The superradiant stability of dyonic RN-like black holes

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Abstract

The effect of magnetic fields on black hole superradiance is an interesting topic with possible astrophysical applications. A dyonic RN-like black hole is not asymptotically flat, it describes a black hole immersed in an asymptotically uniform magnetic field. In this paper, we discuss the superadditive stability of a class of asymptotically flat, band-like black holes, the binary RN black holes. In this article, we introduce the above condition into dyonic RN-like black holes. If a dyonic RN-like black hole satisfies the condition of μ = yω, when μ [?] [?] 2(mH + qΦH),so the dyonic RN-like black hole is superradiantly stable at that time.

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Keywords: superradiantly stable, a new variable y, dyonic RN-like black hole

1. INTRODUCTION

The research on the stability of black holes can be traced back to 1957 when Regge and Wheeler found that Schwarzschild black holes are stable under small perturbations of the metric. In 1970, Zerilli further studied Schwarzschild black holes and RN black holes and reduced the perturbation problem to a process of solving the Schrödinger-like equation (wave equation) [1–4]. In 1972 Teukolsky studied the perturbations of various matter fields (gravitational, electromagnetic, neutrino fields) in Kerr space-time and decoupled the field equations into independent wave equations[5, 6], laying the foundation for the study of the external field disturbance of black holes. In 1983, Chandrasekhar's "Mathematical Theory of Black Holes" systematically expounded the perturbation theory of black holes. Superradiance is essentially the process of radiation enhancement, which plays an important role in optics, quantum mechanics, especially relativity, and astrophysics. Dicke, who coined the term "superradiance" in the context of quantum optics coherent emission [7], achieved the first high-resolution superradiance measurements using coherent synchrotron radiation [7, 8]. Zeldovich believed that the dissipative rotating body amplifies the human radiation, and Starobinsky recognized the super-radiation phenomenon of black holes on his basis: when the frequency of the human radiation satisfies the super-radiation condition, The rotational energy can be extracted from the black hole [7, 8]. Black hole superradiation is closely related to the black hole area theorem, the Penrose process, tidal forces, and even Hawking radiation [9]. In the general theory of relativity, the superradiation of a black hole is to extract energy, charge, and angular momentum in a vacuum[9, 10]. It can be known from the scattering problem of root quantum mechanics: the plane wave whose eigenfrequency is ω moves toward the center of the black hole, and is scattered to infinity under the action of the black hole, and the scattered particles obey a certain angular distribution. Taking the scalar wave under the background of static spherical symmetry as an example, we will see, at this time, the scalar field satisfies the Schrödinger-like equation of the following form:

$$\frac{d^2\psi_{lm}}{dx^2} + V_{\text{eff}}\,\psi_{lm} = 0, \tag{1}$$

where ψ_{lm} is the radial component of the field after decomposing the variables, x is the turtle coordinate, and V_{eff} depends on the theoretical model and the space-time background. In the case of spherical symmetry, we consider the scattering of monochromatic plane waves. Assuming that V_{eff} is constant on the boundary, the asymptotic solution satisfies

$$\psi_{lm} \sim \begin{cases} \mathcal{T}e^{-ik_H r}, & r \to r_+ \\ \mathcal{I}e^{-ik_\infty r} + \mathcal{R}e^{ik_\infty r}, & r \to \infty, \end{cases} \tag{2}$$

in

$$\begin{aligned} k_H &= \omega - \omega_c, \\ k_\infty &= \sqrt{\omega^2 - \mu^2}. \end{aligned} \tag{3}$$

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ω_c is the critical frequency, for a charged and rotating black hole, the critical frequency is

$$\omega_c = q\Phi_H + m\Omega_H = \frac{qQr_p + ma}{r_p^2 + a^2}, \tag{4}$$

where r_p is the event horizon of the black hole (namely the outer horizon), Ω_H is the angular velocity of the black hole and the electric potential at the event horizon, m is the magnetic quantum number of the scalar field, q is the charge of the scalar field. When the black hole is not rotating, the critical frequency degenerates to

$$\omega_c = q\Phi_H = \frac{qQ}{r_p}. \tag{5}$$

$$W \equiv \begin{vmatrix} \psi & \psi^* \\ \psi' & \psi^{*\prime} \end{vmatrix} = \psi\psi^{*\prime} - \psi^*\psi' \tag{6}$$

can be obtained at infinity

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Since the Lansky determinant is a constant, we have

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It can be found that when $\frac{k_H}{k_\infty} > 0, |R|^2 < |I|^2$, the reflected wave amplitude is smaller than the human radiation wave amplitude, and the energy of the scalar field decreases; when $\frac{k_H}{k_\infty} < 0, |R|^2 > |I|^2$, the amplitude of the reflected wave is greater than the amplitude of the human radiation, and the energy of the scalar field increases at this time. Therefore, the superradiance generation condition is the increased condition of the scalar field energy, that is,

$$0 < \omega < \omega_c. \tag{10}$$

In the above equation, the critical angular frequency ω_c is defined as

$$\omega_c = q\Psi, \tag{11}$$

where Ψ is the electromagnetic potential of the outer horizon of the dyonic RN black hole, $\Psi = Q/r_+$. The superradiant condition for an electrically charged massive scalar perturbation on the dyonic RN black hole background is

$$\omega < \omega_c = \frac{qQ}{r_+}. \tag{12}$$

The bound state condition at spatial infinity for the scalar perturbation is

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When the radiation wave with the frequency ω satisfies the formula, superradiance scattering occurs, at this time, the reflected wave carries more energy than the human radiation wave, which is the superradiation occurrence condition of the charged black hole in the general theory of relativity. It is worth noting that if the black hole is not charged, that is, the Schwarzschild black hole, there is no super-radiation phenomenon, and only the rotating black hole or the charged black hole has the super-radiation phenomenon[8–10].

