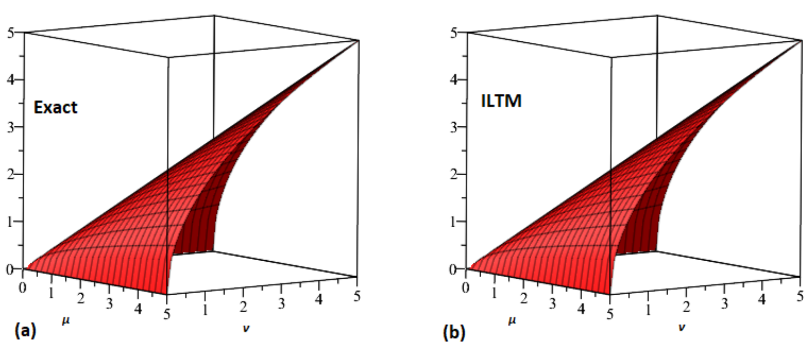

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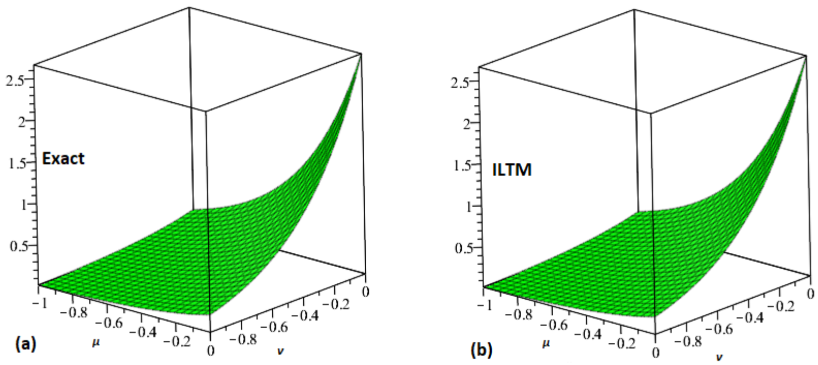
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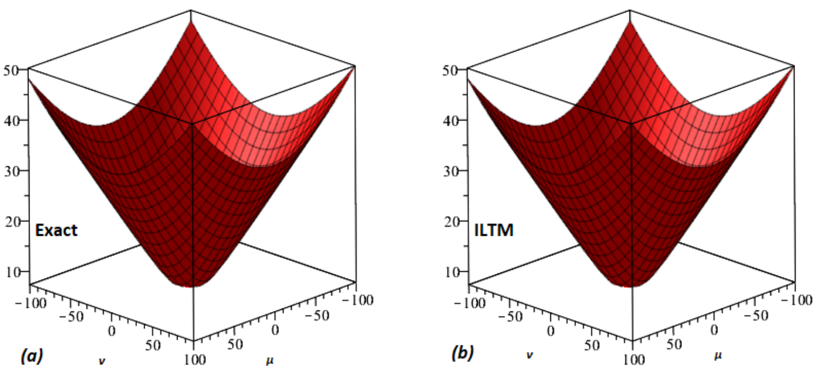
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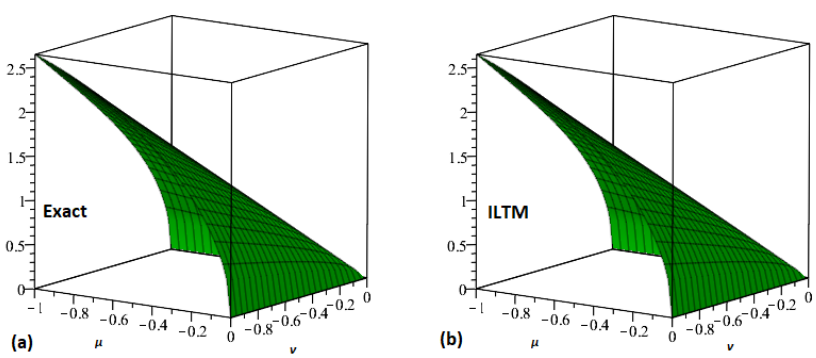
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Received: Added at production | Revised: Added at production | Accepted: Added at production
DOI: xxx/xxxx

ARTICLE TYPE

Fractional analysis of biological population model via novel transform

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This study applied the ZZ transformation and the new iterative transform technique to obtain a fascinating explicit pattern for outcomes of the biological population model. It makes important projections and helps us understand the dynamical method of demographic fluctuations in biological population models. Additionally, ZZ transforms combine various other transformations that already exist. We used a complex fractional transformation to deal with a fractional partial differential equation and a new iterative transform approach to study the nonlinear equations in examine the closed form solutions. In mathematics, the number of equations and their solutions that have been discovered and associated with different innovative properties of the proposed model. A variety of images and tabulations are provided to provide extra context for these notions. The suggested technique accuracy and efficiency imply that it can be applied to a wide range of nonlinear evolutionary problems.

KEYWORDS:
ZZ transform, new iterative method, biological population model, analytical solution, atangana-Baleanu fractional derivative

1 | INTRODUCTION

Fractional calculus (FC), which has been there since classical calculus, has recently received much interest due to its connections to basic ideas. Leibniz and L'Hospital were the first to present fractional calculus, but it has since gained popularity among academics due to its wide range of applications. Following that, it was widely used to examine a variety of occurrences. But several researches emphasized the disadvantages of using this operator specifically, the physical importance of the starting condition and the derivative of a non-zero constant. Then Caputo introduced a novel fractional operator that overcame the earlier limitations. The bulk of models explored and analyzed under the FC framework use the Caputo operator. Some basic works of fractional calculus on different aspects are given by Momani and Shawagfeh, Podlubny, Jafari and Seifi, Kiryakova, Oldham and Spanier, Miller and Ross, Diethelm et al., Trujillo, Kilbas and Kemple and Beyer [1,2,3,4,5].

Biologists consider emigration and dispersal to be crucial processes in the establishment of species populations. Three independent position functions $\Phi = (\zeta, \chi)$ in the area C with ϑ [6] are utilised to represent the transmission of a biological species. Dispersion velocity $u(\Phi, \vartheta)$, population supply $p(\zeta, \vartheta)$, population density $v(\zeta, \vartheta)$ are the three variables. $p(\zeta, \vartheta)$ signifies the rate at which birth and death produce individuals per unit volume, while $v(\zeta, \vartheta)$

[0]**Math. Methods Appl. Sci.:** MMAS, Mathematical Methods in the Applied Sciences

represents the number of people. In addition, $u(\zeta,\vartheta)$ signifies the average speed of persons and population movement from one area to another. The v,u and p for each $D\subset C$ sub-region must be consistent with

$$\frac{d^{ß}}{d\tau^{ß}}\int_{D} vdU+\int_{\partial D} v\mathbf{u}\cdot\hat{n}dA=$$

where $\hat{n}$ is unit normal outward to the boundary ∂D result[7]

$$p=p(v),\ \mathbf{u}=-\lambda(v)\nabla$$

where $\lambda(v)>0$ for $v>0$, and ∇ is the Laplace nonlinear degenerate parabolic PDE can be obtain and which is presented as

$$D_{\tau}^{ß}v=\frac{\partial^2\phi(v)}{\partial\zeta^2}+\frac{\partial^2\phi(v)}{\partial\chi^2}+p(v)$$

