

# Asymptotic periodic solutions of non-densely defined nonautonomous evolution equations

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## Abstract

In this paper, for the bounded solution of the non-densely defined non-autonomous evolution equation, we present the conditions for asymptotic periodicity by using the spectral theory of functions on the half line and the extrapolation theory of nondensely defined evolution equation.

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# ASYMPTOTIC PERIODIC SOLUTIONS OF NON-DENSELY DEFINED NONAUTONOMOUS EVOLUTION EQUATIONS

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**ABSTRACT.** In this paper, for the bounded solution of the non-densely defined non-autonomous evolution equation, we present the conditions for asymptotic periodicity by using the spectral theory of functions on the half line and the extrapolation theory of nondensely defined evolution equation.

## 1. INTRODUCTION

Studying the periodicity of solutions is one of the great problems for the qualitative theory of evolution equations. The existence and uniqueness of periodic solutions have been proved for several important classes of densely defined evolution equations by using classical approaches such as the fixed point method [10, 19, 22], the use of ultimate boundedness of solutions and the compactness of Poincaré map over compact embedding [17, 18, 20], the spectral theory of functions [11, 15, 16], ergodic approach [14]. As indicated in [6], we sometimes need to deal with non-densely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on  $[0, \pi]$  and consider  $A = \frac{\partial^2}{\partial x^2}$  in  $C([0, \pi], \mathbb{R})$ , in order to measure the solutions in the sup-norm, then the domain

$$\mathcal{D}(A) = \{u \in C^2([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\}$$

is not dense in  $C([0, \pi], \mathbb{R})$  with the sup-norm since

$$\overline{\mathcal{D}(A)} = \{u \in C([0, \pi], \mathbb{R}) : u(0) = u(\pi) = 0\} \neq C([0, \pi], \mathbb{R}).$$

Many results on the existence and uniqueness of periodic solutions of nondensely defined evolution equations are obtained [7, 8, 1, 10]. Especially, in [9] K. Ezzinbi and M. Jazar gave a new criterion related to Massera's approach which is more general than the known exponential dichotomy for the existence of periodic and almost periodic solutions for some evolution equations in a Banach space of the form

$$\begin{cases} \frac{d}{dt}x(t) = (A + B(t))x(t) + f(t), & \text{for } t \geq 0 \\ x(0) = x_0 \end{cases} \quad (1.1)$$

where  $A : \mathcal{D}(A) \subset X \rightarrow X$  is a nondensely defined linear operator on a Banach space  $X$  which satisfies the Hille - Yosida condition:

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Thank to.

( $\mathbf{M}_1$ ): there exist  $M_0 \geq 1$  and  $\omega_0 \in \mathbb{R}$  such that  $(\omega_0, +\infty) \subset \rho(A)$  and

$$|R(\xi, A)^n| \leq \frac{M_0}{(\xi - \omega_0)^n}, \text{ for } n \in \mathbb{N} \text{ and } \xi > \omega_0,$$

where  $\rho(A)$  is the resolvent set of  $A$  and  $R(\xi, A) = (\xi - A)^{-1}$ ; the function  $f : \mathbb{R}^+ \rightarrow X$  is bounded continuous, 1-periodic or almost periodic ( $f$  is not identically zero); for every  $t \geq 0$ ,  $B(t)$  is a bounded linear operator on  $X$ .

Recently, in [4], Luong et al studied the densely defined case of Eq. (1.1) when  $A(t) := A + B(t)$  generates a 1-periodic strongly continuous evolutionary process  $(U(t, s))_{t \geq s \geq 0}$  defined on the whole space  $X$  and  $f$  is asymptotic 1-periodic in the sense that  $f$  is bounded, continuous and  $\lim_{t \rightarrow \infty} (f(t+1) - f(t)) = 0$ . (see e.g. [2] and its references). We recall that a function  $x(\cdot)$  is an asymptotic solution to Eq.(1.1) if there is a continuous function  $\epsilon(\cdot)$  such that  $\lim_{t \rightarrow \infty} \epsilon(t) = 0$  and

