A Necessary and Sufficient Condition for the Existence of Global Solutions to Semilinear Parabolic Equations on Bounded Domains

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Abstract

The purpose of this paper is to give a necessary and sufficient condition for the existence and non-existence of global solutions of the following semilinear parabolic equations u t = Δ u + ψ (t) f (u) , in Ω × (0 , t *) , under the Dirichlet boundary condition on a bounded domain. In fact, this has remained as an open problem for a few decades, even for the case f (u) = u p . As a matter of fact, we prove: there is no global solution for any initial data if and only if [?] 0 [?] ψ (t) f (‖ S (t) u 0 ‖ [?]) ‖ S (t) u 0 ‖ [?] dt = [?] for every nonnegative nontrivial initial data u 0 [?] C 0 (Ω) . Here, (S (t)) t [?] 0 is the heat semigroup with the Dirichlet boundary condition.

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A Necessary and Sufficient Condition for the Existence of Global Solutions to Semilinear Parabolic Equations on Bounded Domains

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Summary

The purpose of this paper is to give a necessary and sufficient condition for the existence and non-existence of global solutions of the following semilinear parabolic equations

$$u_t = \Delta u + \psi(t) f(u), \quad \text{in } \Omega \times (0, t^*),$$

under the Dirichlet boundary condition on a bounded domain. In fact, this has remained as an open problem for a few decades, even for the case $f(u) = u^p$. As a matter of fact, we prove:

there is no global solution for any initial data if and only if

$$\int_0^\infty \psi(t) \frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty} dt = \infty$$

for every nonnegative nontrivial initial data $u_0 \in C_0(\Omega)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition.

KEYWORDS:
[MSC2020] 35B44; 35K57; 35K61
Semilinear parabolic equation, Fujita blow-up, Critical exponent

1 | INTRODUCTION

In his seminal paper[7], Fujita firstly studied the reaction-diffusion equation

$$u_t = \Delta u + u^p, \quad \text{in } \mathbb{R}^N \times (0, t^*),$$

where $p > 1$, and obtained that

(i) if $1 < p < p^*$, then there is no global solution for any initial data,

(ii) if $p > p^*$, then there exists a global solution whenever the initial data is sufficiently small,

where $p^* = 1 + \frac{2}{N}$ is called the critical exponent. For the case $p = p^*$, the following researchers obtained that there is no global solution for any nonnegative and nontrivial initial data (see[9] for the case $N = 1$ or 2 and[1] for the case $N \geq 3$).

It is easy to see that the critical exponent p^* leads a necessary and sufficient condition for the existence of the global solutions as above. It is important to find conditions (especially necessary and sufficient condition) for the global existence of solutions to the reaction-diffusion equation, since the reaction-diffusion equations describe various physical and chemical phenomena. Therefore, lots of researchers have been studied the critical exponent for various reaction-diffusion equations to find necessary and sufficient conditions for the existence of the global solutions (see the survey articles[6,11]).

In this paper, we discuss the existence and nonexistence of the global solutions to the reaction-diffusion equation for a general source term $\psi(t)f(u)$:

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + \psi(t)f(u(x,t)), & (x,t) \in \Omega \times (0,t^*), \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,t^*), \\ u(x,0) = u_0(x) \geq 0, & x \in \Omega, \end{cases} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with a smooth boundary $\partial\Omega$, u_0 is a nonnegative and nontrivial $C_0(\Omega)$-function, and ψ is a nonnegative and continuous function on $[0,\infty)$. Throughout this paper, we always assume that the function f satisfies

- f is a locally Lipschitz continuous function on $[0,\infty)$,
- $f(0) = 0$ and $f(s) > 0$ for all $s > 0$.

Then it is well-known that the local existence of the solutions of the equation (1) and the comparison principle are guaranteed.

In his pioneering paper[13], Meier studied the global existence and nonexistence of the solutions to the reaction-diffusion equations (1) and obtained the following result:

Theorem 1 ([13]). Assume that $\psi \in C[0,\infty)$ and $f(u) = u^p$ for $p > 1$. Also, Ω is a general domain in $\mathbb{R}^N$. Then the following statements are true.

