Improved two-dimensional DOA estima-tion of Non-circular Signals for Parallel Sparse Array

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Abstract

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Introduction: In the field of arrays signals processing, analyzing the covariance matrix (CM) of signals can improve the array DOFs and the DOA estimation resolution [1]. In practical application, binary phase shift keying (BPSK) and amplitude shift keying (ASK) signals are widely applied [2]. Different from conventional signals, BPSK and ASK signals have noncircularity property, which makes them be considered as NC signals, and its pseudo covariance matrix (PCM) is not zero, which can be used to extend virtual array as CM. In view of this, researchers proposed a series of effective algorithms, such as spatial smoothing multiple signal classification (SS-MUSIC) [3] and Lasso [4] for 1-D DOA estimation of NC signals, and a good result has been obtained.

However, if the above 1-D algorithms are directly extended to 2-D domain, it will have a large amount of computation and low accuracy. Specifically, due to the existence of unknown NC phase, it is actually dealing with 3-D problem when carrying out 2-D DOA estimation, which contains a great deal of complexity.

In view of this, this letter proposes a low complexity algorithm, which minimizes the impact of unknown NC phase in the whole estimation process. Specifically, by arranging two sparse subarrays symmetrically with the coordinate axis, this letter simplifies the 3-D problem into a 2-D problem, which is regarded as two 1-D estimation with the help of the rank-reduction (RARE) algorithm [5]. Besides, by processing CM and PCM based on two subarrays, the following three superiorities are obtained: the DOFs of virtual array are further effectively expanded; due to the noise of different subarray is uncorrelated, the effective denoising process is realized; the estimated result is contained in it orderly, thus realizing the auto pair-matching process of two angles.

Notations: Throughout this paper, we use lower-case (upper-case) bold characters to denote vectors (matrices). In particular, $(.)^*$, $(.)^{\mathrm{T}}$, $(.)^{\mathrm{H}}$ and $(.)^{-}$ respectively denote the complex conjugation, transpose, conjugate transpose and pseudo inverse operation of a matrix or a vector. $\mathrm{vec}(.)$ denotes the vectorization operation. $\mathrm{diag}(.)$ denotes a diagonalization of a diagonal matrix or a vector. $\det(.)$ denotes the operator for determinant calculation. $\arg(\cdot)$ denotes take phase angle. $\mathrm{E}(.)$ is the statistical expectation operator. $\otimes$ and $\odot$ denote the Kronecker product and Khatri-rao product. $(.)^{a:b}$ denotes the a-th to b-th terms of intercepting a vector. $\mathbf{I}_{J\times R}$ and $\mathbf{O}_{J\times R}$ denote the zero matrix and one matrix of row J and column R. d denotes the unit inter-element spacing which is set to half wavelength $\lambda/2$.

System model: Assuming that K far-field, mutually independent non-Gaussian narrowband NC signals $\{s_k(t)\}_{k=1}^{K}$ impinging a PSA, which consists of two identical sparse linear arrays (SLAs) with a distance to *Y*-axis is d, from directions $\{(\alpha_k,\beta_k)\}_{k=1}^{K}$ which denote the angles between the incident direction of the k-th signal with *Y*-axis and *X*-axis in turn.

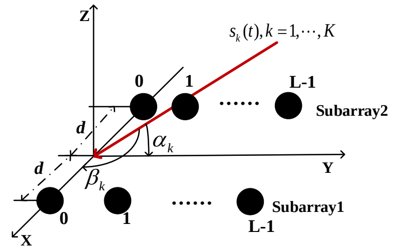


Fig.1 *PSA model*

The position of SLA with L sensors can be recorded as $\mathbb{P}=\{p_l d \mid 1\le l\le L\}$. For convenience, we define $f_k^{\alpha}=j2\pi\cos\alpha_k/\lambda$, where $j=\sqrt{-1}$, and the received signal of two subarrays can be written as [6],

$$\mathbf{x}_i(t)=\mathbf{A}\mathbf{B}_i\mathbf{\Phi}\mathbf{s}_R(t)+\mathbf{n}_i(t), i=1,2 \tag{1}$$

