A Novel Method to Prove the Four Color Theorem

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Abstract

The four-color problem was first posed by Francis Guthrie in 1852. It remained unsolved for over a century. Although computer-assisted proofs emerged in last 50 years, they were somewhat the ”machine-checkable proof”, which were hardly checked by human readers. In this paper, a novel method of rotation inspired by a rotation principle from Zhuan Falun book of Falun Dafa has been developed to prove the Four Color Theorem.

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Abstract

The four-color problem was first posed by Francis Guthrie in 1852. It remained unsolved for over a century. Although computer-assisted proofs emerged in last 50 years, they were somewhat the “machine-checkable proof”, which were hardly checked by human readers. In this paper, a novel method of rotation inspired by a rotation principle from Zhuan Falun book of Falun Dafa has been developed to prove the Four Color Theorem.

Keywords

Novel method of rotation; Four Color Theorem

1. Introduction

The four-color problem was first posed by Francis Guthrie in 1852 [6], it indicate that no more than four colors are required to color the regions of any map so that no two adjacent regions have the same color. The two adjacent regions need to have a shared boundary line not just shared discretized points.

However, the four-color problem is very challenging to be proved mathematically. Over a century, many researchers tried many ways and obtained some useful results but it kept unsolved. There were some progress since 1852. It can be proved that every map has at least one country with five or fewer neighbors by using Euler's formula. One proposed proof was given by Alfred Kempe using Kempe Chain in 1879, but Percy Heawood found counterexample of Kempe’s proof in 1890 [6]. Heawood proved the five color theorem.

There are also some researchers who used computer technology to assist their work effort such as “a computer-assisted proof” by Kenneth Appel and Wolfgang Haken in 1976 [1, 2], and “a simpler one” was in 1997 by Robertson, Sanders, Seymour, and Thomas [5, 7]. However, they were somewhat the “machine-checkable proof”, which were hardly checked by human readers.

This paper will be organized in a few sections: firstly to introduce some challenging features of the color problem; secondly to prove some relevant Lemmas; thirdly to convert the maps and preparation; fourthly to provide the new proof method. In this paper, I also avoid to use the ideas from some researchers to add patches first and remove patches later for map conversion, which will bring an issue to make color problem to increase country number first and reduce it later.

2. The challenging features of the problem

2.1 No-clear-layered structure, which made it more difficult to be four colored

A map may have counties with their locations and connections to form no-clear-layered structure. An example is given in Fig. 1. A map has no-clear-layered structure has two main features: 1) We cannot tell how many layers in the map; 2) We cannot tell which layer any chosen country belongs to. That is, we cannot tell what the ring size of the map is in Fig. 1. Because the complex feature of the no-clear-layered structure, which made it more difficult to be four colored. This made the four-color problem more challenging.

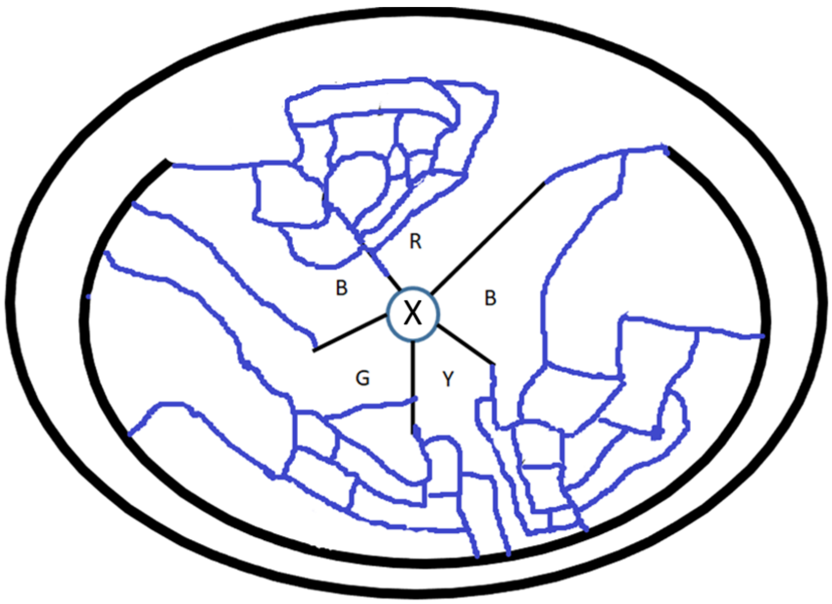


Fig. 1. A map has no-clear-layered structure.

2.2 A map may have more than 3 countries join in a single point

As shown in Fig. 2, a map may have more than three countries join in a single point; however, the color problem concerns the boundary lines rather than on the shared discretized points.

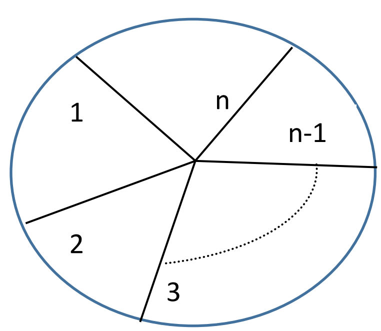


Fig. 2. A map has more than three countries join in a single point.

2.3 A map may have holes or discontinuous regions

In order to be comprehensive, holes or discontinuous regions may not be avoidable. These kinds of cases will also be considered in this research.

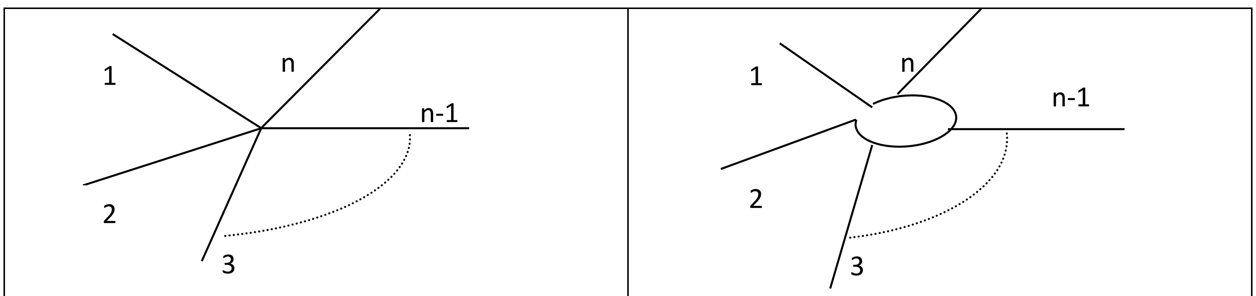
3. Some relevant Lemmas

3.1. ***LEMMA 1:*** *To change the point of intersection in a map to arc of intersection, which makes all the points be joined by no more than three countries (if some parts of this new added arc are placed inside one country, these parts will be vanished); then to color this map modification in four colors is an enhanced version of the four color theorem.*

Proof:

Here, I will not use the way of other researchers to add patches first and remove patches later, and I will keep the number of countries the same as the original.

In the map (such as in Fig. 3a), there are N_m countries, and N_t vertices where more than 3 countries are joined together. We take anyone of vertices V_i as an example to do the transformation of *LEMMA1*.This vertex is joined by N edges. Note that the number of countries around this vertex V_i can be less than the number of regions N here because some regions can be in one country.



(a) (b)

Fig. 3. a) A map has more than three countries join in a single point; b) To choose any one of the countries (e.g. country 1 in the map) and change the point to arc, after this transformation no more than three countries are joined in one point.

Firstly, we draw a small circle (noted C_i here) around vertex V_i (such as in Fig. 3b). There are two requirements for this circle are: 1) there is no other vertex except V_i in the circle; 2) the diameter of the circle is small enough to guarantee the circle is separated into N segments (noted S_1, S_2, …, S_n corresponding with region 1 to region N) by the above N edges.

Secondly, we can choose anyone of the countries (*e.g.* region j here, and the country for region j is country j) to do transformation of *LEMMA1*. There will be two possible situations: 1) no other segment except S_j inside country j, then we erase segment S_j and all the edges and vertex V_i inside this circle C_i. 2) If there are m segments (such as S_j, S_p, …, in total m segments) inside country j, then we erase all these m segments, and erase all edges and vertex V_i inside this circle C_i.

From the above two steps, there is no new country has been added in the map, and there is no country has been removed from the map. It means that the number of countries keep unchanged.

Here, we define some functions, U_k, as the region of the country k; $T(U_k)$, as the colour of country k; $G(U_k, U_h)$, as the relation indicator of shared points between country K and country h; and $F(T(U_k), T(U_h), G(U_k, U_h))$, as the indicator of the conflict between country K and country h; The value of function G and F are as follows.

$$G(U_k, U_h) = \begin{cases} 0 & (U_k \cap U_h = \emptyset) \\ 1 & (\exists X_1 \in (U_k \cap U_h)) \\ 2 & (\exists X_1, X_2, \dots, X_l \in (U_k \cap U_h)) \\ 3 & (\exists X_1, X_2, \dots, X_k \in (U_k \cap U_h), k \to \infty) \end{cases}$$

$$F(T(U_k), T(U_h), G(U_k, U_h)) = \begin{cases} 0 & ((G(U_k, U_h) \neq 3) \cup \left(T(U_k) \neq T(U_h)\right)) \\ 1 & ((G(U_k, U_h) = 3) \cap \left(T(U_k) = T(U_h)\right)) \end{cases}$$

After the above map transformation, if the new map can be colored by 4 colors, that means, $F_{new}\left(T(U_k), T(U_h), G(U_k, U_h)\right) = 0$ for all countries $(U_k, U_h \in N_m)$ in the new map.

Let us see what will happen to keep all countries in the old map the same color as new map, then work out all values of $F_{old}\left(T(U_k), T(U_h), G(U_k, U_h)\right)$. Here, we must note that the actual number of countries joined in vertex V_i can be less than the number of region N in the old map.

There are two cases:

1) As long as one of country k and country h is not joined in vertex V_i, then their $F_{old}\left(T(U_k), T(U_h), G(U_k, U_h)\right) = 0$ in the old map because there is nothing change among countries which are not joined in vertex V_i.

2) Both country k and country h are joined in vertex V_i in the old map. Because the only difference between old map and new map is C_i circle, all other edges outside C_i circle will be no conflict between countries, it means $F_{old}\left(T(U_k), T(U_h), G(U_k, U_h)\right)$ can be held to zero outside C_i circle. Inside C_i circle, edges outside circle C_i are extended and joined in vertex V_i in the old map.

Three situations will be happened here,

(A) in new map, there is no point inside the circle C_i between country k and country h, but in old map there is one point (vertex V_i) has been located between country k and country h, in this situation, $F_{old}\left(T(U_k), T(U_h), G(U_k, U_h)\right)$ will equal to zero because that only one more point is added cannot change value of function G and so as to function F;

(B) arc segments are disappeared in the new map, but there are edges and edges joined to one point (vertex V_i) in the old map, therefore, if country k and country h are not neighbor, the edge connection (arc segment in new map) will become point connection (vertex V_i), in this situation, $F_{old}\left(T(U_k), T(U_h), G(U_k, U_h)\right)$ will equal to zero because function G has been changed from 3 to 1 regarding country k and country h in this small circle C_i, it means that $F_{old}\left(T(U_k), T(U_h), G(U_k, U_h)\right)$ won't equal to 1 because function G changed from 3 to 1 here;

(C) the edges between country k and country h in the new map has been extended to V_i in the old map, that is $G(U_k, U_h) = 3$ for the extended edge, but in the new map, in order to have

$F_{new}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$,there must be $T(U_k) \neq T(U_h)$, therefore $F_{old}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$ will be held for this situation.