(i) If $\limsup_{t\to\infty} \|S(t)w_0\|_\infty^{p-1} \int_0^t \psi(\tau)d\tau = \infty$ for every nonnegative and nontrivial $w_0 \in C_0(\Omega)$, then there is no global solution for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(ii) If $\int_0^\infty \psi(\tau)\|S(\tau)w_0\|_\infty^{p-1} d\tau < \infty$ for some nonnegative and nontrivial $w_0 \in C_0(\Omega)$, then there exists a global solution for sufficiently small nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition.

Meier give two sufficient conditions for the existence and nonexistence of the global solutions. However, a necessary and sufficient condition for the existence of the global solutions to the equation (1) is unknown, even for the case $f(u) = u^p$ and has remained as an open problem for a few decades. To our best knowledge, researches of necessary and sufficient conditions for the global existence of solutions for the reaction-diffusion equations in the current literature consider several specific source terms such as $\psi(t)f(u) = t^\sigma u^p$, $\psi(t)f(u) = e^{\beta t}u^p$, and so on (see[2,14,15]).
Moreover, recent studies of the existence and nonexistence of the global solutions discuss only sufficient conditions for the blow-up solutions and global solutions (for example, see[4,5,12]). In conclusion, the open problem has faced methodological limitations and there has been no progress in research on necessary and sufficient conditions.

From the above point of view, the purpose of this paper is to give the necessary and sufficient condition for the existence of the global solutions, for more general source term $\psi(t)f(u)$.

In order to solve the open problem for more general reaction term $\psi(t)f(u)$ instead of $\psi(t)u^p$, we need to use the minorant function f_m and the majorant function f_M as follows:

$$f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)} \text{ and } f_M(u) := \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}.$$

In fact, the minorant function f_m and the majorant function f_M were firstly discussed in[3]. Then the following condition is naturally necessary to use the minorant function f_m (see[8,10]):

$$\int_1^\infty \frac{ds}{f_m(s)} < \infty. \tag{2}$$

Finally, we state the main theorem as follows: obtained the following results to see ‘completely’ whether or not we have global solutions:

Theorem 2. Let f be a convex function satisfying the assumption (2) and ψ be a nonnegative continuous function. Then the following statements are equivalent.

(i) there is no global solution u to the equation (1) for every nonnegative and nontrivial initial data u_0.

(ii)
$$\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt = \infty$$
for every $\epsilon > 0$.

(iii)
$$\int_0^\infty \psi(t)\frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty} dt = \infty$$
for every nonnegative nontrivial initial data $u_0 \in C_0(\Omega)$.

Here, $(S(t))_{t\geq 0}$ is the heat semigroup with the Dirichlet boundary condition and λ_0 is the first Dirichlet eigenvalue of the Laplace operator Δ.

Theorem 2 is the form of a necessary and sufficient condition for global solutions of the equation (1). Therefore, we can see 'completely' whether or not we have global solutions. Also, the open problem mentioned above is solved with more general source term $\psi(t)f(u)$.

In general, the case $p = p^*$ and $p \neq p^*$ are dealt in a different way and cannot be solved at the same time (see [1,7,9,10]). However, we prove the cases all at once.

As far as authors know, there is no paper which discuss the necessary and sufficient condition (or Fujita's blow-up solutions) on the source term $f(u)$ instead of u^p. From this point of view, the minorant method will be clue to study long-time behaviors (especially, necessary and sufficient conditions) for solutions to PDEs with general type functions.

We organized this paper as follows. In Section 2, we discuss Meier's criterion. We introduce the minorant method and discuss main results in Section 3.

2 | DISCUSSION ON MEIER'S CONDITIONS

The purpose of this section is to discuss the necessary and sufficient condition for the existence of the global solutions, which has not been known and remained as an open problem. Let us deal with sufficient conditions for the blow-up solutions and global solutions to check that these conditions can be a necessary and sufficient condition.