where $\mathbf{A}=[\mathbf{a}(\alpha_1),\cdots,\mathbf{a}(\alpha_k),\cdots,\mathbf{a}(\alpha_K)]$ and $\mathbf{a}(\alpha_k)=[e^{f_k^{\alpha}p_1 d},\cdots,e^{f_k^{\alpha}p_L d}]^{\mathrm{T}}$. $\mathbf{B}_1=\mathrm{diag}(e^{j\pi\cos\beta_1},\cdots,e^{j\pi\cos\beta_K})$ and $\mathbf{B}_2=\mathbf{B}_1^*$. $\mathbf{s}_R(t)=[s_{R1}(t),\cdots,s_{RK}(t)]^{\mathrm{T}}$ and $\mathbf{\Phi}=\mathrm{diag}(e^{j\varphi_1},\cdots,e^{j\varphi_K})$ are the real-valued signals and theirs NC phase, respectively. $\mathbf{n}_1(t),\mathbf{n}_2(t)$ denote two zero-mean, mutually irrelevant additive white Gaussian noise vector independent of signal.

In order to extend original array, the CM of subarray1 is calculated and carry on the denoising processing as like [7]

$$\mathbf{R}_D=\mathbf{R}_{11}-\sigma_n^2\mathbf{I}_{2K}=E[\mathbf{x}_1(t)\mathbf{x}_1^{\mathrm{H}}(t)]=\mathbf{A}\mathbf{R}_K\mathbf{A}^{\mathrm{H}} \tag{2}$$

where $\mathbf{R}_K=\mathrm{diag}(\sigma_1^2,\cdots,\sigma_k^2,\cdots,\sigma_K^2)$. σ_k^2 denotes the power of the k-th signal. σ_n^2 denotes the power of noise.

By vectorizing (2) and removing the repeated positions in it, we can get the received signals of DC [8].

$$\mathrm{vec}(\mathbf{R}_D)=(\mathbf{A}^*\odot\mathbf{A})\mathbf{F}\xrightarrow{remove}\mathbf{y}_1=\mathbf{U}_1\mathbf{F} \tag{3}$$

where $\mathbf{F}=\mathrm{diag}(\mathbf{R}_K)$. $\mathbf{U}_1=[\mathbf{u}_1(\alpha_1),\cdots,\mathbf{u}_1(\alpha_k),\cdots,\mathbf{u}_1(\alpha_K)]$, and $\mathbf{u}_1(\alpha_k)=[e^{f_k^{\alpha}p_1^1 d},\cdots,e^{f_k^{\alpha}p_u^1 d},\cdots,e^{f_k^{\alpha}p_{U_1}^1 d}]^{\mathrm{T}}$. $p_u^1 d\in\mathbb{U}_1=\{(p_i-p_j)d \mid p_i,p_j\in\mathbb{P}\}$ denotes the position of virtual sensors based on DC.

Similarly, by calculating the PCM based on two subarrays and constructing the virtual extend matrix based on sum co-array (SC),

$$\tilde{\mathbf{R}}_{12}=\mathrm{E}[\mathbf{x}_1(t)\mathbf{x}_2^{\mathrm{T}}(t)]=\mathbf{A}\mathbf{\Phi}^2\mathbf{R}_K\mathbf{A}^{\mathrm{T}} \tag{4}$$

$$\mathbf{R}_S=[\tilde{\mathbf{R}}_{12},\tilde{\mathbf{R}}_{12}^{\mathrm{H}}] \tag{5}$$

Obviously, because of the noise received by the two subarrays is uncorrelated, so that (4) will not have noise terms.

By vectorizing (5) and removing the repeated positions in it, we can get the received signals of sum co-array (SC).

$$vec(\mathbf{R}_S)=\begin{bmatrix}\mathbf{A}\odot\mathbf{A}\\(\mathbf{A}\odot\mathbf{A})^{\mathrm{H}}\end{bmatrix}\mathbf{C}\xrightarrow{remove}\mathbf{y}_2=\mathbf{U}_2\mathbf{C} \tag{6}$$

where $\mathbf{C}=\mathrm{diag}(\mathbf{\Phi}^2\mathbf{R}_K)$. $\mathbf{U}_2=[\mathbf{u}_2(\alpha_1),\cdots,\mathbf{u}_2(\alpha_k),\cdots,\mathbf{u}_2(\alpha_K)]$, $\mathbf{u}_2(\alpha_k)=[e^{f_k^{\alpha}p_1^2 d},\cdots,e^{f_k^{\alpha}p_u^2 d},\cdots,e^{f_k^{\alpha}p_{U_2}^2 d}]^{\mathrm{T}}$ and $p_u^2 d\in\mathbb{U}_2=\{\pm(p_i+p_j)d \mid p_i,p_j\in\mathbb{P}\}$ denotes the position of virtual sensors based on SC.

