Wave process in viscoelastic media using fractional derivatives with non singular kernels

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Abstract

We consider the equations of motion of a bar, with given density, infinite in both directions, subjected to longitudinal vibrations under the action of an external load, and a stress-strain relation represented by a fractional order operator. Using three types of fractional operators, the initial-boundary value problems associated with the described phenomenon are posed and solved. Through the bivariate Mittag-Leffer function, which has been recently introduced, we find the fundamental solution of these problems and calculate their moments.

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Abstract

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Keywords
Wave propagation process; linear viscoelasticity; Caputo-Fabrizio fractional derivative, Atangana-Baleanu fractional derivative; Bivariate Mittag-Leffler function.

Mathematics Subject Classification
33E12, 74D05, 74J05

1 Introduction

There are many situations in industry that depend on the ability to characterize and understand how a material deforms when exposed to external influences. The behavior of materials has two typical limits. Elastic materials are rigid materials whose deformation is related to their mechanical stress relative to their stress-free reference geometry and are fully stored as recoverable elastic energy. On the other hand, Newtonian fluids are viscous substances whose deformation speed depends on their mechanical stress; here the mechanical stress necessary to deform the material is completely lost [1].

The study of the behavior of viscoelastic materials is addressed in various contexts and applications, such as materials science, metallurgy, and solid-state physics. In addition, any complex or composite construction containing embedded polymers exhibits viscoelastic behavior under static and dynamic stress conditions. Viscoelastic materials have the characteristic of their time-dependent behavior. When viscoelastic materials are constantly deformed, their internal stress diminishes over time - this is called relaxation. If viscoelastic material is constantly loaded (stressed) its deformation (strain) increases over time - this is called creep [2].

In the field of biology, medicine, petroleum, chemical, and civil engineering, viscoelastic materials, such as rubbers, elastomers, resins, concrete, and skeletons, are common. They have become one of the potential options for novel multifunctional materials due to their excellent intrinsic rheological properties [2–4].

Analytical methods play an important pillar in the study of non-stationary wave processes in linear viscoelastic media [5, 10]. In the description of the inherited properties of a viscoelastic medium, stress and strain are related by means of a Volterra integral operator with a specific kernel or, alternatively, by fractional calculus operators (see [6, 10] for details). For a historical overview of early contributions to this topic, see [7].

The theory of viscoelasticity describes processes in which the state of a mechanical system depends on a complete description of all the actions or forces that acted on it. Currently the accelerated development of this theory has been stimulated by the various technological applications related to the study of sliding or dragging of various bodies, such as metal, concrete, rocks, among others. Through mechanical tests, such as those of traction, compression or torsion, carried out on certain materials, it has been discovered that there is a one-to-one correspondence between tension and strain, therefore, it is necessary to introduce time into the constitutive equations in one way or another [8]. The classical viscoelastic models are consisted of parallel or series with elastic and viscous elements. The exponential material functions of these models encounter difficulties in characterizing the power-law phenomena, which are widely observed for various viscoelastic materials [9]. Gemant was the first scientist to justify the need to use fractional differential operators to describe phenomena in some viscoelastic fluids. Scott-Blair considered the viscoelastic material as the intermediate state between the elastic solid and the viscous fluid and introduced the fractional derivative of the deformation in the constitutive equation, called the Scott-Blair model. The fractional viscoelastic model is validated to predict well power-law type phenomena. Thereafter, the fractional theory of linear viscoelasticity has been gradually improved by Rabotnov, Bagley, Caputo and Mainardi, et al. [10]. Some of the first works to deal with interesting aspects of the application of fractional calculus in viscoelasticity are considered in [34–37]

A wide variety of linear and nonlinear constitutive models are proposed to define the viscoelastic deformation process of viscoelastic materials in order to explore their mechanical behavior. The models most used by Young Operator include the Kelvin-Voigt, Maxwell derivative relation, and the standard linear solid model to describe viscoelastic objects such as beams, plates, and shells, considering the Poisson ratio as continuous for viscoelastic materials [2, 11, 12].

However, the integer-order constitutive stress-strain coupling for conventional viscoelasticity has been questioned because the expected early stages of creep and relaxation do not match the experimental evidence [2, 13]. This is mainly due to the fact that the integer-order differential equations of traditional viscoelastic models may fail in some situations, such as stress relaxation in polymer foams [2, 14] and ultra-slow relaxation in polymer thin films [2, 15]. Fractional order calculus provides a suitable alternative, in order to replace the relation of the classical viscoelastic foundational model, by the theoretical modification of mechanical deformation [2, 16, 17]. The fractional viscoelastic model uses fractional order derivatives to correlate stress fields with areas of deformation in viscoelastic materials. It has been shown that constitutive equations using fractional derivatives are related to molecular theories, which describe the macroscopic behavior of viscoelastic media.
Fractional viscoelastic models have been widely used to describe complex dynamics such as relaxation, oscillation and wave for various types of real materials. More detailed information can be found, e.g. in [7, 18, 19].

Modern fractional calculus relies on the exponential function (Caputo-Fabrizio operator) and the Mittag-Leffler function (Atangana-Baleanu derivative) as memory kernels. However, there are controversial applications and discussions about their origin, correctness and applications. In [23, 24] we dispel many questions and try to show their clear physical basis with the appropriate mathematical correction, as well as that they appear naturally in the constitutive equations of linear viscoelasticity. It is also shown in a very natural way that both the classical power law function (the memory kernel of the classical fractional derivatives) and the Mittag-Leffler function can be approximated by sums of exponentials, and in particular by the Prony series [28].
There are many real physical situations where monotonically decreasing responses cannot be modeled (approximated) by simple power law functions to apply the classical fractional operators (Riemann-Liouville or Caputo). When monotonically decreasing responses cannot be modeled (approximated) by a simple power law function, the main idea is to approximate by finite sums of elementary functions. Studies in linear viscoelasticity are carried out in [25–27] that justify the above mentioned.

One issue that arises when using fractional operators with non-singular kernel is that the model problems in some cases do not satisfy the initial conditions. The aforementioned inconsistency between the solutions and the initial conditions has already been observed in [30, 31]. This has led to the invention of modifications to overcome this drawback. One can review some of the following references for proposals [32, 33].

In the present manuscript we will consider the equations of motion of a bar, infinite in both directions, subjected to longitudinal vibrations under the action of an external load, and a stress-strain type relationship, which is modeled by means of a fractional order operator. We use three types of fractional operators, two non-singular, one with exponential kernel and one of Mittag-Leffler type. And a singular one with power-law type kernel. We pose and solve initial-boundary value problems in viscoelastic media, resulting from the modeling of the aforementioned wave processes. Using the recently appeared bivariate Mittag-Leffler function, we establish the fundamental solution of these initial-boundary value problems and calculate their moments.

2 Preliminares and fractional operators

Let

$$[a,b] \subset \mathbb{R}, \quad \alpha > 0, \quad n = -[-\alpha] = \lceil \alpha \rceil,$$

where $[.]$ is the entire real part and $\lceil . \rceil$ is the upper integer.

Definition 2.1 *The left Riemann-Liouville fractional integral operator of order α, on the real line $\mathbb{R}^+ := \{t \in \mathbb{R} \mid t > 0\}$ of the function u: $\mathbb{R}_x \times \mathbb{R}_t^+ \longrightarrow \mathbb{R}$ is define by*

$$({}_tI_{0+}^{\alpha}u)(x,t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} u(x,\xi) d\xi, \tag{2.1}$$

provides that the integral convergent. Here $\Gamma(\cdot)$ denotes Euler's Gamma function.

Definition 2.2 *Let $f \in L^1(\mathbb{R})$. The integrals*

$$(I_+^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{-\infty}^{x} (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad -\infty < x < \infty, \tag{2.2}$$

$$(I_-^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{\infty} (x-\xi)^{\alpha-1} f(\xi) d\xi, \quad -\infty < x < \infty, \tag{2.3}$$

define Liouville's fractional integrals of order α, from the left and from the right, respectively.

Definition 2.3 *Let the functions $f(x)$ be defined in $\mathbb{R}$, the equalities*

$$\left({}^L D_+^{\alpha} f\right)(x) = [D^n I_+^{n-\alpha} f](x), \tag{2.4}$$

$$\left({}^L D_-^{\alpha} f\right)(x) = (-1)^n [D^n I_-^{n-\alpha} f](x), \tag{2.5}$$

define the fractional Liouville derivatives of order α, on the left and on the right, respectively.

A sufficient condition for the expressions (2.4) and (2.5) to exist is that $f \in AC^{[\alpha]}(\mathbb{R})$; if $\alpha = n$ they coincide with the usual definitions of derivative. Liouville fractional integrals and fractional partial derivatives are defined in a straightforward and trivial way, provided that the function f verifies the necessary conditions posed for the one-dimensional case, with respect to the integration variable, for the existence of the corresponding fractional partial operators.

Definition 2.4 *The Riemann-Liouville fractional partial operator of order α of function $u(x,t)$ on the half axis $\mathbb{R}_t^+$ is defined as*

$$
{}^{RL}D_t^{\alpha}u(x,t) = \begin{cases} \dfrac{\partial^n}{\partial t^n}u(x,t), & \alpha = n \in \mathbb{N}, \\ \dfrac{1}{\Gamma(n-\alpha)}\dfrac{\partial^n}{\partial t^n}\displaystyle\int_0^t \dfrac{u(x,\tau)}{(t-\tau)^{\alpha+1-n}}d\tau, & n-1 < \alpha < n \in \mathbb{N}, \\ ({}_tI_{0+}^{-\alpha}u)(x,t), & \alpha < 0, \end{cases} \tag{2.6}
$$

where $n = \lceil \alpha \rceil$.

Remarks 2.5 *If $u(\cdot,t) \in C(\mathbb{R}_t^+) \cap L^1(\mathbb{R}_t^+)$ for each fixed $x \in \mathbb{R}$, then $({}_0^{RL}D_t^{\beta}\, {}_tI_{0+}^{\beta}u)(x,t) = u(x,t)$*

Definition 2.6 (see [38]) *The Liouville-Caputo fractional partial operator of order β, with $\beta > 0$ and $\beta \neq \mathbb{N}$ on $\mathbb{R}^+$ of function $u(x,t)$ is defined as*

$$
{}^{C}D_t^{\beta}u(x,t) = \frac{1}{\Gamma(n-\beta)}\int_0^t \frac{\frac{\partial^n}{\partial \tau^n}u(x,\tau)}{(t-\tau)^{\beta+1-n}}d\tau, \quad n-1<\beta<n, \quad n \in \mathbb{N}. \tag{2.7}
$$

When $0 < \beta < 1$, expression (2.7) take the following form

$$
{}^{C}D_t^{\beta}u(x,t) = \frac{1}{\Gamma(1-\beta)}\int_0^t \frac{\frac{\partial}{\partial \tau}u(x,\tau)}{(t-\tau)^{\beta}}d\tau.
$$

The validity of this definition is limited for functions u such that $\frac{\partial^n}{\partial t^n}u(\cdot,t) \in L^1(\mathbb{R}_t)$ for each x fixed.

Definition 2.7 (see [39]) *Let $u(\cdot,t) \in H^1(\mathbb{R}_t^+)$. The Caputo-Fabrizio fractional operator of order α in the Caputo sense of function $u(x,t)$ is given as follows*

$$
{}^{CF}D_t^{\alpha}u(x,t) = \frac{M(\alpha)}{n-\alpha}\int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-\tau)\right)\frac{\partial^n}{\partial \tau^n}u(x,\tau)d\tau, \tag{2.8}
$$

where $n-1<\alpha<n$, $n \in \mathbb{N}$ with $M(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

This definition has no singularities at $t = s$.