Now, we have proved that as long as there is no conflict in new map, the old map will be no conflict as well. That is, as long as the new map can be 4 colored, then the old map will be 4 colored.

Finally, we do this transformation for all other vertices one by one, and then the whole map will meet the requirement of *LEMMA1*. Therefore, *LEMMA 1* has been proved. □

3.2. ***LEMMA 2:*** *To encircle another country outside the existing map, then to color this map modification in four colors is an enhanced version of the four color theorem.*

It is easily to be proved that as long as the new map can be colored in four colors then the old one without the encircled country can be colored by the same color arrangement. Actually using the function of conflict, as long as $F_{new}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$, it is easy to see that $F_{old}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$.

3.3. ***LEMMA 3****: To change the planar map to a map on a sphere (or skewed sphere) surface then the map boundary form a region (to treat it as a new country), to color this $N + 1$ countries is an enhanced version of the four color theorem.*

It is easy to project a planar map up onto a sphere (or skewed sphere), and then if there is boundary in planar map, the boundary will form a region on the sphere surface. It will follow the same way of the LEMMA 2 that to color this $N + 1$ countries is an enhanced version of the four color theorem. Actually using the function of conflict, as long as $F_{new}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$, it is easy to see that $F_{old}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$.

3.4. ***LEMMA 4****: To change the planar map to a map on a sphere (or skewed sphere) surface then the map boundary form a region (to treat it as a new country called W country), to break one of countries (called P country) that share an edge with W country to make those two counties become one country (called PW country). The number of total countries does not change by the above operations. To color this map modification in four colors is an enhanced version of the four color theorem.*

The conditions of the LEMMA 4 are including more operations after the LEMMA 3 modification of the original map. However, the total number of countries in LEMMA 4 keep unchanged from the original map, while the total number of countries in LEMMA 3 increased. To use the function of conflict, as long as $F_{new}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$, it is easy to see that $F_{old}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$.

Please note that the final four color arrangement of the LEMMA 4 will be definitely different from that of the LEMMA 3.

*3.5. **LEMMA 5**: Based on LEMMA 3 to deal with the boundary, and fill holes or discontinuous regions with new countries (to assume as M new countries) to fully cover the whole sphere (or skewed sphere) surface, to color this $N + 1 + M$ countries is an enhanced version of the four color theorem.*

It will follow the same way of the LEMMA 2 that to color this $N + 1 + M$ countries is an enhanced version of the four color theorem.

*3.6. **LEMMA 6**: Based on LEMMA 4, and to use similar modifications for holes or discontinuous regions to make them become parts of the existing countries. The number of total countries does not change by the above operation. To color this map modification in four colors is an enhanced version of the four color theorem.*

It is easy to use the function of conflict, as long as $F_{new}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$, it is easy to see that $F_{old}\big(T(U_k), T(U_h), G(U_k, U_h)\big) = 0$.

4. Main proof

Based on modifications from LEMMA1, 4 and 6, a map will fully cover the whole sphere (or skewed sphere) surface, and all the points will be joined by no more than three countries. I will start to provide the main proof. My method is to find whether minimal counterexamples exist in a map with N countries by assuming any map with $N - 1$ countries can be four-colored.

4.1. "Only Five Neighbors" theorem

This theorem states that every map has at least one country with five or fewer neighbors [6]. It is easy to be proved by Euler's polyhedron formula [3]. If any readers are interested in how to prove the "Only Five Neighbors" theorem, please read the book by Wilson [6].

If minimal counterexamples exist, the map

a) Cannot contain a two-sided country (a digon). If a digon exists, because any map with $N - 1$ countries can be four-colored, all other countries can be four-colored first and then to paint the digon country with a different color (in the four colors) from its two neighbors.

b) Cannot contain a three-sided country (a triangle). If a triangle exists, because any map with $N - 1$ countries can be four-colored, all other countries can be four-colored first and then to paint the triangle country with a last different color (in the four colors) from its three neighbors.

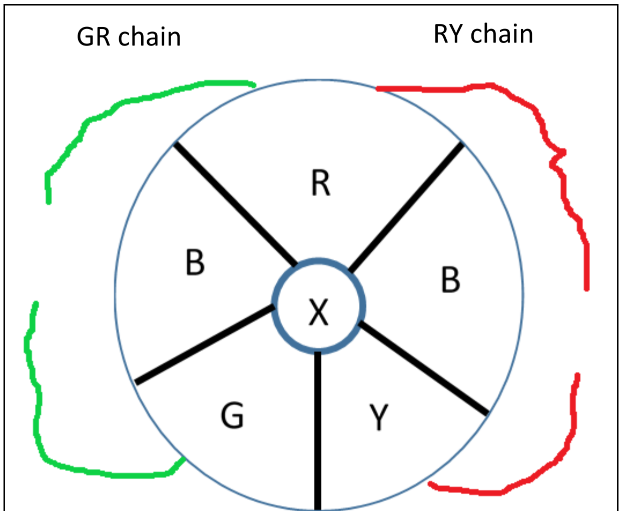
c) Cannot contains a four-sided country (a square). Let us use four colors (G-green, B-blue, Y-yellow, R-red, and assume G and Y are opposite while B and R are opposite from the angle of the square country) to paint all other countries and then work on the square country. To use Kempe's chain method to start from G country to have G-Y color interchange. If this interchange cannot reach Y country, which means that four neighbors only have three colors now, then the square country can become color G. G-Y chain is a complete chain. If this interchange can reach Y country, which means G-Y chain is a complete chain, then B-R chain will be broken due to B-R chain cannot cross the G-Y

Chain. That is, if we start from B country to perform B-R chain color interchange, it will not reach R country, then the square country can become color B.

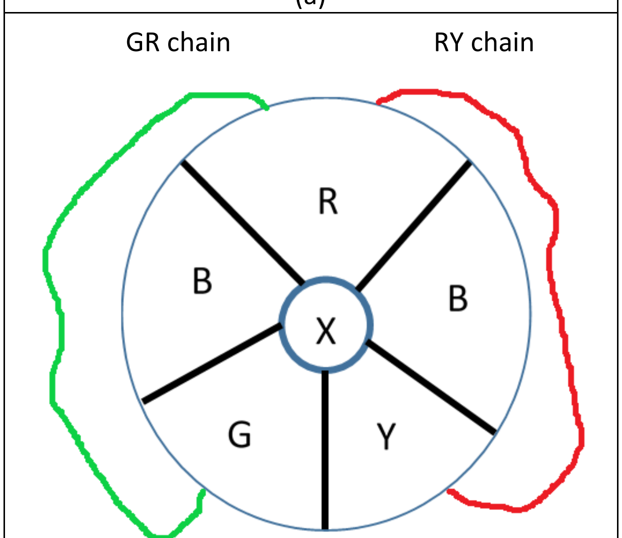
d) Contains a five-sided country (a pentagon), which is a very difficult situation and need to be proved further.

4.2. Some easy cases of the map with a five-sided country (a pentagon)

It can be easily converted by exchanging color to the colors of 5-neighbor countries being R (Red) –B (Blue) – G (Green) – Y (Yellow) – B (Blue). There are easy cases that the color of the pentagon can be determined, such as the three cases in Fig. 4. *Note: one thing must be clarified that the two color chains (eg. GR chain here) represent all kinds of connected R and G color countries with circles and/or branches linking to the GR chain as shown in Fig. 6.



(a)



(b)

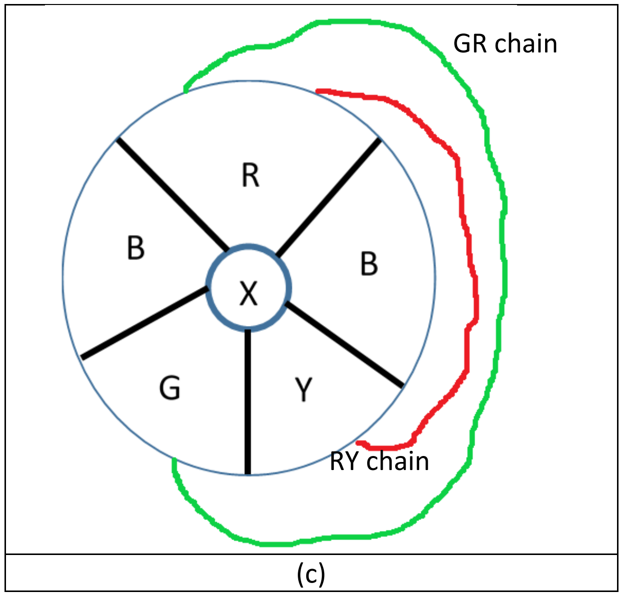


Fig. 4. a) There is no complete GR chain (or complete RY Chain). b) There are GR chains and RY Chains, but no GR-RY crossing. GR chains and RY Chains are located in different side of the map. c) There are GR chains and RY Chains, but no GR-RY crossing. GR chains and RY Chains are located in a same side of the map.

In Fig. 4a, there is no GR chain (or RY Chain), then R can be interchanged to G (or Y), so the pentagon X will be R.

In Fig. 4b and 4c, there are GR chains and RY Chains, but no GR-RY crossing, and then the left B can be interchanged to Y and right B can be interchange to G, so X will be B.

4.3. Some difficult cases of the map with a five-sided country (a pentagon)

The pentagon color cannot be directly determined for the following cases in Fig. 5. The map can be very complicated and the number of countries can be an extremely large value. This is why it is so difficult to prove four color theorem. The GR chains and RY chains can have circles and/or branches, as demonstrated by an example in Fig. 6.

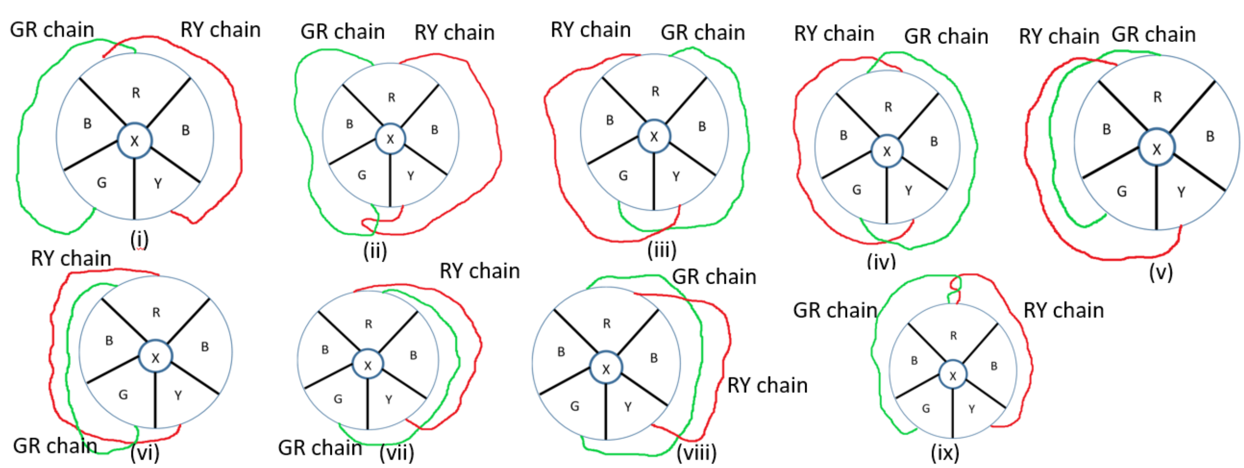


Fig. 5. There are nine cases in which the pentagon color cannot be directly determined.

