CONTINUOUS NON-ARCHIMEDEAN AND p-ADIC WELCH BOUNDS

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Abstract: We prove the continuous non-Archimedean (resp. p-adic) Banach space and Hilbert space versions of non-Archimedean (resp. p-adic) Welch bounds proved by M. Krishna. We formulate continuous non-Archimedean and p-adic functional Zauner conjectures.
Keywords: Non-Archimedean valued field, Non-Archimedean Banach space, p-adic number field, p-adic Banach space, Welch bound, Zauner conjecture.
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1. Introduction

Prof. L. Welch proved the following bounds in 1974 which appears in everyday life [84].

Theorem 1.1. *[84] (**Welch Bounds**) Let $n > d$. If $\{\tau_j\}_{j=1}^n$ is any collection of unit vectors in $\mathbb{C}^d$, then*

$$\sum_{1\leq j,k\leq n}|\langle \tau_j,\tau_k\rangle|^{2m}=\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j,\tau_k\rangle|^{2m}\geq \frac{n^2}{\binom{d+m-1}{m}},\quad \forall m\in \mathbb{N}.$$

In particular,

$$\sum_{1\leq j,k\leq n}|\langle \tau_j,\tau_k\rangle|^{2}=\sum_{j=1}^{n}\sum_{k=1}^{n}|\langle \tau_j,\tau_k\rangle|^{2}\geq \frac{n^2}{d}.$$

Further,

$$\textbf{(Higher order Welch bounds)}\quad \max_{1\leq j,k\leq n, j\neq k}|\langle \tau_j,\tau_k\rangle|^{2m}\geq \frac{1}{n-1}\left[\frac{n}{\binom{d+m-1}{m}}-1\right],\quad \forall m\in \mathbb{N}.$$

In particular,

$$\textbf{(First order Welch bound)}\quad \max_{1\leq j,k\leq n, j\neq k}|\langle \tau_j,\tau_k\rangle|^{2}\geq \frac{n-d}{d(n-1)}.$$

Theorem 1.1 is a fundamental result in many areas such as in the study of root-mean-square (RMS) absolute cross relation of unit vectors [71], frame potential [9, 14, 18], correlations [70], codebooks [27], numerical search algorithms [85, 86], quantum measurements [73], coding and communications [76, 80], code division multiple access (CDMA) systems [49, 50], wireless systems [68], compressed/compressive sensing [1, 6, 29, 32, 72, 78, 79, 81], 'game of Sloanes' [45], equiangular tight frames [77], equiangular lines [7, 8, 15, 17, 20, 23, 26, 30, 31, 34, 35, 37–40, 44, 46, 47, 57, 60, 62, 63, 87], digital fingerprinting [59] etc.
Theorem 1.1 has been upgraded/different proofs were given in [19, 24, 25, 28, 42, 69, 76, 82, 83]. In 2021 M. Krishna derived continuous version of Theorem 1.1 [51]. In 2022 M. Krishna obtained Theorem 1.1

for Hilbert C*-modules [53], Banach spaces [52], non-Archimedean Hilbert spaces [55], p-adic Hilbert spaces [56] non-Archimedean Banach spaces and p-adic Banach spaces [54].

In this paper we derive continuous non-Archimedean (resp. p-adic) Banach space version of non-Archimedean (resp. p-adic) functional Welch bounds in Theorems 2.4 and (resp. Theorems 3.3). We formulate continuous non-Archimedean Zauner conjectures (Conjectures 2.8 and 2.9) and continuous p-adic Zauner conjectures (Conjectures 3.7 and 3.8).

2. Continuous non-Archimedean Welch bounds

In this section we derive continuous non-Archimedean Banach space version of results derived in [54]. Let $\mathbb{K}$ be a non-Archimedean (complete) valued field satisfying

$$\left|\sum_{j=1}^{n}\lambda_j^2\right| = \max_{1\leq j\leq n}|\lambda_j|^2, \quad \forall \lambda_j \in \mathbb{K}, 1\leq j\leq n, \forall n\in\mathbb{N}. \tag{1}$$

For examples of such fields, we refer [64]. Throughout this section, we assume that our non-Archimedean field $\mathbb{K}$ satisfies Equation (1). Letter $\mathcal{X}$ stands for a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. Identity operator on $\mathcal{X}$ is denoted by $I_{\mathcal{X}}$. The dual of $\mathcal{X}$ is denoted by $\mathcal{X}^*$. In the entire paper, G denotes a locally compact group and μ_G denotes a left Haar measure on G. We assume throughout the paper that the diagonal $\Delta \coloneqq \{(g,g) : g\in G\}$ is measurable in $G\times G$. All our integrals are weak non-Archimedean Riemann integrals (see [67]).

Theorem 2.1. ***(First Order Continuous Non-Archimedean Functional Welch Bound)*** *Let $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ and $\{f_g\}_{g\in G}$ be a collection in $\mathcal{X}^*$ satisfying following conditions.*

(i) *For every $x\in\mathcal{X}$ and for every $\phi\in\mathcal{X}^*$, the map*
$$G\ni g\mapsto f_g(x)\phi(\tau_g)\in\mathcal{X}$$
is measurable and integrable.

(ii) *The map*
$$S_{f,\tau} : \mathcal{X}\ni x\mapsto \int_G f_g(x)\tau_g\, d\mu_G(g)\in\mathcal{X}$$
is a well-defined bounded linear operator.