The Mittag-Leffler function of one parameter $E_\gamma(z)$ is given by [38]

$$
E_\gamma(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\gamma k+1)}, \quad z \in \mathbb{C}, \quad \mathrm{Re}(\gamma) > 0. \tag{2.1}
$$

Another important function in the fractional calculus is represented by the Wright function and gives by [29]

$$
W(z;\alpha,\beta) = \sum_{n=0}^{\infty}\frac{z^n}{\Gamma(\alpha n+\beta)n!}, \tag{2.2}
$$

$$
\text{or,}\;\; W(z;\alpha,\beta) = \frac{1}{2\pi i}\int_{H_a}\frac{\mathrm{e}^{\sigma+z\sigma^{-\alpha}}}{\sigma^{\beta}}d\sigma, \quad z \in \mathbb{C}, \quad \alpha > -1, \quad \beta > 0. \tag{2.3}
$$

Definition 2.8 (see [40]) *Let $u(\cdot,t) \in H^1(\mathbb{R}_+)$. The Atangana-Baleanu fractional derivative in the Liouville-Caputo sense (ABC) is defined as follows*

$$
{}^{ABC}D_t^{\alpha}u(x,t) = \frac{B(\alpha)}{n-\alpha}\int_0^t \frac{\partial^n}{\partial t^n}u(x,\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{n-\alpha}\right)d\tau, \tag{2.4}
$$

with $n-1<\alpha<n$, $n\in\mathbb{N}$ and where, $B(\alpha)$ is a normalization function, $B(0) = B(1) = 1$.

Definition 2.9 *Let $u(\cdot,t) \in H^1(\mathbb{R}^+)$. The Atangana-Baleanu fractional derivative in the Riemann-Liouville sense (ABR) is defined as follows*

$$
{}^{ABR}D_t^{\alpha}u(x,t) = \frac{B(\alpha)}{n-\alpha}\frac{\partial^n}{\partial t^n}\int_0^t u(x,\tau)E_{\alpha}\left(-\alpha\frac{(t-\tau)^{\alpha}}{n-\alpha}\right)d\tau, \tag{2.5}
$$

with $n-1<\alpha<n$, $n\in\mathbb{N}$ and $B(\alpha)$ as in Definition 2.8.

The Laplace transform of a function $u(x,t)$ with respect to t is given by

$$
u(x,s) \equiv (\mathcal{L}_t u)(x,s) = \int_0^{\infty} \mathrm{e}^{-st}u(x,t)dt, \tag{2.6}
$$

for any fixed x. And the Fourier transform with respect to x

$$
u(k,t) \equiv (\mathcal{F}_x u)(k,t) = \int_{-\infty}^{\infty} \mathrm{e}^{ikx}u(x,t)dx, \tag{2.7}
$$

for any fixed t.

The inverse Laplace transform with respect to s is given by

$$
(\mathcal{L}_s^{-1}u)(x,t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \mathrm{e}^{st}u(x,s)ds, \tag{2.8}
$$

for any fixed k. And the inverse Fourier transform with respect to k

$$
(\mathcal{F}_k^{-1}u)(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{e}^{-ikx}u(k,t)dx. \tag{2.9}
$$

The following relationships are fulfilled

$$
(\mathcal{L}_s^{-1}(\mathcal{L}_t u)(x,t)u)(x,t) = u(x,t), \text{ y } (\mathcal{F}_k^{-1}(\mathcal{F}_x u)(x,t)u)(x,t) = u(x,t). \tag{2.10}
$$

Let $\mathcal{L}_t(\mathbb{R}^+)$ be the space of functions such that they possess Laplace transform. We denote by $\mathcal{F}(\mathbb{R})$ the space of all functions $f(x)$ such that their Fourier transform exists and use the notation $f(k) := (\mathcal{F}_x f)(k)$.

Definition 2.10 *We write $\mathcal{L}F := \mathcal{L}(\mathbb{R}_+)\times\mathcal{F}(\mathbb{R})$ as the space of functions $u(x,t)$ such that their Laplace transform and Fourier transform exist.*

• If $0 \leq n-1 < \alpha \leq n$, then

$$\mathcal{L}\left({}^{RL}D_t^{\alpha}u\right)(x,s) = s^{\alpha}\left(\mathcal{L}f\right)(s) - \sum_{k=0}^{n-1} s^k \left({}^{RL}D_t^{\alpha-k-1}u\right)(x,0), \tag{2.11}$$

for $\mathrm{Re}(s) > m$, with $m = \max\{\nu_k \mid k = 0,1,2,...,n-1\}$.
• If $0 < \alpha < 1$, then

$$\mathcal{L}\left({}^{CFC}D_t^{\alpha}u\right)(x,s) = \frac{s\mathcal{L}(u)(x,s) - u(x,0)}{s+\alpha(1-s)}. \tag{2.12}$$

• If, $0 < \alpha \leq 1$, then we define the Laplace transform for the Atangana-Baleanu fractional derivative in the Liouville-Caputo sense as follows [40]

$$\mathcal{L}\left({}^{ABC}D_t^{\alpha}u\right)(x,s) = \frac{s^{\alpha}\mathcal{L}(u)(x,s) - s^{\alpha-1}u(x,0)}{s^{\alpha}(1-\alpha)+\alpha}. \tag{2.13}$$

• If, $0 < \alpha \leq 1$, then we define the Laplace transform for the Atangana-Baleanu fractional derivative in the Riemann-Liouville sense as follows [40]

$$\mathcal{L}\left({}^{ABR}D_t^{\alpha}f(t)\right)(s) = \frac{s^{\alpha}\mathcal{L}(u)(x,s)}{s^{\alpha}(1-\alpha)+\alpha}. \tag{2.14}$$

2.1 Lizorkin space and fractional derivatives

Let us denote by $\mathcal{S}$ the Schwartz space of infinitely derivable functions on the entire real line, which, like all their derivatives, tend to zero for $|x| \to \infty$ more quickly than any power of x^{-1}. In other words, if $C_0^{\infty}(\mathbb{R})$ is the space of all infinitely derivable functions on the entire real line such that $f(x) \equiv 0$ in a neighborhood of $x = -\infty$ and $x = +\infty$, then the set $\mathcal{S}$, with $C_0^{\infty}(\mathbb{R}) \subset \mathcal{S}$ consists of those, and only those, infinitely differentiable functions φ, for which the condition is satisfied

$$\mathcal{S} = \left\{\varphi \in C_0^{\infty}(\mathbb{R}) \;:\; \lim_{|x|\to\infty} x^n \varphi^{(m)}(x) = 0\right\}, \quad \text{for all } n, m \in \mathbb{N}. \tag{2.15}$$

Within the Schwartz space, Lizorkin introduced a subspace of functions $\Phi \subset \mathcal{S}$, invariant with respect to integration and fractional derivative, in the sense that the function resulting from the application of these fractional operators continued to belong to the space Φ.

Definition 2.11 *Given the space*

$$\Psi = \left\{\psi \;:\; \psi \in \mathcal{S},\ \psi^{(m)}(0) = 0,\ m = 0,1,2,\ldots\right\}, \tag{2.16}$$

the set of functions of $\mathcal{S}$ whose Fourier transform belongs to the space Ψ is called Lizorkin space and is defined by

$$\Psi = \left\{\varphi \;:\; \varphi \in \mathcal{S},\ (\mathcal{F}\varphi) \in \Psi\right\}. \tag{2.17}$$

The Lizorkin space can be equivalently characterized as the space of Schwartz functions $\varphi \in \mathcal{S}$ that happen to be orthogonal to all polynomials

$$\int_{-\infty}^{+\infty} t^r \varphi(t)dt = 0, \quad r = 0,1,2,\ldots \tag{2.18}$$

Proposition 2.12 *Given a function $f(x)$ such that $f(x) \in \Phi$ and α, with $\alpha > 0$, it follows*

$$\begin{aligned}
\left(\mathcal{F}_x(I^\alpha f)\right)(k) &= (-ik)^{-\alpha}(\mathcal{F}_x f)(k), && (2.19)\\
\Big(\mathcal{F}_x({}^L D^\alpha f)\Big)(k) &= (-ik)^{\alpha}(\mathcal{F}_x f)(k). && (2.20)
\end{aligned}$$

In fact, the existence conditions of the Fourier transform of the fractional derivative can be smoothed as follows

Proposition 2.13 *Given a function $f(x) \in C^{n-1}(\mathbb{R})$, $n-1 < \alpha \leq n$ and such that*

$$\lim_{|x|\to\infty} f^{(r)}(x) = 0, \quad r = 0, 1, \dots n-1, \tag{2.21}$$

then,

$$\left(\mathcal{F}_x\left({}_{+}^{L}D^\alpha f\right)\right)(k) = (-ik)^{\alpha}(\mathcal{F}_x f)(k). \tag{2.22}$$

Due to the validity of this last property, it will be very useful to introduce explicitly a new space, which contains the Schwartz space, and which we define as follows

$$\overline{\mathcal{S}} = \left\{\varphi \in C_0^\infty(\mathbb{R}) \;:\; \lim_{|x|\to\infty} \varphi^{(m)}(x) = 0,\; m = 0, 1, 2\right\}. \tag{2.23}$$

In this space one can define the Fourier transform of the Liouville fractional derivative of order α, for all $\alpha > 0$.

The Mittag-Leffler function of two and three parameters (Prabhakar's classical function) [43–45] are defined respectively by

$$E_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0,\; \beta \in \mathbb{C}. \tag{2.24}$$

$$E^{\gamma}_{\alpha,\beta}(z) := \sum_{k=0}^{\infty} \frac{(\gamma)_k}{k!\Gamma(\alpha k + \beta)} z^k, \quad \operatorname{Re}\alpha, \operatorname{Re}\beta > 0,\; \gamma > 0, \tag{2.25}$$

where $(\gamma)_k := \gamma(\gamma+1)\cdots(\gamma+k-1) = \frac{\Gamma(\gamma+k)}{\Gamma(\gamma)}$, $(\gamma)_0 := 1$ and $(1)_k = \frac{\Gamma(1+k)}{\Gamma(1)} = k!$.

The definition of the Prabhakar function in its most general form is given below.