$$x'(t) = (A + B(t))x(t) + f(t) + \epsilon(t), \quad \forall t \geq 0.$$

By using the spectral theory of functions on the half line and the induced evolution semigroups in various spectral function spaces. Luong et al introduced the new condition for the unique existence bounded solution to be asymptotic 1-periodic on the half line. More precisely, they showed that a bounded and continuous function  $g : \mathbb{R} \rightarrow X$  is asymptotic 1-periodic if and only if its *circular spectrum*  $\sigma(g)$  (see [11] for more detail of this notion) satisfies  $\sigma(g) \subset \{1\}$ . Therefore, the existence of asymptotic 1-periodic solutions is reduced to that of solutions  $x(\cdot)$  such that  $\sigma(x(\cdot)) \subset \{1\}$ . The search for asymptotic solutions  $x(\cdot)$  with  $\sigma(x(\cdot)) \subset \{1\}$  can be done by using the evolution semigroup associated with the homogeneous equations  $x'(t) = A(t)x(t)$  in appropriate function spaces. In the case that the operator  $A$  is not densely defined, the linear part  $A + B(t)$  does not generate a strongly continuous evolutionary process on the whole space  $X$ , so the results obtained in [4] are not guaranteed. Moreover, the inhomogeneous part  $f(\cdot)$  taking value in the whole space  $X$  while the values of mild solution  $x(\cdot)$  is exactly in  $X_0 = \overline{\mathcal{D}(A)}$ . To overcome these difficulties, in this paper we first use the theory of extrapolation spaces to express the mild solution of Eq. (1.1) in terms of an evolution process  $(\mathcal{U}_B(t, s))_{t \geq s \geq 0}$  defined on closed subspace  $X_0$  (see [1] and the references therein for more detail). Then, by using the periodicity and boundedness of  $(\mathcal{U}_B(t, s))$  combined with the circular spectrum of functions we state the conditions for the unique bounded solution of (1.1) to be asymptotic periodic which fit the case of densely defined of non-autonomous linear part.

Before concluding this introduction section we give an outline of the paper. We briefly list the main notations in Section 2. This section also contains the definitions as well as properties of circular spectra of functions on the half line and extrapolation spaces. Section 3 contains the main result of the paper that deals with the asymptotic periodicity of solutions to non-densely defined nonautonomous evolution equations of the form (1.1).

## 2. PRELIMINARIES

**2.1. Notations.** In this paper  $\mathbb{R}, \mathbb{R}^+$  and  $\mathbb{C}$  stand for the real line, its positive half line, and the complex plane. If  $X$  denotes a (complex) Banach space, then  $\mathcal{L}(X)$  stands for the space of all bounded linear operators in  $X$ . The spectrum of a linear operator  $T$  in a Banach space is denoted by  $\sigma(T)$ , and  $\rho(T) := \mathbb{C} \setminus \sigma(T)$ . We denote by  $BC(\mathbb{R}^+, X)$

the space of all bounded continuous functions from  $\mathbb{R}^+$  to a Banach space  $X$  with supremum norm, and  $C_0(\mathbb{R}^+, X)$  is the space  $\left\{g \in BC(\mathbb{R}^+, X) : \lim_{t \rightarrow \infty} g(t) = 0\right\}$ . Finally,  $\Gamma$  will stand for the unit circle  $\{z \in \mathbb{C} : |z| = 1\}$ .

**2.2. Circular spectra of functions on the half line.** Many of the concepts and results in this subsection are discussed and proved in [4, 5].

We consider the translation operator  $S$  in  $BC(\mathbb{R}^+, X_0)$  defined as

$$[Sx](\xi) := x(1 + \xi), \quad \xi \geq 0, \quad x \in BC(\mathbb{R}^+, X_0).$$

Furthermore, we also consider the quotient spaces

$$Y := BC(\mathbb{R}^+, X_0) / C_0(\mathbb{R}^+, X_0).$$

Then,  $S$  induces operators in  $Y$  that will be denoted by  $\bar{S}$ . It is well known that  $\bar{S}$  is an isometry, so  $\sigma(\bar{S}) \subset \Gamma$ .

For each  $x \in BC(\mathbb{R}^+, X_0)$  let us consider the complex function  $[Sx](\lambda)$  in  $\lambda \in \mathbb{C} \setminus \Gamma$  defined as

$$[Sx](\lambda) := R(\lambda, \bar{S})\bar{x}, \quad \lambda \in \mathbb{C} \setminus \Gamma.$$

**Definition 2.2.1** ([5]). The circular spectrum of a function  $x \in BC(\mathbb{R}^+, X_0)$  is defined to be the set of all  $\xi_0 \in \Gamma$  such that  $[Sx](\lambda)$  has no analytic extension into any neighborhood of  $\xi_0$  in the complex plane. This spectrum of  $x$  is denoted by  $\sigma(x)$ . We will denote by  $\rho(x)$  the set  $\Gamma \setminus \sigma(x)$ .

The following lemma justifies the introduction of these concepts of spectra.