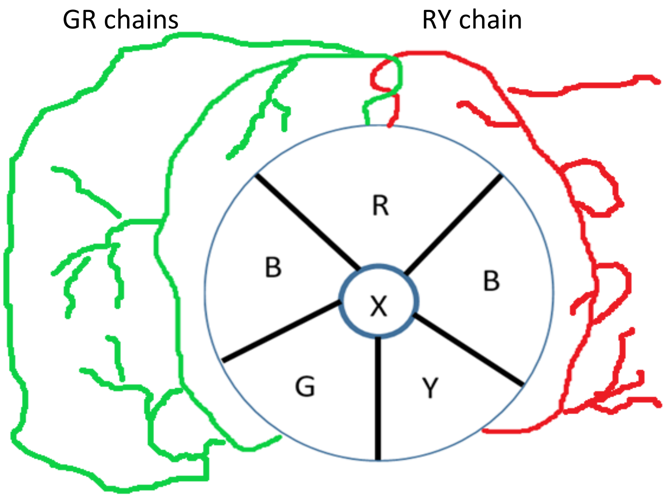


Fig. 6. The GR chains and RY chains in a map can have circles and/or branches

4.4. A generalization can be done for all different cases

LEMMA 7: *All different cases in Fig. 6 can be converted into Case 9 (note: it can have many more twisted GR and RY crossings, not just limited in 1 or 2 crossings) via some ways of two colors exchange, some countries enlarge, different angle view on sphere (or skewed sphere) surface, sphere change to skewed sphere as long as the relative connections of countries do not change.*

It is important to present that the maps on sphere (or skewed sphere) surface can have the following conversion without changing the relative connections of countries, as shown in Fig. 7.

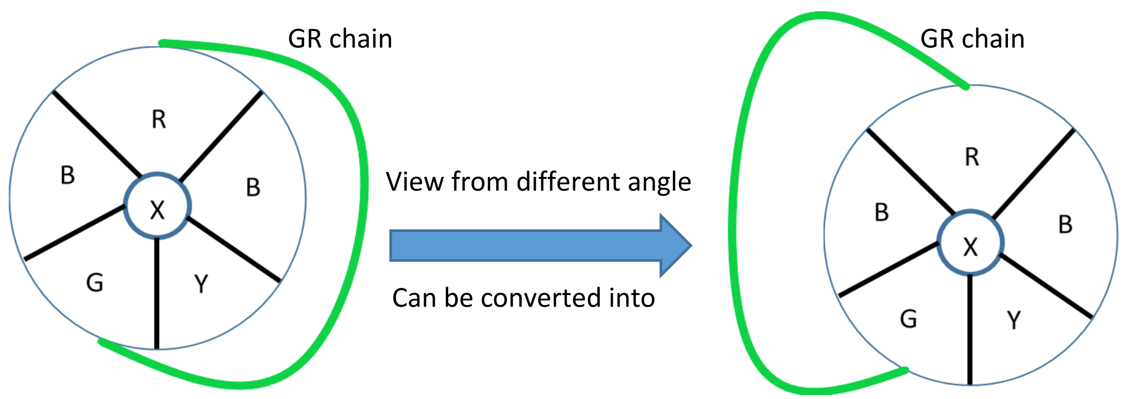


Fig. 7. A map on sphere (or skewed sphere) surface can be viewed from different angle and be converted to the right-hand-side map without changing the relative connections of countries.

It can be more easily understood by using irregular shape balloon with the fully covered map. To take the case 2 as an example, the bottom part of the map can be expanded like a balloon and all other parts keep unchanged, then this newly skewed sphere surface map will look similar as the case 9. To use different angle view and balloon expansion method, the case 5 will become a similar map as the case 1.

The case 1 is actually a special case of the case 9 with just one GR-RY crossing. All others can also be converted into the Case 9 maps.

Note: The case 9 can have many twisted GR and RY crossings. The case 9 will be used for further proof. More crossings or less crossings do not affect the proof by using my method.

In order to make my proof clearer, there are two features to be used for further proof:

a) **Feature 1**: There maybe many GR chains (or RY chains) with circles. The GR chain with least number of countries in the section of R-B(left side)-G as shown in Fig. 8 will be chosen for further proof. The RY chain with least number of countries in the section of R-B(right side)-Y as shown in Fig. 8 will be chosen for further proof.

b) **Feature 2**: The chosen GR chain and RY chain can have branches both inside and outside, but here just simply presented without branches. They may have circles but circles can only be outside the chain because the inside circles will violate the feature 1.

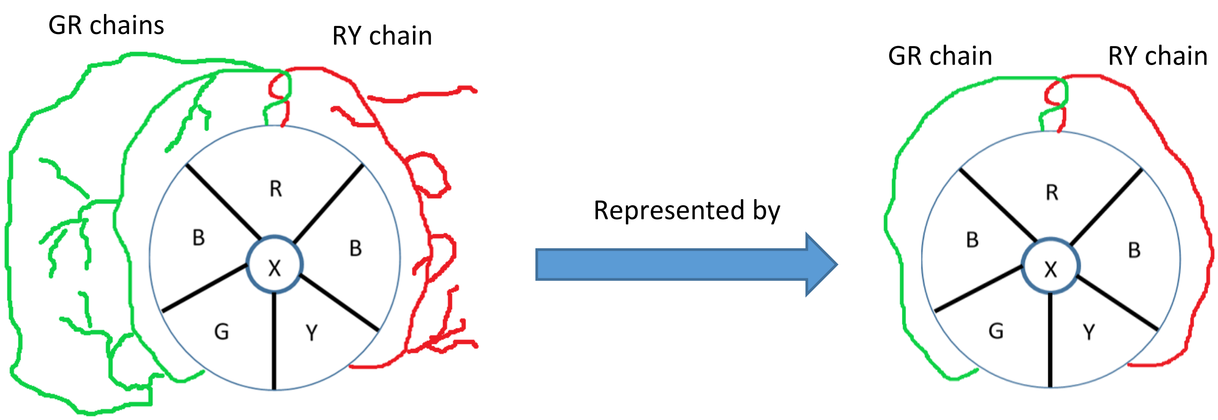


Fig. 8. The map presentation by choosing the GR chain with least number of countries in the section of R-B(left side)-G, and the RY chain with least number of countries in the section of R-B(right side)-Y.

The further proof will be based on the map of the case 9 as shown in Fig. 9. The N-countries map of the case 9, except a five-sided country (named country X), all other countries can be colored using four colors and no neighbor countries have a same color. All the curves are not smooth or straight because the real map may have much more complicated windy curve. I did not use a boundary curve to close the five neighbor countries of the pentagon because the map can be very complicated as in Fig. 1 with no-clear-layered structure, the important of the proof is just use Kempe chains, which will just be drawn into the relevant countries in the middle part of the map.

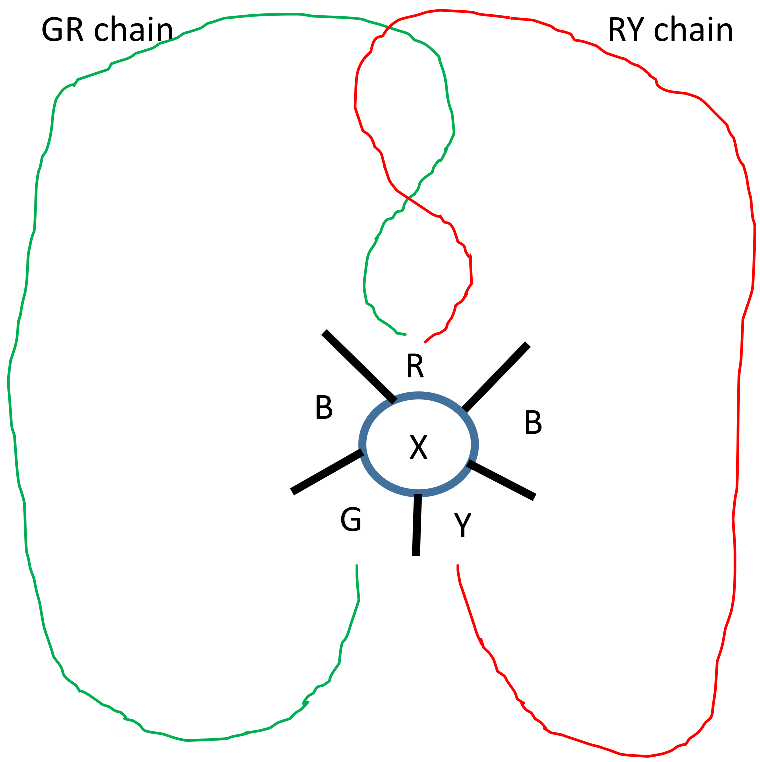


Fig. 9. A map of the case 9 will be used for further proof.

4.5. Inspired by the teaching from Zhuan Falun book of Falun Dafa [4]

I have been inspired by the teaching from Zhuan Falun book of Falun Dafa, which teaches the characteristics of the universe. There are three main principles:

法輪常轉 (English translation: Constantly turns the Falun)

佛法無邊 (English translation: The Buddha Fa is boundless)

旋法至極 (English translation: The revolving Fa reaches the Extreme)

The idea of the "The Buddha Fa is boundless" has been applied in previous sections and the map fully cover the sphere (or skewed sphere) surface.

The idea of the "Constantly turns the Falun" will be shown in the following section.

The idea of the "The revolving Fa reaches the Extreme" is the most important part and will be applied in my proof.

4.6. A rotation method and its rotation table

The rotation table is shown in Table 1. *Note: after 16 rotates, the relative color and location of the 5-neighbor countries back to the initial one, however, we can just keep increasing the number of the subscript for more rotates. Every rotation will include: a) Two chain groups with features of shared country and double (the same color) inside the 5-neighbors of the X country; b) 2-Color linked chain(s) interchange inside the corresponding section in the map using the dotted lines; c) After the interchange, if the new chain (which can be just a single country for certain special cases) can be formed, they will be the last column in the table, otherwise, if new chain cannot be formed, then the color of X can be determined. The new chain will use the solid lines.