2.2 Multivariate Mittag-Leffler function

Definition 2.14 (see [48]) *Let $z_j \in \mathbb{C}$, $\alpha_j, \beta, \delta \in \mathbb{R}$ with $\alpha_j > 0$, $j = 1, \cdots, m$. Then, the multinomial function of Prabhakar is defined as*

$$E^{\delta}_{\alpha_1,\dots,\alpha_n,\beta}\big(z_1,\dots,z_n\big) := \sum_{k=0}^{\infty} \sum_{\substack{k_1+k_2+\cdots+k_n=k\\ k_1,\dots,k_n\geq 0}} \frac{(\delta)_k}{\prod\limits_{j=1}^{n} k_j!} \cdot \frac{\prod\limits_{j=1}^{n} z_j^{k_j}}{\Gamma\left(\sum\limits_{j=1}^{n}\alpha_j k_j + \beta\right)}. \tag{2.26}$$

Furthermore, since the double sum in (2.26) converges absolutely and locally uniformly for any z_j with $j = 1, \ldots, n$, then this expression can be replaced by the following multiple sum

$$E^{\delta}_{\alpha_1,\ldots,\alpha_n,\beta}\big(z_1,\ldots,z_n\big) := \sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}\frac{(\delta)_{k_1+\cdots+k_n}}{\prod\limits_{j=1}^{n}k_j!}\cdot\frac{\prod\limits_{j=1}^{n}z_j^{k_j}}{\Gamma\left(\sum\limits_{j=1}^{n}\alpha_j k_j+\beta\right)}. \tag{2.27}$$

Some known particular cases of the expression (2.27), are the following

$$E^{1}_{\alpha_1,1}\big(z_1\big) = \sum_{k_1=0}^{\infty}\frac{(1)_{k_1}}{k_1!}\cdot\frac{z_1^{k_1}}{\Gamma\big(\alpha_1 k_1+1\big)} = E_{\alpha_1,1}\big(z_1\big), \quad \text{if} \quad \delta = n = \beta = 1,$$

$$E^{1}_{\alpha_1,\beta}\big(z_1\big) = \sum_{k_1=0}^{\infty}\frac{(1)_{k_1}}{k_1!}\cdot\frac{z_1^{k_1}}{\Gamma\big(\alpha_1 k_1+\beta\big)} = E_{\alpha_1,\beta}\big(z_1\big), \quad \text{if} \quad \delta = n = 1.$$

In the case that $\delta = 1$, the Pochhammer symbol yields $(1)_k = k!$ and the expression (2.27) is the multinomial Mittag-Leffler function defined in [47]

$$E^{1}_{\alpha_1,\ldots,\alpha_n,\beta}\big(z_1,\ldots,z_n\big) = E_{\alpha_1,\ldots,\alpha_n,\beta}\big(z_1,\ldots,z_n\big). \tag{2.28}$$

This function plays an essential role in the study of fractional order diffusion equations with respect to time with multiple terms. For some useful properties of the multinomial Mittag-Leffler function (2.28) in this context see [46]. In the case of $m = 2$ and $m = 3$, the Mittag-Leffler Bivariate (MLB) and Mittag-Leffler Trivariate (MLT) function (2.27) were recently introduced in [49, 50], and are written as follows

$$E^{\delta}_{\alpha_1,\alpha_2,\beta}\big(z_1,z_2\big) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{(\delta)_{k_1+k_2}}{k_1!\cdot k_2!}\cdot\frac{z_1^{k_1}\cdot z_2^{k_2}}{\Gamma\big(\alpha_1 k_1+\alpha_2 k_2+\beta\big)}, \tag{2.29}$$

$$E^{\delta}_{\alpha_1,\alpha_2,\alpha_3,\beta}\big(z_1,z_2,z_3\big) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{k_3=0}^{\infty}\frac{(\delta)_{k_1+k_2+k_3}}{k_1!\cdot k_2!\cdot k_3!}\cdot\frac{z_1^{k_1}\cdot z_2^{k_2}\cdot z_3^{k_3}}{\Gamma\big(\alpha_1 k_1+\alpha_2 k_2+\alpha_3 k_3+\beta\big)}. \tag{2.30}$$

In applications it turns out to be very important the case when we write $z_1 = \omega_1 t^{\alpha_1}$ and $z_2 = \omega_2 t^{\alpha_2}$ for a single variable t, and (optionally) multiply by an extra power function of t (see [49])

$$u(t) := t^{\gamma-1}E^{\delta}_{\alpha,\beta,\gamma}\left(\omega_1 t^{\alpha_1},\omega_2 t^{\alpha_2}\right) = \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\delta)_{r+n}}{\Gamma(\alpha r+\beta n+\gamma)}\frac{\omega_1^r\omega_2^n}{r!n!}t^{\alpha r+\beta n+\gamma-1}. \tag{2.31}$$

This function is formally univariate, since it depend on a single variable t in addition to the parameters $\alpha, \beta, \gamma, \delta$.

Theorem 2.15 (see [49]) *The univariate form (2.31) of the bivariate Mittag-Leffler function (2.29), with $\delta = 1$, is a solution of fractional ODE*

$${}^{RL}D_t^{\alpha+\beta}u(t) - \omega_2\,{}^{RL}D_t^{\alpha}u(t) - \omega_1\,{}^{RL}D_t^{\beta}u(t) = \frac{t^{\gamma-\alpha-\beta-1}}{\Gamma(\gamma-\alpha-\beta)}.$$

And if the extra assumption $0 < \mathrm{Re}(\alpha+\beta) < 1$ $u(t)$ *is solution of the fractional initial value problem*

$$ {}^{C}D_t^{\alpha+\beta}u(t) - \omega_2\, {}^{C}D_t^{\alpha}u(t) - \omega_1\, {}^{C}D_t^{\beta}u(t) = \frac{\omega_1 t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{\omega_2 t^{-\beta}}{\Gamma(1-\beta)}, \quad u(0) = 1. $$

3 Fractional calculus and viscoelasticity

The use of fractional calculus in linear viscoelasticity leads us to generalize the classical mechanical models, the basic Newton element (dashpot) is substituted by the more general Scott-Blair element (of order ν), sometimes referred to as *pot*. In fact, we can construct the class of these generalized models from Hooke and Scott- Blair elements, disposed singly and in branches of two (in series or in parallel). The material functions are obtained using the combination rule; their determination is made easy if we take into account the *correspondence principle* between the classical and fractional mechanical models [20].

Definition 3.1 *An element whose stress* $\sigma(t)$ *is proportional to the fractional order derivative of the deformation* $\varepsilon(t)$ *is called a fractional calculus element.*

Let F be the coefficient of viscosity of a dissipator (damper), E be the modulus of elasticity of a spring, and $E = F/E$. In terms of the fractional calculus element the constitutive law

$$ \sigma(t) = E\eta^{\beta} D^{\beta}\varepsilon(t), \quad 0 \le \beta \le 1, \tag{3.32} $$

it calls a spring-pot.

The expression $\eta = F/E$ is a characteristic time, known as relaxation time or creep time, which depends on the specific characteristics of the model under consideration.

Problems of impact waves essentially concern the response of a long viscoelastic rod of uniform small cross-section to dynamical (uniaxial) loading conditions. According to the elementary theory, the rod is taken to be homogeneous (of density ρ), for example, semi-infinite in extent ($x \ge 0$), and undisturbed for $t < 0$. For $t \le 0$ the end of the rod (at $x = 0$) is subjected to a disturbance (the input) denoted by $r_0(t)$. The response variable (the output) denoted by $r(x,t)$ may be either the displacement $u(x,t)$, the particle velocity $v(x,t) = \frac{\partial}{\partial x}u(x,t)$, the stress $\sigma(x,t)$, or the strain $\epsilon(x,t)$ (see [20] for more details).

In the following section we raise a mathematical problem that defined on the real line satisfying the field equations: Eq. of motion, kinematic Eq. and a certain stress-strain relation.

3.1 Wave processes in viscoelastic media

Consider the equations of motion of a homogeneous bar, infinite in both directions, of certain density ρ, which undergoes longitudinal vibrations under the action of an external load $f(x,t)$ (calculated per unit volume) (see [8])

$$ \left\{ \begin{aligned} \frac{\partial}{\partial x}\sigma(x,t) + f(x,t) &= \rho\frac{\partial^2}{\partial t^2}u(x,t), \\ \varepsilon(x,t) &= \frac{\partial}{\partial x}u(x,t). \end{aligned} \right. \quad \Bigg| \quad x \in \mathbb{R}, \quad t \ge 0. \tag{3.33} $$

Here, x is the coordinate of a point on the bar, t is the time, σ is the stress, ε is the strain and u is the displacement of the bar material element.

We relate the above equations of motion through the constitutive (stress-strain) relation

$$\sigma(x,t) = ED^{\alpha}\varepsilon(x,t), \quad 0 < \alpha < 1, \tag{3.34}$$

where, E and α are specific to the material constituting the bar, and D^{α} denotes the fractional order differentiation with respect to the variable t.

Koeller [21] emphasizes that the fractional calculus element did not have a mechanical treatment until then, but this concept is fundamental to describe the relaxation and creep functions. For the physical treatment of fractional derivatives, see the article [22].

3.2 Statement of problem

By eliminating $\sigma(x,t)$ and $\varepsilon(x,t)$ using the equations of motion (3.33) and the constitutive relation (3.34) we get

$$\frac{\partial^2}{\partial t^2}u(x,t) = a^2 D^{\alpha}\frac{\partial^2}{\partial x^2}u(x,t) + \frac{1}{\rho}f(x,t), \quad x \in \mathbb{R}, \quad t \geq 0, \tag{3.35}$$

where $a^2 = E/\rho$.

Assume the following cases for the charge $f(x,t)$

- $f(x,t)$ depends only on the spatial variable x, i.e. $f(x,t) \equiv f(x)$;
- $f(x,t) = \delta(x)\delta(t)$.

Consider the following initial-boundary value problem

$$\begin{cases} \dfrac{\partial^2}{\partial t^2}u(x,t) = a^2 D^{\alpha}\dfrac{\partial^2}{\partial x^2}u(x,t) + \dfrac{1}{\rho}f(x), \quad x \in \mathbb{R}, \quad t \geq 0, \\ \lim\limits_{|x|\to\infty} u(x,t) = 0, \quad t > 0, \\ u(x,0+) = 0 = \dfrac{\partial}{\partial t}u(x,0+), \quad x \in \mathbb{R}. \end{cases} \tag{3.36}$$

Here for D^{α} we will use the Caputo-Fabrizio operator ${}^{CF}D_t^{\alpha}$ and the Atangana-Baleanu derivative ${}^{ABC}D_t^{\alpha}$, ${}^{ABR}D_t^{\alpha}$ in the Caputo and Riemann-Liouville sense, respectively. And the case Riemann-Liouville derivative, ${}^{RL}D_t^{\alpha}$. The fundamental solution of these problems is established and its moments are calculated.

Remark 3.2 *For $\alpha = 0$ (or $\alpha = 1$), the FPDE of the problem (3.36), either with the Riemann-Liouville, Caputo-Fabrizio or Atangana-Baleanu derivatives, is of hyperbolic (or parabolic) type and the known classical solutions to this equation are recovered if we substitute $\alpha = 0$ (or $\alpha = 1$). For example if we use the Riemann-Liouville derivative we obtain the known result [8], which by completeness of the manuscript we develop at the end of the next section, and we add the formula of its moments.*

4 Main results

4.1 Homogeneous boundary conditions

In this section we will discuss the problem (3.36), when the ends of the infinite length bar, remains unchanged or fixed, at time $t = 0$. We will solve this problem using the following fractional order derivatives

- Caputo-Fabrizio sense,
- Atangana-Baleanu in the Caputo and Riemann-Liouville sense,
- Riemann-Liouville sense.

And we will present for the first time, as far as we know, the solutions of the problem (3.36) using various types of derivatives of fractional order and the bivariate Mittag-Leffler function (2.29). This form of the solutions allows us to calculate in a simple way the fundamental solutions of these problems and its moments.

We shall denote by $\mathcal{F}(\mathbb{R})$ the space of functions $f(x)$ such that there exists the Fourier transform (2.7), and we shall use the notation $f(k) := (\mathcal{F}_x f)(k)$.