From this point of view, let us discuss Meier's conditions. If the domain Ω is bounded, then it is well-known that $\|S(t)u_0\|_\infty \sim e^{-\lambda_0 t}$ for $t \geq 0$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$. *i.e.* for a nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$, there exist positive constants c_1 and c_2 such that $c_1\|S(t)u_0\|_\infty \leq e^{-\lambda_0 t} \leq c_2\|S(t)u_0\|_\infty$. Therefore, Theorem 1 can be understood as follows:

· If
$$(C1)\ :\ \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t \psi(\tau)d\tau = \infty,$$
then there is no global solution to the equation (1) for any nonnegative and nontrivial initial data.

· If
$$(C2)\ :\ \int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}dt < \infty$$
then there exists a global solution to the equation (1) for sufficiently small initial data.

Let us consider the function ψ defined by $\psi(t) := (t+1)^{-\delta}e^{(p-1)\lambda_0 t}$ for $0 \le \delta \le 1$. Then it follows that

$$\begin{aligned}\limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t \psi(\tau)d\tau &= \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t (\tau+1)^{-\delta}e^{(p-1)\lambda_0\tau}d\tau\\ &\le \limsup_{t\to\infty} e^{-(p-1)\lambda_0 t}\int_0^t e^{(p-1)\lambda_0\tau}d\tau\\ &= \frac{1}{(p-1)\lambda_0} < \infty\end{aligned}$$

and

$$\int_0^\infty \psi(t)e^{-(p-1)\lambda_0 t}dt = \int_0^\infty (t+1)^{-\delta}dt = \infty.$$

This implies that if the function $\psi(t) := (t+1)^{-\delta}e^{(p-1)\lambda_0 t}$ for $0 \le \delta \le 1$, then we don't know whether or not the solution exists globally.

Now, we are going to check whether or not the solution exists globally, by considering the simple example. Let's consider the functions ψ and f defined by $\psi(t) := (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}$ and $f(u) := u^2$ in the equation (1). Then the equation (1) follows that

$$\begin{cases} u_t(x,t) = \Delta u(x,t) + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}u^2, & (x,t)\in\Omega\times(0,t^*),\\ u(x,t) = 0, & (x,t)\in\partial\Omega\times(0,t^*),\\ u(x,0) = u_0(x) \ge 0, & x\in\Omega. \end{cases} \tag{3}$$

Now, we consider the eigenfunction ϕ_0 to be $\sup_{x\in\Omega}\phi_0 dx = 1$, corresponding to the first Dirichlet eigenvalue λ_0. Suppose that the solution u to the equation (3) exists globally. Multiplying the equation (3) by ϕ_0 and integrating over Ω, we use Green's theorem and Jensen's inequality to obtain

$$\begin{aligned}&\int_\Omega u_t(x,t)\phi_0(x)dx\\ &= \int_\Omega \phi_0(x)\Delta u(x,t)dx + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}\int_\Omega u^2\phi_0(x)dx\\ &= -\lambda_0\int_\Omega u(x,t)\phi_0(x)dx + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}\left(\int_\Omega u(x,t)\phi_0(x)dx\right)^2,\end{aligned}$$

for all $t > 0$. Putting $y(t) := \int_\Omega u(x,t)\phi_0(x)dx$, for $t \ge 0$, then $y(t)$ exists for all time t and satisfies the following inequality

$$\begin{cases} y'(t) \ge -\lambda_0 y(t) + (t+1)^{-\frac{1}{2}}e^{\lambda_0 t}y^2(t), \ t > 0,\\ y(0) = y_0 := \int_\Omega u_0(x)\phi_0(x)dx > 0. \end{cases} \tag{4}$$

Multiplying $e^{\lambda_0 t}$ by the inequality (4), then we have

$$\left[e^{\lambda_0 t}y(t)\right]' \ge (t+1)^{-\frac{1}{2}}\left[e^{\lambda_0 t}y(t)\right]^2 \ge 0,$$

for all $t > 0$, which implies that

$$\frac{d}{dt}\left[e^{\lambda_0 t}y(t)\right]^{-1} \le -(t+1)^{-\frac{1}{2}}$$

for all $t > 0$. Solving the differential inequality, then we obtain that

$$y(t) \ge \frac{e^{-\lambda_0 t}}{y_0^{-1} - \int_0^t(\tau+1)^{-\frac{1}{2}}d\tau},$$

for all $t > 0$, which leads a contradiction. Hence, the solution u to the equation (3) blows up at finite time.