Table 1. The rotation table with the first 17 rotates

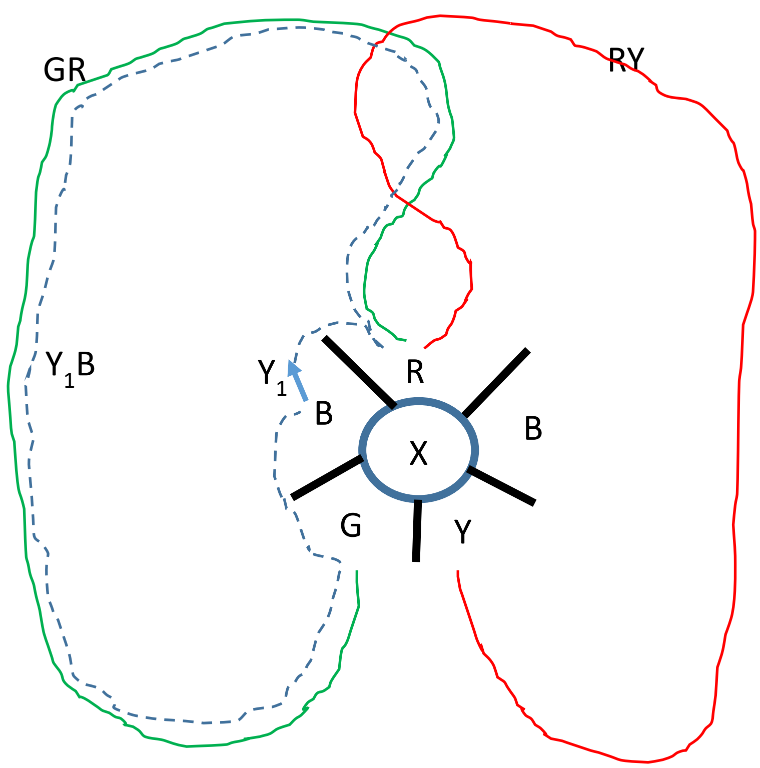
Rotates	Chain 1	Chain 2	Share	Doubles	interchange	Form new chains
1	GR	RY	R	B, B	Inside GR, B→Y_1	BG
2	BG	GR	G	Y, Y_1	Inside BG, Y→R_1	Y_1B
3	Y_1B	BG	B	R, R_1	Inside Y_1B, R→G_1	R_1Y_1
4	R_1Y_1	Y_1B	Y_1	G, G_1	Inside R_1Y_1, G→B_1	G_1R_1
5	G_1R_1	R_1Y_1	R_1	B, B_1	Inside G_1R_1, B→Y_2	B_1G_1
6	B_1G_1	G_1R_1	G_1	Y_1, Y_2	Inside B_1G_1, Y_1→R_2	Y_2B_1
7	Y_2B_1	B_1G_1	B_1	R_1, R_2	Inside Y_2B_1, R_1→G_2	R_2Y_2
8	R_2Y_2	Y_2B_1	Y_2	G_1, G_2	Inside R_2Y_2, G_1→B_2	G_2R_2
9	G_2R_2	R_2Y_2	R_2	B_1, B_2	Inside G_2R_2, B_1→Y_3	B_2G_2
10	B_2G_2	G_2R_2	G_2	Y_2, Y_3	Inside B_2G_2, Y_2→R_3	Y_3B_2
11	Y_3B_2	B_2G_2	B_2	R_2, R_3	Inside Y_3B_2, R_2→G_3	R_3Y_3
12	R_3Y_3	Y_3B_2	Y_3	G_2, G_3	Inside R_3Y_3, G_2→B_3	G_3R_3
13	G_3R_3	R_3Y_3	R_3	B_2, B_3	Inside G_3R_3, B_2→Y_4	B_3G_3
14	B_3G_3	G_3R_3	G_3	Y_3, Y_4	Inside B_3G_3, Y_3→R_4	Y_4B_3
15	Y_4B_3	B_3G_3	B_3	R_3, R_4	Inside Y_4B_3, R_3→G_4	R_4Y_4
16	R_4Y_4	Y_4B_3	Y_4	G_3, G_4	Inside R_4Y_4, G_3→B_4	G_4R_4

17	G_4R_4	R_4Y_4	R_4	B_3, B_4	Inside G_4R_4, $B_3 \rightarrow Y_5$	B_4G_4

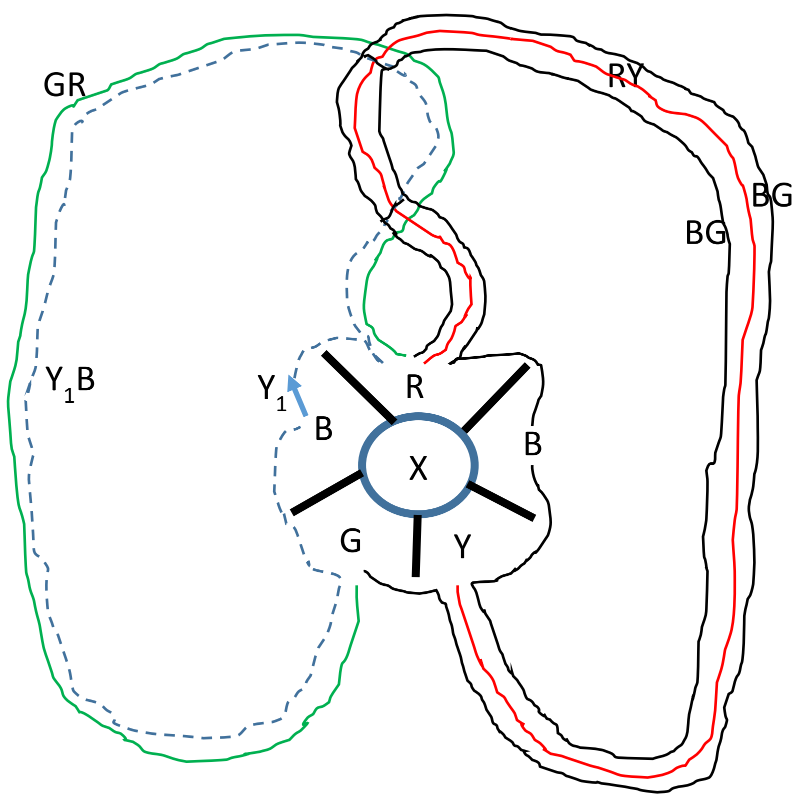
4.7. The steps of the main proof

Step 1: To follow the first row of the table, 2-Color BY_1 linked chain(s) interchange can be done inside the corresponding GR section in Fig. 9, which shows with dotted Y_1B chain in the left-hand-side of Fig. 10a. *Note: The tightest Y_1B chain to GR chain will be chosen for further proof. The chosen Y_1B chain can have branches. It may have circles but circles can only be inside the chain because the outside circles will violate the feature of the tightest Y_1B chain to GR chain.

After the interchange, if the new BG chain (shown in the first row and the last column of the rotation table) can be formed that shown in the right-hand-side of Fig. 10b, otherwise, if new BG chain cannot be formed, then B can be change to G and the color of X can be determined to be B. If new BG chain and GR chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note: The tightest BG chains are using solid lines in Fig. 10b, which just show that they are naturally formed without any actions. The RY chain will be sandwiched by the new BG chains. The new BG chains can have branches. The inside BG chain may have circles but circles can only be inside the chain because the outside circles will violate the feature of the tightest BG chain to RY chain. The outside BG chain may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest BG chain to RY chain.



(a)

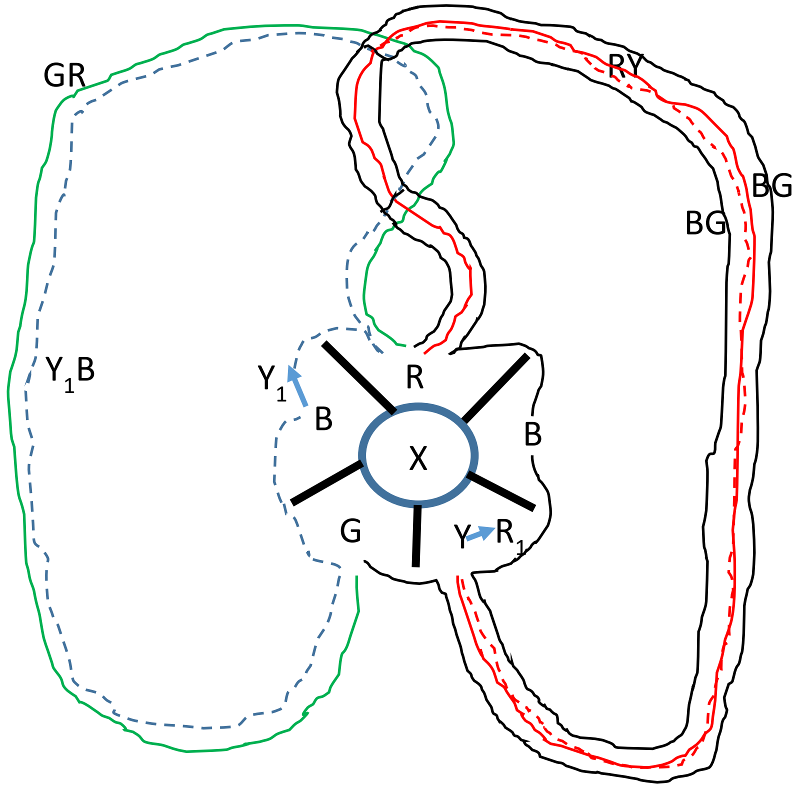


(b)

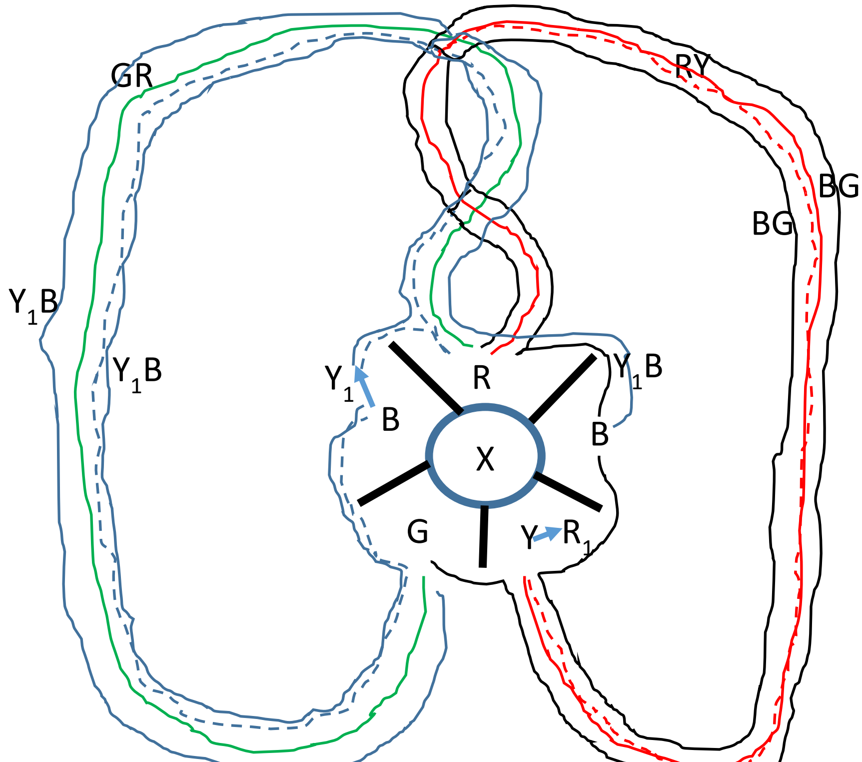
Fig. 10. The first step of the main proof.

Step 2: To follow the second row of the table, 2-Color YR_1 linked chain(s) interchange can be done inside the newly formed BG chain in Fig. 10b, which shows with dotted YR_1 chain in the right-hand-side of Fig. 11a. *Note: The YR_1 chain interchange will be restricted by the BG chains and cannot go out of the BG chains.

After the interchange, if the new Y_1B chain (shown in the second row and the last column of the rotation table) can be formed that shown in the left-hand-side of Fig. 11b, otherwise, if new Y_1B chain cannot be formed, then B can be change to Y and the color of X can be determined to be B. If new Y_1B chain and BG chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note: The tightest Y_1B chains are using solid lines in the middle of Fig. 11b and tightest Y_1B chains in the left-hand-side of Fig. 11b, which just show that they are naturally formed without any actions. The GR chain will be sandwiched by the new Y_1B chains. The new Y_1B chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest Y_1B chain to GR chain.