Then, we can construct the virtual extend matrix based on SDC,

$$\mathbf{R}_{SD}=[\mathbf{R}_D,\mathbf{R}_S]^{\mathrm{T}} \tag{7}$$

By carrying out processes similar to (3) and (6) for (7), it can be noted that since $\mathbb{U}_1$ and $\mathbb{U}_2$ have the same elements, this letter uses the former value here. And in order to carry out the SS-MUSIC algorithm, it is necessary to intercept the continuous virtual sensors of the received signals based on SDC.

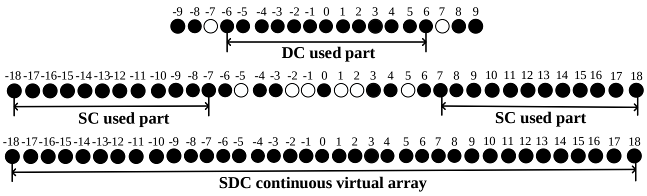


Fig.2 *Set physical array as coprime array with two coprime integers 3, 4. The used continuous virtual array based on DC, SC and SDC with* $(2V_1+1=)13$, $(2V_2=)24$ and $(2V+1=)$ 37 sensors, respectively.

$$\mathrm{vec}(\mathbf{R}_{SD})\xrightarrow[intercept]{remove}\mathbf{y}_3=\begin{bmatrix}\tilde{\mathbf{U}}_{22} & \mathbf{O}_2 & \mathbf{O}_2\\ \mathbf{O}_1 & \tilde{\mathbf{U}}_1 & \mathbf{O}_1\\ \mathbf{O}_2 & \mathbf{O}_2 & \tilde{\mathbf{U}}_{21}\end{bmatrix}\begin{bmatrix}\mathbf{C}\\ \mathbf{F}\\ \mathbf{C}\end{bmatrix}=\tilde{\mathbf{U}}_3\mathbf{D} \tag{8}$$

where $\mathbf{D}=\mathrm{diag}([\mathbf{C},\mathbf{F},\mathbf{C}])$. $\tilde{\mathbf{U}}_1\in\mathbb{C}^{(2V_1+1)\times K}$ and $\tilde{\mathbf{U}}_{22}(=\tilde{\mathbf{U}}_{21}^*)\in\mathbb{C}^{V_2\times K}$ respectively denotes the used continuous virtual array steering matrix of $\mathbf{U}_1$ and $\mathbf{U}_2$, which is shown in Fig.2. $\tilde{\mathbf{U}}_3=[\tilde{\mathbf{U}}_{22},\tilde{\mathbf{U}}_1,\tilde{\mathbf{U}}_{21}]^{\mathrm{T}}=[\mathbf{u}_3(\alpha_1),\cdots,\mathbf{u}_3(\alpha_k),\cdots,\mathbf{u}_3(\alpha_K)]$ and $\mathbf{u}_3(\alpha_k)^{1:2V+1}=[e^{f_k^{\alpha}(-Vd)},\cdots,1,\cdots,e^{f_k^{\alpha}Vd}]^{\mathrm{T}}$. $\mathbf{O}_1$ and $\mathbf{O}_2$ are zero matrices of the same size as $\tilde{\mathbf{U}}_1$ and $\tilde{\mathbf{U}}_{22}$ in turn.

For (8), it can be found that by arranging two subarrays reasonably, angle β_k can be eliminated in the process of calculating α_k , thus the the 3-D problem is simplified to into a 2-D problem.