(iii) *The operator $S_{f,\tau}$ is diagonalizable.*

(iv)
$$\int_{G\times G}|f_g(\tau_h)f_h(\tau_g)|\, d(\mu_G\times\mu_G)(g,h)<\infty.$$

Then
$$\max\left\{\left|\int_\Delta f_g(\tau_g)^2\, d(\mu_G\times\mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}\geq\frac{1}{|d|}\left|\int_G f_g(\tau_g)\, d\mu_G(g)\right|^2.$$

In particular, if $f_g(\tau_g)=1$ for all $g\in G$, then

(First order continnuous non-Archimedean functional Welch bound)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}\geq \frac{|\mu(G)|^2}{|d|}.$$

Proof. We first note that

$$\operatorname{Tra}(S_{f,\tau})=\int\limits_G f_g(\tau_g)\,d\mu_G(g),$$

$$\operatorname{Tra}(S_{f,\tau}^2)=\int\limits_G\int\limits_G f_g(\tau_h)f_h(\tau_g)\,d\mu_G(g)\,d\mu_G(h).$$

Let $\lambda_1,\dots,\lambda_d$ be the diagonal entries in the diagonalization of $S_{f,\tau}$. Then using the diagonalizability of $S_{f,\tau}$ and the non-Archimedean Cauchy-Schwarz inequality (Theorem 2.4.2 [64]), we get

$$\begin{aligned}
\left|\int\limits_G f_g(\tau_g)\,d\mu_G(g)\right|^2 &= |\operatorname{Tra}(S_{f,\tau})|^2=\left|\sum_{k=1}^d\lambda_k\right|^2\leq |d|\left|\sum_{k=1}^d\lambda_k^2\right|=|d||\operatorname{Tra}(S_{f,\tau}^2)|\\
&=|d|\left|\int\limits_G\int\limits_G f_g(\tau_h)f_h(\tau_g)\,d\mu_G(g)\,d\mu_G(h)\right|=|d|\left|\int\limits_{G\times G} f_g(\tau_h)f_h(\tau_g)\,d(\mu_G\times\mu_G)(g,h)\right|\\
&=|d|\left|\int\limits_\Delta f_g(\tau_g)^2\,d(\mu_G\times\mu_G)(g,g)+\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)f_h(\tau_g)\,d(\mu_G\times\mu_G)(g,h)\right|\\
&\leq |d|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^2\,d(\mu_G\times\mu_G)(g,g)\right|,\left|\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)f_h(\tau_g)\,d(\mu_G\times\mu_G)(g,h)\right|\right\}\\
&\leq |d|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^2\,d(\mu_G\times\mu_G)(g,g)\right|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}.
\end{aligned}$$

Whenever $f_g(\tau_g)=1$ for all $g\in G$,

$$|\mu(G)|^2\leq |d|\max\left\{|(\mu_G\times\mu_G)(\Delta)|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}.$$

□

Corollary 2.2. ***(First Order Continuous Non-Archimedean Welch Bound)*** *Let $\mathcal{X}$ be a d-dimensional non-Archimedean Hilbert space over $\mathbb{K}$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ satisfying following conditions.*

(i) *For every $x\in\mathcal{X}$ and for every $\phi\in\mathcal{X}^*$, the map*

$$G\ni g\mapsto \langle x,\tau_g\rangle\phi(\tau_g)\in\mathcal{X}$$

is measurable and integrable.

(ii) *The map*

$$S_\tau:\mathcal{X}\ni x\mapsto\int\limits_G\langle x,\tau_g\rangle\tau_g\,d\mu_G(g)\in\mathcal{X}$$

is a well-defined bounded linear operator.

(iii) *The operator S_τ is diagonalizable.*

(iv)

$$\int\limits_{G\times G} |\langle \tau_g, \tau_h\rangle|^2 \, d(\mu_G \times \mu_G)(g,h) < \infty.$$

Then

$$\max\left\{\left|\int\limits_{\Delta} \langle \tau_g, \tau_g\rangle^2 \, d(\mu_G \times \mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h\rangle|^2\right\} \geq \frac{1}{|d|}\left|\int\limits_{G} \langle \tau_g, \tau_g\rangle \, d\mu_G(g)\right|^2.$$

In particular, if $\langle \tau_g, \tau_g\rangle = 1$ for all $g \in G$, then

(First order continnuous non-Archimedean Welch bound)

$$\max\left\{|(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h\rangle|^2\right\} \geq \frac{|\mu(G)|^2}{|d|}.$$

Next we obtain higher order continuous non-Archimedean functional Welch bounds. We need the following vector space result.

Theorem 2.3. *[13, 21] If $\mathcal{V}$ is a vector space of dimension d and $Sym^m(\mathcal{V})$ denotes the vector space of symmetric m-tensors, then*

$$dim(Sym^m(\mathcal{V})) = \binom{d+m-1}{m}, \quad \forall m \in \mathbb{N}.$$

Theorem 2.4. *(**Higher Order Continuous Non-Archimedean Functional Welch Bounds**) Let $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ and $\{f_g\}_{g\in G}$ be a collection in $\mathcal{X}^*$ satisfying following conditions.*

(i) *For every $x \in Sym^m(\mathcal{X})$ and for every $\phi \in Sym^m(\mathcal{X})^*$, the map*

$$G \ni g \mapsto f_g^{\otimes m}(x)\phi(\tau_g^{\otimes m}) \in Sym^m(\mathcal{X})$$

is measurable and integrable.

(ii) *The map*

$$S_{f,\tau} : Sym^m(\mathcal{X}) \ni x \mapsto \int\limits_{G} f_g^{\otimes m}(x)\tau_g^{\otimes m} \, d\mu_G(g) \in Sym^m(\mathcal{X})$$

is a well-defined bounded linear operator.