Definition 4.1 *We define $\mathcal{LF} := \mathcal{L}(\mathbb{R}^+) \times \mathcal{F}(\mathbb{R})$ as functions space $u(x,t)$ such that there exist the Laplace transform (2.6) and the Fourier transform (2.7).*

Theorem 4.2 *Let be $f \in \mathcal{F}(\mathbb{R})$ and $0 < \alpha < 1$. Then, the solution $u(x,t) \in \mathcal{LF}$ of the problem (3.36), using $D^\alpha \equiv^{CF} D_t^\alpha$, is given by*

$$u(x,t) = \frac{1}{\rho} \int_{-\infty}^{\infty} G^\alpha(x-\xi,t) f(\xi) d\xi, \tag{4.37}$$

where

$$G^\alpha(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(t^2 E^1_{2,1,3}\left(-\frac{(ak)^2}{1-\alpha}t^2, -bt\right) + bt^3 E^1_{2,1,4}\left(-\frac{(ak)^2}{1-\alpha}t^2, -bt\right)\right] \mathrm{e}^{-ikx}\, dk.$$

Proof. Applying the Fourier-Laplace transform to the FPDE of the problem (3.36) and using the respective initial conditions, we obtain

$$\begin{aligned} u(k,s) &= \frac{f(k)}{\rho \cdot s^2} \cdot \frac{(1-\alpha)s + \alpha}{(1-\alpha)s^2 + \alpha s + (ak)^2}, \\ &= \frac{f(k)}{\rho} \cdot \left[\frac{1}{s^2} \cdot \frac{(1-\alpha)s}{(1-\alpha)s^2 + \alpha s + (ak)^2} \right. \\ &\qquad \left. + \frac{1}{s^2} \cdot \frac{\alpha}{(1-\alpha)s^2 + \alpha s + (ak)^2} \right]. \end{aligned}$$

Or equivalently

$$u(k,s) = \frac{f(k)}{\rho} \cdot \left[\frac{s^{-1}}{s^2 + bs + \frac{(ak)^2}{1-\alpha}} + b\frac{s^{-2}}{s^2 + as + \frac{(ak)^2}{1-\alpha}} \right], \quad b := \frac{\alpha}{1-\alpha}. \tag{4.38}$$

Now using the method of series we have

$$\begin{aligned}
\frac{1}{s^2+bs+\frac{(ak)^2}{1-\alpha}} &= \frac{1}{s^2+bs}\cdot\frac{1}{1+\frac{\frac{(ak)^2}{1-\alpha}}{s^2+bs}} = \frac{1}{s^2+bs}\sum_{m=0}^{\infty}\left(\frac{-\frac{(ak)^2}{1-\alpha}}{s^2+bs}\right)^m,\\
&= \sum_{m=0}^{\infty}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m}{(s^2+bs)^{m+1}} = \sum_{m=0}^{\infty}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m}{(s^2)^{m+1}}\frac{1}{\left(1+\frac{bs}{s^2}\right)^{m+1}},
\end{aligned}$$

provided that

$$\left|\frac{-(ak)^2}{s^2+bs}\right| < (1-\alpha).$$

Now using the formula $\frac{1}{(1+z)^{k+1}} = \sum\limits_{n=0}^{\infty}\binom{n+k}{n}(-z)^n$,

$$\begin{aligned}
\frac{1}{s^2+bs+\frac{(ak)^2}{1-\alpha}} &= \sum_{m=0}^{\infty}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m}{(s^2)^{m+1}}\frac{1}{\left(1+\frac{bs}{s^2}\right)^{m+1}},\\
&= \sum_{m=0}^{\infty}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m}{(s^2)^{m+1}}\sum_{n=0}^{\infty}\binom{m+n}{n}\left(-\frac{b}{s}\right)^n, \quad \left|\frac{-b}{s}\right|<1,\\
&= \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n}{s^{2m+n+2}}.
\end{aligned}$$

Thus, using the above and the expression (4.38), we have

$$\begin{aligned}
u(k,s) &= \frac{f(k)}{\rho}\cdot\left[\frac{s^{-1}}{s^2+bs+\frac{(ak)^2}{1-\alpha}} + b\frac{s^{-2}}{s^2+bs+\frac{(ak)^2}{1-\alpha}}\right],\\
&= \frac{f(k)}{\rho}\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n}{s^{2m+n+3}}\\
&\qquad + \frac{f(k)}{\rho}\cdot b\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n}{s^{2m+n+4}}.
\end{aligned}$$

Then, using the fact that $\mathcal{L}^{-1}\{\frac{1}{s^{\alpha+1}}\}(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ we obtain

$$\begin{aligned}
u(k,t) = \frac{f(k)}{\rho}&\cdot\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n t^{2m+n+2}}{\Gamma(2m+n+3)}\\
&+ \frac{f(k)}{\rho}\cdot b\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n t^{2m+n+3}}{\Gamma(2m+n+4)},\\
= \frac{f(k)}{\rho}&\cdot t^2E^1_{2,1,3}\left(-\frac{(ak)^2}{1-\alpha}t^2,-bt\right) + \frac{f(k)}{\rho}\cdot bt^3E^1_{2,1,4}\left(-\frac{(ak)^2}{1-\alpha}t^2,-bt\right).
\end{aligned}$$

Finally, applying the inverse Fourier transform we get

$$u(x,t) = \frac{t^2}{2\pi\rho}\int_{-\infty}^{\infty} E^1_{2,1,3}\left(-\frac{(ak)^2}{1-\alpha}t^2, -bt\right) f(k)\,\mathrm{e}^{-ikx}\,dk$$
$$+ \frac{bt^3}{2\pi\rho}\int_{-\infty}^{\infty} E^1_{2,1,4}\left(-\frac{(ak)^2}{1-\alpha}t^2, -bt\right) f(k)\,\mathrm{e}^{-ikx}\,dk. \tag{4.39}$$

Note that the argument to simplify the double series we have requerid extra conditions on s, namely $\left|\frac{-(ak)^2}{s^2+bs}\right| < (1-\alpha)$ and $\left|\frac{-b}{s}\right| < 1$, for proper convergence of the series. But these conditions can be removed at the end by analytic continuation, to give the desired result for all s. The theorem is proved. ■

Let $f \in \overline{\mathcal{S}}$. The solution expressed in (4.37) can be written, in explicit form, according to the following result.

Corollary 4.3 *Let $f(x) \in \overline{\mathcal{S}}$ and $0 < \alpha < 1$. Then, the solution $u(x,t) \in \mathcal{LF}$ of the problem (3.36) assumes the form*

$$u(x,t) = \frac{t^2}{\rho}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(\frac{a^2}{1-\alpha}\right)^m(-b)^n t^{2m+n}}{\Gamma(2m+n+3)} f^{(2m)}(x)$$
$$+ \frac{bt^3}{\rho}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(\frac{a^2}{1-\alpha}\right)^m(-b)^n t^{2m+n}}{\Gamma(2m+n+4)} f^{(2m)}(x). \tag{4.40}$$

provided that the series in (4.40) converges for all $x \in \mathbb{R}$ and $t > 0$.

Proof. Considering the expression (4.39), substituting (2.29) and using (2.20), we have

$$u(x,t) = \frac{1}{2\pi\rho}\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n t^{2m+n+2}}{\Gamma(2m+n+3)} f(k)e^{-ikx}dk$$
$$+ \frac{b}{2\pi\rho}\int_{-\infty}^{\infty}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(-\frac{(ak)^2}{1-\alpha}\right)^m(-b)^n t^{2m+n+3}}{\Gamma(2m+n+4)} f(k)e^{-ikx}dk,$$
$$= \frac{t^2}{\rho}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(\frac{a^2}{1-\alpha}\right)^m(-b)^n t^{2m+n}}{\Gamma(2m+n+3)}\frac{1}{2\pi}\int_{-\infty}^{\infty}(-ik)^{2m} f(k)e^{-ikx}dk$$
$$+ \frac{bt^3}{\rho}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(\frac{a^2}{1-\alpha}\right)^m(-b)^n t^{2m+n}}{\Gamma(2m+n+4)}\frac{1}{2\pi}\int_{-\infty}^{\infty}(-ik)^{2m} f(k)e^{-ikx}dk,$$
$$= \frac{t^2}{\rho}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(\frac{a^2}{1-\alpha}\right)^m(-b)^n t^{2m+n}}{\Gamma(2m+n+3)} f^{(2m)}(x)$$
$$+ \frac{bt^3}{\rho}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\binom{m+n}{n}\frac{\left(\frac{a^2}{1-\alpha}\right)^m(-b)^n t^{2m+n}}{\Gamma(2m+n+4)} f^{(2m)}(x). \tag{4.41}$$

The exchange of the symbols of integral and sums is guaranteed under uniform convergence bivariate Mittag-Leffler function. Moreover, recall that the Fourier transform of the classical derivative of a function in $\overline{\mathcal{S}}$, we obtain the expression for the solution $u(x,t)$. ■

Remarks 4.4 *According to the condition (2.21), the formula (4.40) provides the solution $u(x,t)$ of the problem (3.36) provided that $f(x)$ is an analytical function of $x \in \mathbb{R}$, that the series (4.40) converges for any $x \in \mathbb{R}$ and $t > 0$, and that $f(x)$ together with all its derivatives $f^{(2m)}(x)$ tend to zero at infinity.*

Let us now consider the Atangana-Baleanu fractional order derivative in the Caputo sense given by the expression (2.4).

Theorem 4.5 *Let $f \in \mathcal{F}(\mathbb{R})$ and $0 < \alpha < 1$. Then, the solution $u(x,t)$ of the problem (3.36), using $D^\alpha \equiv {}^{ABC}D_t^\alpha$, is given by*

$$u(x,t) = \frac{1}{\rho} \int_{-\infty}^{\infty} G^\alpha(x-\xi,t) f(\xi) d\xi, \tag{4.42}$$

where

$$\begin{aligned} G^\alpha(x,t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Bigg[& t^2 E^1_{\alpha,2,\alpha+3}\left(\frac{-\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) \\ & + \left(\frac{\alpha}{1-\alpha}\right) t^{2+\alpha} E^1_{\alpha,2,3}\left(\frac{-\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\Bigg] \mathrm{e}^{-ikx} dk. \end{aligned} \tag{4.43}$$

Proof. Applying the Fourier-Laplace transform to the FPDE of the problem (3.36), using the formula (2.13) and the initial conditions, we obtain

$$\begin{aligned} u(k,s) &= \frac{f(k)}{\rho s} \frac{(1-\alpha)s^\alpha + \alpha}{(1-\alpha)s^{2+\alpha} + \alpha s^2 + (ak)^2 s^\alpha} \\ &= \frac{f(k)}{\rho s^3}\left[\frac{s^\alpha}{s^\alpha + \frac{(ak)^2}{1-\alpha}s^{\alpha-2} + \frac{\alpha}{1-\alpha}} + \frac{\alpha}{1-\alpha} \frac{1}{s^\alpha + \frac{(ak)^2}{1-\alpha}s^{\alpha-2} + \frac{\alpha}{1-\alpha}}\right]. \end{aligned}$$

Since both terms of the previous equation have the same denominator, using the method of series this can be rewritten as

$$\begin{aligned} \frac{1}{s^\alpha + \frac{(ak)^2}{1-\alpha}s^{\alpha-2} + \frac{\alpha}{1-\alpha}} &= \frac{1}{s^\alpha + \frac{(ak)^2}{1-\alpha}s^{\alpha-2}} \frac{1}{1 + \frac{\frac{\alpha}{1-\alpha}}{s^\alpha + \frac{(ak)^2}{1-\alpha}s^{\alpha-2}}}, \\ &= \frac{s^{2-\alpha}}{s^2 + \frac{(ak)^2}{1-\alpha}} \frac{1}{1 + \frac{\alpha s^{2-\alpha}}{(1-\alpha)\left(s^2 + \frac{(ak)^2}{1-\alpha}\right)}}, \\ &= \frac{s^{2-\alpha}}{s^2 + \frac{(ak)^2}{1-\alpha}} \sum_{r=0}^{\infty} \left(-\frac{\alpha}{1-\alpha}\right)^r \left(\frac{s^{2-\alpha}}{s^2 + \frac{(ak)^2}{1-\alpha}}\right)^r, \\ &= \sum_{r=0}^{\infty} \left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-\alpha(r+1)}}{\left(s^2 + \frac{(ak)^2}{1-\alpha}\right)^{r+1}}, \end{aligned} \tag{4.44}$$

provided that $\left|\dfrac{s^{2-\alpha}}{s^2+\frac{(ak)^2}{1-\alpha}}\right| < \frac{1-\alpha}{\alpha}$. Using the previous result in the equation (4.44) we obtain

$$u(k,s)$$
$$= \frac{f(k)}{\rho}\left[\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+3)}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}} + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-[\alpha(r+1)+3]}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}}\right],$$

$$= \frac{f(k)}{\rho}\left[\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+3)}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \right.$$
$$\left. + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-[\alpha(r+1)+3]}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}}\right].$$