Hod proved[10] that the Kerr black hole should be superradiant stable under massive scalar perturbation when $\mu \geq \sqrt{2}m\Omega_H$, where μ is the mass.

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The structure of this paper is as follows. In Section 2, we describe dyonic RN-like black holes' new class of action and field equations. In Section 3, we derive the radial equation of motion and effective potential. In Section 4, we carefully analyze the effective potential's shape and obtain the system's superradiant stability parameter region. Section 5 is dedicated to the conclusion.

2. NEW CLASS OF ACTION AND FIELD EQUATIONS

We set the system to be in the interval from 0 to 1. Since the interval from 0 to 1 can be mapped to the interval from 0 to infinity, the size sequence this paper discusses is unchanged. The purpose of this theory is that find that the modified Einstein gravitational equation has a Reissner-Nordstrom solution in a vacuum. First, we can consider the following equation (modified Einstein's gravitational equation).

The proper time of spherical coordinates is[13](The metric which is in exponential form)

$$ds^2 = -e^{G(t,r)}dt^2 + e^{-G(t,r)}dr^2 + \left[r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \tag{14}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda(\left(g^{\theta\theta}\right)^2)g_{\mu\nu} = -\frac{8\pi G}{C^4}T_{\mu v} \tag{15}$$

In this work, the action(we set $8G = c = 1$) is given by the following relation which in the special case, reduces to the Einstein-Maxwell dilaton gravity:[10]

$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(-2\Lambda(\left(g^{\theta\theta}\right) + R) \tag{16}$$

where Λ is a function of the Ricci scalar R and Φ is the representation of the dilatonic field, also similar to f(R)(We will now consider non-pathological functional forms of $f(R)$ that can be expanded in a Taylor series of the form $f(R) = a_0 + R + a_2R^2 + a_3R^3 + \ldots a_nR^n + \ldots$ where we have normalized all coefficients concerning the coefficient of the linear term). Variation of the action for the metric $g_{\mu\nu}$, the gauge A_μ and dilaton field Φ gives the following field equations:

This leads to:

$$\frac{1}{2}R\Lambda'(R) - \Lambda = 0. \tag{17}$$

$$\begin{gathered} R_{\mathrm{kl}}\Lambda'(R) - \frac{1}{2}g_{kl}\Lambda = 0 \\ \nabla_\sigma\left[\sqrt{-g}\Lambda'(R)g^{\mu\nu}\right] = 0. \end{gathered} \tag{18}$$

In this relationship, we get

$$\mathbf{\Lambda} = B(\mathbf{p}\times\mathbf{r})/r^4, \tag{19}$$

B is an algebraic parameter, and p is a momentum or momentum operator.

A dyonic-like RN black hole is a stationary spherically symmetric space-time geometry, which is the solution of the Einstein-Maxwell theory [14]. Using spherical coordinates (t, r, θ, ϕ), the line element can be expressed as (we use natural units, where $G = c = \hbar = 1$).We set up a geometric entity, and B takes a certain value for the parametric algebra so that the following formula holds.

$$ds^2 = -\frac{\square}{r^2}\mathrm{d}t^2 + \frac{r^2}{\square}\mathrm{d}r^2 + r^2\ \mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \tag{20}$$

Where

$$\square = -2Mr + r^2 + Q^2 + B^2, \tag{21}$$

M is the mass of the black hole, and Q and B are the electric and magnetic charges of the black hole, respectively. The dynamic RN black hole has an outer horizon in r_+ and an inner horizon in r_-,

$$r_+ = M + \sqrt{M^2 - Q^2 - B^2}, \quad r_- = M - \sqrt{M^2 - Q^2 - B^2}. \tag{22}$$

They satisfy the following relation

$$\square = (r - r_+)(r - r_-), \quad r_+r_- = Q^2 + B^2, \quad r_+ + r_- = 2M. \tag{23}$$

The equation of motion of the charged massive scalar perturbation Φ in the dynamic RN black hole background is described by the covariant Klein-Gordon (KG) equation

$$\left(D^\nu D_\nu - \mu^2\right)\Phi = 0, \tag{24}$$

where $D^{\nu} = \nabla^{\nu} - iqA^{\nu}$ and $D_{\nu} = \nabla_{\nu} - iqA_{\nu}$ are covariant Derivatives, q and μ are the charge and mass of the scalar field, respectively. The electromagnetic field of a dynamic black hole is described by the following vector potential

$$A_{\nu} = \left(-\frac{Q}{r}, 0, 0, B(\cos\theta \mp 1)\right), \tag{25}$$

The upper minus sign applies to the northern hemisphere of the black hole, and the lower plus sign applies to the southern hemisphere.

The solution of the KG equation can be decomposed into the following form

$$\Phi(t, r, \theta, \phi) = R(r)Y(\theta)e^{im\phi}e^{-i\omega t}, \tag{26}$$

where ω is the angular frequency of the scalar perturbation and m is the azimuthal harmonic index. $Y(\theta)$ is the angular part of the solution and $R(r)$ is the radial part of the solution. Substituting the above solution into the KG equation, we can get the radial and angular parts of the equation of motion. Considering the different electromagnetic potentials in the northern and southern hemispheres, the equation of motion angle is discussed below in two cases.

3. THE RADIAL EQUATION OF MOTION AND EFFECTIVE POTENTIAL

A new radial wave function is defined as[11–14]

$$\psi_{lm} \equiv \mathit{\Delta}^{\frac{1}{2}} R_{lm}. \tag{27}$$

to substitute the radial equation of motion for a Schrodinger-like equation

$$\frac{d^2\Psi_{lm}}{dr^2} + (\omega^2 - V)\Psi_{lm} = 0, \tag{28}$$

where

$$\omega^2 - V = \frac{U + M^2 - a^2 - Q^2}{\mathit{\Delta}^2}, \tag{29}$$

in which V denotes the effective potential.