In this instance, the fractional order is considered in the sense of Caputo. In addition, Nisbet and Gurney[8] used (v) as an unique case to simulate and evaluate the animal population. Usually, the preparations are made by young animals who wish to establish their own breeding region after attaining maturity and moving away from their parental territory, or by adult species who are endangered by mature invaders. In either of these two scenarios, it is much more probable that they will be directed toward the adjacent uninhabited territory. The probability distribution is resolved on the side of the mesh affected by the population density gradient between these two possibilities.
Now, Eq. (3) with $\phi(v)=v^2$ leads to

$$D_{\tau}^{ß}v=\frac{\partial^2 v^2}{\partial\zeta^2}+\frac{\partial^2 v^2}{\partial\chi^2}+p(v),\ \tau\geq 0,\zeta,\chi\in R,$$

with the given initial condition $v(\zeta,\chi,0)$. For $\mu=1$, Eq. (III) simplifies the classical biological population model (NBPM):

$$\frac{\partial v}{\partial\tau}=\frac{\partial^2 v^2}{\partial\zeta^2}+\frac{\partial^2 v^2}{\partial\chi^2}+p(v),\ \tau\geq 0,\zeta,\chi\in R.$$

For $p(v)$, three examples of constitutive equations are presented as follows(i) $p(v)=cv, c=$ constant, Malthusian law[6].
(ii) $p(v)=c_1v-c_2v^2, c_1,c_2=$ positive constants, Verhulst law[7].
(iii) $p(v)=cv^{\gamma}, (c>0, 0<\gamma<1)$, porous media[9,10].
Numerous academicians have recent times devised more specific and efficient ways for locating and analyzing solutions to nonlinear and intricate problems. In response to this, George Adomian, an American scientist and aerospace engineer, developed the Adomian decomposition method[11]. ADM has been utilized effectively to examine the behavior of nonlinear dynamic systems without the necessity for perturbation or linearization. ADM, on the other hand, necessitates a considerable amount of time and computer memory for analytical work. Rawashdeh and Maitama conceived and cultivated the FNDM[12,13], a hybrid of NTM and ADM, to satisfy these needs. Because FNDM is an improved form of ADM, it does not require pertubation, linearization, or discretization. Due to its accuracy and effectiveness, several mathematics and scientists have recently utilized FNDM to appreciate physical behavior in a wide range of complex scenarios[14,15]. The considered technique is distinctive in that it employs a simple algorithm to evaluate the solution and is based on Adomian polynomials, enabling rapid convergence of the obtained solution for the nonlinear portion of the problem. These polynomials generalize to a Maclaurin series when a free external parameter is introduced. In this study, FNDM was utilized to solve a fractional order, two-dimensional biological population model. Numerous authors have solved the given biological population model using a variety of numeric and analytic techniques in order to analyze the behavior and demonstrate the efficacy of the techniques[16,17,18].

2 | PRELIMINARIES

Definition 1. The Aboodh transform set of function is given as

$$B=\left\{g(\tau):\exists M,n_1,n_2>0,|g(\tau)|<Me^{-s\tau}\right\}$$

and is expressed as[19,20]

$$A\{g(\tau)\} = \frac{1}{s}\int_0^\infty g(\tau)e^{-s\tau}d\tau, \quad \tau > 0 \text{ and } n_1 \le s \le n_2$$

Theorem 1. When G and F are used as $g(\tau) \in B$ Aboodh and Laplace transformations, respectively[21,22]

$$G(s) = \frac{F(s)}{s}. \tag{1}$$

Zain[23] was developed first time the ZZ transform. It mixture the integral transforms of Aboodh and Laplace. The ZZ transformation is given as

Definition 2. Suppose that $g(\tau)\forall\tau \ge 0$ is a function then the ZZ transformation $Z(v,s)$ of $g(\tau)$ is defined as[23]

$$ZZ(g(\tau)) = Z(v,s) = s\int_0^\infty g(v\tau)e^{-s\tau}d\tau.$$

The ZZ transformation is linear, just as the Aboodh and Laplace transforms.

$$E_{ß}(z) = \sum_{m=0}^{\infty}\frac{z^m}{\Gamma(1+mß)}, \quad \mathrm{Re}(ß) > 0.$$

Definition 3. The Atangana-Baleanu (AB) Caputo derivative of a function $\nu(\varphi,\tau) \in H^1(a,b)$, then for $ß \in (0,1)$, is given as[24]

$$ABC_a D_\tau^{ß}\nu(\varphi,\tau) = \frac{\psi(ß)}{1-ß}\int_a^\tau \nu'(\varphi,\tau)E_{ß}\left(\frac{-ß(\tau-\eta)^{ß}}{1-ß}\right)d\eta.$$

Definition 4. Let the AB Riemann-Liouville derivative $\nu(\varphi,\tau) \in H^1(a,b)$, then for $ß \in (0,1)$, is defined as[24]

$${}_a^{ABR}D_\tau^{ß}\nu(\varphi,\eta) = \frac{\psi(ß)}{1-ß}\frac{d}{d\tau}\int_a^\tau \nu(\varphi,\eta)E_{ß}\left(\frac{-ß(\tau-\eta)^{ß}}{1-ß}\right)d\eta,$$

where with the condition $\psi(0) = \psi(1) = 1$, $\psi(ß)$ is a term and $b > a$.

Theorem 2. The AB Caputo and Riemann-Liouville derivative of Laplace transformation are, respectively, defined as[24]

$$\mathrm{L}\left\{{}_a^{ABC}D_\tau^{ß}\nu(\varphi,\tau)\right\}(s) = \frac{\psi(ß)}{1-ß}\frac{s^{ß}L\{\nu(\varphi,\tau)\} - s^{ß-1}\nu(\varphi,0)}{s^{ß}+\frac{ß}{1-ß}} \tag{2}$$

and

$$L\left\{\begin{matrix}ABR\\a\end{matrix}D_\tau^{ß}\nu(\varphi,\tau)\right\}(s) = \frac{\psi(ß)}{1-ß}\frac{s^{ß}L\{\nu(\varphi,\tau)\}}{s^{ß}+\frac{ß}{1-ß}} \tag{3}$$

The following theorem have been suggested, with the supposition that $g(\tau) \in H^1(a,b), b > a$ and $ß \in (0,1)$.

Theorem 3. The AB Riemann-Liouville of Aboodh transformation of derivative is defined as[22]

$$G(s) = A\left\{\begin{matrix}ABR\\a\end{matrix}D_\tau^{ß}\nu(\varphi,\tau)\right\}(s) = \frac{1}{s}\left[\frac{\psi(ß)}{1-ß}\frac{s^{ß}L\{\nu(\varphi,\tau)\}}{s^{ß}+\frac{ß}{1-ß}}\right] \tag{4}$$

Proof. We obtained at the required result applying the Theorem 2.1 and eq. (3). In the theorem below, the relationships link the transforms of ZZ and Aboodh is defined. □

Theorem 4. The AB Caputo derivative of Aboodh transform is defined as[22]

$$G(s) = A\left\{\begin{matrix}ABC\\a\end{matrix}D_\tau^{ß}\nu(\varphi,\tau)\right\}(s) = \frac{1}{s}\left[\frac{\psi(ß)}{1-ß}\frac{s^{ß}L\{\nu(\varphi,\tau)\} - s^{ß-1}\nu(\varphi,0)}{s^{ß}+\frac{ß}{1-ß}}\right] \tag{5}$$