Now, using the formula $\frac{1}{(1+z)^{k+1}} = \sum\limits_{n=0}^{\infty}\binom{n+k}{n}(-z)^n$, we have

$$u(k,s) = \frac{f(k)}{\rho}\left[\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r s^{-(\alpha r+3)}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(-\frac{(ak)^2}{(1-\alpha)s^2}\right)^n \right.$$
$$\left. + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r s^{-[\alpha(r+1)+3]}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(-\frac{(ak)^2}{(1-\alpha)s^2}\right)^n\right],$$
$$= \frac{f(k)}{\rho}\left[\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha r+2n+3}}\right.$$
$$\left. + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha(r+1)+2n+3}}\right].$$

Then, using the fact that $\mathcal{L}^{-1}\{\frac{1}{s^{\alpha+1}}\}(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ we obtain

$$u(k,t) = \frac{f(k)}{\rho}\left[\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{t^{\alpha r+2n+2}}{\Gamma(\alpha r+2n+3)}\right.$$
$$\left. + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{t^{\alpha(r+1)+2n+2}}{\Gamma(\alpha(r+1)+2n+3)}\right],$$
$$= \frac{f(k)}{\rho}\left[t^2\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}\right.$$
$$\left. + \frac{\alpha}{1-\alpha}t^{2+\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}\right],$$
$$= \frac{f(k)}{\rho}\left[t^2E^1_{\alpha,2,3}\left(\frac{-\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) + \frac{\alpha}{1-\alpha}t^{2+\alpha}E^1_{\alpha,2,\alpha+3}\left(\frac{-\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right].$$

Finally applying the inverse Fourier transform to the previous equation, the result is obtained

$$u(x,t) = \frac{t^2}{2\pi\rho}\int_{-\infty}^{\infty} E^1_{\alpha,2,3}\left(\frac{-\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) f(k)\mathrm{e}^{-ikx}dk$$
$$+\left(\frac{\alpha}{1-\alpha}\right)\frac{t^{2+\alpha}}{2\pi\rho}\int_{-\infty}^{\infty} E^1_{\alpha,2,\alpha+3}\left(\frac{-\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) f(k)\mathrm{e}^{-ikx}dk. \quad (4.45)$$

Note that the argument to simplify the double series we have requerid extra conditions on s, namely $\left|\frac{s^{2-\alpha}}{s^2+\frac{(ak)^2}{1-\alpha}}\right| < \frac{1-\alpha}{\alpha}$, for proper convergence of the series. But these conditions can be removed at the end by analytic continuation, to give the desired result for all s. The theorem is proven. ■

Corollary 4.6 *Let $f(x) \in \overline{\mathcal{S}}$ and $0 < \alpha < 1$. Then, the solution $u(x,t) \in \mathcal{LF}$ of the problem (3.36) assumes the form*

$$u(x,t) = \frac{t^2}{\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}f^{(2n)}(x)$$
$$+\frac{\alpha t^{2+\alpha}}{(1-\alpha)\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}f^{(2n)}(x).$$

Proof. Considering the expression (4.39), substituting (2.29) and using (2.20), we have

$$u(x,t) = \frac{1}{\rho 2\pi}\int_{-\infty}^{\infty}\left[t^2\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}\right.$$
$$\left.+\frac{\alpha}{1-\alpha}t^{2+\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}\right]\mathrm{e}^{-ikx}dk,$$
$$= \frac{t^2}{\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}\frac{1}{2\pi}\int_{-\infty}^{\infty}(-ik)^{2n}f(k)\mathrm{e}^{-ikx}dk$$
$$+\frac{\alpha t^{2+\alpha}}{(1-\alpha)\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}$$
$$\frac{1}{2\pi}\int_{-\infty}^{\infty}(-ik)^{2n}f(k)\mathrm{e}^{-ikx}dk,$$
$$= \frac{t^2}{\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}f^{(2n)}(x)$$
$$+\frac{\alpha t^{2+\alpha}}{(1-\alpha)\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}f^{(2n)}(x).$$

The result is proven. ■

We use the well-known space of Fourier transform of tempered distributions $\mathcal{F}'(\mathbb{R}) = \mathcal{S}'$ [53].

Definition 4.7 *We define $\mathcal{LF}' := \mathcal{L}(\mathbb{R}^+) \times \mathcal{F}'(\mathbb{R})$ as a generalized functions space $u(x,t)$ such that there exist the Laplace transform (2.6) and the Fourier transform of generalized functions.*

Corollary 4.8 *Let $u(x,t) \in \mathcal{LF}'$ be a fundamental solution to the problem (3.36), with $0 < \alpha < 1$ and $f(x) = \delta(x)$. Then, $u(x,t)$ is given by*

$$u(x,t) = \frac{t^2}{\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}\delta^{(2n)}(x)$$
$$+\frac{\alpha t^{2+\alpha}}{(1-\alpha)\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}\delta^{(2n)}(x).$$

Or alternatively using the representation of the derivative of the Dirac delta function [54]

$$u(x,t)$$
$$=\frac{1}{\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n+2}}{\Gamma(\alpha r+2n+3)}\frac{1}{\pi}\int_0^{\infty}\frac{d^{2n}}{dx^{2n}}\cos(kx)dk$$
$$+\frac{\alpha}{(1-\alpha)\rho}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n+2+\alpha}}{\Gamma(\alpha r+2n+3+\alpha)}\frac{1}{\pi}\int_0^{\infty}\frac{d^{2n}}{dx^{2n}}\cos(kx)dk.$$

Now we calculate the moments of the fundamental solution of the problem (3.36). For the end we use the well-know property

$$\int_{-\infty}^{\infty}x^l u(x,t;\alpha,\beta)dx = (-i)^l\left[\frac{d^l}{dk^l}(\mathcal{F}_x u)(k,t)\right]_{k=0},\ l=0,1,2\ldots \tag{4.46}$$

and the expression

$$u(k,t) = \frac{f(k)}{\rho}\left[t^2E^1_{\alpha,2,3}\left(\frac{-\alpha}{1-\alpha}t^\alpha,-\frac{(ak)^2}{1-\alpha}t^2\right)+\frac{\alpha}{1-\alpha}t^{2+\alpha}E^1_{\alpha,2,\alpha+3}\left(\frac{-\alpha}{1-\alpha}t^\alpha,-\frac{(ak)^2}{1-\alpha}t^2\right)\right], \tag{4.47}$$

with $f(k) = 1$.

Proposition 4.9 *The moments of fundamental solution $u(x,t) \in \mathcal{LF}'$ of the problem (3.36) gives by*

$$\int_{-\infty}^{\infty}x^{2l}u(x,t)dx = \frac{(-i)^{2l}}{\rho}\left[\sum_{r=0}^{\infty}\binom{2l+r}{2l}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{a^2}{1-\alpha}\right)^{2l}\frac{\Gamma(2l+1)t^{\alpha r+2l+2}}{\Gamma(\alpha r+2l+3)}\right]$$
$$+\frac{(-i)^{2l}\alpha}{(1-\alpha)\rho}\left[\sum_{r=0}^{\infty}\binom{2l+r}{2l}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{a^2}{1-\alpha}\right)^{2l}\frac{\Gamma(2l+1)t^{\alpha r+2l+2+\alpha}}{\Gamma(\alpha r+2l+3+\alpha)}\right],\ l=0,1,2,\ldots$$

Proof. Applying the relations (4.46), (4.47) and the Bivariant Mittag-Leffler function (2.29) (with $\delta = 1$), we obtain

$$
\begin{aligned}
&\int_{-\infty}^{\infty} x^{2l} u(x,t)dx \\
&= (-i)^{2l}\left[\frac{1}{\rho}\frac{d^{2l}}{dk^{2l}}\left[t^2\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}\right.\right. \\
&\qquad \left.\left.+\ \frac{\alpha}{1-\alpha}t^{2+\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}\right]\right]_{k=0}, \\
&= (-i)^{2l}\frac{t^2}{\rho}\left[\sum_{r=0}^{\infty}\sum_{n=2l}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3)}\frac{\Gamma(n+1)}{\Gamma(n-l+1)}k^{n-2l}\right]_{k=0} \\
&+(-i)^{2l}\frac{\alpha t^{2+\alpha}}{(1-\alpha)\rho}\left[\sum_{r=0}^{\infty}\sum_{n=2l}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{a^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+3+\alpha)}\frac{\Gamma(n+1)}{\Gamma(n-l+1)}k^{n-2l}\right]_{k=0}, \\
&= \frac{(-i)^{2l}}{\rho}\left[\sum_{r=0}^{\infty}\binom{2l+r}{2l}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{a^2}{1-\alpha}\right)^{2l}\frac{\Gamma(2l+1)t^{\alpha r+2l+2}}{\Gamma(\alpha r+2l+3)}\right] \\
&\quad +\frac{(-i)^{2l}\alpha}{(1-\alpha)\rho}\left[\sum_{r=0}^{\infty}\binom{2l+r}{2l}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{a^2}{1-\alpha}\right)^{2l}\frac{\Gamma(2l+1)t^{\alpha r+2l+2+\alpha}}{\Gamma(\alpha r+2l+3+\alpha)}\right], \quad l=0,1,2,\ldots
\end{aligned}
$$

The result is proven. ■

4.2 No homogeneous boundary conditions

In this section, we will solve the problem (3.36) using the Atangana-Baleanu derivative in the Caputo sense and in the Riemann-Liouville sense, with the peculiarity that the ends of the bar, of infinite length, are they are affecte by some kind of force at time $t = 0$. Next, we present the solution of the problem (3.36) together with the aforementioned initial conditions

$$
\left\{\begin{array}{r} u(x,0) = \varphi(x), \\ \dfrac{\partial}{\partial t}u(x,0) = \psi(x), \end{array}\right. \Bigg|\ x \in \mathbb{R}, \tag{4.48}
$$

where, φ, ψ are sufficiently well-behaved functions, in particular that $\varphi, \psi \in \mathcal{F}(\mathbb{R})$.