The above example implies that the condition $(C1)$ is no longer necessary condition for the nonexistence of global solution. In fact, the main part of this paper is focused on the condition $(C2)$ to see whether $(C2)$ is necessary and sufficient condition of the existence of the global solution.

On the other hand, if the function f in the equation (1) doesn't have a multiplicative property, then we cannot apply Meier's results. For example, let us consider the function $f(u) = \frac{u^2+u}{2}$. Then it is easy to see that $u \le f(u) \le u^2$ for $u \ge 1$ and $u^2 \le f(u) \le u$ for $0 \le u \le 1$. Therefore, we cannot determine the parameter p in Theorem 1, in the case of $f(u) = \frac{u^2+u}{2}$. From this point of view, we have to consider a new method, so called the minorant method to deal with a function f which is not multiplicative. In conclusion, we provide a formula $\frac{f(\|S(t)u_0\|_\infty)}{\|S(t)u_0\|_\infty}$ instead of $\|S(t)u_0\|_\infty^{p-1}$ to give a criterion of the existence of the global solution when the source term is $\psi(t)f(u)$.

3 | MAIN RESULTS

In this section, we briefly introduce the minorant function and the majorant function. We note that the minorant function and the majorant function will play an important role. Next, we prove the main theorem by using the minorant function and majorant function.

First of all, we introduce the definition of the minorant function and majorant function of the function f.

Definition 1. For a function f, the minorant function $f_m : [0,\infty) \to [0,\infty)$ and the majorant function $f_M : [0,\infty) \to [0,\infty)$ are defined by

$$f_m(u) := \inf_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \qquad u \ge 0,$$
$$f_M(u) := \sup_{0<\alpha<1} \frac{f(\alpha u)}{f(\alpha)}, \qquad u \ge 0.$$

Then the functions f_m and f_M satisfy that $f(\alpha)f_m(u) \le f(\alpha u) \le f(\alpha)f_M(u)$, $0 < \alpha < 1$, $u > 0$. Then it is easy to see that the functions f, f_m, and f_M satisfy that

$$f_m(u)f(\alpha) \le f(u\alpha) \le f_M(u)f(\alpha), \;\; u > 0, \;\; 0 < \alpha < 1.$$

In fact, the properties of the minorant function and the majorant function were discussed in[3].

Now, we introduce the definition of the blow-up solutions and global solutions.

Definition 2. We say that a solution u blows up at finite time t^*, if there exists $0 < t^* < \infty$ such that $\|u(\cdot,t)\|_\infty \to \infty$ as $t \to t^*$. On the other hand, a solution u exists globally whenever $\|u(\cdot,t)\|_\infty$ is bounded for each time $t \ge 0$.

Now, we prove Theorem 2.

Proof of Theorem 2. $(ii) \Leftrightarrow (iii)$: It is well-known that $\|S(t)u_0\|_\infty \sim e^{-\lambda_0 t}$ for $t \ge 0$, for every nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$. That is, for each nontrivial initial data $u_0 \in C_0(\Omega)$, there exist positive constants c_1 and c_2 such that $c_1 e^{-\lambda_0 t} \le \|S(t)u_0\|_\infty \le c_2 e^{-\lambda_0 t}$ for $t \ge 0$. This implies that

$$\frac{1}{c_2} e^{\lambda_0 t} f(c_1 e^{-\lambda_0 t}) \le \frac{f\left(\|S(t)u_0\|_\infty\right)}{\|S(t)u_0\|_\infty} \le \frac{1}{c_1} e^{\lambda_0 t} f(c_2 e^{-\lambda_0 t})$$

for $t \ge 0$. This completes the proof by considering $\epsilon = c_1$ and $\epsilon = c_2$, respectively.