(a)

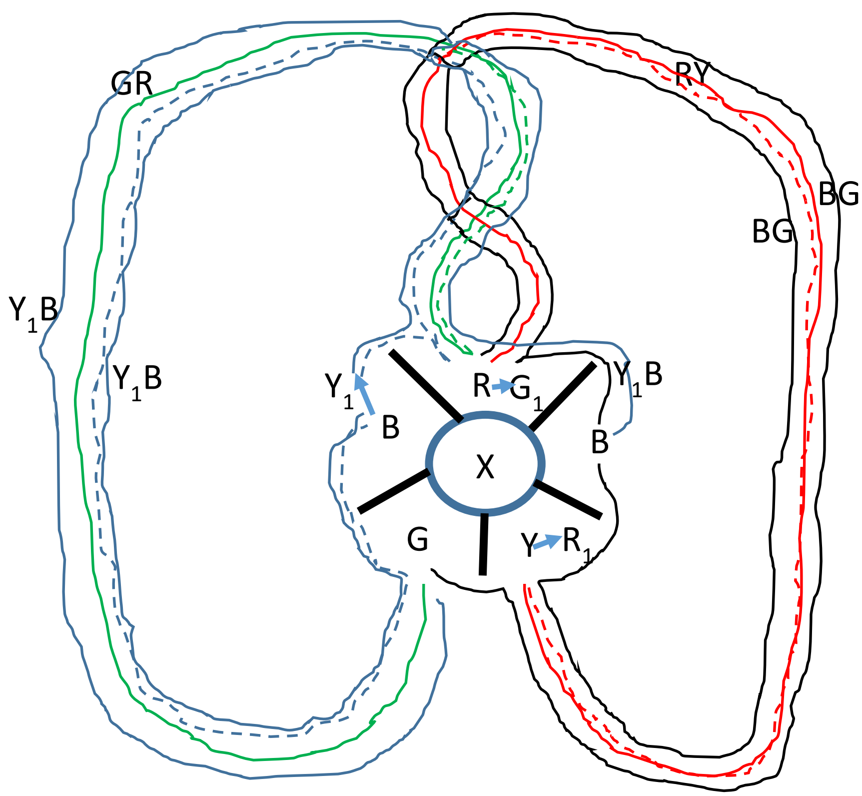


(b)

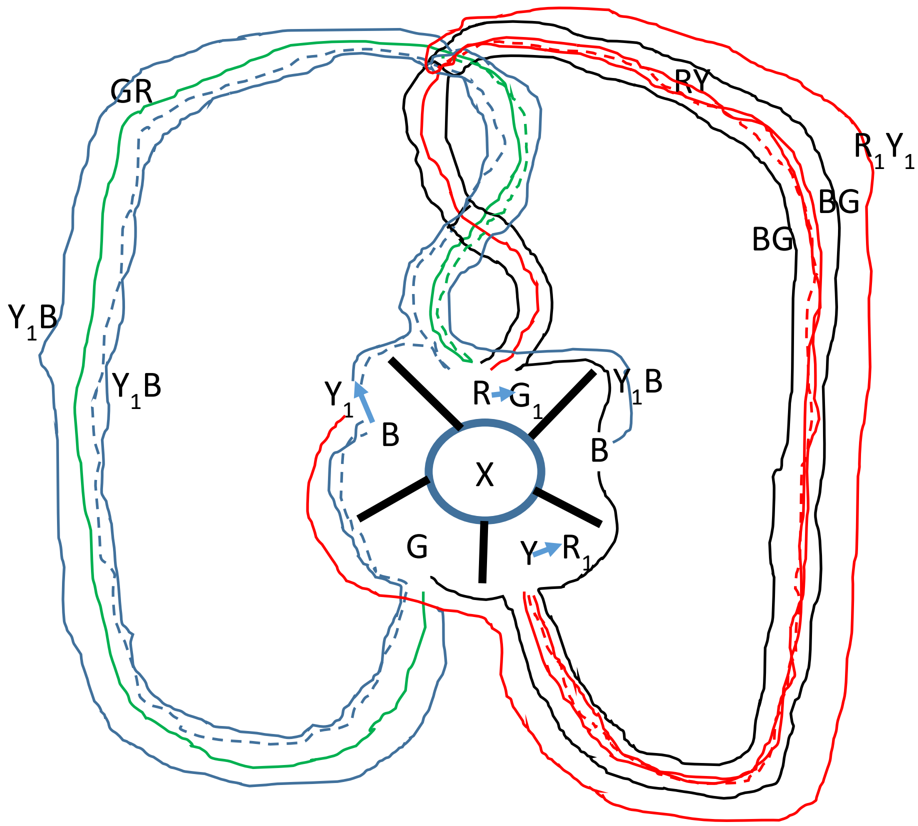
Fig. 11. The second step of the main proof.

Step 3: To follow the third row of the table, 2-Color RG_1 linked chain(s) interchange can be done inside the newly formed Y_1B chain in the middle of Fig. 11b, which shows with dotted RG_1 chain in the middle of Fig. 12a. *Note: The RG_1 chain interchange will be restricted by the Y_1B chains and cannot go out of the Y_1B chains.

After the interchange, if the new R_1Y_1 chain (shown in the third row and the last column of the rotation table) can be formed that shown in the right-hand-side of Fig. 12b, otherwise, if new R_1Y_1 chain cannot be formed, then R_1 can be change to Y and the color of X can be determined to be R. If new R_1Y_1 chain and Y_1B chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note: The tightest R_1Y_1 chains are using solid lines in the right-hand-side of Fig. 12b, which just show that they are naturally formed without any actions. The right-hand-side BG chain (formed in Step 1) will be sandwiched by the new R_1Y_1 chains. The new R_1Y_1 chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest R_1Y_1 chain to BG chain.



(a)

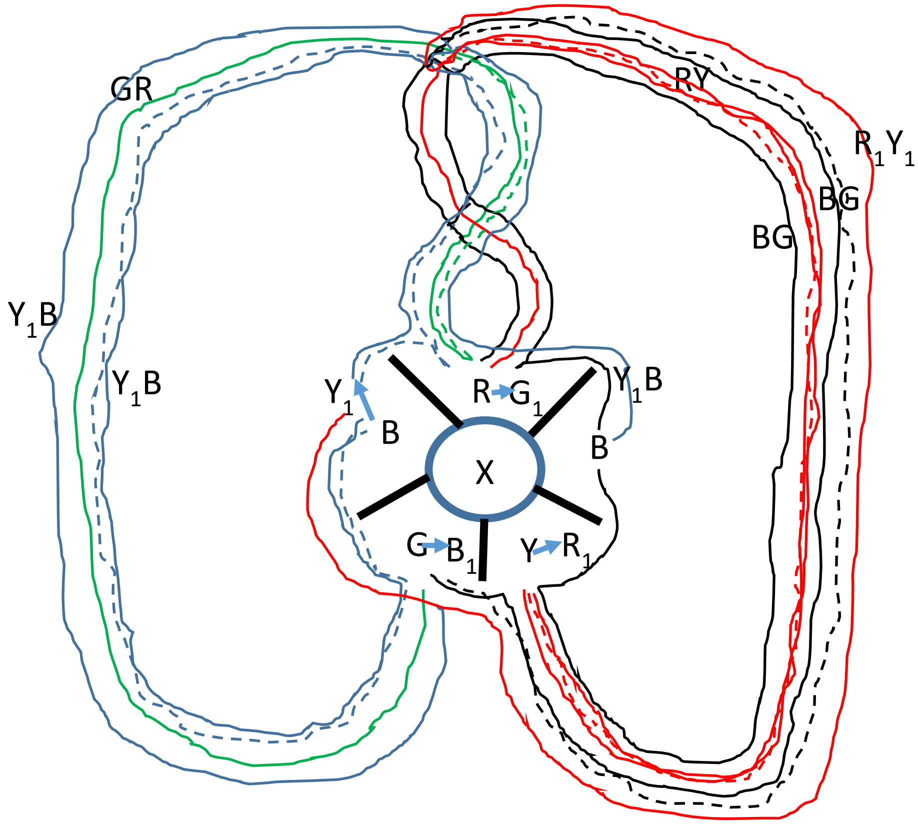


(b)

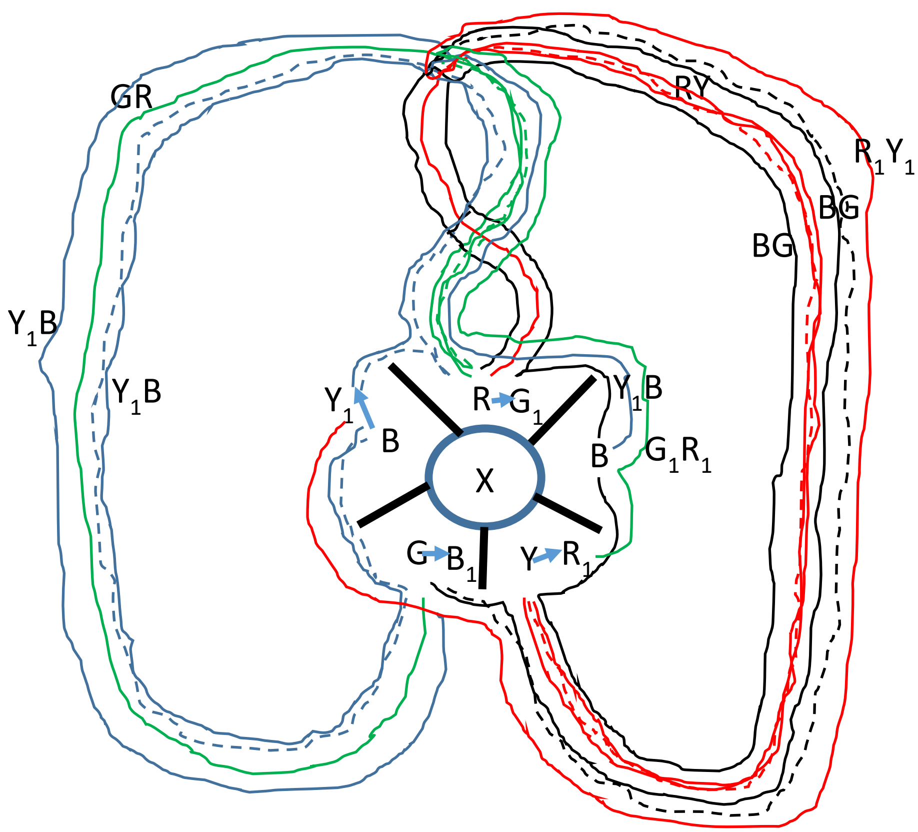
Fig. 12. The third step of the main proof.

Step 4: To follow the fourth row of the table, 2-Color GB_1 linked chain(s) interchange can be done inside the newly formed R_1Y_1 chain in the right-hand-side of Fig. 12b, which shows with dotted GB_1 chain in the right-hand-side of Fig. 13a. *Note: The GB_1 chain interchange will be restricted by the R_1Y_1 chains and cannot go out of the R_1Y_1 chains.