Theorem 4.10 *Let $f \in \mathcal{F}(\mathbb{R})$ and $0 < \alpha < 1$. Then, the solution $u(x,t)$ of the problem (3.36) together with the boundary conditions (4.48), using $D^\alpha \equiv {}^{ABC}D_t^\alpha$, in the Caputo sense is given by*

$$
u(x,t) = \frac{1}{\rho}\int_{-\infty}^{\infty} G_1^\alpha(x-\xi,t)f(\xi)d\xi + \frac{1}{\rho}\int_{-\infty}^{\infty} G_2^\alpha(x-\xi,t)\varphi(\xi)d\xi + \frac{1}{\rho}\int_{-\infty}^{\infty} G_3^\alpha(x-\xi,t)\psi(\xi)d\xi, \tag{4.49}
$$

where

$$G_1^\alpha(x,t) = \frac{t^2}{2\pi} \int\limits_{-\infty}^{\infty} \left[E^1_{\alpha,2,3}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) \right.$$
$$\left. + \frac{\alpha}{1-\alpha}t^\alpha E^1_{\alpha,2,3+\alpha}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right] f(k)\mathrm{e}^{-ikx}dk,$$
$$G_2^\alpha(x,t) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \left[E^1_{\alpha,2,1}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) \right.$$
$$+ \frac{\alpha}{1-\alpha}t^\alpha E^1_{\alpha,2,\alpha+1}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)$$
$$\left. + \frac{(ak)^2}{1-\alpha}t^\alpha E^1_{\alpha,2,\alpha+1}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right] \varphi(k)\,\mathrm{e}^{-ikx}dk,$$

$$G_3^\alpha(x,t) = \frac{t}{2\pi} \int\limits_{-\infty}^{\infty} \left[E^1_{\alpha,2,2}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right) \right.$$
$$\left. + \frac{\alpha}{1-\alpha}t^\alpha E^1_{\alpha,2,\alpha+2}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right] \psi(k)\mathrm{e}^{-ikx}dk.$$

Proof. Applying the Fourier-Laplace transform to the FPDE of the problem (3.36), using the formula (2.13) and the initial conditions (4.48) we obtain

$$s^2u(k,s) - s\varphi(k) - \psi(k) = -\frac{s^\alpha(ak)^2}{(1-\alpha)s^\alpha+\alpha}u(k,s) + \frac{(ak)^2s^{\alpha-1}}{(1-\alpha)s^\alpha+\alpha}\varphi(k) + \frac{1}{\rho s}f(k),$$

from where it follows that

$$u(k,s) = H_1(k,s)\frac{f(k)}{\rho} + H_2(k,s)\varphi(k) + H_3(k,s)\psi(k), \tag{4.50}$$

where

$$H_1(k,s) = \frac{\left[s^\alpha + \frac{\alpha}{1-\alpha}\right]s^{-1-\alpha}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}}, \tag{4.51}$$
$$H_2(k,s) = \frac{\left[s^\alpha + \frac{\alpha}{1-\alpha}\right]s^{1-\alpha}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}} + \frac{\frac{(ak)^2}{1-\alpha}s^{-1}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}}, \tag{4.52}$$
$$H_3(k,s) = \frac{\left[s^\alpha + \frac{\alpha}{1-\alpha}\right]s^{-\alpha}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}}. \tag{4.53}$$

Note that each of the functions $H_1(k,s)$, $H_2(k,s)$ and $H_3(k,s)$ have the same denominator, which using the series method can be rewritten

$$\frac{1}{s^\alpha + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}} = \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2r-\alpha r}}{\left(s^2 + \frac{(ak)^2}{1-\alpha}\right)^{r+1}}, \tag{4.54}$$

provided that $\left|\dfrac{s^{2-\alpha}}{s^2+\frac{(ak)^2}{1-\alpha}}\right| < \frac{1-\alpha}{\alpha}$. Using (4.54) and expressions (4.51)-(4.53), we obtain

$$
\begin{aligned}
H_1(k,s) \quad &= \quad \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2r-\alpha r-1}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}} + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2r-\alpha r-1-\alpha}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}}, \\
&= \quad \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+3)}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\qquad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r + \frac{s^{2(r+1)-[\alpha(r+1)+3]}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}},
\end{aligned}
$$

$$
\begin{aligned}
H_2(k,s) \quad &= \quad \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{(2-\alpha)r+1}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}} + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{(2-\alpha)r+1-\alpha}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}} \\
&\qquad + \frac{(ak)^2}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{(2-\alpha)r-1}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}}, \\
&= \quad \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-\alpha r-1}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\qquad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-[\alpha(r+1)+1]}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\qquad + \frac{(ak)^2}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+3)}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}},
\end{aligned}
$$

$$
\begin{aligned}
H_3(k,s) \quad &= \quad \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{(2-\alpha)r}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}} + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{(2-\alpha)r-\alpha}}{\left(s^2+\frac{(ak)^2}{1-\alpha}\right)^{r+1}}, \\
&= \quad \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+2)}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\qquad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-[\alpha(r+1)+2]}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}}.
\end{aligned}
$$

Now, using the formula $\frac{1}{(1+z)^{k+1}} = \sum_{n=0}^{\infty} \binom{n+k}{n}(-z)^n$ in the expressions for $H_1(s,k), H_2(s,k), H_3(s,k)$ above, we obtain

$$\begin{aligned}
H_1(k,s) &= \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+3)}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\quad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r + \frac{s^{2(r+1)-[\alpha(r+1)+3]}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}}, \\
&= \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r s^{-(\alpha r+3)} \sum_{n=0}^{\infty}\binom{n+r}{n}\left(-\frac{(ak)^2}{(1-\alpha)s^2}\right)^n \\
&\quad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r s^{-[\alpha(r+1)+3]} \sum_{n=0}^{\infty}\binom{n+r}{n}\left(-\frac{(ak)^2}{(1-\alpha)s^2}\right)^n, \\
&= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha r+2n+2+1}} \\
&\quad + \frac{1}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha(r+1)+2n+2+1}}.
\end{aligned}$$

Then, using the fact that $\mathcal{L}^{-1}\{\frac{1}{s^{\alpha+1}}\}(t) = \frac{t^\alpha}{\Gamma(\alpha+1)}$ we obtain

$$\begin{aligned}
H_1(k,t) = &\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{t^{\alpha r+2n+2}}{\Gamma(\alpha r+2n+3)} \\
&+ \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r\binom{n+r}{n}\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{t^{\alpha(r+1)+2n+2}}{\Gamma(\alpha(r+1)+2n+3)}. \quad (4.55)
\end{aligned}$$

Proceeding completely analogously, we obtain for $H_2(k,s)$

$$\begin{aligned}
H_2(k,s) &= \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-\alpha r-1}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\quad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-[\alpha(r+1)+1]}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
&\quad + \frac{(ak)^2}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r \frac{s^{2(r+1)-(\alpha r+3)}}{(s^2)^{r+1}} \frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}}, \\
&= \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha r+2n+1}} \\
&\quad + \frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha(r+1)+2n+1}} \\
&\quad + \frac{(ak)^2}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n \frac{1}{s^{\alpha r+2n+2+1}},
\end{aligned}$$

using the fact that $\mathcal{L}^{-1}\{\frac{1}{s^{\alpha+1}}\}(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}$ we obtain

$$
\begin{aligned}
H_2(k,t) = & \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+1)} \\
& +\frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha(r+1)+2n}}{\Gamma(\alpha(r+1)+2n+1)} \\
& +\frac{(ak)^2}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{(1-\alpha)}\right)^n\frac{t^{\alpha(r+1)+2n}}{\Gamma(\alpha(r+1)+2n+1)}. \quad (4.56)
\end{aligned}
$$

And for $H_3(k,s)$ we have

$$
\begin{aligned}
H_3(k,s) \quad = \quad & \sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{s^{2(r+1)-(\alpha r+2)}}{(s^2)^{r+1}}\frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
& +\frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\left(\frac{-\alpha}{1-\alpha}\right)^r\frac{s^{2(r+1)-[\alpha(r+1)+2]}}{(s^2)^{r+1}}\frac{1}{\left(1+\frac{(ak)^2}{(1-\alpha)s^2}\right)^{r+1}} \\
& \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{1}{s^{\alpha r+2n+1+1}} \\
& +\frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{1}{s^{\alpha(r+1)+2n+1+1}},
\end{aligned}
$$

using the fact that $\mathcal{L}^{-1}\{\frac{1}{s^{\alpha+1}}\}(t) = \frac{t^{\alpha+1}}{\Gamma(\alpha+1)}$ we obtain

$$
\begin{aligned}
H_3(k,t) = & \sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha r+2n+1}}{\Gamma(\alpha r+2n+2)} \\
& +\frac{\alpha}{1-\alpha}\sum_{r=0}^{\infty}\sum_{n=0}^{\infty}\binom{n+r}{n}\left(\frac{-\alpha}{1-\alpha}\right)^r\left(-\frac{(ak)^2}{1-\alpha}\right)^n\frac{t^{\alpha(r+1)+2n+1}}{\Gamma(\alpha(r+1)+2n+2)}. \quad (4.57)
\end{aligned}
$$

Finally applying the inverse Fourier transform to the expression (4.50), using (4.55), (4.56) and (4.57), we obtain

$$
\begin{aligned}
u(x,t) = \frac{1}{2\pi\rho}\int_{-\infty}^{\infty} H_1(k,t)f(k)e^{-ikx}dk + \frac{1}{2\pi}\int_{-\infty}^{\infty} H_2(k,t)\varphi(k)e^{-ikx}dk \\
+\frac{1}{2\pi}\int_{-\infty}^{\infty} H_3(k,t)\psi(k)e^{-ikx}dk,
\end{aligned}
$$

$$= \frac{1}{2\pi\rho} \int_{-\infty}^{\infty} \Bigg[\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{n} \left(\frac{-\alpha}{1-\alpha} \right)^r \left(-\frac{(ak)^2}{1-\alpha} \right)^n \frac{t^{\alpha r+2n+2}}{\Gamma(\alpha r+2n+3)}$$

$$+ \frac{\alpha}{1-\alpha} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \left(\frac{-\alpha}{1-\alpha} \right)^r \binom{n+r}{n} \left(-\frac{(ak)^2}{1-\alpha} \right)^n \frac{t^{\alpha(r+1)+2n+2}}{\Gamma(\alpha(r+1)+2n+3)} \Bigg] f(k) e^{-ikx} dk$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Bigg[\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{n} \left(\frac{-\alpha}{1-\alpha} \right)^r \left(-\frac{(ak)^2}{1-\alpha} \right)^n \frac{t^{\alpha r+2n}}{\Gamma(\alpha r+2n+1)}$$

$$+ \frac{\alpha}{1-\alpha} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{n} \left(\frac{-\alpha}{1-\alpha} \right)^r \left(-\frac{(ak)^2}{1-\alpha} \right)^n \frac{t^{\alpha(r+1)+2n}}{\Gamma(\alpha(r+1)+2n+1)}$$

$$+ \frac{(ak)^2}{1-\alpha} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{n} \left(\frac{-\alpha}{1-\alpha} \right)^r \left(-\frac{(ak)^2}{(1-\alpha)} \right)^n \frac{t^{\alpha(r+1)+2n}}{\Gamma(\alpha(r+1)+2n+1)} \Bigg] \varphi(k) e^{-ikx} dk$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Bigg[\sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{n} \left(\frac{-\alpha}{1-\alpha} \right)^r \left(-\frac{(ak)^2}{1-\alpha} \right)^n \frac{t^{\alpha r+2n+1}}{\Gamma(\alpha r+2n+2)}$$

$$+ \frac{\alpha}{1-\alpha} \sum_{r=0}^{\infty} \sum_{n=0}^{\infty} \binom{n+r}{n} \left(\frac{-\alpha}{1-\alpha} \right)^r \left(-\frac{(ak)^2}{1-\alpha} \right)^n \frac{t^{\alpha(r+1)+2n+1}}{\Gamma(\alpha(r+1)+2n+2)} \Bigg] \psi(k) e^{-ikx} dk.$$

Or equivalently

$$u(x,t) = \frac{1}{2\pi\rho} \int_{-\infty}^{\infty} \Bigg[t^2 E^1_{\alpha,2,3} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right)$$

$$+ \frac{\alpha}{1-\alpha} t^{2+\alpha} E^1_{\alpha,2,\alpha+3} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right) \Bigg] f(k) e^{-ikx} dk$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Bigg[E^1_{\alpha,2,1} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right)$$

$$+ \frac{\alpha}{1-\alpha} t^\alpha E^1_{\alpha,2,\alpha+1} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right)$$

$$+ \frac{(ak)^2}{1-\alpha} t^\alpha E^1_{\alpha,2,\alpha+1} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right) \Bigg] \varphi(k) e^{-ikx} dk$$

$$+ \frac{1}{2\pi} \int_{-\infty}^{\infty} \Bigg[t E^1_{\alpha,2,2} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right)$$

$$+ \frac{\alpha}{1-\alpha} t^{1+\alpha} E^1_{\alpha,2,\alpha+2} \left(-\frac{\alpha}{1-\alpha} t^\alpha, -\frac{(ak)^2}{1-\alpha} t^2 \right) \Bigg] \psi(k) e^{-ikx} dk.$$

Note that the argument to simplify the double series we have requerid extra conditions on s, namely $\left| \dfrac{s^{2-\alpha}}{s^2 + \frac{(ak)^2}{1-\alpha}} \right| < \frac{1-\alpha}{\alpha}$, for proper convergence of the series. But these conditions can be removed at the end by analytic continuation, to give the desired result for all s. The theorem is proven. ■

In the following theorem we present the solution of the problem (3.36) together with the boundary conditions (4.48), using the Atangana-Baleanu derivative in the Riemann-Liouville sense.