$(ii) \Rightarrow (i)$: Suppose that

$$\int_0^\infty \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt = \infty$$

for every $\epsilon > 0$. First of all, we consider the eigenfunction ϕ_0 to be $\sup_{x\in\Omega} \phi_0 dx = 1$, corresponding to the first Dirichlet eigenvalue λ_0. Suppose that the solution u exists globally, on the contrary. Multiplying the equation (1) by ϕ_0 and integrating over Ω, we use Green's theorem and Jensen's inequality to obtain

$$\int_\Omega u_t(x,t)\phi_0(x)dx = \int_\Omega \phi_0(x)\Delta u(x,t)dx + \psi(t)\int_\Omega f(u(x,t))\phi_0(x)dx$$
$$\geq -\lambda_0 \int_\Omega u(x,t)\phi_0(x)dx + \psi(t) f\left(\int_\Omega u(x,t)\phi_0(x)dx\right),$$

for all $t > 0$. Putting $y(t) := \int_\Omega u(x,t)\phi_0(x)dx$, for $t \geq 0$, then $y(t)$ exists for all time t and satisfies the following inequality

$$\begin{cases} y'(t) \geq -\lambda_0 y(t) + \psi(t) f(y(t)), \ t > 0, \\ y(0) = y_0 := \int_\Omega u_0(x)\phi_0(x)dx > 0. \end{cases}$$

Then the inequality can be written as

$$\left[e^{\lambda_0 t} y(t)\right]' \geq \psi(t) e^{\lambda_0 t} f(y(t)) \geq 0, \tag{5}$$

for $t > 0$ so that $e^{\lambda_0 t} y(t)$ is nondecreasing on $[0, \infty)$. On the other hand, by the definition of f_m, we can find $v_1 \in [0, 1]$ such that $f_m = 0$ on $[0, v_1)$ and $f_m > 0$ on (v_1, ∞). Then there exists $\epsilon > 0$ such that $y(0) > \epsilon v_1$. *i.e.* $v_1 < \frac{y(0)}{\epsilon} \leq \frac{e^{\lambda_0 t} y(t)}{\epsilon}$ for $t \geq 0$. Combining all these arguments, it follows from (5) and the definition of f_m that

$$\frac{\frac{e^{\lambda_0 t} y(t)}{\epsilon}}{f_m\left(\frac{e^{\lambda_0 t} y(t)}{\epsilon}\right)} \geq \frac{1}{\epsilon}\psi(t) e^{\lambda_0 t} f(\epsilon e^{-\lambda_0 t}),$$

for all $t > 0$. Now, define a function $F_m : (v_1, \infty) \to (0, v_\infty)$ by

$$F_m(v) := \int_v^\infty \frac{dw}{f_m(w)}, \ v > v_1$$

where $v_\infty := \lim_{v \to v_1} \int_v^\infty \frac{dw}{f_m(w)}$. Then it is easy to see that F_m is well-defined and continuous function. Also, F_m is a bijection and strictly decreasing with its inverse F_m^{-1} and $\lim_{v\to\infty} F_m(v) = 0$. Integrating the inequality (5) over $[0, t]$, we obtain

$$F_m\left(\frac{y(0)}{\epsilon}\right) - F_m\left(\frac{e^{\lambda_0 t} y(t)}{\epsilon}\right) \geq \frac{1}{\epsilon}\int_0^t \psi(\tau) e^{\lambda_0 \tau} f(\epsilon e^{-\lambda_0 \tau}) d\tau,$$

for all $t \geq 0$. Hence, we obtain

$$y(t) \geq \epsilon e^{-\lambda_0 t} F_m^{-1}\left[F_m\left(\frac{y(0)}{\epsilon}\right) - \frac{1}{\epsilon}\int_0^t \psi(\tau) e^{\lambda_0 \tau} f(\epsilon e^{-\lambda_0 \tau}) d\tau\right]$$

for all $t \geq 0$, which implies that $y(t)$ cannot be global.