Considering the superradiation condition, i.e. $\omega < \omega_c$, and the bound state condition, when the potential is captured, the Kerr-Newman black hole and the charged massive scalar perturbation system are superradiation stable. Beyond the outer event horizon of a Kerr-Newman black hole, it doesn't exist. Therefore, the shape of the effective potential V is next analyzed to investigate the presence of trapping wells.

The asymptotic behaviors of the effective potential V around the inner and outer horizons and at spatial infinity can be expressed as

$$V(r \to +\infty) \to \mu^2 - \frac{2(2M\omega^2 - qQ\omega - M\mu^2)}{r} + \mathcal{O}(\frac{1}{r^2}), \tag{30}$$

$$V(r \to r_+) \to -\infty, \quad V(r \to r_-) \to -\infty. \tag{31}$$

If a Kerr black hole satisfies the condition of $\mu = y\omega$, it will be superradiantly stable when $\mu < \sqrt{2}m\Omega_H$. In this article, we introduce the above condition into Kerr-Newman black holes. Therefore, the formula of the asymptotic behaviors is written as

$$V(r \to +\infty) \to y^2\omega^2 - \frac{2[M(2 - y^2)\omega^2 - qQ\omega]}{r} + \mathcal{O}(\frac{1}{r^2}), \tag{32}$$

$$V(r \to r_+) \to -\infty, \quad V(r \to r_-) \to -\infty. \tag{33}$$

It is concluded from the equations above that the effective potential approximates a constant at infinity in space, and the extreme between its inner and outer horizons cannot be less than one. The asymptotic behavior of the derivative of the effective potential V at spatial infinity can be expressed as

$$V'(r \to +\infty) \to \frac{2[M(2 - y^2)\omega^2 - qQ\omega]}{r^2} + \mathcal{O}(\frac{1}{r^3}), \tag{34}$$

The derivative of the effective potential has to be negative to satisfy the no trapping well condition,

$$2M(2 - y^2)\omega^2 - 2Qq\omega < 0. \tag{35}$$

4. ANALYSIS OF SUPERRADIANT STABILITY

In this section, we will find the regions in the parameter space where the system of dyonic RN black hole and massive scalar perturbation is superradiantly stable. We determine the parameter regions by considering the extremes of the effective potential in the range $r_- < r < +\infty$

Now, we define a new variable $z, z = r - r_-$. The expression of the derivative of the effective potential V is

$$\begin{aligned} V'(r) = & \frac{-2\left(ar^4 + br^3 + cr^2 + dr + e\right)}{\triangle^3} \\ = V'(z) = & \frac{-2\left(a_1 z^4 + b_1 z^3 + c_1 z^2 + d_1 z + e_1\right)}{\triangle^3}, \end{aligned} \tag{36}$$

where

$$\begin{gathered} a_1 = a; \quad b_1 = (4r_-)\, a_1 + b, \\ c_1 = \left(6r_-^2\right) a_1 + (3r_-)\, b_1 + c, \\ d_1 = \left(4r_-^3\right) a_1 + \left(3r_-^2\right) b_1 + (2r_-)\, c_1, \\ e_1 = \left(r_-^2\right) a_1 + \left(r_-^3\right) b_1 + \left(r_-^2\right) c_1 + (r_-)\, d_1 + e. \end{gathered} \tag{37}$$

Explicitly,

$$\begin{aligned} a_1 = & -2M\omega^2 + qQ\omega + M\mu^2, \\ b_1 = & -2\left(8M^2 - 6Mr_+ + r_+^2\right)\omega^2 + 2qQ\,(5M - 2r_+)\,\omega \\ & + \mu^2\left(6M^2 - 6Mr_+ + r_+^2\right) - q^2\left(Q^2 + B^2\right) + \lambda, \\ c_1 = & -6\,(2M - r_+)^3\,\omega^2 + 9qQ\,(2M - r_+)^2\,\omega \\ & + 3\left((M - r_+)\left(\mu^2\,(2M - r_+)^2 - q^2Q^2 - q^2B^2 + \lambda\right) - Mq^2Q^2\right), \end{aligned} \tag{38}$$

$$\begin{aligned} d_1 = & -2\,(4M - 3r_+)\,(2M - r_+)^3\,\omega^2 + 2qQ\,(7M - 5r_+)\,(2M - r_+)^2\,\omega \\ & + 2q^2\left(-B^2\left(M^2 - 5Q^2\right) + 4Q^4 + B^4\right) - 2q^2Q^2\,(3M\,(r_+ - 2M) \\ & + 2\mu^2\left(2M^2 - 3Mr_+ + r_+^2\right)^2 + 2\left(Q^2 + B^2\right)\Big) - 12Mq^2Q^2 r_- \\ & + 2\,(M - r_+)^2\,(\lambda - 1) \\ e_1 = & (r_+ - r_-)\,(qQ - \omega r_-)^2\, r_-^2 + \frac{1}{4}\,(r_+ - r_-)^3, \end{aligned} \tag{39}$$

where $\lambda = l(l+1), \quad l > qB$[14].Since we set the system to range from 0 to 1, $qB > q^2Q^2$.

In this paper, we denote the numerator of the derivative of the effective potential $V'(z)$. This quartic polynomial of z allows us to study the existence of trapped wells beyond the horizon by analyzing the properties of the roots of the equation. We use z_1, z_2, z_3 and z_4 to represent the four roots of $f_1(z) = 0$. The relationship between them conforms to Vieta's theorem.

$$z_1 z_2 z_3 z_4 = \frac{e_1}{a_1}, z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 = \frac{c_1}{a_1}. \tag{40}$$

When $z > 0$, from the asymptotic behavior of the effective potential of the inner and outer horizons and space infinity, it can be inferred that the equation $V'(z) = 0$(or $f_1(z) = 0$) cannot be less on two. So the two positive roots are written as z_1, z_2.