Proof. We may investigate the given result by applying Theorem 2.1 and equation (2). □

Theorem 5. If $G(s)$ and $Z(v,s)$ are the Aboodh and ZZ transforms of $g(\tau) \in B$. Then, we achieved[22]

$$Z(v,s) = \frac{s^2}{v^2}G\left(\frac{s}{v}\right)$$

Proof. The ZZ transform definition, we get

$$Z(v,s) = s\int_0^\infty g(v\tau)e^{-s\tau}\,d\tau \tag{6}$$

Putting $v\tau = \tau$ in eq. (6) we get

$$Z(v,s) = \frac{s}{v}\int_0^\infty g(\tau)e^{-\frac{\pi z}{v}}\,d\tau \tag{7}$$

The right-hand side of the above eq. (7) is given as

$$Z(v,s) = \frac{s}{v}F\left(\frac{s}{v}\right), \tag{8}$$

where $F(.)$ express the Laplace transform of $g(\tau)$. Using the theorem 2.1, eq. (8) can be define as

$$Z(v,s) = \frac{s}{v}\frac{F\left(\frac{s}{v}\right)}{\left(\frac{s}{v}\right)} \times \left(\frac{s}{v}\right) = \left(\frac{s}{v}\right)^2 G\left(\frac{s}{v}\right), \tag{9}$$

where $G(.)$ represent as the Aboodh transform of $g(\tau)$. □

Theorem 6. ZZ transform of $g(\tau) = \tau^{ß-1}$ is defined as

$$Z(v,s) = \Gamma(ß)\left(\frac{v}{s}\right)^{ß-1} \tag{10}$$

Proof. The Aboodh transform of $g(\tau) = \tau^{ß}$, $ß \geq 0$ is

$$G(s) = \frac{\Gamma(ß)}{s^{ß+1}}$$

$$\text{Now,}\quad G\left(\frac{s}{v}\right) = \frac{\Gamma(ß)v^{ß+1}}{s^{ß+1}}.$$

Using eq. (10), we achieved

$$Z(v,s) = \frac{s^2}{v^2}G\left(\frac{s}{v}\right) = \frac{s^2}{v^2}\frac{\Gamma(ß)v^{ß+1}}{s^{ß+1}} = \Gamma(ß)\left(\frac{v}{s}\right)^{ß-1}$$

□

Theorem 7. Let $ß, \omega \in C$ and $\mathrm{Re}(ß) > 0$, then the ZZ transform of $E_ß\left(\omega\tau^{ß}\right)$ is defined as[22]

$$ZZ\left\{\left(E_ß\left(\omega\tau^{ß}\right)\right)\right\} = Z(v,s) = \left(1 - \omega\left(\frac{v}{s}\right)^{ß}\right)^{-1} \tag{11}$$

Proof. We know that Aboodh transform of $E_ß\left(\omega\tau^{ß}\right)$ is defined as

$$G(s) = \frac{F(s)}{s} = \frac{s^{ß-1}}{s\left(s^{ß} - \omega\right)} \tag{12}$$

So,

$$G\left(\frac{s}{v}\right) = \frac{\left(\frac{s}{v}\right)^{ß-1}}{\left(\frac{s}{v}\right)\left(\left(\frac{s}{v}\right)^{ß} - \omega\right)}, \tag{13}$$

Using the Theorem 2.9, we achieved

$$Z(v,s) = \left(\frac{s}{v}\right)^2 G\left(\frac{s}{v}\right) = \left(\frac{s}{v}\right)^2 \frac{\left(\frac{s}{v}\right)^{ß-1}}{\left(\frac{s}{v}\right)\left(\left(\frac{s}{v}\right)^{ß} - \omega\right)}$$

$$= \frac{\left(\frac{s}{v}\right)^{ß}}{\left(\frac{s}{v}\right)^{ß} - \omega} = \left(1 - \omega\left(\frac{v}{s}\right)^{ß}\right)^{-1}$$

□

Theorem 8. If $G(s)$ and $Z(v,s)$ are the Aboodh and ZZ transforms of $g(\tau)$. Then the ZZ transformation of AB Caputo derivative is defined as[22]

$$ZZ\left\{\begin{matrix} ABC \\ 0 \end{matrix} D_{\tau}^{ß} g(\tau)\right\} = \left[\frac{\psi(ß)}{1-ß}\frac{\frac{s^{a+2}}{v^{ß+2}}G\left(\frac{s}{v}\right) - \frac{s^{ß}}{v^{ß}}f(0)}{\frac{s^{ß}}{v^{ß}} + \frac{ß}{1-ß}}\right] \tag{14}$$

Proof. USing the eqs. (1) and (5), we get

$$G\left(\frac{s}{v}\right) = \frac{v}{s}\left[\frac{\psi(ß)}{1-ß}\frac{\left(\frac{s}{v}\right)^{ß+1}G\left(\frac{s}{v}\right) - \left(\frac{s}{v}\right)^{ß-1}f(0)}{\left(\frac{s}{v}\right)^{ß} + \frac{ß}{1-ß}}\right] \tag{15}$$

So, the ZZ transform of AB Caputo is defined as

$$Z(v,s) = \left(\frac{s}{v}\right)^2 G\left(\frac{s}{v}\right) = \left(\frac{s}{v}\right)^2 \frac{v}{s}\left[\frac{\psi(ß)}{1-ß}\frac{\left(\frac{s}{v}\right)^{ß+1}G\left(\frac{s}{v}\right) - \left(\frac{s}{v}\right)^{ß-1}f(0)}{\left(\frac{s}{v}\right)^{ß} + \frac{ß}{1-ß}}\right]$$
$$= \left[\frac{\psi(ß)}{1-ß}\frac{\left(\frac{s}{v}\right)^{ß+2}G\left(\frac{s}{v}\right) - \left(\frac{s}{v}\right)^{ß}f(0)}{\left(\frac{s}{v}\right)^{ß} + \frac{ß}{1-ß}}\right]$$

□

Theorem 9. Let us suppose that $G(s)$ and $Z(v,s)$ are the Aboodh and ZZ transforms of $g(\tau)$. Then the ZZ transformation of AB Riemann-Liouville derivative is defined as[22]

$$ZZ\left\{\begin{matrix} ABR \\ 0 \end{matrix} D_{\tau}^{ß} f(\tau)\right\} = \left[\frac{\psi(ß)}{1-ß}\frac{\frac{5^{ß+2}}{v^{ß+2}}G\left(\frac{s}{v}\right)}{\frac{s^{\mu}}{v^{\mu}} + \frac{ß}{1-ß}}\right] \tag{16}$$

Proof. Using the eqs. (1) and (4), we get

$$G\left(\frac{s}{v}\right) = \frac{v}{s}\left[\frac{\psi(ß)}{1-ß}\frac{\left(\frac{s}{v}\right)^{ß+1}G\left(\frac{s}{v}\right)}{\left(\frac{s}{v}\right)^{ß} + \frac{ß}{1-ß}}\right] \tag{17}$$

From the eq. (9), the ZZ transform of AB Riemann-Liouville is defined as.