Theorem 4.11 *Let $f \in \mathcal{F}(\mathbb{R})$ and $0 < \alpha < 1$. Then, the solution $u(x,t)$ of the problem (3.36) together with the boundary conditions (4.48), using $D^\alpha \equiv {}^{ABR}D_t^\alpha$, in the Riemann-Liouville sense is given by*

$$u(x,t) = \frac{1}{\rho}\int_{-\infty}^{\infty} G_1^\alpha(x-\xi,t)f(\xi)d\xi + \frac{1}{\rho}\int_{-\infty}^{\infty} G_2^\alpha(x-\xi,t)\varphi(\xi)d\xi + \frac{1}{\rho}\int_{-\infty}^{\infty} G_3^\alpha(x-\xi,t)\psi(\xi)d\xi, \tag{4.58}$$

where

$$G_1^\alpha(x,t) = \frac{t^2}{2\pi}\int_{-\infty}^{\infty}\left[E^1_{\alpha,2,3}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right.$$
$$\left. + \frac{\alpha}{1-\alpha}t^\alpha E^1_{\alpha,2,3+\alpha}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right] f(k)\mathrm{e}^{-ikx}dk,$$

$$G_2^\alpha(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[E^1_{\alpha,2,1}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right.$$
$$\left. + \frac{\alpha}{1-\alpha}t^\alpha E^1_{\alpha,2,\alpha+1}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right] \varphi(k)\,\mathrm{e}^{-ikx}dk,$$

$$G_3^\alpha(x,t) = \frac{t}{2\pi}\int_{-\infty}^{\infty}\left[E^1_{\alpha,2,2}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right.$$
$$\left. + \frac{\alpha}{1-\alpha}t^\alpha E^1_{\alpha,2,\alpha+2}\left(-\frac{\alpha}{1-\alpha}t^\alpha, -\frac{(ak)^2}{1-\alpha}t^2\right)\right] \psi(k)\mathrm{e}^{-ikx}dk.$$

Proof. Applying the Fourier-Laplace transform to the EDPF of the problem (3.36), using the formula (2.14) and the initial conditions (4.48) we obtain

$$s^2u(k,s) - s\varphi(k) - \psi(k) = -\frac{s^\alpha k^2}{(1-\alpha)s^\alpha + \alpha}u(k,s) + \frac{1}{\rho s}f(k),$$

from where it follows that

$$u(k,s) = H_1(k,s)\frac{f(k)}{\rho} + H_2(k,s)\varphi(k) + H_3(k,s)\psi(k), \tag{4.59}$$

where

$$H_1(k,s) = \frac{\left[s^\alpha + \frac{\alpha}{1-\alpha}\right]s^{-1-\alpha}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}}, \tag{4.60}$$

$$H_2(k,s) = \frac{\left[s^\alpha + \frac{\alpha}{1-\alpha}\right]s^{1-\alpha}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}}, \tag{4.61}$$

$$H_3(k,s) = \frac{\left[s^\alpha + \frac{\alpha}{1-\alpha}\right]s^{-\alpha}}{s^2 + \frac{\alpha}{1-\alpha}s^{2-\alpha} + \frac{(ak)^2}{1-\alpha}}. \tag{4.62}$$

The expressions (4.60) and (4.62) are the same as (4.51) and (4.53) respectively, (4.61) differs by the term $\dfrac{\frac{(ak)^2}{1-\alpha}s^{-1}}{s^2+\frac{k^2}{1-\alpha}+\frac{\alpha}{1-\alpha}s^{2-\alpha}}$ when compared with (4.52), with which, we can use the same strategy that was used in the proof of the previous theorem. The theorem is proven. ■

Theorem 4.12 (see [8]) *Let be $f \in \mathcal{F}(\mathbb{R})$ and $0<\alpha<1$. Then, the solution $u(x,t)$ of the problem (3.36) using the Riemann-Liouville derivative for $D^\alpha \equiv {}^{RL}D_t^\alpha$, is given by*

$$u(x,t) = \frac{1}{\rho}\int\limits_{-\infty}^{\infty} G(x-\xi,t)f(\xi)d\xi, \tag{4.63}$$

where

$$G^\alpha(x,t) = \frac{t^2}{2\pi}\int\limits_{-\infty}^{\infty} E_{2-\alpha,3}(-a^2k^2t^{2-\alpha})\,\mathrm{e}^{-ikx}\,dk = \frac{t^2}{\pi}\int\limits_{0}^{\infty} E_{2-\alpha,3}(-a^2k^2t^{2-\alpha})\,\cos(kx)dk.$$

Proof. Applying the Fourier-Laplace transform to the EDPF of the problem (3.36) and using the respective initial conditions, we obtain

$$u(k,s) = \frac{f(k)}{\rho s}\cdot\frac{1}{s^2+(ak)^2s^\alpha} = \frac{f(k)}{\rho}\cdot\frac{s^{-\alpha-1}}{s^{2-\alpha}+(ak)^2} = \frac{f(k)}{\rho}\cdot\frac{s^{(2-\alpha)-3}}{s^{2-\alpha}+(ak)^2}. \tag{4.64}$$

Using the formula $\mathcal{L}(t^{\beta-1}E_{\alpha,\beta}(\lambda t^\alpha))(s) = \frac{s^{\alpha-\beta}}{s^\alpha-\lambda}$, $\operatorname{Re} s>0$, $\lambda\in\mathbb{C}$, $\left|\lambda s^{-\alpha}\right|<1$ and applying the inverse Laplace transform we have

$$u(k,t) = \frac{f(k)}{\rho}\cdot t^2E_{2-\alpha,3}\left(-a^2k^2\,t^{2-\alpha}\right). \tag{4.65}$$

Finally, by applying the inverse Fourier transform, the desired result is obtained

$$u(x,t) = \frac{t^2}{2\pi\rho}\int\limits_{-\infty}^{\infty} E_{2-\alpha,3}\left(-a^2k^2\,t^{2-\alpha}\right)f(k)\,\mathrm{e}^{-ikx}dk.$$

which is equivalent to the expression (4.63). The theorem is proved. ■

Corollary 4.13 *The fundamental solution $u(x,t)\in\mathcal{LF}$ of problem (3.36) is given by*

$$u(x,t) = \frac{1}{2a}t^{1+\alpha/2}W\left(-\frac{|x|}{at^{1-\alpha/2}};\frac{\alpha}{2}-1,2+\frac{\alpha}{2}\right), \tag{4.66}$$

where $W(z;\alpha,\beta)$ is as in (2.2).

Proof. Applying the inverse Fourier transform to the expression (4.64)

$$u(x,s) = \frac{1}{a^2}\frac{1}{2\pi i}\int\limits_{-\infty}^{\infty}\frac{s^{(2-\alpha)-3}}{\frac{s^{2-\alpha}}{a^2}+k^2}e^{-ikx}dk,$$

using the well-known formula $\mathcal{F}_k^{-1}\left\{\frac{a}{b+k^2}\right\}(x) = \frac{a}{2b^{1/2}}\mathrm{e}^{-|x|b^{1/2}}$ $a,b>0$, we have

$$u(x,s) = \frac{1}{a}\frac{s^{-\alpha-1}}{2s^{(2-\alpha)/2}}e^{-|x|\sqrt{\frac{s^{2-\alpha}}{a^2}}}.$$

Finally, apply the inverse Laplace transform to the above equation, performing the substitution $st=\sigma$ (transforming the Bromwich contour B_r into the Hankel contour) and using the expressions (2.3) and (2.2), we obtain

$$\begin{aligned} u(x,t) &= \frac{1}{2a}\frac{1}{2\pi i}\int\limits_{\gamma-i\infty}^{\gamma+i\infty} e^{st}\frac{s^{-\alpha-1}}{s^{1-\alpha/2}}e^{-\frac{|x|}{a}s^{1-\alpha/2}}ds = \frac{t^{1+\alpha/2}}{2a}\frac{1}{2\pi i}\int\limits_{H_a}\frac{e^{\sigma-\frac{|x|}{at^{1-\alpha/2}}\sigma^{1-\alpha/2}}}{\sigma^{2+\alpha/2}}d\sigma, \\ &= \frac{1}{2a}t^{1+\alpha/2}W\left(-\frac{|x|}{at^{1-\alpha/2}};\frac{\alpha}{2}-1,2+\frac{\alpha}{2}\right). \end{aligned}$$

The result is proven. ■

We calculate the even moments of the fundamental solution of the problem (3.36).

Proposition 4.14 *The moments of fundamental solution $u(x,t)\in\mathcal{LF}'$ of the problem (3.36) gives by*

$$\int\limits_{-\infty}^{\infty} x^{2n}u(x,t)dx = (-i)^{2n}(-a^2)^{2n}t^{(2-\alpha)2n+2}\frac{\Gamma(2n+1)}{\Gamma((2-\alpha)2n+3)},\quad n=0,1,2,\ldots$$

Proof. Applying the relations (4.46), (4.65) and taking into account of definition of Mittag-Leffler function (2.24), we obtain

$$\begin{aligned} \int\limits_{-\infty}^{\infty} x^{2n}u(x,t)dx &= (-i)^{2n}\left[\frac{t^2}{\rho}\frac{d^{2n}}{dk^{2n}}\sum_{j=0}^{\infty}\frac{(-a^2k^2)^j t^{(2-\alpha)j}}{\Gamma((2-\alpha)j+3)}\right]_{k=0}, \\ &= (-i)^l\left[\frac{t^2}{\rho}\sum_{j=2n}^{\infty}\frac{(-a^2)^j k^{j-2n}t^{(2-\alpha)j}}{\Gamma((2-\alpha)j+3)}\frac{\Gamma(j+1)}{\Gamma(j-2n+1)}\right]_{k=0}, \\ &= \frac{(-i)^{2n}(-a^2)^{2n}t^{(2-\alpha)2n+2}}{\Gamma((2-\alpha)2n+3)}\Gamma(2n+1), \\ &= (-i)^{2n}(-a^2)^{2n}t^{(2-\alpha)2n+2}\frac{\Gamma(2n+1)}{\Gamma((2-\alpha)2n+3)},\quad n=0,1,2,\ldots \end{aligned}$$

5 Conclusions

The propagation of waves in homogeneous media with viscoelastic characteristics, allows answering questions of the so-called *waves produced by impact*, named for the fact that they are generated by a blow or external load, exerted on a material that is initially found resting. The use of the integral transforms technique (of the Laplace-Fourier type) allows to obtain integral representations of the aforementioned waves and leads naturally to the concepts of wavefront velocity and complex refractive index. The integral representation of the solution also allows us to discuss the dispersion and dissipation phenomena that accompany the evolution of these waves.

The main goal of this manuscript is the creation links to time-dependent fractional operators with non-singular (exponential memories) involved in time-fractional equations. As we already explained in the Introduction, there are many viscoelastic materials whose experimental behaviors exhibit strong deviations from the power law. In such cases, it is natural to raise the questions about the proper modeling of dynamic processes in such media and to call for new fractional operators. This is, in fact, the same question raised by Bagley and Torvik. Therefore, we believe that our models can help to amplify the range of mathematical objects that help to describe the behaviors of viscoelastic materials that depart from the classical power law.