(iii) *The operator $S_{f,\tau}$ is diagonalizable.*

(iv)

$$\int\limits_{G\times G} |f_g(\tau_h)f_h(\tau_g)|^m \, d(\mu_G \times \mu_G)(g,h) < \infty.$$

Then

$$\max\left\{\left|\int\limits_{\Delta} f_g(\tau_g)^{2m} \, d(\mu_G \times \mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h} |f_g(\tau_h)f_h(\tau_g)|^m\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\int\limits_{G} f_g(\tau_g)^m \, d\mu_G(g)\right|^2.$$

In particular, if $f_g(\tau_g)=1$ for all $g\in G$, then

(Higher order continuous non-Archimedean functional Welch bounds)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|^m\right\}\geq \frac{|\mu(G)|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Let $\lambda_1,\dots,\lambda_{\dim(\text{Sym}^m(\mathcal{X}))}$ be the diagonal entries in the diagonalization of $S_{f,\tau}$. We note that

$$b\dim(\text{Sym}^{\text{m}}(\mathcal{X}))=\text{Tra}(bI_{\text{Sym}^m(\mathcal{X})})=\text{Tra}(S_{f,\tau})=\int\limits_G f_g^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g),$$

$$b^2\dim(\text{Sym}^{\text{m}}(\mathcal{X}))=\text{Tra}(b^2I_{\text{Sym}^m(\mathcal{X})})=\text{Tra}(S_{f,\tau}^2)=\int\limits_G\int\limits_G f_g^{\otimes m}(\tau_h^{\otimes m})f_h^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\,d\mu_G(h).$$

Then

$$\begin{aligned}
\left|\int\limits_G f_g(\tau_g)^m\,d\mu_G(g)\right|^2&=\left|\int\limits_G f_g^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\right|^2=|\operatorname{Tra}(S_{f,\tau})|^2=\left|\sum_{k=1}^{\dim(\text{Sym}^m(\mathcal{X}))}\lambda_k\right|^2\\
&\leq|\dim(\text{Sym}^m(\mathcal{X}))|\left|\sum_{k=1}^{\dim(\text{Sym}^m(\mathcal{X}))}\lambda_k^2\right|=|\dim(\text{Sym}^m(\mathcal{X}))||\operatorname{Tra}(S_{f,\tau}^2)|\\
&=\left|\binom{d+m-1}{m}\right||\operatorname{Tra}(S_{f,\tau}^2)|=\left|\binom{d+m-1}{m}\right|\left|\int\limits_G\int\limits_G f_g^{\otimes m}(\tau_h^{\otimes m})f_h^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\,d\mu_G(h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_G\int\limits_G f_g(\tau_h)^m f_h(\tau_g)^m\,d\mu_G(g)\,d\mu_G(h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_{G\times G} f_g(\tau_h)^m f_h(\tau_g)^m\,d(\mu_G\times\mu_G)(g,h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)+\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)^m f_h(\tau_g)^m\,d(\mu_G\times\mu_G)(g,h)\right|\\
&\leq|d|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)\right|,\left|\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)^m f_h(\tau_g)^m\,d(\mu_G\times\mu_G)(g,h)\right|\right\}\\
&\leq\left|\binom{d+m-1}{m}\right|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)\right|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)^m f_h(\tau_g)^m|\right\}\\
&=\left|\binom{d+m-1}{m}\right|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)\right|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)f_h(\tau_g)|^m\right\}.
\end{aligned}$$

Whenever $f_g(\tau_g)=1$ for all $g\in G$,

$$|\mu(G)|^2\leq\left|\binom{d+m-1}{m}\right|\max\left\{|(\mu_G\times\mu_G)(\Delta)|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)f_h(\tau_g)|^m\right\}.$$

∎

Corollary 2.5. ***(Higher Order Continuous Non-Archimedean Welch Bounds)*** *Let $\mathcal{X}$ be a d-dimensional non-Archimedean Hilbert space over $\mathbb{K}$. Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ satisfying following conditions.*

(i) *For every $x \in Sym^m(\mathcal{X})$ and for every $\phi \in Sym^m(\mathcal{X})^*$, the map*

$$G \ni g \mapsto \langle x, \tau_g^{\otimes m}\rangle \phi(\tau_g^{\otimes m}) \in Sym^m(\mathcal{X})$$

is measurable and integrable.

(ii) *The map*

$$S_\tau : Sym^m(\mathcal{X}) \ni x \mapsto \int\limits_G \langle x, \tau_g^{\otimes m}\rangle \tau_g^{\otimes m}\, d\mu_G(g) \in Sym^m(\mathcal{X})$$

is a well-defined bounded linear operator.

(iii) *The operator S_τ is diagonalizable.*

(iv)

$$\int\limits_{G\times G} |\langle \tau_g, \tau_h\rangle|^{2m}\, d(\mu_G \times \mu_G)(g,h) < \infty.$$

Then

$$\max\left\{\left|\int\limits_\Delta \langle \tau_g, \tau_g\rangle^{2m}\, d(\mu_G \times \mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h\rangle|^{2m}\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\int\limits_G \langle \tau_g, \tau_g\rangle^m\, d\mu_G(g)\right|^2.$$

In particular, if $\langle \tau_g, \tau_g\rangle = 1$ for all $g \in G$, then

(Higher order continuous non-Archimedean Welch bounds)

$$\max\left\{|(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h\rangle|^{2m}\right\} \geq \frac{|\mu(G)|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Motivated from Theorem 2.1 and Corollary 2.2 we formulate the following questions.