After the interchange, if the new G_1R_1 chain (shown in the fourth row and the last column of the rotation table) can be formed that shown in the middle of Fig. 13b, otherwise, if new G_1R_1 chain cannot be formed, then G_1 can be change to R and the color of X can be determined to be G. If new G_1R_1 chain and R_1Y_1 chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note: The tightest G_1R_1 chains are using solid lines in the middle of Fig. 13b, which just show that they are naturally formed without any actions. The middle Y_1B chain (formed in Step 2) will be sandwiched by the new G_1R_1 chains. The new G_1R_1 chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest G_1R_1 chain to Y_1B chain.


GR
RY
R_1Y_1
BG
BG
Y_1B
Y_1B
Y_1B
Y_1
B
$R \to G_1$
B
X
$G \to B_1$
$Y \to R_1$

(a)

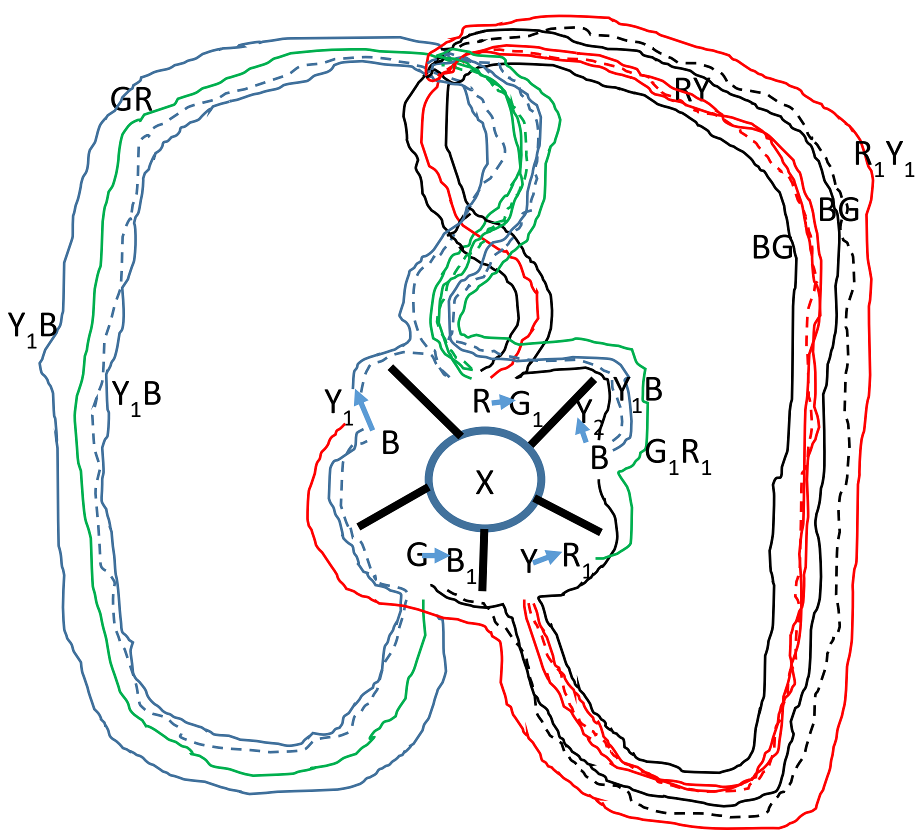

GR
RY
R_1Y_1
BG
BG
Y_1B
Y_1B
Y_1B
Y_1
B
$R \to G_1$
B
G_1R_1
X
$G \to B_1$
$Y \to R_1$

(b)

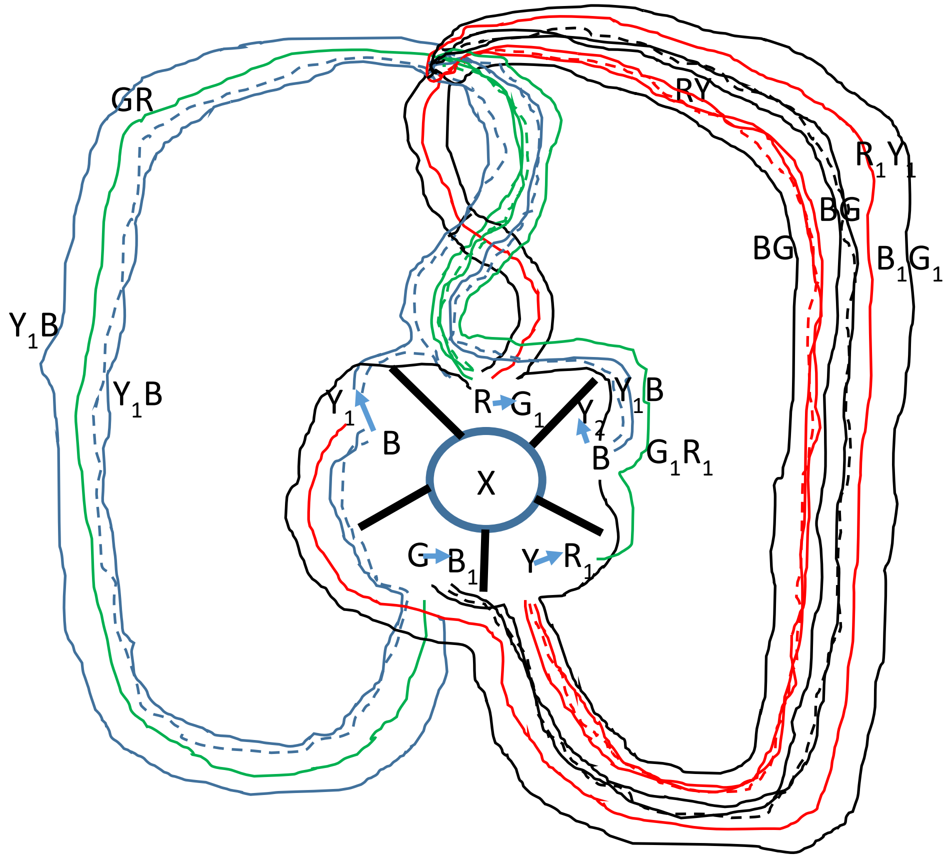
Fig. 13. The fourth step of the main proof.

Step 5: To follow the fifth row of the table, 2-Color BY_2 linked chain(s) interchange can be done inside the newly formed G_1R_1 chain in the middle of Fig. 13b, which shows with dotted BY_2 chain in the middle of Fig. 14a. *Note: The BY_2 chain interchange will be restricted by the G_1R_1 chains and cannot go out of the G_1R_1 chains.

After the interchange, if the new B_1G_1 chain (shown in the fifth row and the last column of the rotation table) can be formed that shown in the right-hand-side of Fig. 14b, otherwise, if new B_1G_1 chain cannot be formed, then B_1 can be change to G and the color of X can be determined to be B. If new B_1G_1 chain and G_1R_1 chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note: The tightest B_1G_1 chains are using solid lines in the right-hand-side of Fig. 14b, which just show that they are naturally formed without any actions. The right-hand-side R_1Y_1 chain (formed in Step 3) will be sandwiched by the new B_1G_1 chains. The new B_1G_1 chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest B_1G_1 chain to R_1Y_1 chain.



(a)

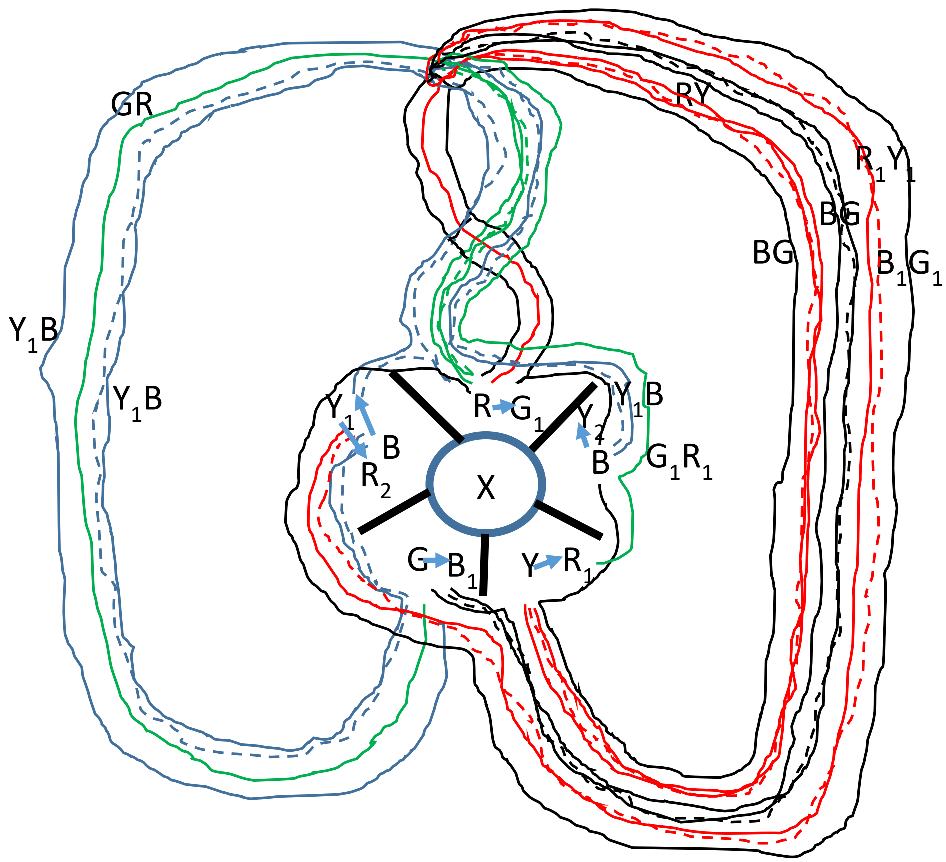


(b)

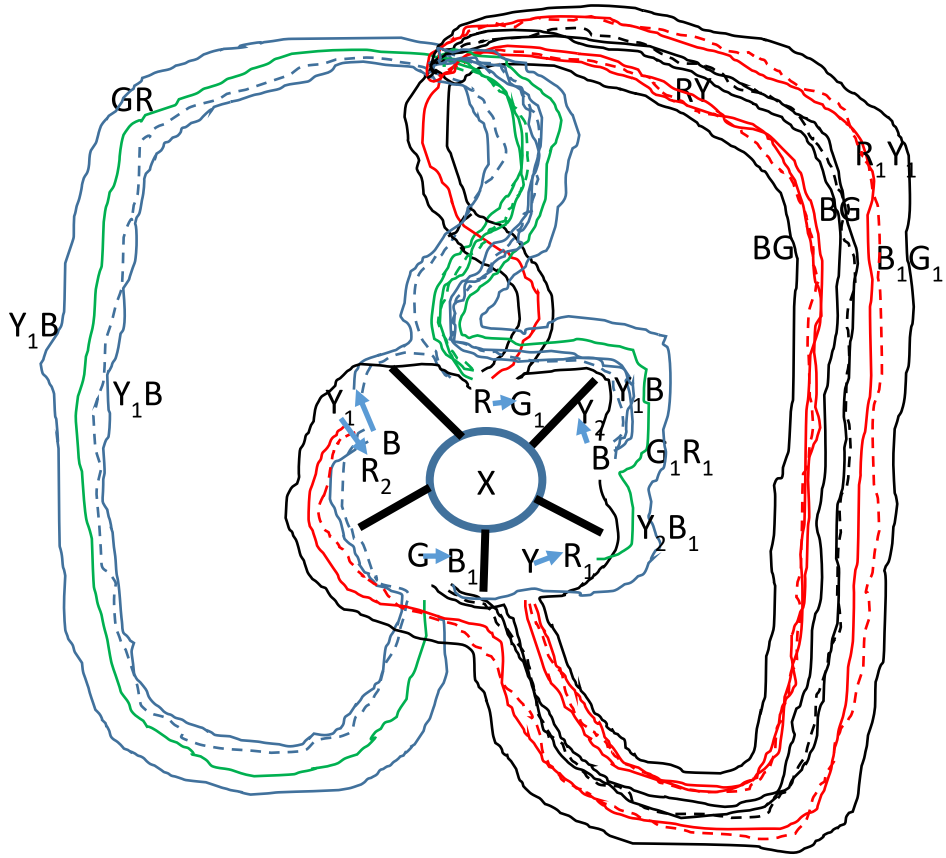
Fig. 14. The fifth step of the main proof.