Then, the estimated value of covariance matrix is obtained by performing spatial smoothing process,

$$\mathbf{R}_{SS}=\mathbf{y}_3^{i:i+V}(\mathbf{y}_3^{i:i+V})^{\mathrm{H}},i=1,\cdots,V+1 \tag{9}$$

$$\hat{\mathbf{R}}_{SS}=\left[\sum\nolimits_{v=1}^{V+1}\mathbf{R}_{SS}(i)\right]\Big/(V+1) \tag{10}$$

Based on $\hat{\mathbf{R}}_{SS}$, the peak search function can be defined as [3]

$$v(\alpha_{\mathrm{g}_1},\varphi_{\mathrm{g}_2})=1\Big/\left[\mathbf{H}^{\mathrm{H}}(\alpha_{\mathrm{g}_1},\varphi_{\mathrm{g}_2})\mathbf{E}_n\mathbf{E}_n^{\mathrm{H}}\mathbf{H}(\alpha_{\mathrm{g}_1},\varphi_{\mathrm{g}_2})\right] \tag{11}$$

where $\mathbf{E}_n\in\mathbb{C}^{(V+1)\times(V+1-K)}$ denotes the noise subspace of $\hat{\mathbf{R}}_{SS}$. $\mathrm{g}_1=1,\cdots,G_\alpha$, and $\mathrm{g}_2=1,\cdots,G_\varphi$ denotes the search grids for angles α and NC phases φ . With defining,

$$\mathbf{H}(\alpha_{\mathrm{g}_1},\varphi_{\mathrm{g}_2})=\mathbf{h}_1(\alpha_{\mathrm{g}_1})\mathbf{T}\mathbf{h}_2(\varphi_{\mathrm{g}_2}) \tag{12}$$

where $\mathbf{h}_1(\alpha_{\mathrm{g}_1})=\mathrm{diag}[\mathbf{u}_4(\alpha_{\mathrm{g}_1})]$ and $\mathbf{u}_4(\alpha_{\mathrm{g}_1})=\mathbf{u}_3(\alpha_{\mathrm{g}_1})^{V+1:2V+1}=[1,\cdots,e^{jV\pi\cos\alpha_{\mathrm{g}_1}}]^{\mathrm{T}}\in\mathbb{C}^{V+1}$, $\mathbf{h}_2(\varphi_{\mathrm{g}_2})=[1,e^{j2\varphi_{\mathrm{g}_2}}]^{\mathrm{T}}$.

$$\mathbf{T}=\begin{bmatrix}\mathbf{I}_{1\times(V_1+1)} & \mathbf{O}_{1\times V_2}\\ \mathbf{O}_{1\times(V_1+1)} & \mathbf{I}_{1\times V_2}\end{bmatrix}^{\mathrm{T}} \tag{13}$$

where $V_1+V_2=V$, and can transform (11) into

$$\begin{aligned}v(\alpha_{\mathrm{g}_1},\varphi_{\mathrm{g}_2})&=1\Big/\left[\mathbf{h}_2^{\mathrm{H}}(\varphi_{\mathrm{g}_2})\mathbf{T}^{\mathrm{H}}\mathbf{h}_1^{\mathrm{H}}(\alpha_{\mathrm{g}_1})\mathbf{E}_n\mathbf{E}_n^{\mathrm{H}}\mathbf{h}_1(\alpha_{\mathrm{g}_1})\mathbf{T}\mathbf{h}_2(\varphi_{\mathrm{g}_2})\right]\\&=1\Big/\left[\mathbf{h}_2^{\mathrm{H}}(\varphi_{\mathrm{g}_2})\mathbf{G}(\alpha_{\mathrm{g}_1})\mathbf{h}_2(\varphi_{\mathrm{g}_2})\right]\end{aligned} \tag{14}$$

where $\mathbf{G}(\alpha_{\mathrm{g}_1})=\mathbf{T}^{\mathrm{H}}\mathbf{h}_1^{\mathrm{H}}(\alpha_{\mathrm{g}_1})\mathbf{E}_n\mathbf{E}_n^{\mathrm{H}}\mathbf{h}_1(\alpha_{\mathrm{g}_1})\mathbf{T}$. According to RARE algorithm, it can be solved as

$$v(\alpha_{\mathrm{g}_1})=\arg\max\{1/\det[\mathbf{G}(\alpha_{\mathrm{g}_1})]\} \tag{15}$$