Question 2.6. ***Let $\mathbb{K}$ be a non-Archimedean field satisfying Equation (1) and $\mathcal{X}$ be a d-dimensional non-Archimedean Banach space over $\mathbb{K}$. For which locally compact group G with measurable diagonal Δ, there exist a collection $\{\tau_g\}_{g\in G}$ in $\mathcal{X}$ and a collection $\{f_g\}_{g\in G}$ in $\mathcal{X}^*$ satisfying the following.***

(i) ***For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map***

$$G \ni g \mapsto f_g(x)\phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) ***The map***

$$S_{f,\tau} : \mathcal{X} \ni x \mapsto \int\limits_G f_g(x)\tau_g\, d\mu_G(g) \in \mathcal{X}$$

is a well-defined bounded linear operator.

(iii) ***The operator $S_{f,\tau}$ is diagonalizable.***

(iv)

$$\int\limits_{G\times G} |f_g(\tau_h)f_h(\tau_g)|\, d(\mu_G\times\mu_G)(g,h)<\infty.$$

(v) $f_g(\tau_g)=1$ ***for all*** $g\in G$***.***

(vi)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}=\frac{|\mu(G)|^2}{|d|}.$$

(vii) $\|f_g\|=1$***,*** $\|\tau_g\|=1$ ***for all*** $g\in G$***.***

Question 2.7. ***Let*** $\mathbb{K}$ ***be a non-Archimedean field satisfying Equation (1) and*** $\mathcal{X}$ ***be a*** d***-dimensional non-Archimedean Hilbert space over*** $\mathbb{K}$***. For which locally compact group*** G ***with measurable diagonal*** Δ***, there exists a collection*** $\{\tau_g\}_{g\in G}$ ***in*** $\mathcal{X}$ ***satisfying the following.***

(i) ***For every*** $x\in\mathcal{X}$ ***and for every*** $\phi\in\mathcal{X}^*$***, the map***

$$G\ni g\mapsto \langle x,\tau_g\rangle\phi(\tau_g)\in\mathcal{X}$$

is measurable and integrable.

(ii) ***The map***

$$S_\tau:\mathcal{X}\ni x\mapsto \int\limits_{G}\langle x,\tau_g\rangle\tau_g\, d\mu_G(g)\in\mathcal{X}$$

is a well-defined bounded linear operator.

(iii) ***The operator*** S_τ ***is diagonalizable.***

(iv)

$$\int\limits_{G\times G} |\langle\tau_g,\tau_h\rangle|^2\, d(\mu_G\times\mu_G)(g,h)<\infty.$$

(v) $\langle\tau_g,\tau_g\rangle=1$ ***for all*** $g\in G$***.***

(vi)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|\langle\tau_g,\tau_h\rangle|^2\right\}=\frac{|\mu(G)|^2}{|d|}.$$

A particular case of Questions 2.6 and 2.7 are following continuous non-Archimedean versions of Zauner conjecture (see [2–5, 10–12, 33, 36, 43, 48, 51, 58, 66, 74, 88] for Zauner conjecture in Hilbert spaces, [53] for Zauner conjecture in Hilbert C*-modules, [52] for Zauner conjecture in Banach spaces, [55] for Zauner conjecture in non-Archimedean Hilbert spaces, [56] for Zauner conjecture in p-adic Hilbert spaces and [54] for Zauner conjecture for non-Archimedean Banach spaces and p-adic Banach spaces).

Conjecture 2.8. ***(Continuous Non-Archimedean Functional Zauner Conjecture) Let*** $\mathbb{K}$ ***non-Archimedean field satisfying Equation (1). Given locally compact group*** G ***and for each*** $d\in\mathbb{N}$***, there exist a collection*** $\{\tau_g\}_{g\in G}$ ***in*** $\mathbb{K}^d$ ***(w.r.t. any non-Archimedean norm) and a collection*** $\{f_g\}_{g\in G}$ ***in*** $(\mathbb{K}^d)^*$ ***satisfying the following.***

(i) ***For every*** $x\in\mathbb{K}^d$ ***and for every*** $\phi\in(\mathbb{K}^d)^*$***, the map***

$$G\ni g\mapsto f_g(x)\phi(\tau_g)\in\mathbb{K}^d$$

is measurable and integrable.

(ii) ***The map***

$$S_{f,\tau} : \mathbb{K}^d \ni x \mapsto \int\limits_{G} f_g(x)\tau_g \, d\mu_G(g) \in \mathbb{K}^d$$

is a well-defined bounded linear operator.

(iii) ***The operator*** $S_{f,\tau}$ ***is diagonalizable.***

(iv)

$$\int\limits_{G\times G} |f_g(\tau_h)f_h(\tau_g)| \, d(\mu_G \times \mu_G)(g,h) < \infty.$$

(v) $f_g(\tau_g) = 1$ ***for all*** $g \in G$***.***

(vi)

$$|(\mu_G \times \mu_G)(\Delta)| = |f_g(\tau_h)f_h(\tau_g)| = \frac{|\mu(G)|^2}{|d|}, \quad \forall g,h \in G, g \neq h.$$

(vii) $\|f_g\| = 1$***,*** $\|\tau_g\| = 1$ ***for all*** $g \in G$***.***

Conjecture 2.9. ***(Continuous Non-Archimedean Zauner Conjecture) Let*** $\mathbb{K}$ ***non-Archimedean field satisfying Equation (1). Given locally compact group*** G ***and for each*** $d \in \mathbb{N}$***, there exists a collection*** $\{\tau_g\}_{g\in G}$ ***in*** $\mathbb{K}^d$ ***satisfying the following.***

(i) ***For every*** $x \in \mathbb{K}^d$ ***and for every*** $\phi \in (\mathbb{K}^d)^*$***, the map***

$$G \ni g \mapsto \langle x, \tau_g\rangle\phi(\tau_g) \in \mathbb{K}^d$$

is measurable and integrable.