$$Z(v,s) = \left(\frac{s}{v}\right)^2 G\left(\frac{s}{v}\right) = \left(\frac{s}{v}\right)^2 \left(\frac{v}{s}\right)\left[\frac{\psi(ß)}{1-ß}\frac{\left(\frac{s}{v}\right)^{ß+1}G\left(\frac{s}{v}\right)}{\left(\frac{s}{v}\right)^{ß} + \frac{ß}{1-ß}}\right]$$
$$= \left[\frac{\psi(ß)}{1-ß}\frac{\left(\frac{s}{v}\right)^{ß+2}G\left(\frac{s}{v}\right)}{\left(\frac{s}{v}\right)^{ß} + \frac{ß}{1-ß}}\right]$$

□

3 | THE GENERAL IMPLEMENTATION OF ITERATIVE TRANSFORM METHOD

This section will cover briefly iterative transform method, which is used to solve fractional non-linear partial differential equations.

$$D_{\tau}^{ß}\Phi(\zeta,\chi,\tau) + R\Phi(\zeta,\chi,\tau) + N\Phi(\zeta,\chi,\tau) = g(\zeta,\chi,\tau), \quad n-1 < ß \le n, n \in N, \tag{18}$$

$$\Phi^{(k)}(\zeta,\chi,0) = h_k(\zeta,\chi), \quad k = 0,1,2,...,n-1, \tag{19}$$

where $D_{\tau}^{ß}\Phi(\zeta,\chi,\tau)$ is the fractional Caputo operator of order ß, $n-1 < ß \le n$, denoted by Eq. 18, R and N are linear and nonlinear operators. The $g(\zeta,\chi,\tau)$ is source function.
Applying ZZ transformation of Eq. 18, we have

$$Z[D_{\tau}^{ß}\Phi(\zeta,\chi,\tau)] + Z[R\Phi(\zeta,\chi,\tau) + N\Phi(\zeta,\chi,\tau)] = Z[g(\zeta,\chi,\tau)]. \tag{20}$$

$$\begin{aligned} Z[\Phi(\zeta,\chi,\tau)] =& \frac{v}{s}\Phi(\zeta,\chi,0) + \frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z[g(\zeta,\chi,\tau)] \\ & - \frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z[R\Phi(\zeta,\chi,\tau) + N\Phi(\zeta,\chi,\tau)]. \end{aligned} \tag{21}$$

Apply inverse ZZ transformation of Eq. 21, we get

$$\begin{aligned} \Phi(\zeta,\chi,\tau) =& Z^{-1}\left[\frac{v}{s}\Phi(\zeta,\chi,0) + Z[g(\zeta,\chi,\tau)]\right] \\ & - Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z[R\Phi(\zeta,\chi,\tau) + N\Phi(\zeta,\chi,\tau)]\right]. \end{aligned} \tag{22}$$

From iterative method,

$$\Phi(\zeta,\chi,\tau) = \sum_{i=0}^{\infty}\Phi_i(\zeta,\chi,\tau). \tag{23}$$

Since R is a linear operator

$$R\left(\sum_{i=0}^{\infty}\Phi_i(\zeta,\chi,\tau)\right) = \sum_{i=0}^{\infty} R\left[\Phi_i(\zeta,\chi,\tau)\right], \tag{24}$$

and the non-linear operator N is splitted as

$$N\left(\sum_{i=0}^{\infty}\Phi_i(\zeta,\chi,\tau)\right) = N[\Phi_0(\zeta,\chi,\tau)] + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}\Phi_k(\zeta,\chi,\tau\right) - N\left(\sum_{k=0}^{i-1}\Phi_k(\zeta,\chi,\tau)\right)\right\}. \tag{25}$$

Putting equations 23, 24 and 25 in equation 22, we obtain

$$\begin{aligned} \sum_{i=0}^{\infty}\Phi_i(\zeta,\chi,\tau) = Z^{-1}\left[\frac{v}{s}\Phi(\zeta,\chi,0) + Z[g(\zeta,\chi,\tau)]\right] - Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z\right. \\ \left.\left[\sum_{i=0}^{\infty} R[\Phi_i(\zeta,\chi,\tau)] + N[\Phi_0(\zeta,\chi,\tau)] + \sum_{i=1}^{\infty}\left\{N\left(\sum_{k=0}^{i}\Phi_k\left(\zeta,\chi,\tau\right) - N\left(\sum_{k=0}^{i-1}\Phi_k(\zeta,\chi,\tau)\right)\right)\right\}\right]\right]. \end{aligned} \tag{26}$$

Applying Eq. 26, we defined the following iterative producer

$$\Phi_0(\zeta,\chi,\tau) = Z^{-1}\left[\frac{v}{s}\Phi(\zeta,\chi,0) + \frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z(g(\zeta,\chi,\tau))\right], \tag{27}$$

$$\Phi_1(\zeta,\chi,\tau) = -Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z[R[\Phi_0(\zeta,\chi,\tau)] + N[\Phi_0(\zeta,\chi,\tau)]\right], \tag{28}$$

$$\Phi_{m+1}(\zeta,\chi,\tau) = -Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)} Z\left[R(\Phi_m(\zeta,\chi,\tau)) - \left\{N\left(\sum_{k=0}^{m}\Phi_k(\zeta,\chi,\tau)\right) - N\left(\sum_{k=0}^{m-1}\Phi_k(\zeta,\chi,\tau)\right)\right\}\right]\right], \quad m \geq 1 \tag{29}$$

The approximates m-terms result of Eq. 28 and Eq. 29 in form of series as

$$\Phi(\zeta,\chi,\tau) \cong \Phi_0(\zeta,\chi,\tau) + \Phi_1(\zeta,\chi,\tau) + \Phi_2(\zeta,\chi,\tau) + \cdots, +\Phi_m(\zeta,\chi,\tau), \quad m = 1, 2\cdots, \tag{30}$$

4 | NUMERICAL RESULTS

4.1 | Example

Consider fractional order Biological population model is given as

$$\frac{\partial^{ß}\Phi}{\partial\tau^{ß}} = \frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + h\Phi^{-1}(1-r\Phi), \quad 0 < ß \leq 1, \zeta, \chi \in \Re, \tau > 0, \tag{31}$$

with the initial condition

$$\Phi(\zeta,\chi,0) = \sqrt{\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5}, \tag{32}$$

The ZZ transformation to Eq. 31 is define as

$$\begin{aligned}
&\frac{\psi(ß)}{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)} Z[\Phi(\zeta,\chi,\tau)] - \frac{v}{s}\Phi(\zeta,\chi,0) = Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + h\Phi^{-1}(1-r\Phi)\right),\\
&\frac{\psi(ß)}{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)} Z[\Phi(\zeta,\chi,\tau)] = \frac{v}{s}\Phi(\zeta,\chi,0) + Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + h\Phi^{-1}(1-r\Phi)\right),
\end{aligned} \tag{33}$$

$$Z[\Phi(\zeta,\chi,\tau)] = \frac{v}{s}\sqrt{\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5} + \frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + h\Phi^{-1}(1-r\Phi)\right)\right]. \tag{34}$$

Applying inverse ZZ transformation of Eq. 34, we get

$$\Phi(\zeta,\chi,\tau) = \sqrt{\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5} + Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + h\Phi^{-1}(1-r\Phi)\right)\right]\right]. \tag{35}$$