In order to further simplify the calculation process, bring $z_{g_1}=e^{j\pi\cos\alpha_{g_1}}$ into (15), and it can be transformed into finding the K roots of the following formula

$$\det[\mathbf{G}(z_{\mathrm{g}_1})]=0 \tag{16}$$

Consequently,

$$\hat{\alpha}_k=\cos^{-1}[\arg(z_{\mathrm{g}_1})/\pi],k=1,2,\cdots,K \tag{17}$$

After that, to realize the estimation of angle β , we need to calculate the CM based on two subarrays, and carry out processes similar to (6),

$$\mathbf{R}_{12}=\mathrm{E}[\mathbf{x}_1(t)\mathbf{x}_2^{\mathrm{H}}(t)]=\mathbf{A}\mathbf{B}_1^2\mathbf{R}_K\mathbf{A}^{\mathrm{H}} \tag{18}$$

$$\mathrm{vec}(\mathbf{R}_{12})=(\mathbf{A}^*\odot\mathbf{A})\mathbf{J}\xrightarrow{remove}\mathbf{y}_4=\mathbf{U}_1\mathbf{J} \tag{19}$$

where $\mathbf{J}=\mathrm{diag}(\mathbf{B}_1^2\mathbf{R}_K)$.

In order to realize estimation of β_k and complete the automatic pairing process, we only need to process with (19) as follows,

$$(\beta_k)_{k=1}^K=\cos^{-1}[\arg(\mathbf{U}_1^-\mathbf{y}_4)/2\pi)] \tag{20}$$

where $\mathbf{U}_1^-$ can be obtained by $\hat{\alpha}_k$ and because angle α_k in it has sequential information, so that the automatic pair-matching process is completed.

Simulations: This section uses numerical results to verify the superiority of the proposed algorithm, and it is mainly divided into the following three parts: showing the expansion effect of the continuous DOFs under the proposed algorithm; comparing the computational complexity and running time between the algorithms; comparing the DOA estimation performance between algorithms by Monte Carlo experiments.

For convenience, we briefly introduce four common SLAs that will be used below (the order is coprime array (CA) [9], augmented coprime array (ACA) [10], coprime array with compressed inter-element spacing (CACIS) [10], two-level nested array (TNA) [11]),

$$\mathbb{P}_1=\{mNd\mid 0\le m\le M\text{-}1\}\cup\{nMd|0\le n\le N\text{-}1\} \tag{21}$$

$$\mathbb{P}_2=\{mNd|0\le m\le 2M\text{-}1\}\cup\{nMd|0\le n\le N\text{-}1\} \tag{22}$$

$$\mathbb{P}_3=\{mNd\mid 0\le m\le M\text{-}1\}\cup\{n\tilde{M}d|0\le n\le N\text{-}1\} \tag{23}$$

$$\mathbb{P}_4=\{m_1d|0\le m_1\le M_1-1\}\cup\{[m_2(M_1+1)-1]d|1\le m_2\le M_2\} \tag{24}$$

where m,n,M,N,M_1,M_2 are all integers, $M(=p\tilde{M},2\le p\le M)$ and N are coprime numbers.

By setting different initial values, we verify the increase effect of the continuous DOFs under the proposed algorithm, and the result is shown in Table 1. From this, we can clearly see the advantage of the proposed algorithm in expanding continuous DOFs.

Table 1: expansion effect of the continuous DOFs

Array Type	Initial Value	DC	SDC
CA	$M=3,N=4$	13	37
	$M=4,N=5$	17	21
	$M=6,N=7$	25	29
ACA	$M=3,N=4$	29	53
	$M=4,N=5$	47	87
	$M=6,N=7$	95	179
CACIS ($p=2$)	$M=4,N=5$	23	43
	$M=6,N=5$	35	65
	$M=6,N=7$	47	89
TNA	$M_1=4,M_2=3$	29	37
	$M_1=4,M_2=5$	49	57
	$M_1=6,M_2=7$	97	109