(ii) ***The map***

$$S_{\tau} : \mathbb{K}^d \ni x \mapsto \int\limits_{G} \langle x, \tau_g\rangle\tau_g \, d\mu_G(g) \in \mathbb{K}^d$$

is a well-defined bounded linear operator.

(iii) ***The operator*** S_τ ***is diagonalizable.***

(iv)

$$\int\limits_{G\times G} |\langle \tau_g, \tau_h\rangle|^2 \, d(\mu_G \times \mu_G)(g,h) < \infty.$$

(v) $\langle \tau_g, \tau_g\rangle = 1$ ***for all*** $g \in G$***.***

(vi)

$$|(\mu_G \times \mu_G)(\Delta)| = |\langle \tau_g, \tau_h\rangle|^2 = \frac{|\mu(G)|^2}{|d|}, \quad \forall g,h \in G, g \neq h.$$

There are four bounds which are companions of Welch bounds in Hilbert spaces. To recall them we need the notion of Gerzon's bound.

Definition 2.10. *[45] Given* $d \in \mathbb{N}$*, define* ***Gerzon's bound***

$$\mathcal{Z}(d,\mathbb{K}) := \begin{cases} d^2 & \text{if} \quad \mathbb{K} = \mathbb{C} \\ \frac{d(d+1)}{2} & \text{if} \quad \mathbb{K} = \mathbb{R}. \end{cases}$$

Theorem 2.11. *[16, 22, 41, 45, 61, 65, 75, 85] Define* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$ *and* $m := \dim_{\mathbb{R}}(\mathbb{K})/2$*. If* $\{\tau_j\}_{j=1}^n$ *is any collection of unit vectors in* $\mathbb{K}^d$*, then*

(i) ***(Bukh-Cox bound)***

$$\max_{1\leq j,k \leq n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geq \frac{\mathcal{Z}(n-d, \mathbb{K})}{n(1+m(n-d-1)\sqrt{m^{-1}+n-d})-\mathcal{Z}(n-d, \mathbb{K})} \quad \textit{if} \quad n>d.$$

(ii) ***(Orthoplex/Rankin bound)***

$$\max_{1\leq j,k \leq n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geq \frac{1}{\sqrt{d}} \quad \textit{if} \quad n>\mathcal{Z}(d, \mathbb{K}).$$

(iii) ***(Levenstein bound)***

$$\max_{1\leq j,k \leq n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geq \sqrt{\frac{n(m+1)-d(md+1)}{(n-d)(md+1)}} \quad \textit{if} \quad n>\mathcal{Z}(d, \mathbb{K}).$$

(iv) ***(Exponential bound)***

$$\max_{1\leq j,k \leq n, j\neq k}|\langle \tau_j, \tau_k\rangle| \geq 1-2n^{\frac{-1}{d-1}}.$$

Motivated from Theorem 2.11, Theorem 2.1 and Corollary 2.2 we ask the following problems.

Question 2.12. ***Whether there is a continuous non-Archimedean functional version of Theorem 2.11? In particular, does there exists a version of***

(i) ***continuous non-Archimedean functional Bukh-Cox bound?***
(ii) ***continuous non-Archimedean functional Orthoplex/Rankin bound?***
(iii) ***continuous non-Archimedean functional Levenstein bound?***
(iv) ***continuous non-Archimedean functional Exponential bound?***

Question 2.13. ***Whether there is a continuous non-Archimedean version of Theorem 2.11? In particular, does there exists a version of***

(i) ***continuous non-Archimedean Bukh-Cox bound?***
(ii) ***continuous non-Archimedean Orthoplex/Rankin bound?***
(iii) ***continuous non-Archimedean Levenstein bound?***
(iv) ***continuous non-Archimedean Exponential bound?***

3. Continuous p-adic Welch bounds

In this section we derive p-adic Banach space version of results done in [56]. Let p be a prime and $\mathbb{Q}_p$ be the field of p-adic numbers. In this section, $\mathcal{X}$ is a d-dimensional p-adic Banach space over $\mathbb{Q}_p$.

Theorem 3.1. ***(First Order Continuous p-adic Functional Welch Bound)*** *Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Banach space over $\mathbb{Q}_p$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ and $\{f_g\}_{g\in G}$ be a collection in $\mathcal{X}^*$ satisfying following conditions.*

(i) *For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map*

$$G \ni g \mapsto f_g(x)\phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) *There exists $b \in \mathbb{Q}_p$ such that*

$$\int\limits_{G} f_g(x)\tau_g \, d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)
$$\int\limits_{G\times G} |f_g(\tau_h)f_h(\tau_g)|\, d(\mu_G\times\mu_G)(g,h)<\infty.$$

Then
$$\max\left\{\left|\int\limits_{\Delta} f_g(\tau_g)^2\, d(\mu_G\times\mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}\geq \frac{1}{|d|}\left|\int\limits_{G} f_g(\tau_g)\, d\mu_G(g)\right|^2.$$

In particular, if $f_g(\tau_g)=1$ for all $g\in G$, then

(First order continuous p-adic functional Welch bound)
$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}\geq \frac{|\mu(G)|^2}{|d|}.$$

Proof. Define $S_{f,\tau}:\mathcal{X}\ni x\mapsto \int\limits_G f_g(x)\tau_g\, d\mu_G(g)\in\mathcal{X}$. Then
$$bd=\operatorname{Tra}(bI_{\mathcal{X}})=\operatorname{Tra}(S_{f,\tau})=\int\limits_G f_g(\tau_g)\, d\mu_G(g),$$
$$b^2d=\operatorname{Tra}(b^2I_{\mathcal{X}})=\operatorname{Tra}(S_{f,\tau}^2)=\int\limits_G\int\limits_G f_g(\tau_h)f_h(\tau_g)\, d\mu_G(g)\, d\mu_G(h).$$