$(i) \Rightarrow (ii)$: Suppose that

$$\int_0^\infty \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt < \infty$$

for some $\epsilon > 0$. We note that there exists a maximal interval $[0, m^*)$ such that f_M is finite. Then it is true that the integral $\int_v^{m^*} \frac{dw}{f_M(w)}$ is finite for each $v \in (0, m^*)$, $\lim_{v\to 0} \int_v^{m^*} \frac{dw}{f_M(w)} = \infty$, and $\lim_{v\to m^*} \int_v^{m^*} \frac{dw}{f_M(w)} = 0$. Then a function $F_M : (0, m^*) \to (0, \infty)$ defined by

$$F_M(v) := \int_v^{m^*} \frac{dw}{f_M(w)}, \ v \in (0, m^*)$$

is a well-defined and continuous function. Also, F_m is a bijection and strictly decreasing with its inverse F_M^{-1}. Now, take a number z_0 such that

$$0 < z_0 < F_M^{-1}\left[\frac{1}{\epsilon}\int_0^\infty \psi(t) e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) dt\right]$$

and define a nondecreasing function $z : [0,\infty) \to [z_0,\infty)$ by

$$z(t) := F_M^{-1}\left[F_M(z_0) - \frac{1}{\epsilon}\int_0^t \psi(\tau)e^{\lambda_0\tau} f\left(\epsilon e^{-\lambda_0\tau}\right)d\tau\right],\ t \geq 0.$$

Then $z(t)$ is a bounded solution of the following ODE problem:

$$\begin{cases} z'(t) = \frac{1}{\epsilon}\psi(t)e^{\lambda_0 t} f\left(\epsilon e^{-\lambda_0 t}\right) f_M\left(z(t)\right),\ t > 0, \\ z(0) = z_0. \end{cases}$$

Now, consider a function $v(x,t) := e^{-\lambda_0 t}\phi_0(x)$ on $\overline{\Omega}\times[0,\infty)$ which is a solution to the heat equation $v_t = \Delta v$ under the Dirichlet boundary condition. Let $\overline{u}(x,t) := z(t)v(x,t)$ for $(x,t) \in \overline{\Omega}\times[0,\infty)$. Since f is convex, $\frac{f(u)}{u}$ is nondecrasing. Then it follows that

$$\begin{aligned} \overline{u}_t(x,t) =& \Delta\overline{u}(x,t) + \psi(t)v(x,t)\left[\frac{f(\epsilon e^{-\lambda_0 t})}{\epsilon e^{-\lambda_0 t}}\right] f_M(z(t)) \\ \geq& \Delta\overline{u}(x,t) + \psi(t)v(x,t)\left[\frac{f(v(x,t))}{v(x,t)}\right] f_M(z(t)) \\ \geq& \Delta\overline{u}(x,t) + \psi(t)f(\overline{u}(x,t)) \end{aligned}$$

for all $(x,t) \in \Omega\times(0,\infty)$. It follows that $\overline{u}$ is the supersolution to the equation (1), which implies that u exists globally. □

Corollary 1. Let the function ψ be a nonnegative continuous function and the function f be a nonnegative continuous and quasi-multiplitive function. *i.e.* there exist $\gamma_2 \geq \gamma_1 > 0$ such that

$$\gamma_1 f(\alpha)f(u) \leq f(\alpha u) \leq \gamma_2 f(\alpha)f(u), \tag{6}$$

for $0 < \alpha < 1$ and $u > 0$. Then the following statements are equivalent:

(i) $\int_0^\infty \psi(t)e^{\lambda_0 t} f\left(e^{-\lambda_0 t}\right)dt = \infty$.

(ii) $\int_0^\infty \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}dt = \infty$ for every nonnegative and nontrivial $w_0 \in C_0(\Omega)$.

(iii) $\int_0^\infty \psi(t)\frac{dt}{F(e^{-\lambda_0 t})} = \infty$, where $F(v) := \int_v^\infty \frac{dw}{f(w)}$.

(iv) There is no global solution to the equation (1) for any initial data.