Applying iterative method define in Eqs. 24 and 25 we achieved the following results of example 4.1

$$\begin{aligned}
&\Phi_0(\zeta,\chi,\tau) = \sqrt{\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5}.\\
&\Phi_1(\zeta,\chi,\tau) = Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi_0^2) + \frac{\partial^2}{\partial\chi^2}(\Phi_0^2) + h\Phi_0^{-1}(1-r\Phi_0)\right)\right]\right].
\end{aligned} \tag{36}$$

$$= h\left(\left(\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5\right)^{-\frac{1}{2}}\right)\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right], \tag{37}$$

$$\begin{aligned}
&\Phi_2(\zeta,\chi,\tau) = Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi_1^2) + \frac{\partial^2}{\partial\chi^2}(\Phi_1^2) + h\Phi_1^{-1}(1-r\Phi_1)\right)\right]\right],\\
&= -2h^2\left(\left(\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5\right)^{-\frac{3}{2}}\right)\frac{1}{(\mathrm{B}(ß))^2}\left[(1-ß)^2 + \frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)} + \frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right],
\end{aligned} \tag{38}$$

$$\Phi_3(\zeta,\chi,\tau) = Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi_2^2) + \frac{\partial^2}{\partial\chi^2}(\Phi_2^2) + h\Phi_2^{-1}(1-r\Phi_2)\right)\right]\right]. \tag{39}$$

$$= 3h^3\left(\left(\frac{hr}{4}\zeta^2 + \frac{hr}{4}\chi^2 + \chi + 5\right)^{-\frac{5}{2}}\right)\frac{1}{(\mathrm{B}(ß))^3}\left[(1-ß)^3 + \frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)} + \frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)} + \frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)} + \frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\right]. \tag{40}$$

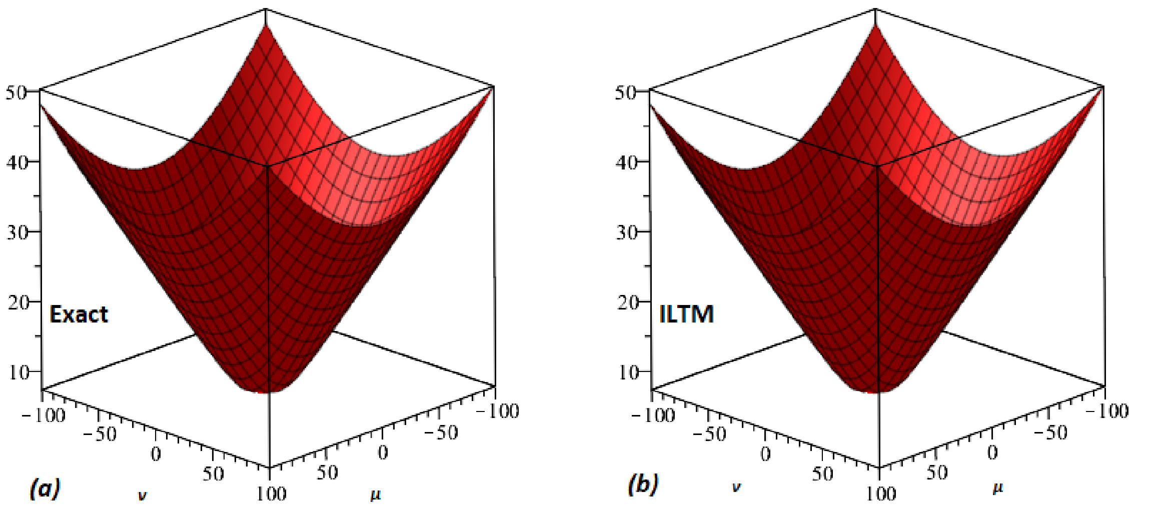

Figure 1 The first figure show the exact and second the analytical solution graph of example 4.1 at $ß = 1$, $h = 0.01$ and $r = 48$.

The series type of approximate result is provided as

$$\begin{aligned}\Phi(\zeta,\chi,\tau) &= \Phi_0(\zeta,\chi,\tau)+\Phi_1(\zeta,\chi,\tau)+\Phi_2(\zeta,\chi,\tau)+\Phi_3(\zeta,\chi,\tau)+\cdots,\\ &= (\frac{hr}{4}\zeta^2+\frac{hr}{4}\chi^2+\chi+5)^{\frac{1}{2}}+h((\frac{hr}{4}\zeta^2+\frac{hr}{4}\chi^2+\chi+5)^{-\frac{1}{2}})\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right]\\ &- 2h^2((\frac{hr}{4}\zeta^2+\frac{hr}{4}\chi^2+\chi+5)^{-\frac{3}{2}})\frac{1}{(\mathrm{B}(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right]+3h^3((\frac{hr}{4}\zeta^2+\frac{hr}{4}\chi^2\\ &+\chi+5)^{-\frac{5}{2}})\frac{1}{(\mathrm{B}(ß))^3}\left[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\right]+\cdots,\end{aligned} \tag{41}$$

The exact result is

$$\Phi(\zeta,\chi,\tau)=\sqrt{\frac{hr}{4}\zeta^2+\frac{hr}{4}\chi^2+\chi+2h\tau+5}. \tag{42}$$

4.2 | Example

Consider the biological population model is define as

$$\frac{\partial^{ß}\Phi}{\partial\tau^{ß}}=\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+h\Phi, \tag{43}$$

with initial condition

$$\Phi(\zeta,\chi,0)=\sqrt{\zeta\chi}, \tag{44}$$

ZZ transformation apply to Eq. 43 is define as

$$\begin{aligned}\frac{\psi(ß)}{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}Z[\Phi(\zeta,\chi,\tau)]-\frac{v}{s}\Phi(\zeta,\chi,0)&=Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+h\Phi),\\ \frac{\psi(ß)}{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}Z[\Phi(\zeta,\chi,\tau)]&=\frac{v}{s}\Phi(\zeta,\chi,0)+Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+h\Phi),\end{aligned} \tag{45}$$

$$Z[\Phi(\zeta,\chi,\tau)]=\frac{v}{s}\sqrt{\zeta\chi}+\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+h\Phi)\right]. \tag{46}$$

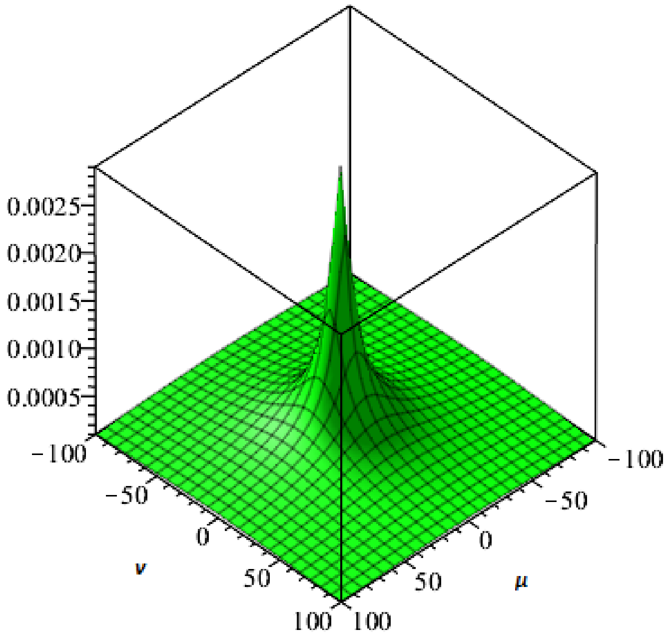

Figure 2 Example 4.1 error graph at ß = 1

Applying inverse ZZ transformation of Eq. 46

$$\Phi(\zeta,\chi,\tau) = \sqrt{\zeta\chi} + Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+h\Phi)\right]\right]. \tag{47}$$