Therefore
$$\begin{aligned}\left|\int\limits_G f_g(\tau_g)\, d\mu_G(g)\right|^2 &=|\operatorname{Tra}(S_{f,\tau})|^2=|bd|^2=|d||b^2d|=|d|\left|\int\limits_G\int\limits_G f_g(\tau_h)f_h(\tau_g)\, d\mu_G(g)\, d\mu_G(h)\right|\\
&=|d|\left|\int\limits_{G\times G} f_g(\tau_h)f_h(\tau_g)\, d(\mu_G\times\mu_G)(g,h)\right|\\
&=|d|\left|\int\limits_{\Delta} f_g(\tau_g)^2\, d(\mu_G\times\mu_G)(g,g)+\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)f_h(\tau_g)\, d(\mu_G\times\mu_G)(g,h)\right|\\
&\leq |d|\max\left\{\left|\int\limits_{\Delta} f_g(\tau_g)^2\, d(\mu_G\times\mu_G)(g,g)\right|, \left|\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)f_h(\tau_g)\, d(\mu_G\times\mu_G)(g,h)\right|\right\}\\
&\leq |d|\max\left\{\left|\int\limits_{\Delta} f_g(\tau_g)^2\, d(\mu_G\times\mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}.\end{aligned}$$

Whenever $f_g(\tau_g)=1$ for all $g\in G$,
$$|\mu(G)|^2\leq |d|\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}.$$

□

Corollary 3.2. ***(First Order Continuous p-adic Welch Bound)*** *Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Hilbert space over $\mathbb{Q}_p$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ satisfying following conditions.*

(i) *For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map*

$$G \ni g \mapsto \langle x, \tau_g \rangle \phi(\tau_g) \in \mathcal{X}$$

is measurable and integrable.

(ii) *There exists $b \in \mathbb{Q}_p$ such that*

$$\int\limits_G \langle x, \tau_g \rangle \tau_g \, d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)

$$\int\limits_{G\times G} |\langle \tau_g, \tau_h \rangle|^2 \, d(\mu_G \times \mu_G)(g,h) < \infty.$$

Then

$$\max\left\{\left|\int\limits_\Delta \langle \tau_g, \tau_g \rangle^2 \, d(\mu_G \times \mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h \rangle|^2\right\} \geq \frac{1}{|d|}\left|\int\limits_G \langle \tau_g, \tau_g \rangle \, d\mu_G(g)\right|^2.$$

In particular, if $\langle \tau_g, \tau_g \rangle = 1$ for all $g \in G$, then

(First order continuous p-adic Welch bound)

$$\max\left\{|(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h \rangle|^2\right\} \geq \frac{|\mu(G)|^2}{|d|}.$$

Now we derive higher order version of Theorem 3.1.

Theorem 3.3. ***(Higher Order Continuous p-adic Functional Welch Bounds)*** *Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Banach space over $\mathbb{Q}_p$. Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ and $\{f_g\}_{g\in G}$ be a collection in $\mathcal{X}^*$ satisfying following conditions.*

(i) *For every $x \in \text{Sym}^m(\mathcal{X})$ and for every $\phi \in \text{Sym}^m(\mathcal{X})^*$, the map*

$$G \ni g \mapsto f_g^{\otimes m}(x)\phi(\tau_g^{\otimes m}) \in \text{Sym}^m(\mathcal{X})$$

is measurable and integrable.

(ii) *There exists $b \in \mathbb{Q}_p$ satisfying*

$$\int\limits_G f_g^{\otimes m}(x)\tau_g^{\otimes m} \, d\mu_G(g) = bx, \quad \forall x \in \text{Sym}^m(\mathcal{X}).$$

(iii)

$$\int\limits_{G\times G} |f_g(\tau_h) f_h(\tau_g)|^m \, d(\mu_G \times \mu_G)(g,h) < \infty.$$

Then

$$\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m} \, d(\mu_G \times \mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h} |f_g(\tau_h) f_h(\tau_g)|^m\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\int\limits_G f_g(\tau_g)^m \, d\mu_G(g)\right|^2.$$

In particular, if $f_g(\tau_g)=1$ for all $g\in G$, then

(Higher order p-adic continuous functional Welch bound)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|^m\right\}\geq \frac{|\mu(G)|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Proof. Define $S_{f,\tau}:\operatorname{Sym}^m(\mathcal{X})\ni x\mapsto \int\limits_G f_g^{\otimes m}(x)\tau_g^{\otimes m}\,d\mu_G(g)\in \operatorname{Sym}^m(\mathcal{X})$. Then

$$b\dim(\mathrm{Sym^m}(\mathcal{X}))=\operatorname{Tra}(bI_{\operatorname{Sym}^m(\mathcal{X})})=\operatorname{Tra}(S_{f,\tau})=\int\limits_G f_g^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g),$$

$$b^2\dim(\mathrm{Sym^m}(\mathcal{X}))=\operatorname{Tra}(b^2I_{\operatorname{Sym}^m(\mathcal{X})})=\operatorname{Tra}(S_{f,\tau}^2)=\int\limits_G\int\limits_G f_g^{\otimes m}(\tau_h^{\otimes m})f_h^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\,d\mu_G(h).$$