Statements & Declarations

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Competing Interests

The authors have no relevant financial or non-financial interests to disclose.

Conflicts of Interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Author Contributions

M.A. Taneco-Hernández: Writing - original draft, Formal analysis, Investigation, Writing - review & editing. J.F. Gómez-Aguilar: Conceptualization, Methodology, Validation, Writing - review & editing. B. Cuahutenango-Barro: Conceptualization, Methodology, Validation, review & editing.

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

[1] Bonfanti, A. and Kaplan, J. L. Charras, G. and Kabla, A. Fractional viscoelastic models for power-law materials. Soft Matter, 2020; vol. 16, issue 26, pp. 6002-6020.

[2] Khalil, Khalil M., Ahmed E. Abouelregal, and Doaa Atta. A new fractional viscoelastic model for an infinitely thermoelastic body with a spherical cavity including Caputo-Fabrizio operator without singular kernel. Chinese Journal of Physics (2021). doi: https://doi.org/10.1016/j.cjph.2021.11.001

[3] C. Li, H. Guo, X. Tian and T. He, Generalized thermoviscoelastic analysis with fractional order strain in a thick viscoelastic plate of infinite extent. J Therm Stresses 2019; 42(8): 1051-1070.

[4] Wineman A. Nonlinear Viscoelastic Solids—A Review. Mathematics and Mechanics of Solids. 2009;14(3):300-366.

[5] Atanacković, T., Pilipović, S., Stanković, B., Zorica, D. Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, London (2014).

[6] Rossikhin, Yu.A., Shitikova, M.V. Application of fractional calculus for dynamic problems of solid mechanics: Novel trends and recent results. Appl. Mech. Rev. 63 (2010), Art. # 010801-1-25.

[7] Mainardi, F. An historical perspective on fractional calculus in linear viscoelasticity, Fract. Calc. Appl. Anal., 15, No 4 (2012), 712-717.

[8] Surguladze, T.A. On certain applications of fractional calculus of viscoelasticity. J. Math. Sci., Vol. 112, No. 5, (2002), 4517-4557.

[9] Hong Guang Sun, Yong Zhang, Baleanu Dumitru, Wen Chen, YangQuan Chen. A new collection of real world applications of fractional calculus in science and engineering. Communications in Nonlinear Science and Numerical Simulation, Volume 64, 2018, Pages 213-231.

[10] Mainardi, F. Fractional Calculus and Waves in Linear Viscoelasticity. An Introduction to Mathematical Models. British Library Cataloguing-Publication Data. Imperial Collage Press, 2010.

[11] Katsikadelis, J. T. Nonlinear dynamic analysis of viscoelastic membranes described with fractional differential models. J Theor Appl Mech 2012; 50(3): 743-53.

[12] Pritchard, R. H. and Terentjev, E. M. Oscillation and damping in the fractional Maxwell materials. J. Rheology 2017; 61: 187-203.

[13] Di Paola, M., Pirrotta, A. and Valenza, A. Visco-elastic behavior through fractional calculus: An easier method for best fitting experimental results. Mech Mater 2011; 43(12): 799-806.

[14] X. X. Guo, G. Q. Yan, Benyahia, L. and Sahraoui, S. Fitting stress relaxation experiments with fractional Zener model to predict high frequency moduli of polymeric acoustic foams. Mech Time-Depend Mater 2016; 20(4): 523-533.

[15] Reiter, G., Hamieh, M., Damman, P., Sclavons, S., Gabriele, S., Vilmin, T. and Raphaël, E. Residual stresses in thin polymer films cause rupture and dominate early stages of dewetting. Nat Mater 2005; 4(10); 754-758.

[16] Zhuravkov, M. A. and Romanova, N. S. Review of methods and approaches for mechanical problem solutions based on fractional calculus. Math Mech Solids 2016; 21(5): 595-620.

[17] Mashayekhi, S., Miles, P., Hussaini, M. Y. and Oates, W. S. Fractional viscoelasticity in fractal and non-fractal media: Theory, experimental validation, and uncertainty analysis. J Mech Phys Solids 2018; 111:134-156.

[18] Scott Blair, G.W. Analytical and integrative aspects of the stress-strain-time problem,J. Sci. Instruments,21, No. 5, 80-84 (1944).

[19] Bagley R. L., Torvik, P. J. A theoretical basis for the application of fractional calculus to viscoelasticity. J Rheol 1983;27:201-10.

[20] Caputo, M. and Mainardi, F. Linear models of dissipation in anelastic solids, Riv. Nuovo Cimento (Ser. II) 1, (1971), 161-198.

[21] Koeller, R. C. Applications of fractional calculus to the theory of viscoelasticity, J. Appl. Mech., 51, (1984), 299-307.

[22] Nigmatullin, R. R. Fractional integrals and their physical treatment. Teor. Mat. Fiz., 90, No. 3, 354-368 (1992).

[23] Hristov, J.Y. Chapter 11 - Prony's series and modern fractional calculus: Rheological models with Caputo-Fabrizio operator. Editor(s): Yeliz Karaca, Dumitru Baleanu, Yu-Dong Zhang, Osvaldo Gervasi, Majaz Moonis. Multi-Chaos, Fractal and Multi-Fractional Artificial Intelligence of Different Complex Systems, Academic Press, 2022, Pages 187-200.

[24] Hristov J. Linear Viscoelastic Responses: The Prony Decomposition Naturally Leads Into the Caputo-Fabrizio Fractional Operator. Front. Phys. 6:135, (2018).

[25] Hristov J., Response functions in linear viscoelastic constitutive equations and related fractional operators. Mathematical Modelling Of Natural Phenomena, 14 (2019) 3 (April).

[26] Hristov J. Linear viscoelastic response: The Prony decomposition naturally leads into the Caputo-Fabrizio fractional operator, Frontiers in Physics, December 2018.

[27] Hristov J., Linear viscoelastic responses and constitutive equations in terms of fractional operators with non-singular kernels: Pragmatic approach, Memory kernel correspondence requirement and analyses. Eur. Phys. J. Plus (2019) 134: 283.

[28] de Prony G.R. Essai Experimentale at analitique. J. Ecole Polytechnique, 1 (1975) 24–76.

[29] Erdelyi, A., Magnus, W., Oberhettinger F., y Tricomi, F. G. (1953). Higher transcendental functions, vol.1,3, McGraw-Hill Book Corp., New York.

[30] H. Yépez-Martínez, J.F. Gómez-Aguilar. A new modified definition of Caputo-Fabrizio fractional-order derivative and their applications to the multi step homotopy analysis method (MHAM), J. Comput. Appl. Math. 346 (2019) 247–260.

[31] Sabatier, J., Farges, C. Initial Value Problem Should Not Be Associated to a Fractional Model Description Whatever the Derivative Definition Used. International workshop on Advanced Theory and Applications of Fractional Calculus. Springer, Cham, 2022.

[32] H. Yépez-Martínez, J.F. Gómez-Aguilar, Laplace variational iteration method for modified fractional derivatives with non-singular kernel, J. Appl. Comput. Mech. 6 (3) (2020) 684–69 .

[33] Lorenzo, C.F., and Hartley, T.T. Initialized Fractional Calculus, 2000. `https://ntrs.nasa.gov/citations/20000031631`

[34] Rossikhin, Yu. A., Shitikova, M. V. , and Kolesnikov, M. A. Application of viscoelastic models with two fractional parameters to problems of vibrating 2-DOF mechanical systems. Deposited at VINITI, November 19, 1997, No. 3383-97.

[35] Rossikhin, Yu. A., Shitikova, M. V. , and Kolesnikov, M. A. Application of the generalized model of a standard linear body with two fractional parameters to the description of the impact of a viscoelastic bar on a rigid obstacle. Deposited at VINITI, June 19, 1998, No. 1885-B98.

[36] Enelund, M. Fenander, A. andOlson, P. Fractional integral formulation of constitutive equations of viscoelasticity. AIAA J., 35, No. 8, 1356–1362 (1997).

[37] Enelund, M. and osefson, B. L. Time-domain finite element analysis of viscoelastic structures with fractional derivatives constitutive relations. AIAA J., 35, No. 10, 1630–1637 (1997).

[38] Podlubny, I. Fractional Differential Equations: An Introduction to Fractional Derivatives, Fractional Differential Equations, to Methods of Their Solution and some of Their Applications. Academic Press, San Diego, Calif, USA, (1999).

[39] Caputo, M. and Fabricio, M. A New Definition of Fractional Derivative without Singular Kernel. Progr. Fract. Differ. Appl. 1(2), (2015), 73-85.

[40] Atangana, A. D. Baleanu. New Fractional Derivatives with Nonlocal and Non-Singular Kernel. Theory and Application to Heat Transfer Model. Therm Sci. 20(2), (2016), 763-769.

[41] Lozada, J. and Nieto J. J. Properties of a New Fractional Derivative without Singular Kernel. Progr. Fract. Differ. Appl. 1(2), (2015), 87-92.

[42] Kilbas, A. A. Srivastava, H. M. and Trujillo, J. J. Theory and Applications of Fractional Differential Equations. North-Holland Mathematics Studies 204. Editor: Jan van Mill. Elsevier, Amsterdam 2006.

[43] Gorenflo, R. Kilbas, A. A. Mainardi, F. andRogosin, S. V. Mittag-Leffler Functions, Related Topics and Applications. Springer Monographs in Mathematics. Heidelberg, 2014.

[44] Giusti, A., Colombaro, I., Garra, R., Garrappa, R., Polito, F., Popolizio, M., and Mainardi, F. A practical guide to Prabhakar fractional calculus. Fractional Calculus and Applied Analysis, 23(1), 9-54. (2020).

[45] Prabhakar, T. R. A singular integral equation with a generalized Mittag Leffler function in the kernel. Yokohama Mathematical Journal. 19(1), 7-15. (1971).

[46] Li, Z., Liu, Y., and Yamamoto, M. Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients. Applied Mathematics and Computation, 257, 381-397. (2015).

[47] Luchko, Y. U. R. I. I., and Gorenflo, R. . An operational method for solving fractional differential equations with the Caputo derivatives. Acta Math. Vietnam, 24(2), 207-233. (1999).

[48] Bazhlekova, E., and Bazhlekov, I. . Identification of a space-dependent source term in a nonlocal problem for the general time-fractional diffusion equation. Journal of Computational and Applied Mathematics, 386, 113213. (2021).

[49] Fernandez, A., Kürt, C., and Özarslan, M. A. A naturally emerging bivariate Mittag-Leffler function and associated fractional-calculus operators. Computational and Applied Mathematics, 39(3), 1-27. (2020).

[50] Ahmadova, A., Huseynov, I. T., Fernandez, A., and Mahmudov, N. I. Trivariate Mittag-Leffler functions used to solve multi-order systems of fractional differential equations. *Communications in Nonlinear Science and Numerical Simulation*, 97, 105735. (2021).

[51] M. Caputo, M. Fabrizio. 3D Memory Constitutive Equations for Plastic Media. Journal of Engineering Mechanics, 143(5), (2017), D4016008.

[52] Atangana, A. Koca, I. Chaos in a simple nonlinear system with Atangana-Baleanu derivatives with fractional order. Chaos Soliton & Fractals, 89, (2016), 447-454.

[53] Vladimirov, V. S. (2002). *Methods of the theory of generalized functions*. CRC Press.

[54] Li, Ming. Integral Representation of Fractional Derivative of Delta Function. Fractal and Fractional, 4(3), 47 (2020).