Applying iterative method expressed in Eqs. 24 and 25, we achieved the following result of example 4.2

$$\Phi_0(\zeta,\chi,\tau) = \sqrt{\zeta\chi},$$
$$\Phi_1(\zeta,\chi,\tau) = Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_0^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_0^2)+h\Phi_0))\right]\right]. \tag{48}$$

$$= h\sqrt{\zeta\chi}\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right], \tag{49}$$

$$\Phi_2(\zeta,\chi,\tau) = Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_1^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_1^2)+h\Phi_1)\right]\right],$$
$$= h^2\sqrt{\zeta\chi}\frac{1}{(\mathrm{B}(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right], \tag{50}$$

$$\Phi_3(\zeta,\chi,\tau) = Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_2^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_2^2)+h\Phi_2)\right]\right]. \tag{51}$$

$$= h^3\sqrt{\zeta\chi}\frac{1}{(\mathrm{B}(ß))^3}\Big[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\Big], \tag{52}$$

The analytical result of series form is define as

$$\Phi(\zeta,\chi,\tau) = \Phi_0(\zeta,\chi,\tau)+\Phi_1(\zeta,\chi,\tau)+\Phi_2(\zeta,\chi,\tau)+\Phi_3(\zeta,\chi,\tau)+,...,$$
$$= \sqrt{\zeta\chi}+h\sqrt{\zeta\chi}\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right]+h^2\sqrt{\zeta\chi}\frac{1}{(\mathrm{B}(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right]+ \tag{53}$$
$$h^3\sqrt{\zeta\chi}\frac{1}{(\mathrm{B}(ß))^3}\Big[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\Big]+,\cdots,$$

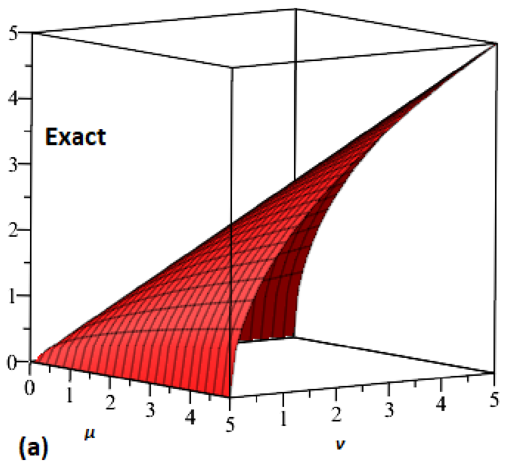

Figure 3 The first figure show that exact and second the analytical solution of example 4.2 at $ß = 1$, $h = 0.01$ and $r = 48$.

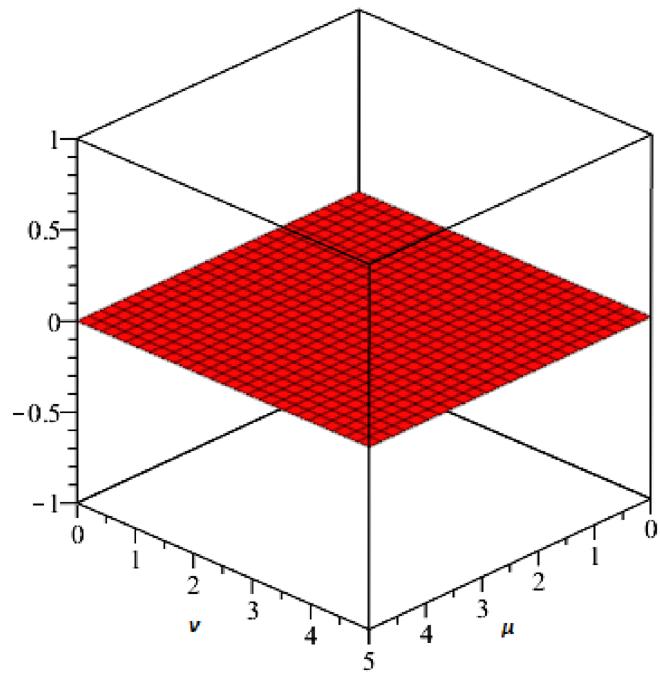

Figure 4 Example 4.2 error at $ß = 1$

The exact result is

$$\Phi(\zeta, \chi, \tau) = \sqrt{\zeta\chi} e^{h\tau}, \tag{54}$$

4.3 | Example

Consider fractional order biological population model is define as

$$\frac{\partial^{ß}\Phi}{\partial\tau^{ß}} = \frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + \Phi, \tag{55}$$

with initial condition

$$\Phi(\zeta,\chi,0)=\sqrt{\sin\zeta\sinh\chi}, \tag{56}$$

ZZ transformation apply to Eq. 55 is define as

$$\begin{aligned}
&\frac{\psi(ß)}{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}Z[\Phi(\zeta,\chi,\tau)]-\frac{v}{s}\Phi(\zeta,\chi,0)=Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+\Phi),\\
&\frac{\psi(ß)}{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}Z[\Phi(\zeta,\chi,\tau)]=\frac{v}{s}\Phi(\zeta,\chi,0)+Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+\Phi),
\end{aligned} \tag{57}$$

$$Z[\Phi(\zeta,\chi,\tau)]=\frac{v}{s}\sqrt{\zeta\chi}+\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+\Phi)\right]. \tag{58}$$

Applying inverse ZZ transformation of Eq. 58, we achieved

$$\Phi(\zeta,\chi,\tau)=\sqrt{\zeta\chi}+Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+\Phi)\right]\right]. \tag{59}$$

Applying iterative method define in Eqs. 24 and 25, we achieved the following result of example 4.3

$$\begin{aligned}
&\Phi_0(\zeta,\chi,\tau)=\sqrt{\sin\zeta\sinh\chi},\\
&\Phi_1(\zeta,\chi,\tau)=Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_0^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_0^2)+\Phi_0))\right]\right].
\end{aligned} \tag{60}$$

$$=\sqrt{\sin\zeta\sinh\chi}\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right], \tag{61}$$

$$\begin{aligned}
&\Phi_2(\zeta,\chi,\tau)=Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_1^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_1^2)+\Phi_1)\right]\right].\\
&=\sqrt{\sin\zeta\sinh\chi}\frac{1}{(B(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right],
\end{aligned} \tag{62}$$

$$\Phi_3(\zeta,\chi,\tau)=Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_2^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_2^2)+\Phi_2)\right]\right]. \tag{63}$$

$$=\sqrt{\sin\zeta\sinh\chi}\frac{1}{(B(ß))^3}\Big[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\Big]. \tag{64}$$

The analytical series type solution is define as

$$\begin{aligned}
&\Phi(\zeta,\chi,\tau)=\Phi_0(\zeta,\chi,\tau)+\Phi_1(\zeta,\chi,\tau)+\Phi_2(\zeta,\chi,\tau)+\Phi_3(\zeta,\chi,\tau)+\cdots,\\
&=\sqrt{\sin\zeta\sinh\chi}+\sqrt{\sin\zeta\sinh\chi}\frac{\tau^{ß}}{\Gamma ß+1}+\sqrt{\sin\zeta\sinh\chi}\frac{1}{(B(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right]\\
&+\sqrt{\sin\zeta\sinh\chi}\frac{1}{(B(ß))^3}\Big[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\Big]+\cdots,
\end{aligned} \tag{65}$$