Therefore

$$\begin{aligned}
&\left|\int\limits_G f_g(\tau_g)^m\,d\mu_G(g)\right|^2=\left|\int\limits_G f_g^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\right|^2=|\operatorname{Tra}(S_{f,\tau})|^2=|b\dim(\mathrm{Sym^m}(\mathcal{X}))|^2\\
&=|\dim(\mathrm{Sym^m}(\mathcal{X}))|\left|b^2\dim(\mathrm{Sym^m}(\mathcal{X}))\right|\\
&=|\dim(\mathrm{Sym^m}(\mathcal{X}))|\left|\int\limits_G\int\limits_G f_g^{\otimes m}(\tau_h^{\otimes m})f_h^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\,d\mu_G(h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_G\int\limits_G f_g^{\otimes m}(\tau_h^{\otimes m})f_h^{\otimes m}(\tau_g^{\otimes m})\,d\mu_G(g)\,d\mu_G(h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_G\int\limits_G f_g(\tau_h)^m f_h(\tau_g)^m\,d\mu_G(g)\,d\mu_G(h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_{G\times G} f_g(\tau_h)^m f_h(\tau_g)^m\,d(\mu_G\times\mu_G)(g,h)\right|\\
&=\left|\binom{d+m-1}{m}\right|\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)+\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)^m f_h(\tau_g)^m\,d(\mu_G\times\mu_G)(g,h)\right|\\
&\leq\left|\binom{d+m-1}{m}\right|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)\right|,\left|\int\limits_{(G\times G)\setminus\Delta} f_g(\tau_h)^m f_h(\tau_g)^m\,d(\mu_G\times\mu_G)(g,h)\right|\right\}\\
&\leq\left|\binom{d+m-1}{m}\right|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)\right|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)^m f_h(\tau_g)^m|\right\}\\
&=\left|\binom{d+m-1}{m}\right|\max\left\{\left|\int\limits_\Delta f_g(\tau_g)^{2m}\,d(\mu_G\times\mu_G)(g,g)\right|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)f_h(\tau_g)|^m\right\}.
\end{aligned}$$

Whenever $f_g(\tau_g)=1$ for all $g\in G$,

$$|\mu(G)|^2\leq\left|\binom{d+m-1}{m}\right|\max\left\{|(\mu_G\times\mu_G)(\Delta)|,\sup_{g,h\in G,g\neq h}|f_g(\tau_h)f_h(\tau_g)|^m\right\}.$$

□

Corollary 3.4. ***(Higher Order Continuous p-adic Welch Bounds)*** *Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Hilbert space over $\mathbb{Q}_p$. Let $m \in \mathbb{N}$. Let $\{\tau_g\}_{g\in G}$ be a collection in $\mathcal{X}$ satisfying following conditions.*

(i) *For every $x \in Sym^m(\mathcal{X})$ and for every $\phi \in Sym^m(\mathcal{X})^*$, the map*
$$G \ni g \mapsto \langle x, \tau_g^{\otimes m}\rangle \phi(\tau_g^{\otimes m}) \in Sym^m(\mathcal{X})$$
is measurable and integrable.

(ii) *There exists $b \in \mathbb{Q}_p$ satisfying*
$$\int\limits_G \langle x, \tau_g^{\otimes m}\rangle \tau_g^{\otimes m}\, d\mu_G(g) = bx, \quad \forall x \in Sym^m(\mathcal{X}).$$

(iii)
$$\int\limits_{G\times G} |\langle \tau_g, \tau_h\rangle|^{2m}\, d(\mu_G \times \mu_G)(g,h) < \infty.$$

Then
$$\max\left\{\left|\int\limits_\Delta \langle \tau_g, \tau_g\rangle^{2m}\, d(\mu_G \times \mu_G)(g,g)\right|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h\rangle|^m\right\} \geq \frac{1}{\left|\binom{d+m-1}{m}\right|}\left|\int\limits_G \langle \tau_g, \tau_g\rangle^m\, d\mu_G(g)\right|^2.$$

In particular, if $\langle \tau_g, \tau_g\rangle = 1$ for all $g \in G$, then

(Higher order continuous p-adic Welch bound)
$$\max\left\{|(\mu_G \times \mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h} |\langle \tau_g, \tau_h\rangle|^{2m}\right\} \geq \frac{|\mu(G)|^2}{\left|\binom{d+m-1}{m}\right|}.$$

Using Theorem 3.1 and Corollary 3.2 we ask the following questions.

Question 3.5. ***Let p be a prime and $\mathcal{X}$ be a d-dimensional p-adic Banach space over $\mathbb{Q}_p$. For which locally compact group G with measurable diagonal Δ, there exist a collection $\{\tau_g\}_{g\in G}$ in $\mathcal{X}$ and a collection $\{f_g\}_{g\in G}$ in $\mathcal{X}^*$ satisfying the following.***

(i) ***For every $x \in \mathcal{X}$ and for every $\phi \in \mathcal{X}^*$, the map***
$$G \ni g \mapsto f_g(x)\phi(\tau_g) \in \mathcal{X}$$
is measurable and integrable.

(ii) ***There exists $b \in \mathbb{Q}_p$ such that***
$$\int\limits_G f_g(x)\tau_g\, d\mu_G(g) = bx, \quad \forall x \in \mathcal{X}.$$

(iii)
$$\int\limits_{G\times G} |f_g(\tau_h)f_h(\tau_g)|\, d(\mu_G \times \mu_G)(g,h) < \infty.$$

(iv) $f_g(\tau_g) = 1$ ***for all*** $g \in G$.

(v)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|f_g(\tau_h)f_h(\tau_g)|\right\}=\frac{|\mu(G)|^2}{|d|}.$$

(vi) $\|f_g\|=1$, $\|\tau_g\|=1$ ***for all*** $g\in G$.