Research shows that for any ω

$$e_1 > 0. \tag{41}$$

and in

$$e_1 > 0, \quad c_1 < 0, \tag{42}$$

$f_1(z) = 0$, that is, z_3, z_4 are all negative numbers.

When $y^2 > 2(a_1 > 0)$ for $e_1 > 0$, $c_1 < 0$ at this time, then $f(\omega) < 0$, and we can know that the equation $\mathrm{V_1{}'(z)} = 0$ cannot have more than two positive roots. So the dyonic RN-like black hole is superradiantly stable at that time.

5. SUMMARY

In this paper, we introduce $\mu = y\omega$[12, 13] into dyonic RN-like black holes and discuss the superradiation stability of dyonic RN-like black holes. We adopt the method of variable separation to divide the motion equations of the least coupled scalar perturbation in dynamical RN black holes into two forms: angular and radial.

Hod proved [10] that when $\mu \geq \sqrt{2}m\Omega_H$ (where μ is the mass), Kerr black holes should be superradiantly stable under large-scale scalar perturbations. In this post, a new variable y is added here to extend the results of the previous post.

When $\mu \geq \sqrt{2}(q\Phi_H)$, this dyonic RN-like black hole was superradiantly stable at that time.

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The effect of magnetic fields on black hole superradiance is an interesting topic with possible astrophysical applications. A dyonic RN-like black hole is not asymptotically flat, it describes a black hole immersed in an asymptotically uniform magnetic field. In this paper, we discuss the superadditive stability of a class of asymptotically flat, band-like black holes, the binary RN black holes. In this article, we introduce the above condition into dyonic RN-like black holes. If a dyonic RN-like black hole satisfies the condition of $\mu = y\omega$[11, 12], when $\mu \geq \sqrt{2}(q\varPhi_H)$, so the dyonic RN-like black hole is superradiantly stable at that time.

The structure of this paper is as follows. In Section 2, we describe dyonic RN-like black holes' new class of action and field equations. In Section 3, we derive the radial equation of motion and effective potential. In Section 4, we carefully analyze the shape of the effective potential and obtain the superadditive stability parameter region of the system. Section 5 is dedicated to the conclusion.

2. NEW CLASS OF ACTION AND FIELD EQUATIONS

We set the system to be in the interval from 0 to 1. Since the interval from 0 to 1 can be mapped to the interval from 0 to infinity, the size sequence of the system value remains unchanged. The purpose of this theory is that find that the modified Einstein gravitational equation has a Reissner-Nordstrom solution in a vacuum. First, we can consider the following equation (modified Einstein's gravitational equation).

The proper time of spherical coordinates is[13](The metric which is in exponential form)

$$ds^2 = -e^{G(t,r)}dt^2 + e^{-G(t,r)}dr^2 + \left[r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2\right] \tag{14}$$

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda(\left(g^{\theta\theta}\right)^2)g_{\mu\nu} = -\frac{8\pi G}{C^4}T_{\mu v} \tag{15}$$

In this work, the action(we set $8G = c = 1$) is given by the following relation which in the special case, reduces to the Einstein-Maxwell dilaton gravity:[10]

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where Λ is a function of the Ricci scalar R and Φ is the representation of the dilatonic field, also similar to f(R)(We will now consider non-pathological functional forms of $f(R)$ that can be expanded in a Taylor series of the form $f(R) = a_0 + R + a_2R^2 + a_3R^3 + \dots a_nR^n + \dots$ where we have normalized all coefficients with respect to the coefficient of the linear term). Variation of the action with respect to the metric $g_{\mu\nu}$, the gauge A_μ and dilaton field Φ gives the following field equations:

This leads to:

$$\frac{1}{2}R\Lambda'(R) - \Lambda = 0 \tag{17}$$

$$\begin{gathered} R_{\mathrm{kl}}\Lambda'(R) - \frac{1}{2}g_{kl}\Lambda = 0 \\ \nabla_\sigma\left[\sqrt{-g}\Lambda'(R)g^{\mu\nu}\right] = 0 \end{gathered} \tag{18}$$

In this relation, y and z are just two integration constants and are assumed to be positive from avoiding non-physical ambiguity.

$$\mathbf{\Lambda} = B(\mathbf{p}\times\mathbf{r})/r^4 \tag{19}$$

B is an algebraic parameter, and p is a momentum or momentum operator.

A dyonic-like RN black hole is a stationary spherically symmetric space-time geometry, which is the solution of the Einstein-Maxwell theory [14]. Using spherical coordinates (t, r, θ, ϕ), the line element can be expressed as (we use natural units, where $G = c = \hbar = 1$).We set up a geometric entity, and B takes a certain value for the parametric algebra so that the following formula holds.

$$ds^2 = -\frac{\square}{r^2}\mathrm{dt}^2 + \frac{r^2}{\square}\mathrm{dr}^2 + r^2\ \mathrm{d}\theta^2 + r^2\sin^2\theta\mathrm{d}\phi^2, \tag{20}$$

Where

$$\square = -2Mr + r^2 + Q^2 + B^2, \tag{21}$$

M is the mass of the black hole, Q_e and B are the electric and magnetic charges of the black hole, respectively. The dynamic RN black hole has an outer horizon in r_+ and an inner horizon in r_-,

$$r_+ = M + \sqrt{M^2 - Q^2 - B^2}, \quad r_- = M - \sqrt{M^2 - Q^2 - B^2}. \tag{22}$$

Obviously, they satisfy the following relation

$$\square = (r - r_+)(r - r_-), \quad r_+r_- = Q^2 + B^2, \quad r_+ + r_- = 2M. \tag{23}$$