The exact result is

$$\Phi(\zeta,\chi,\tau)=\sqrt{\sin\zeta\sinh\chi}e^{\tau}, \tag{66}$$

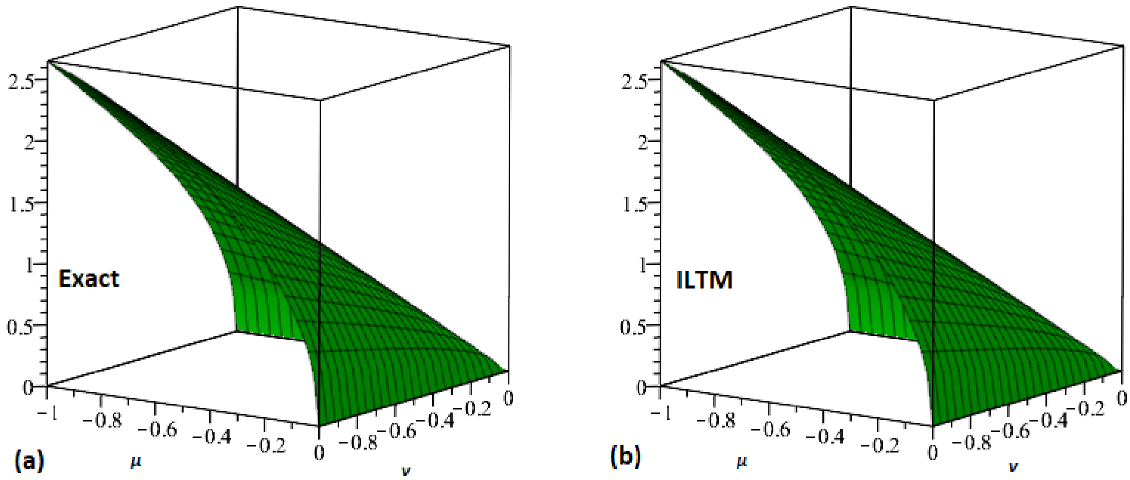

Figure 5 The first graph of exact and second analytical solution graph of example 4.3 at $ß = 1$ $h = 0.01$ and $r = 48$.

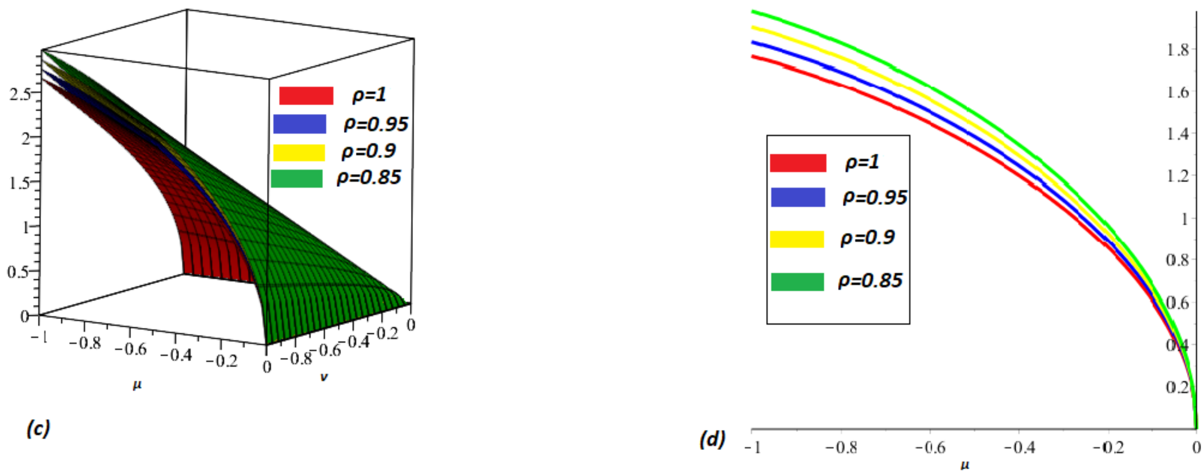

Figure 6 The various fractional-order of ß at example **4.3**

4.4 | Example

Consider the fractional order biological population model is define as

$$\frac{\partial^{ß}\Phi}{\partial\tau^{ß}} = \frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + \Phi(1 - r\Phi), \tag{67}$$

with initial condition

$$\Phi(\zeta, \chi, 0) = \exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}. \tag{68}$$

ZZ transformation apply to Eq. 67 is define as

$$\begin{aligned} &\frac{\psi(ß)}{\left(1 - ß + ß\left(\frac{v}{s}\right)^{ß}\right)} Z[\Phi(\zeta,\chi,\tau)] - \frac{v}{s}\Phi(\zeta,\chi,0) = Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + \Phi(1 - r\Phi)\right), \\ &\frac{\psi(ß)}{\left(1 - ß + ß\left(\frac{v}{s}\right)^{ß}\right)} Z[\Phi(\zeta,\chi,\tau)] = \frac{v}{s}\Phi(\zeta,\chi,0) + Z\left(\frac{\partial^2}{\partial\zeta^2}(\Phi^2) + \frac{\partial^2}{\partial\chi^2}(\Phi^2) + \Phi(1 - r\Phi)\right), \end{aligned} \tag{69}$$

$$Z[\Phi(\zeta,\chi,\tau)]=\frac{v}{s}\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}+\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+\Phi(1-r\Phi))\right]. \tag{70}$$

Applying inverse ZZ transformation of Eq. 70, we get

$$\Phi(\zeta,\chi,\tau)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}+Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi^2)+\frac{\partial^2}{\partial\chi^2}(\Phi^2)+\Phi(1-r\Phi))\right]\right]. \tag{71}$$

Applying iterative method define in Eqs. 24 and 25, we achieve the following result of example 4.4

$$\begin{aligned}\Phi_0(\zeta,\chi,\tau)&=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)},\\ \Phi_1(\zeta,\chi,\tau)&=Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_0^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_0^2)+\Phi_0(1-r\Phi_0)))\right]\right].\end{aligned} \tag{72}$$

$$=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right], \tag{73}$$

$$\begin{aligned}\Phi_2(\zeta,\chi,\tau)&=Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_1^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_1^2)+\Phi_1(1-r\Phi_1))\right]\right],\\ &=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}\frac{1}{(\mathrm{B}(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right],\end{aligned} \tag{74}$$

$$\Phi_3(\zeta,\chi,\tau)=Z^{-1}\left[\frac{\left(1-ß+ß\left(\frac{v}{s}\right)^{ß}\right)}{\psi(ß)}\left[Z(\frac{\partial^2}{\partial\zeta^2}(\Phi_2^2)+\frac{\partial^2}{\partial\chi^2}(\Phi_2^2)+\Phi_2(1-r\Phi_2))\right]\right], \tag{75}$$

$$=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}\frac{1}{(\mathrm{B}(ß))^3}\Big[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\Big]. \tag{76}$$