Question 3.6. ***Let*** p ***be a prime and*** $\mathcal{X}$ ***be a*** d***-dimensional p-adic Hilbert space over*** $\mathbb{Q}_p$***. For which locally compact group*** G ***with measurable diagonal*** Δ***, there exists a collection*** $\{\tau_g\}_{g\in G}$ ***in*** $\mathcal{X}$ ***satisfying the following.***

(i) ***For every*** $x\in\mathcal{X}$ ***and for every*** $\phi\in\mathcal{X}^*$***, the map***

$$G\ni g\mapsto \langle x,\tau_g\rangle\phi(\tau_g)\in\mathcal{X}$$

is measurable and integrable.

(ii) ***There exists*** $b\in\mathbb{Q}_p$ ***such that***

$$\int\limits_G \langle x,\tau_g\rangle\tau_g\, d\mu_G(g)=bx,\quad \forall x\in\mathcal{X}.$$

(iii)

$$\int\limits_{G\times G}|\langle\tau_g,\tau_h\rangle|^2\, d(\mu_G\times\mu_G)(g,h)<\infty.$$

(iv) $\langle\tau_g,\tau_g\rangle=1$ ***for all*** $g\in G$.

(v)

$$\max\left\{|(\mu_G\times\mu_G)(\Delta)|, \sup_{g,h\in G, g\neq h}|\langle\tau_g,\tau_h\rangle|^2\right\}=\frac{|\mu(G)|^2}{|d|}.$$

(vi) $\|\tau_g\|=1$ ***for all*** $g\in G$.

A particular case of Questions 3.5 and 3.6 are following continuous p-adic Zauner conjectures.

Conjecture 3.7. ***(Continuous p-adic Functional Zauner Conjecture) Let*** p ***be a prime. Given locally compact group*** G ***and for each*** $d\in\mathbb{N}$***, there exist a collection*** $\{\tau_g\}_{g\in G}$ ***in*** $\mathbb{Q}_p^d$ ***(w.r.t. any non-Archimedean norm) and a collection*** $\{f_g\}_{g\in G}$ ***in*** $(\mathbb{Q}_p^d)^*$ ***satisfying the following.***

(i) ***For every*** $x\in\mathbb{Q}_p^d$ ***and for every*** $\phi\in(\mathbb{Q}_p^d)^*$***, the map***

$$G\ni g\mapsto f_g(x)\phi(\tau_g)\in\mathbb{Q}_p^d$$

is measurable and integrable.

(ii) ***There exists*** $b\in\mathbb{Q}_p$ ***such that***

$$\int\limits_G f_g(x)\tau_g\, d\mu_G(g)=bx,\quad \forall x\in\mathbb{Q}_p^d.$$

(iii)

$$\int\limits_{G\times G}|f_g(\tau_h)f_h(\tau_g)|\, d(\mu_G\times\mu_G)(g,h)<\infty.$$

(iv) $f_g(\tau_g)=1$ ***for all*** $g\in G$.

(v)

$$|(\mu_G \times \mu_G)(\Delta)| = |f_g(\tau_h)f_h(\tau_g)| = \frac{|\mu(G)|^2}{|d|}, \quad \forall g, h \in G, g \neq h.$$

(vi) $\|f_g\| = 1$, $\|\tau_g\| = 1$ ***for all*** $g \in G$.

Conjecture 3.8. ***(Continuous p-adic Zauner Conjecture) Let*** p ***be a prime. Given locally compact group*** G ***and for each*** $d \in \mathbb{N}$, ***there exists a collection*** $\{\tau_g\}_{g\in G}$ ***in*** $\mathbb{Q}_p^d$ ***satisfying the following.***

(i) ***For every*** $x \in \mathbb{Q}_p^d$ ***and for every*** $\phi \in (\mathbb{Q}_p^d)^*$, ***the map***

$$G \ni g \mapsto \langle x, \tau_g\rangle\phi(\tau_g) \in \mathbb{Q}_p^d$$

is measurable and integrable.

(ii) ***There exists*** $b \in \mathbb{Q}_p$ ***such that***

$$\int\limits_G \langle x, \tau_g\rangle\tau_g \, d\mu_G(g) = bx, \quad \forall x \in \mathbb{Q}_p^d.$$

(iii)

$$\int\limits_{G\times G} |\langle \tau_g, \tau_h\rangle|^2 \, d(\mu_G \times \mu_G)(g, h) < \infty.$$

(iv) $\langle \tau_g, \tau_g\rangle = 1$ ***for all*** $g \in G$.

(v)

$$|(\mu_G \times \mu_G)(\Delta)| = |\langle \tau_g, \tau_h\rangle|^2 = \frac{|\mu(G)|^2}{|d|}, \quad \forall g, h \in G, g \neq h.$$

(vi) $\|\tau_g\| = 1$ ***for all*** $g \in G$.

Theorem 2.11, Theorem 3.1 and Corollary 3.2 give the following problems.

Question 3.9. ***Whether there is a continuous p-adic functional version of Theorem 2.11? In particular, does there exists a version of***

(i) ***continuous p-adic functional Bukh-Cox bound?***
(ii) ***continuous p-adic functional Orthoplex/Rankin bound?***
(iii) ***continuous p-adic functional Levenstein bound?***
(iv) ***continuous p-adic functional Exponential bound?***

Question 3.10. ***Whether there is a continuous p-adic version of Theorem 2.11? In particular, does there exists a version of***

(i) ***continuous p-adic Bukh-Cox bound?***
(ii) ***continuous p-adic Orthoplex/Rankin bound?***
(iii) ***continuous p-adic Levenstein bound?***
(iv) ***continuous p-adic Exponential bound?***

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