Step 6: To follow the sixth row of the table, 2-Color Y_1R_2 linked chain(s) interchange can be done inside the newly formed B_1G_1 chain in the right-hand-side of Fig. 14b, which shows with dotted Y_1R_2 chain in the right-hand-side of Fig. 15a. *Note: The Y_1R_2 chain interchange will be restricted by the B_1G_1 chains and cannot go out of the B_1G_1 chains.

After the interchange, if the new Y_2B_1 chain (shown in the sixth row and the last column of the rotation table) can be formed that shown in the middle of Fig. 15b, otherwise, if new Y_2B_1 chain cannot be formed, then Y_2 can be change to B and the color of X can be determined to be Y. If new Y_2B_1 chain and B_1G_1 chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note: The tightest Y_2B_1 chains are using solid lines in the middle of Fig. 15b, which just show that they are naturally formed without any actions. The middle G_1R_1 chain (formed in Step 4) will be sandwiched by the new Y_2B_1 chains. The new Y_2B_1 chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest Y_2B_1 chain to G_1R_1 chain.



(a)

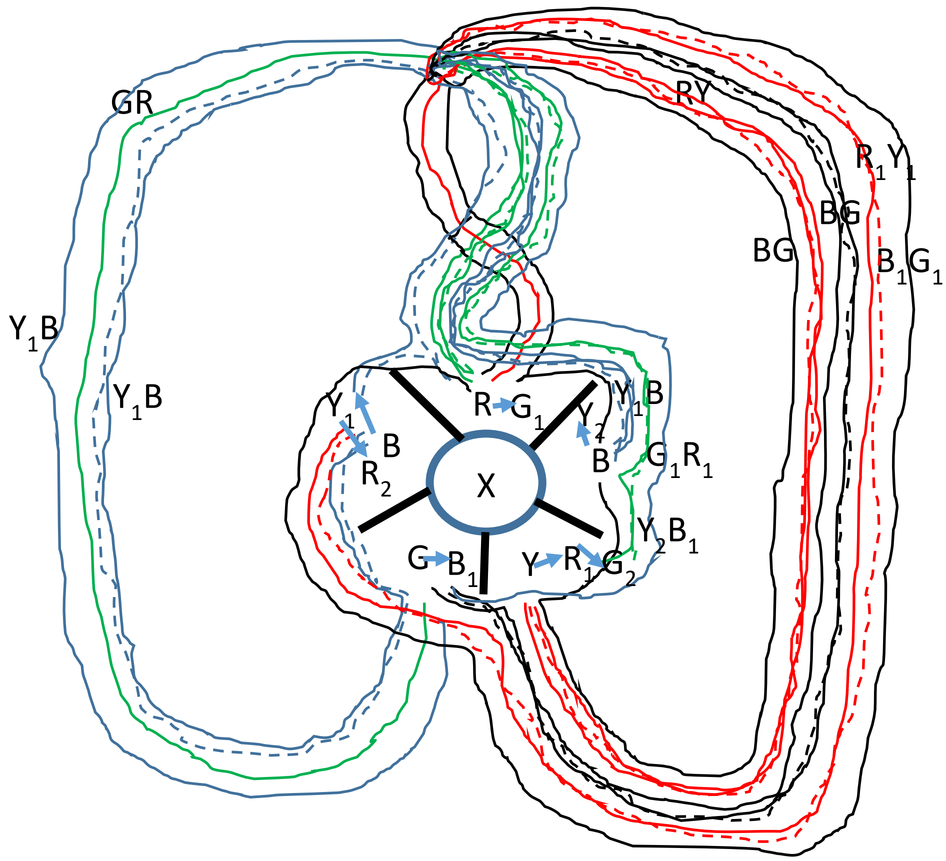


(b)

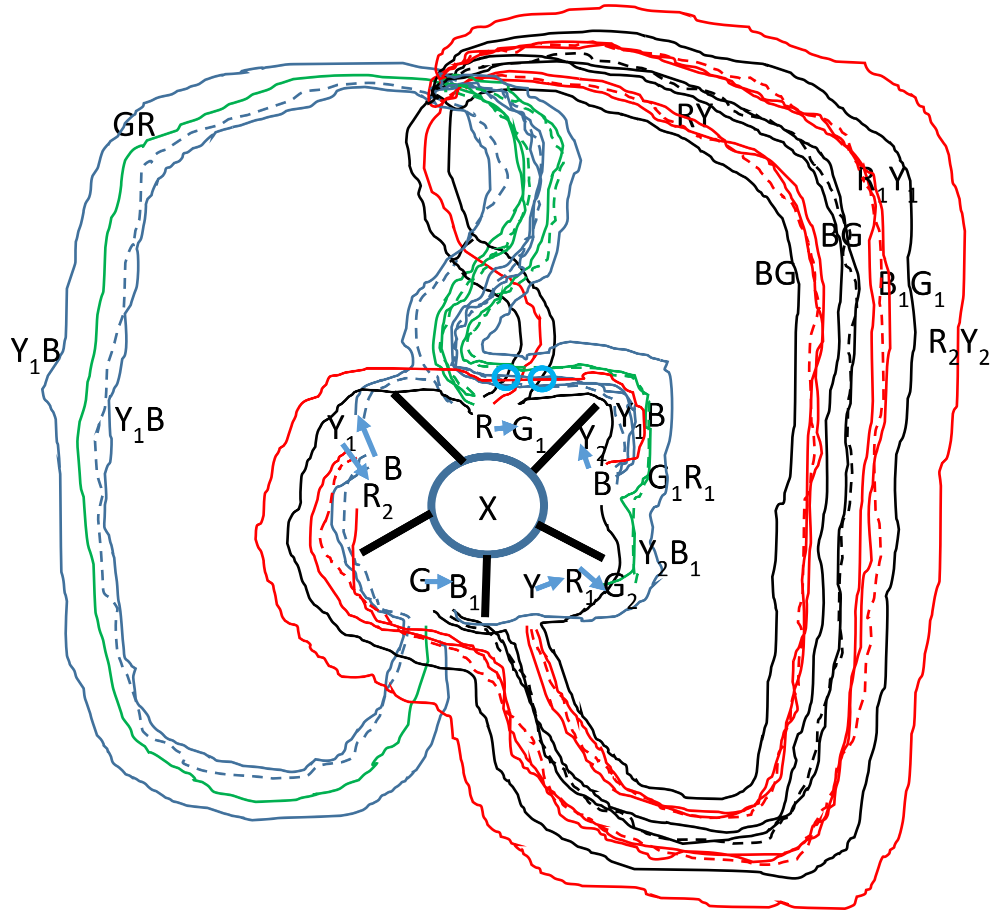
Fig. 15. The sixth step of the main proof.

Step 7: To follow the seventh row of the table, 2-Color R_1G_2 linked chain(s) interchange can be done inside the newly formed Y_2B_1 chain in the middle of Fig. 15b, which shows with dotted R_1G_2 chain in the middle of Fig. 16a. *Note: The R_1G_2 chain interchange will be restricted by the Y_2B_1 chains and cannot go out of the Y_2B_1 chains.

After the interchange, if the new R_2Y_2 chain (shown in the seventh row and the last column of the rotation table) can be formed that shown in the right-hand-side of Fig. 16b, otherwise, if new R_2Y_2 chain cannot be formed, then R_2 can be change to Y and the color of X can be determined to be R. If new R_2Y_2 chain and Y_2B_1 chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note 1: The tightest R_2Y_2 chains are using solid lines in the right-hand-side of Fig. 16b, which just show that they are naturally formed without any actions. The right-hand-side B_1G_1 chain (formed in Step 5) will be sandwiched by the new R_2Y_2 chains. The new R_2Y_2 chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest R_2Y_2 chain to B_1G_1 chain. *Note 2: Please note in the middle of Fig. 16b, there are two light blue circles, in which is R_2Y_2 crossing BG chain formed in Step 1). If BG chain have not been broken during previous steps, the R_2Y_2 chain will not form because it will not reach Y_2. Then, R_2 can be changed to Y and the color of X can be determined to be R. If BG chain have been broken and R_2Y_2 can cross, then carry on for more rotation steps.



(a)

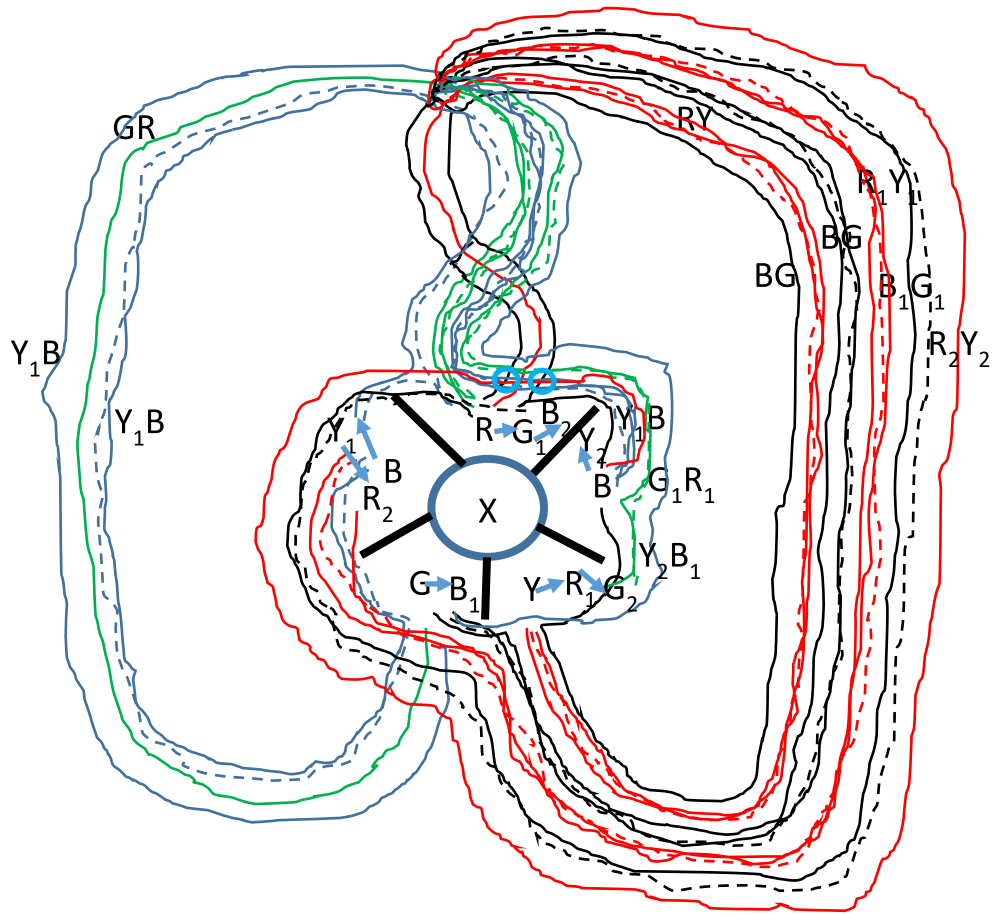


(b)