The equation of motion of the charged massive scalar perturbation Φ in the dynamic RN black hole background is described by the covariant Klein-Gordon (KG) equation

$$\left(D^{\nu} D_{\nu} - \mu^2\right) \Phi = 0, \tag{24}$$

where $D^{\nu} = \nabla^{\nu} - iqA^{\nu}$ and $D_{\nu} = \nabla_{\nu} - iqA_{\nu}$ are covariant Derivatives, q and μ are the charge and mass of the scalar field, respectively. The electromagnetic field of a dynamic black hole is described by the following vector potential

$$A_{\nu} = \left(-\frac{Q}{r}, 0, 0, B(\cos\theta \mp 1)\right), \tag{25}$$

The upper minus sign applies to the northern hemisphere of the black hole, and the lower plus sign applies to the southern hemisphere.

The solution of the KG equation can be decomposed into the following form

$$\Phi(t, r, \theta, \phi) = R(r) Y(\theta) e^{im\phi} e^{-i\omega t}, \tag{26}$$

where ω is the angular frequency of the scalar perturbation and m is the azimuthal harmonic index. $Y(\theta)$ is the angular part of the solution and $R(r)$ is the radial part of the solution. Substituting the above solution into the KG equation, we can get the radial and angular parts of the equation of motion. Considering the different electromagnetic potentials in the northern and southern hemispheres, the equation of motion angle is discussed below in two cases.

3. THE RADIAL EQUATION OF MOTION AND EFFECTIVE POTENTIAL

A new radial wave function is defined as[11–14]

$$\psi_{lm} \equiv \Delta^{\frac{1}{2}} R_{lm}. \tag{27}$$

to substitute the radial equation of motion for a Schrodinger-like equation

$$\frac{d^2 \Psi_{lm}}{dr^2} + (\omega^2 - V)\Psi_{lm} = 0, \tag{28}$$

where

$$\omega^2 - V = \frac{U + M^2 - a^2 - Q^2}{\Delta^2}, \tag{29}$$

in which V denotes the effective potential.

Considering the superadditive condition, i.e. $\omega < \omega_c$, and the bound state condition, when the potential is captured, the Kerr-Newman black hole and the charged massive scalar perturbation system are superadditive stables. Beyond the outer event horizon of a Kerr-Newman black hole, it doesn't exist. Therefore, the shape of the effective potential V is next analyzed to investigate the presence of trapping wells.

The asymptotic behaviors of the effective potential V around the inner and outer horizons and at spatial infinity can be expressed as

$$V(r \to +\infty) \to \mu^2 - \frac{2(2M\omega^2 - qQ\omega - M\mu^2)}{r} + \mathcal{O}(\frac{1}{r^2}), \tag{30}$$

$$V(r \to r_+) \to -\infty, \quad V(r \to r_-) \to -\infty. \tag{31}$$

If a Kerr black hole satisfy the condition of $\mu = y\omega$, it will be superradiantly stable when $\mu < \sqrt{2} m \Omega_H$. In this article, we introduce the above condition into Kerr-Newman black holes. Therefore, the formula of the asymptotic behaviors is written as

$$V(r \to +\infty) \to y^2\omega^2 - \frac{2[M(2 - y^2)\omega^2 - qQ\omega]}{r} + \mathcal{O}(\frac{1}{r^2}), \tag{32}$$

$$V(r \to r_+) \to -\infty, \quad V(r \to r_-) \to -\infty. \tag{33}$$

It is concluded from the equations above that the effective potential approximates a constant at infinity in space, and the extreme between its inner and outer horizons cannot be less than one. The asymptotic behavior of the derivative of the effective potential V at spatial infinity can be expressed as

$$V'(r \to +\infty) \to \frac{2[M(2-y^2)\omega^2 - qQ\omega]}{r^2} + \mathcal{O}(\frac{1}{r^3}), \tag{34}$$

The derivative of the effective potential has to be negative in order to satisfy the no trapping well condition,

$$2M(2-y^2)\omega^2 - 2Qq\omega < 0. \tag{35}$$

4. ANALYSIS OF SUPERRADIANT STABILITY

In this section, we will find the regions in the parameter space where the system of dyonic RN black hole and massive scalar perturbation is superradiantly stable. We determine the parameter regions by considering the extremes of the effective potential in the range $r_- < r < +\infty$

Now, we define a new variable $z, z = r - r_-$. The expression of the derivative of the effective potential V is

$$\begin{aligned} V'(r) &= \frac{-2\left(ar^4 + br^3 + cr^2 + dr + e\right)}{\triangle^3} \\ = V'(z) &= \frac{-2\left(a_1 z^4 + b_1 z^3 + c_1 z^2 + d_1 z + e_1\right)}{\triangle^3}, \end{aligned} \tag{36}$$

where

$$\begin{gathered} a_1 = a; \quad b_1 = (4r_-)\, a_1 + b, \\ c_1 = \left(6r_-^2\right) a_1 + (3r_-)\, b_1 + c, \\ d_1 = \left(4r_-^3\right) a_1 + \left(3r_-^2\right) b_1 + (2r_-)\, c_1, \\ e_1 = \left(r_-^2\right) a_1 + \left(r_-^3\right) b_1 + \left(r_-^2\right) c_1 + (r_-)\, d_1 + e. \end{gathered} \tag{37}$$