Proof. Theorem 2 says that (i), (ii), and (iv) are equivalent. Therefore, we now discuss (iii).
(i)⇔(iii) : Let $F(v) = \int_v^\infty \frac{dw}{f(w)}$. Then the assumption (6) follows that

$$\int_1^\infty \frac{z}{\gamma_2 f(z)f(s)}ds \leq \int_z^\infty \frac{dw}{f(w)} \leq \int_1^\infty \frac{z}{\gamma_1 f(z)f(s)}ds,$$

for $z > 0$. This implies that

$$\frac{F(1)}{\gamma_2}\frac{z}{f(z)} \leq F(z) \leq \frac{F(1)}{\gamma_1}\frac{z}{f(z)}.$$

i.e. $F(z) \sim \frac{z}{f(z)}$, $z > 0$. Therefore, the proof is complete. □

Remark 1. In 2014, Loayza and Paixão[12] studied the conditions for existence and nonexistence of the global solutions to the equation (1) under the general domain and obtained the following statements:

(i) for every $w_0 \in C_0(\Omega)$, there exist $\tau > 0$ such that

$$\int_{\|S(\tau)w_0\|_\infty}^\infty \frac{dw}{f(w)} \leq \int_0^\tau \psi(\sigma)d\sigma, \tag{7}$$

then there is no global solution u for every initial data,

(ii) the solution u exists globally for small initial data, whenever

$$\int_0^\infty \psi(t)\frac{f(\|S(t)w_0\|_\infty)}{\|S(t)w_0\|_\infty}dt < 1, \tag{8}$$

for some $w_0 \in C_0(\Omega)$.

In fact, Corollary 1 imply that the conditions (7) and (8) have a strong relation, even though (7) and (8) have different formulas.

Also, by using Corollary 1, the example in Section 2 can be characterized completely as follows:

***Example* 2.** Let the domain Ω be bounded in $\mathbb{R}^N$, $\psi(t) := (t+1)^{-\sigma}e^{kt}$, and $f(u) := u^p$ where $\sigma \in \mathbb{R}$, $k \in \mathbb{R}$, and $p > 1$. Then the following statements are true.

(i) If $k > (p-1)\lambda_0$, then there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(ii) If $k < (p-1)\lambda_0$, then there exists a global solution to the equation (1) for sufficiently small initial data $u_0 \in C_0(\Omega)$.

(iii) If $k = (p-1)\lambda_0$ and $\sigma \le 1$, then there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$.

(iv) If $k = (p-1)\lambda_0$ and $\sigma > 1$, then there exists a global solution to the equation (1) for sufficiently small initial data $u_0 \in C_0(\Omega)$.

Lastly, we give another example.

***Example* 3.** Let the domain Ω be bounded in $\mathbb{R}^N$, $\psi(t) := e^{rt} + e^{st}$, and $f(u) := u^p + u^q$ where $r \ge s \ge 0$ and $p \ge q > 1$. Then we easily see that

$$f_m(u) = \min\left\{u^q, \frac{u^p + u^q}{2}\right\} \text{ and } f_M(u) = \max\left\{u^q, \frac{u^p + u^q}{2}\right\}.$$

This implies that $\int_1^\infty \frac{ds}{f_m(s)} < \infty$. On the other hand,

$$\int_0^\infty \left(e^{rt} + e^{st}\right)e^{\lambda_0 t}\left(\epsilon^p e^{-\lambda_0 pt} + \epsilon^q e^{-\lambda_0 qt}\right)dt = \int_0^\infty \epsilon^p\left(e^{(r-(p-1)\lambda_0)t} + e^{(s-(p-1)\lambda_0)t}\right) + \epsilon^q\left(e^{(r-(q-1)\lambda_0)t} + e^{(s-(q-1)\lambda_0)t}\right)dt.$$

Therefore, there is no global solution u to the equation (1) for any nonnegative and nontrivial initial data $u_0 \in C_0(\Omega)$ if and only if $r - (q-1)\lambda_0 \ge 0$.

CONFLICT OF INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

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