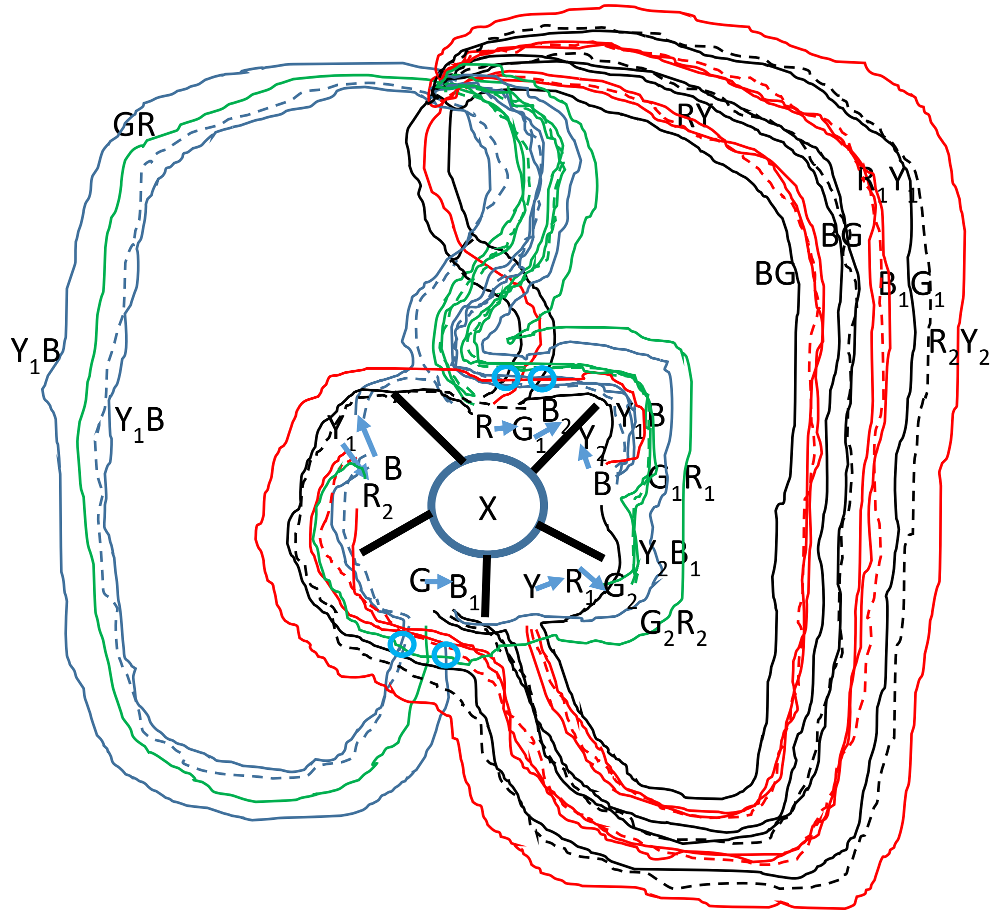
Fig. 16. The seventh step of the main proof.

Step 8: To follow the seventh row of the table, 2-Color G_1B_2 linked chain(s) interchange can be done inside the newly formed R_2Y_2 chain in the right-hand-side of Fig. 16b, which shows with dotted G_1B_2 chain in the right-hand-side of Fig. 17a. *Note: The G_1B_2 chain interchange will be restricted by the R_2Y_2 chains and cannot go out of the R_2Y_2 chains.

After the interchange, if the new G_2R_2 chain (shown in the eighth row and the last column of the rotation table) can be formed that shown in the middle of Fig. 17b, otherwise, if new G_2R_2 chain cannot be formed, then G_2 can be change to R and the color of X can be determined to be G. If new G_2R_2 chain and R_2Y_2 chain become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable. *Note 1: The tightest G_2R_2 chains are using solid lines in the middle of Fig. 17b, which just show that they are naturally formed without any actions. The middle Y_2B_1 chain (formed in Step 6) will be sandwiched by the new G_2R_2 chains. The new G_2R_2 chains can have branches. These chains may have circles but circles can only be outside the chain because the inside circles will violate the feature of the tightest G_2R_2 chain to Y_2B_1 chain. *Note 2: Please note in the middle of Fig. 17b, there are two light blue circles, in which is G_2R_2 crossing BY_1 chain formed in Step 2). If BY_1 chain have not been broken during previous steps, the G_2R_2 chain will not form because it will not reach $R_{2.}$ Then, G_2 can be changed to R and the color of X can be determined to be G. If BY_1 chain have been broken and G_2R_2 can cross, then carry on for more rotation steps.



(a)



(b)

Fig. 17. The eighth step of the main proof.

Steps (from Step 9 to the step before Final Step Z): will include

a) To carry on steps following the rotation table. Most important part is the relevant two-color chain interchange inside the newly formed two-color chain in the previous one-step.

b) The tightest new two-color chains are naturally formed without any actions.

c) It will have counter two-color chains crossing (such as new formed GR chains vs previous BY chains in Step 8). It will need to examine whether old chains (such as previous BY chains in Step 8) have been broken during previous steps or not.

- **The scenario to stop:**
  - If no new two-color chains can be naturally formed, then country X color can be determined.
  - Or, if chain 1 and chain 2 in a row of the Table 1 (or table's expansion) become separate and no cross that is similar to Fig. 4b or 4c, which is 4-colorable.
- **The scenario to continue:**

- If there is no scenario to stop, then to continue next rotation step.
- Every further 16 rotations will make the relative color and location of the 5-neighbor countries back to the initial one; however, we can just keep increasing the number of the subscript for more rotates.

Final Step Z:

For some extreme scenarios, new two-color chains can always be naturally formed in all previous steps, and chain 1 and chain 2 in a row of the Table 1 (or table's expansion) have been always crossing. In this final step, to use Fig. 18 as an example (there will be 16 of these kinds of graphs corresponding to the rotation table repeated every 16 steps), to perform two-color $B_{m-1}Y_{m+1}$ interchange to the join point in the region G_mR_m chain crossing R_mY_m chain. Because the expanding feature of all previous rotation steps, it made RY chains and BG chains expanding to reach the join point, by which the left-hand-side R_mY_m chain will be linked to the right-hand-side R_mY_m chain. Then, B_mG_m chain starting from B_m will not be able to naturally form because it will follow the path sandwiched by R_mY_m chain to go outside the R_mY_m circle and cannot reach G_m. Therefore, B_m can be change to G and X can be determined to be B.

Please note this is an extreme situation that RY chains and BG chains expanding to reach the join point, in reality, because any two-color chain may have branches and circles, which may make two-color chain linked to itself before the final step Z. Then the new two-color chain will not be naturally formed. Then, X can be determined.

Finally, the four color theorem is proved.

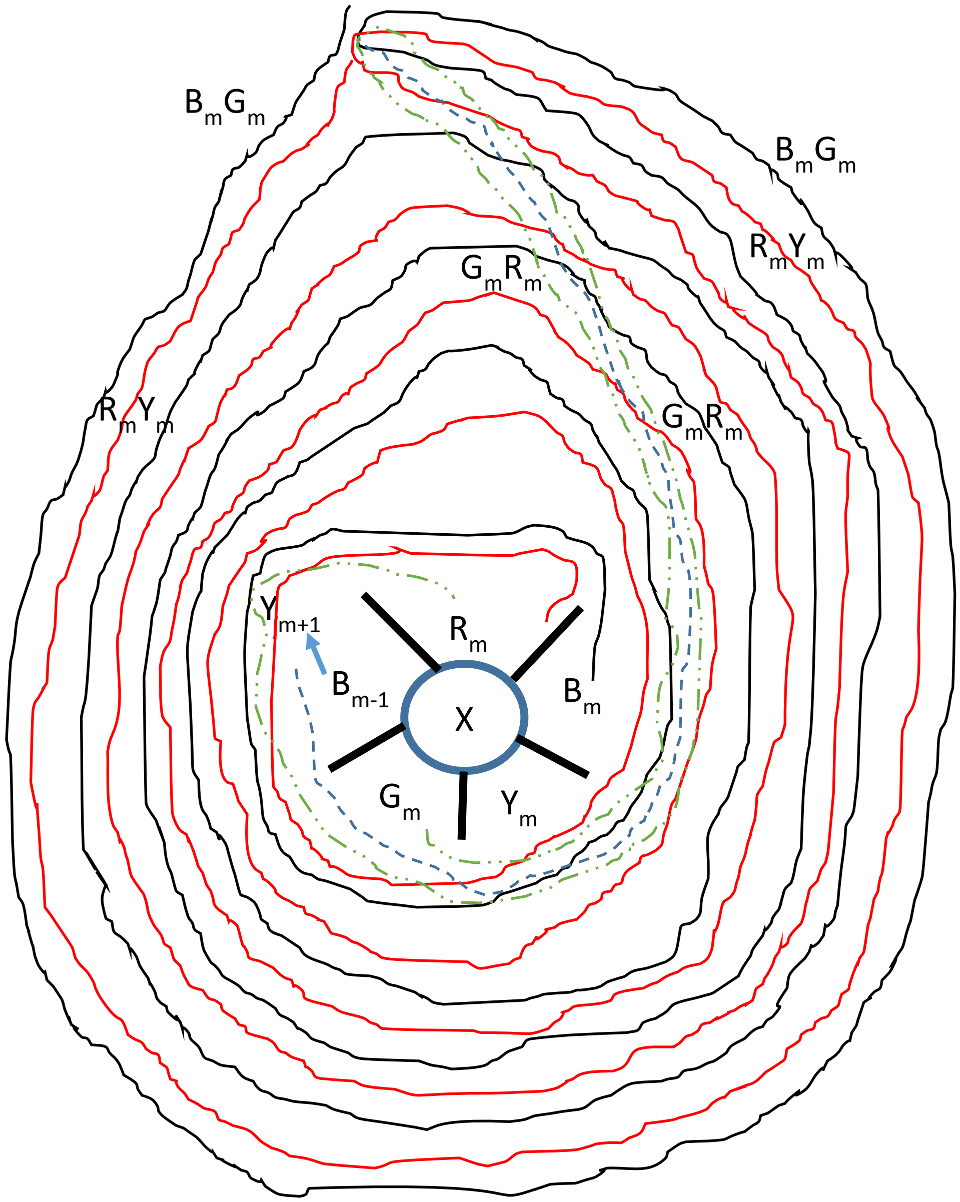


Fig. 18. The final step of the main proof.

5. Conclusions:

Based on modifications from LEMMA1, 4 and 6, a map will fully cover the whole sphere (or skewed sphere) surface, and all the points will be joined by no more than three countries. The proof is based on this new map.

I have been inspired by the teaching from Zhuan Falun book of Falun Dafa, which teaches the characteristics of the universe, and finally, I proved the Four Color Theorem by a novel method of rotation.

Data Availability Statements

The data that support the findings of this study are available from the corresponding author upon reasonable request.

Declarations

On behalf of all authors, the corresponding author states that there is no conflict of interest.

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