Explicitly,

$$\begin{aligned} a_1 =& -2M\omega^2 + qQ\omega + M\mu^2, \\ b_1 =& -2\left(8M^2 - 6Mr_+ + r_+^2\right)\omega^2 + 2qQ\left(5M - 2r_+\right)\omega \\ &+ \mu^2\left(6M^2 - 6Mr_+ + r_+^2\right) - q^2\left(Q^2 + B^2\right) + \lambda, \\ c_1 =& -6\left(2M - r_+\right)^3\omega^2 + 9qQ\left(2M - r_+\right)^2\omega \\ &+ 3\Big(\left(M - r_+\right)\Big(\mu^2\left(2M - r_+\right)^2 - q^2Q^2 - q^2B^2 + \lambda\Big) - Mq^2Q^2\Big), \end{aligned} \tag{38}$$

$$\begin{aligned} d_1 =& -2\left(4M - 3r_+\right)\left(2M - r_+\right)^3\omega^2 + 2qQ_e\left(7M - 5r_+\right)\left(2M - r_+\right)^2\omega \\ &+ 2q^2\left(-B^2\left(M^2 - 5Q^2\right) + 4Q^4 + B^4\right) - 2q^2Q^2\left(3M\left(r_+ - 2M\right)\right. \\ &\left.+2\mu^2\left(2M^2 - 3Mr_+ + r_+^2\right)^2 + 2\left(Q^2 + B^2\right)\right) - 12Mq^2Q^2r_- \\ &+ 2\left(M - r_+\right)^2\left(\lambda - 1\right) \\ e_1 =& \left(r_+ - r_-\right)\left(qQ - \omega r_-\right)^2 r_-^2 + \frac{1}{4}\left(r_+ - r_-\right)^3, \end{aligned} \tag{39}$$

where $\lambda = l(l+1), \quad l > qB$[14].Since we set the system to range from 0 to 1, $qB > q^2Q^2$.

In this paper, we denote the numerator of the derivative of the effective potential $V'(z)$. This quartic polynomial of z allows us to study the existence of trapped wells beyond the horizon by analyzing the properties of the roots of the equation. We use z_1, z_2, z_3 and z_4 to represent the four roots of $f_1(z) = 0$. The relationship between them conforms to Vieta's theorem.

$$z_1 z_2 z_3 z_4 = \frac{e_1}{a_1}, z_1 z_2 + z_1 z_3 + z_1 z_4 + z_2 z_3 + z_2 z_4 + z_3 z_4 = \frac{c_1}{a_1}. \tag{40}$$

When $z > 0$, from the asymptotic behavior of the effective potential of the inner and outer horizons and space infinity, it can be inferred that the equation $V'(z) = 0$(or $f_1(z) = 0$) cannot be less on two. So the two positive roots are written as z_1, z_2.

Research shows that for any ω

$$e_1 > 0. \tag{41}$$

and in

$$e_1 > 0, \quad c_1 < 0, \tag{42}$$

$f_1(z) = 0$, that is, z_3, z_4 are all negative numbers.

When $y^2 > 2(a_1 > 0)$ for $e_1 > 0$, $c_1 < 0$ at this time, then $f(\omega) < 0$, and we can know that the equation $\mathrm{V_1}'(\mathrm{z}) = 0$ cannot have more than two positive roots. So the dyonic RN-like black hole is superradiantly stable at that time.

5. SUMMARY

In this paper, we introduce $\mu = y\omega$[12, 13] into dyonic RN-like black holes and discuss the superadditive stability of dyonic RN-like black holes. We adopt the method of variable separation to divide the motion equations of the least coupled scalar perturbation in dynamical RN black holes into two forms: angular and radial.

Hod proved [10] that when $\mu \geq \sqrt{2}m\Omega_H$ (where μ is the mass), Kerr black holes should be superradiatively stable under large-scale scalar perturbations. In this post, a new variable y is added here to extend the results of the previous post.

When $\mu \geq \sqrt{2}(q\Phi_H)$, this dyonic RN-like black hole was superradiatively stable at that time.

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The superradiant stability of dyonic RN-like black holes

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The effect of magnetic fields on black hole superradiance is an interesting topic with possible astrophysical applications. A dyonic RN-like black hole is not asymptotically flat, it describes a black hole immersed in an asymptotically uniform magnetic field. In this paper, we discuss the superadditive stability of a class of asymptotically flat, band-like black holes, the binary RN black holes. In this article, we introduce the above condition into dyonic RN-like black holes. If a dyonic RN-like black hole satisfies the condition of $\mu = y\omega$, when $\mu \geq \sqrt{2}(m\Omega_H + q\Phi_H)$,so the dyonic RN-like black hole is superradiantly stable at that time.

Keywords: superradiantly stable,a new variable y, dyonic RN-like black hole

1. INTRODUCTION

The research on the stability of black holes can be traced back to 1957 when Regge and Wheeler found that Schwarzschild black holes are stable under small perturbations of the metric. In 1970, Zerilli further studied Schwarzschild black holes and RN black holes and reduced the perturbation problem to a process of solving the Schrödinger-like equation (wave equation) [1–4]. In 1972 Teukolsky studied the perturbations of various matter fields (gravitational, electromagnetic, neutrino fields) in Kerr space-time and decoupled the field equations into independent wave equations[5, 6], laying the foundation for the study of the external field disturbance of black holes. In 1983, Chandrasekhar's "Mathematical Theory of Black Holes" systematically expounded the perturbation theory of black holes. Superradiance is essentially the process of radiation enhancement, which plays an important role in optics, quantum mechanics, especially relativity, and astrophysics. Dicke, who coined the term "superradiance" in the context of quantum optics coherent emission [7], achieved the first high-resolution superradiance measurements using coherent synchrotron radiation [7, 8]. Zeldovich believed that the dissipative rotating body amplifies the human radiation, and Starobinsky recognized the super-radiation phenomenon of black holes on his basis: when the frequency of the human radiation satisfies the super-radiation condition, The rotational energy can be extracted from the black hole [7, 8]. Black hole superradiation is closely related to the black hole area theorem, the Penrose process, tidal forces, and even Hawking radiation [9]. In the general theory of relativity, the superradiation of a black hole is to extract energy, charge, and angular momentum in a vacuum[9, 10]. It can be known from the scattering problem of root quantum mechanics: the plane wave whose eigenfrequency is ω moves toward the center of the black hole, and is scattered to infinity under the action of the black hole, and the scattered particles obey a certain angular distribution. Taking the scalar wave under the background of static spherical symmetry as an example, we will see, at this time, the scalar field satisfies the Schrödinger-like equation of the following form:

$$\frac{d^2\psi_{lm}}{dx^2} + V_{\text{eff}}\,\psi_{lm} = 0, \tag{1}$$

where ψ_{lm} is the radial component of the field after decomposing the variables, x is the turtle coordinate, and V_{eff} depends on the theoretical model and the space-time background. In the case of spherical symmetry, we consider the scattering of monochromatic plane waves. Assuming that V_{eff} is constant on the boundary, the asymptotic solution of satisfies

$$\psi_{lm} \sim \begin{cases} \mathcal{T}e^{-ik_H x}, & r \to r_+ \\ \mathcal{I}e^{-ik_\infty x} + \mathcal{R}e^{ik_\infty x}, & r \to \infty, \end{cases} \tag{2}$$

In

$$\begin{aligned} k_H &= \omega - \omega_c, \\ k_\infty &= \sqrt{\omega^2 - \mu^2}. \end{aligned} \tag{3}$$

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ω_c is the critical frequency, for a charged and rotating black hole, the critical frequency is

$$\omega_c = q\Phi_H + m\Omega_H = \frac{qQr_p + ma}{r_p^2 + a^2}, \tag{4}$$

where r_p is the event horizon of the black hole (namely the outer horizon), Ω_H is the angular velocity of the black hole and the electric potential at the event horizon, m is the magnetic quantum number of the scalar field, q is the charge of the scalar field. When the black hole is not rotating, the critical frequency degenerates to

$$\omega_c = q\Phi_H = \frac{qQ}{r_p}. \tag{5}$$

$$W \equiv \begin{vmatrix} \psi & \psi^* \\ \psi' & \psi^{*\prime} \end{vmatrix} = \psi\psi^{*\prime} - \psi^*\psi' \tag{6}$$

can be obtained at infinity

$$W = 2ik_\infty \left(|\mathcal{R}|^2 - |\mathcal{I}|^2\right) \tag{7}$$

And at the horizon is

$$W = -2ik_H|\mathcal{T}|^2. \tag{8}$$

Since the Lansky determinant is a constant, we have

$$|\mathcal{R}|^2 = |\mathcal{I}|^2 - \frac{k_H}{k_\infty}|\mathcal{T}|^2. \tag{9}$$

It can be found that when $\frac{k_H}{k_\infty} > 0, |R|^2 < |I|^2$, the reflected wave amplitude is smaller than the human radiation wave amplitude, and the energy of the scalar field decreases; when $\frac{k_H}{k_\infty} < 0, |R|^2 > |I|^2$, the amplitude of the reflected wave is greater than the amplitude of the human radiation, and the energy of the scalar field increases at this time. Therefore, the superradiance generation condition is the increased condition of the scalar field energy, that is,

$$0 < \omega < \omega_c. \tag{10}$$

In the above equation, the critical angular frequency ω_c is defined as

$$\omega_c = q\Psi, \tag{11}$$

where Ψ is the electromagnetic potential of the outer horizon of the dyonic RN black hole, $\Psi = Q/r_+$. The superradiant condition for an electrically charged massive scalar perturbation on the dyonic RN black hole background is

$$\omega < \omega_c = \frac{qQ}{r_+}. \tag{12}$$

The bound state condition at spatial infinity for the scalar perturbation is

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When the radiation wave with the frequency ω satisfies the formula, superradiance scattering occurs, at this time, the reflected wave carries more energy than the human radiation wave, which is the superradiation occurrence condition of the charged black hole in the general theory of relativity. It is worth noting that if the black hole is not charged, that is, the Schwarzschild black hole, there is no super-radiation phenomenon, and only the rotating black hole or the charged black hole has the super-radiation phenomenon[8–10].

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$$\omega^2 - V = \frac{U + M^2 - a^2 - Q^2}{\Delta^2}, \tag{29}$$

in which V denotes the effective potential.

Considering the superadditive condition, i.e. $\omega < \omega_c$, and the bound state condition, when the potential is captured, the Kerr-Newman black hole and the charged massive scalar perturbation system are superadditive stables. Beyond the outer event horizon of a Kerr-Newman black hole, it doesn't exist. Therefore, the shape of the effective potential V is next analyzed to investigate the presence of trapping wells.

The asymptotic behaviors of the effective potential V around the inner and outer horizons and at spatial infinity can be expressed as

$$V(r \to +\infty) \to \mu^2 - \frac{2(2M\omega^2 - qQ\omega - M\mu^2)}{r} + \mathcal{O}(\frac{1}{r^2}), \tag{30}$$

$$V(r \to r_+) \to -\infty, \quad V(r \to r_-) \to -\infty. \tag{31}$$

If a Kerr black hole satisfy the condition of $\mu = y\omega$, it will be superradiantly stable when $\mu < \sqrt{2}m\Omega_H$. In this article, we introduce the above condition into Kerr-Newman black holes. Therefore, the formula of the asymptotic behaviors is written as

$$V(r \to +\infty) \to y^2\omega^2 - \frac{2[M(2 - y^2)\omega^2 - qQ\omega]}{r} + \mathcal{O}(\frac{1}{r^2}), \tag{32}$$

$$V(r \to r_+) \to -\infty, \quad V(r \to r_-) \to -\infty. \tag{33}$$

It is concluded from the equations above that the effective potential approximates a constant at infinity in space, and the extreme between its inner and outer horizons cannot be less than one. The asymptotic behavior of the derivative of the effective potential V at spatial infinity can be expressed as

$$V'(r \to +\infty) \to \frac{2[M(2 - y^2)\omega^2 - qQ\omega]}{r^2} + \mathcal{O}(\frac{1}{r^3}), \tag{34}$$