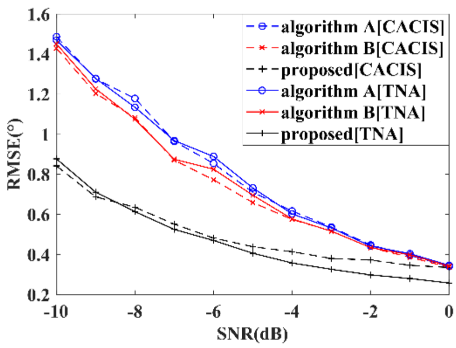
Then, the computational complexity is analyzed, and compared with the algorithm A [7] and algorithm B [11]. The amount of computation of the proposed algorithm is mainly composed of the following sections: the estimation of $\mathbf{R}_{11}$, $\tilde{\mathbf{R}}_{12}$, $\mathbf{R}_{SS}$ and $\mathbf{R}_{12}$; the calculation of $\mathbf{G}(\alpha_{\mathrm{g}_1})$ and $\mathbf{U}_1^-$, so the computational complexity of the proposed algorithm is obtained $O\{3L^2Y+2(V+1)^2(V+4)+U_1^3\}$, where Y denotes the number of snapshots. From [7] and [11], it is known that the computational complexity of algorithm A and B is $O\{2L^2Y+2K^3+(2MN+2M)^3\}$, and $O\{2(M_1+M_2)^2Y+(2M_2(M_1+1))^3\}$, respectively.

Due to the complexity of parameters, and only one special case under a particular SLA is given in [7] and [11], it is not easy to compare directly. Therefore, we conduct 500 experiments to intuitively compare the running time, and it is assumed that the physical array is CACIS $(M=6, N=5, p=3)$, the number of snapshots is 500 and the angles of NC signals are $(40°, 70°), (50°, 65°)$, the result is shown in Table 2.

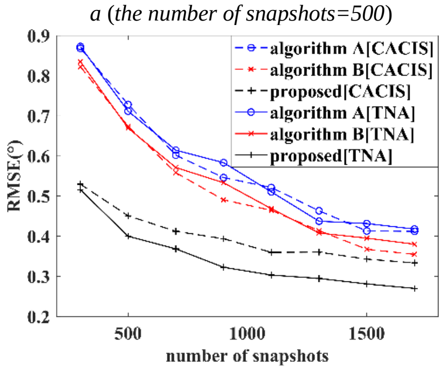
Table 2: running time under three algorithms

Algorithm	A	B	The proposed
Operation time(s)	93.450	105.457	96.200

For Table 2, we can find that the operation time under the three algorithms is almost the same. The proposed algorithm is slightly better than the algorithm B, but worse than the algorithm A.



a (*the number of snapshots=500*)



b (*SNR=-5dB*)

Fig.3 *RMSEs versus SNR and the number of snapshots*

Finally, Monte Carlo experiments are carried out to compare the performance of three algorithms under different SNR and the number of snapshots, the RMSE can be expressed as follows,

$$\text{RMSE}=\frac{1}{K}\sum_{k=1}^{K}\sqrt{\frac{1}{N_m}\sum_{i=1}^{N_m}[(\hat{\alpha}_k^i-\alpha_k)^2+(\hat{\beta}_k^i-\beta_k)^2]} \tag{25}$$

where the number of experiments $N_m=500$, $\hat{\alpha}_k^i$ and $\hat{\beta}_k^i$ denotes the estimated value of the α_k and β_k in the *i*-th experiment. In this experiment, the setting of the signal is consistent with running time experiment, and in order to compare fairly, the number of two used SLAs are the same, that is CACIS $(M=6, N=5, p=3)$ and TNA $(M_1=7, M_2=3)$.

For Fig. 3*a*, we can find that the proposed algorithm performs much better than the other two algorithms in the case of low SNR, which proves that the expand virtual array based on SDC can restrain the influence of noise to a great extent. For Fig. 3*b*, it can be seen that the proposed algorithm can still have good performance with less snapshots, which means that only a short period of experimental data is needed to obtain a more accurate estimation result.

Conclusion: The greatest contribution of this letter is that by symmetrically placing two parallel subarrays, the complex 3-D problem that 2-D DOA estimation based on DSC is reduced to 2-D problem, and further reduced to two 1-D dimensional problems by RARE algorithm, so that a virtual array with more sensors is obtained. Through a series of simulations experiments, we demonstrate the advantages of the proposed algorithm in array expansion, running time, estimation accuracy and so on. In addition, this algorithm has strong generalization value and is suitable for almost all cases of 2-D DOA estimation based on DC of SLAs.

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