The analytical series form solution is define as

$$\begin{aligned}\Phi(\zeta,\chi,\tau)&=\Phi_0(\zeta,\chi,\tau)+\Phi_1(\zeta,\chi,\tau)+\Phi_2(\zeta,\chi,\tau)+\Phi_3(\zeta,\chi,\tau)+\cdots,\\ &=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}+\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}\frac{1}{\psi(ß)}\left[1-ß+\frac{ß\tau^{ß}}{\Gamma(ß+1)}\right]+\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}\frac{1}{(\mathrm{B}(ß))^2}\left[(1-ß)^2+\frac{2ß(1-ß)\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2\tau^{2ß}}{\Gamma(2ß+1)}\right]+\\ &\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)}\frac{1}{(\mathrm{B}(ß))^3}\Big[(1-ß)^3+\frac{3ß(1-ß)^2\tau^{ß}}{\Gamma(ß+1)}+\frac{ß^2(1-ß)\tau^{2ß+1}}{\Gamma(2ß+2)}+\frac{2ß^2(1-ß)\tau^{2ß}}{\Gamma(2ß+1)}+\frac{ß^3\tau^{2ß+1}}{\Gamma(2ß+2)}\Big]+\cdots,\end{aligned} \tag{77}$$

The exact result is

$$\Phi(\zeta,\chi,\tau)=\exp^{\frac{1}{2}\sqrt{\frac{r}{2}}(\zeta+\chi)+\tau}, \tag{78}$$

5 | GRAPHICAL REPRESENTATION

The present work employs iterative transform method to solve several significant non-integer order biological models. The proposed technique's answer is to illustrate through its graphical depiction. The solution graphs for example 4.1 at $ß=1$ are displayed in Figure 1. It is established that the iterative transform method solution is highly similar to the precise solution. The error analysis of iterative transform method for example 4.1 is discussed in Figure 2. It is observed that the considered technique has a sufficient degree of accuracy. Similarly, picture 3 illustrates the solution-plot of iterative transform method and the actual solution for example 4.2. These solution-graphs are extremely close, confirming the proposed technique's reliability. Additionally, as illustrated in Figure 4, a higher degree of precision is achieved. In Figure 5, the actual and ILTM findings for Example 4.3 are examined. Both the precise and iterative transform method solutions are identical, demonstrating the proposed method's dependability. In Figure 6,

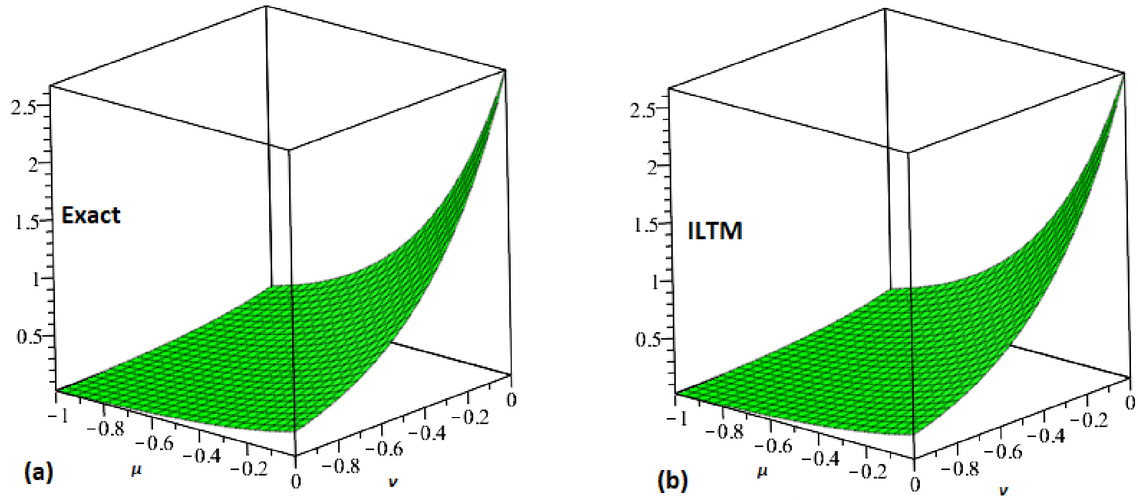

Figure 7 The first graph of exact and second graph of analytical solution of example 4.4 at ß $= 1$, $h = 0.01$ and $r = 48$.

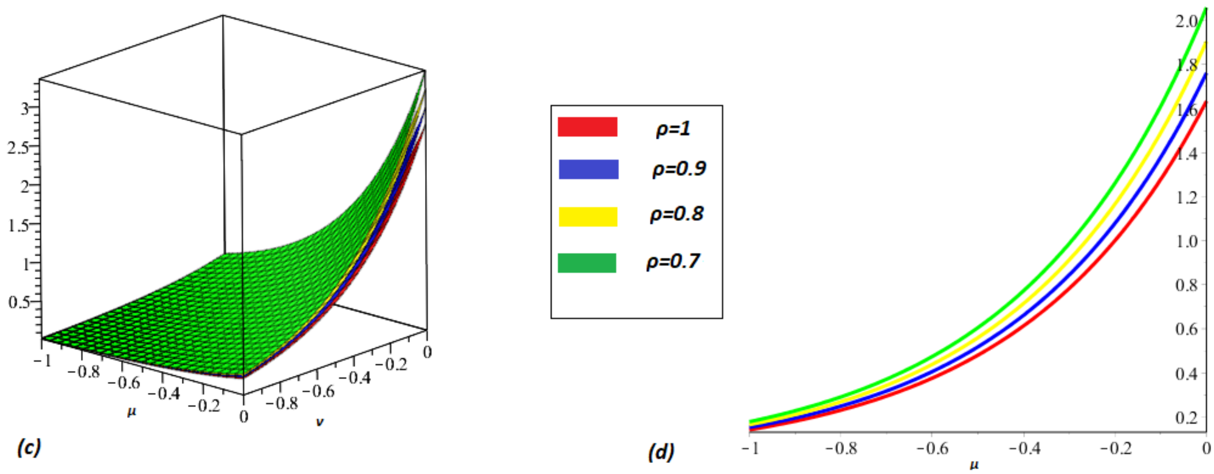

Figure 8 The various fractional order of ß at example **4.4**

the solution to case 4.3 is determined at different fractional-orders. The hypothesis is tested that when fractional-order solutions approach integer-order solutions, fractional-order solutions convergence to integer-order solutions. The precise and iterative transform method solutions to example 4.4 are depicted graphically in Figure 7. Figure 7 illustrates the answer to Example 4 at various fractional-orders. Figure 8 illustrates the convergence of solutions with varying fractional-order.

6 | CONCLUSION

The current study is about solving fractional-order biological population models with an effective analytical method. For integer and fractional-order models, the current technique is used. Plotted are the solution figures of iterative transform method and the exact results to the issues. It is evaluated whether the iterative transform method results are in strong agreement with the current technique's actual solutions. The iterative transform method solutions to the problems at various fractional-orders are also depicted graphically. The phenomenon of fractional-order results convergent to integer-order results is noticed. This behaviour of the generated solution validates the efficacy of the proposed approach. Because of its simple and effective application, the proposed technique can be adjusted for the solution of various fractional partial differential equations that arise in sciences and engineering.

CONFLICTS OF INTEREST

The authors declare that there are no conflicts of interest regarding the publication of this article.

FUNDING

This work was funded by the Deanship of Scientific Research at Jouf University under grant No (DSR-2021-03-03184).

DATA AVAILABILITY

The numerical data used to support the findings of this study are included within the article.

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How to cite this article: M. Benhamed, A. Alshehri, M. Naeem, and R. Shah (2022), Fractional analysis of biological population model via novel transform, *Math. Methods Appl. Sci.*, *2022;00:1–6.*