The derivative of the effective potential has to be negative in order to satisfy the no trapping well condition,

$$2M(2 - y^2)\omega^2 - 2Qq\omega < 0. \tag{35}$$

4. ANALYSIS OF SUPERRADIANT STABILITY

In this section, we will find the regions in the parameter space where the system of dyonic RN black hole and massive scalar perturbation is superradiantly stable. We determine the parameter regions by considering the extremes of the effective potential in the range $r_- < r < +\infty$

Now, we define a new variable $z, z = r - r_-$. The expression of the derivative of the effective potential V is

$$\begin{aligned} V'(r) &= \frac{-2\left(ar^4 + br^3 + cr^2 + dr + e\right)}{\triangle^3} \\ = V'(z) &= \frac{-2\left(a_1 z^4 + b_1 z^3 + c_1 z^2 + d_1 z + e_1\right)}{\triangle^3}, \end{aligned} \tag{36}$$

where

$$\begin{gathered} a_1 = a; \quad b_1 = (4r_-)\, a_1 + b, \\ c_1 = \left(6r_-^2\right) a_1 + (3r_-)\, b_1 + c, \\ d_1 = \left(4r_-^3\right) a_1 + \left(3r_-^2\right) b_1 + (2r_-)\, c_1, \\ e_1 = \left(r_-^2\right) a_1 + \left(r_-^3\right) b_1 + \left(r_-^2\right) c_1 + (r_-)\, d_1 + e. \end{gathered} \tag{37}$$

Explicitly,

$$\begin{aligned} a_1 =& -2M\omega^2 + qQ\omega + M\mu^2, \\ b_1 =& -2\left(8M^2 - 6Mr_+ + r_+^2\right)\omega^2 + 2qQ\left(5M - 2r_+\right)\omega \\ &+ \mu^2\left(6M^2 - 6Mr_+ + r_+^2\right) - q^2\left(Q^2 + B^2\right) + \lambda, \\ c_1 =& -6\left(2M - r_+\right)^3\omega^2 + 9qQ\left(2M - r_+\right)^2\omega \\ &+ 3\left(\left(M - r_+\right)\left(\mu^2\left(2M - r_+\right)^2 - q^2Q^2 - q^2B^2 + \lambda\right) - Mq^2Q^2\right), \end{aligned} \tag{38}$$

$$\begin{aligned} d_1 =& -2\left(4M - 3r_+\right)\left(2M - r_+\right)^3\omega^2 + 2qQ_e\left(7M - 5r_+\right)\left(2M - r_+\right)^2\omega \\ &+ 2q^2\left(-B^2\left(M^2 - 5Q^2\right) + 4Q^4 + B^4\right) - 2q^2Q^2\left(3M\left(r_+ - 2M\right)\right. \\ &\left. + 2\mu^2\left(2M^2 - 3Mr_+ + r_+^2\right)^2 + 2\left(Q^2 + B^2\right)\right) - 12Mq^2Q^2r_- \\ &+ 2\left(M - r_+\right)^2\left(\lambda - 1\right) \\ e_1 =& \left(r_+ - r_-\right)\left(qQ - \omega r_-\right)^2 r_-^2 + \frac{1}{4}\left(r_+ - r_-\right)^3, \end{aligned} \tag{39}$$

where $\lambda = l(l+1), \quad l > qB$[14].

In this paper, we denote the numerator of the derivative of the effective potential $V'(z)$. This quartic polynomial of z allows us to study the existence of trapped wells beyond the horizon by analyzing the properties of the roots of the equation. We use z_1, z_2, z_3 and z_4 to represent the four roots of $f_1(z) = 0$. The relationship between them conforms to Vieta's theorem.

$$z_1z_2z_3z_4 = \frac{e_1}{a_1}, z_1z_2 + z_1z_3 + z_1z_4 + z_2z_3 + z_2z_4 + z_3z_4 = \frac{c_1}{a_1}. \tag{40}$$

When $z > 0$, from the asymptotic behavior of the effective potential of the inner and outer horizons and space infinity, it can be inferred that the equation $V'(z) = 0$(or $f_1(z) = 0$) cannot be less on two. So the two positive roots are written as z_1, z_2.

Research shows that for any ω

$$e_1 > 0. \tag{41}$$

and in

$$e_1 > 0, \quad c_1 < 0, \tag{42}$$

$f_1(z) = 0$, that is, z_3, z_4 are all negative numbers.

When $y^2 > 2(a_1 > 0)$ for $e_1 > 0$, $c_1 < 0$ at this time, then $f(\omega) < 0$, and we can know that the equation $\mathrm{V_1}'(\mathrm{z}) = 0$ cannot have more than two positive roots. So the dyonic RN-like black hole is superradiantly stable at that time.

5. SUMMARY

In this paper, we introduce $\mu = y\omega$[12, 13] into dyonic RN-like black holes and discuss the superadditive stability of dyonic RN-like black holes. We adopt the method of variable separation to divide the motion equations of the least coupled scalar perturbation in dynamical RN black holes into two forms: angular and radial.

Hod proved [10] that when $\mu \geq \sqrt{2}m\Omega_H$ (where μ is the mass), Kerr black holes should be superradiatively stable under large-scale scalar perturbations. In this post, a new variable y is added here to extend the results of the previous post.

When $\mu \geq \sqrt{2}(q\Phi_H)$, this dyonic RN-like black hole was superradiatively stable at that time.

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