**Lemma 2.2.2** ([5]). *Let  $x \in BC(\mathbb{R}^+, X_0)$ . Then, for each  $x \in BC(\mathbb{R}^+, X_0)$ ,*

$$\sigma(Qx) \subset \sigma(x)$$

*provided that  $Q$  is an operator in  $BC(\mathbb{R}^+, X_0)$  that commutes with  $S$  and leaves  $C_0(\mathbb{R}^+, X_0)$  invariant.*

**2.3. Mild solutions and extrapolation spaces.** It is well known that (see [1] and the references therein) the part  $A_0$  of  $A$  in  $X_0$  generates a  $C_0$ -semigroup  $(T_0(t))_{t \geq 0}$  on  $X_0$  satisfying  $\|T_0(t)\| \leq Me^{\omega t}$ ,  $\forall t \geq 0$ . Moreover, for  $\lambda \in \rho(A_0)$  the resolvent  $R(\lambda, A_0)$  is the restriction of  $R(\lambda, A)$  to  $X_0$ . On  $X_0$  we introduce the norm  $\|x\|_{-1} = \|R(\lambda_0, A_0)x\|$ , where  $\lambda_0 \in \rho(A)$  is fixed. A different choice of  $\lambda_0 \in \rho(A)$  leads to an equivalent norm. The completion  $X_{-1}$  of  $X_0$  with respect to  $\|\cdot\|_{-1}$  is called the extrapolation space of  $X_0$  with respect to  $A$ . The extrapolated semigroup  $(T_{-1}(t))_{t \geq 0}$  consists of the unique continuous extensions  $T_{-1}(t)$  of the operators  $T_0(t)$ ,  $t \geq 0$ , to  $X_{-1}$ . The semigroup  $(T_{-1}(t))_{t \geq 0}$  is strongly continuous and its generator  $A_{-1}$  is the unique continuous extension of  $A_0$  to  $\mathcal{L}(X_0, X_{-1})$ . Moreover,  $X$  is continuously embedded in  $X_{-1}$  and  $R(\lambda, A_{-1})$  is the unique continuous extension of  $R(\lambda, A)$  to  $X_{-1}$  for  $\lambda \in \rho(A)$ . Finally,  $A_0$  and  $A$  are the parts of  $A_{-1}$  in  $X_0$  and  $X$ , respectively.

We now give the definition of a mild solution of (1.1) as follows.

**Definition 2.3.1.** Let  $x_0 \in X_0$ . A function  $x \in C(\mathbb{R}^+, X_0)$  is called a mild solution to (1.1) if it satisfies the integral equation

$$x(t) = T_0(t-s)x(s) + \int_s^t T_{-1}(t-h)(B(h)x(h) + f(h))dh. \quad (2.1)$$

for all  $t \geq s \geq 0$ .

We consider the following homogeneous linear equation

$$\begin{cases} \frac{dx}{dt} = (A + B(t))x(t), t \geq 0 \\ x(0) = x_0 \in X_0 \end{cases} \quad (2.2)$$

and assume that

(M<sub>2</sub>):  $t \mapsto B(t)x$  is strongly measurable for every  $x \in X_0$ ,

(M<sub>3</sub>): The operator  $B(\cdot)$  is 1-periodic.

**Proposition 2.3.2** ([1]). *Let (M<sub>1</sub>) - (M<sub>3</sub>) be satisfied. Then, there exists a unique 1-periodic strongly continuous evolutionary process  $(\mathcal{U}_B(t, s))_{t \geq s \geq 0}$  that satisfies*

- i)  $\mathcal{U}_B(t, s) \in \mathcal{L}(X_0)$  for all  $t \geq s \geq 0$ ;
- ii)  $\mathcal{U}_B(t, t) = I$ , for every  $t \in \mathbb{R}$ ;
- iii)  $\mathcal{U}_B(t, s)\mathcal{U}_B(s, r) = \mathcal{U}_B(t, r)$ , for all  $t \geq s \geq r$ ;
- iv)  $\mathcal{U}_B(t+1, s+1) = \mathcal{U}_B(t, s)$  for all  $t \geq s \geq 0$ ;
- v) The function  $(t, s, x) \mapsto \mathcal{U}_B(t, s)x$  is continuous in  $(t, s, x)$ ;
- vi) There are positive constants  $K, \delta$  such that

$$\|\mathcal{U}_B(t, s)\| \leq Ke^{\delta(t-s)}, \text{ for all } t \geq s \geq 0.$$

vii) Furthermore,

$$\mathcal{U}_B(t, s)x = T_0(t-s)x + \int_s^t T_{-1}(t-h)B(h)\mathcal{U}_B(h, s)x dh, \quad t \geq s \geq 0, x \in X_0.$$

i.e.,  $t \mapsto \mathcal{U}_B(t, 0)x_0$  is the unique solution of (2.2).

**Theorem 2.3.3** ([1]). *Let  $f \in L^1_{\text{loc}}(\mathbb{R}^+, X)$  and  $x_0 \in X_0$ . Then there is a unique mild solution  $x(\cdot) \in C(\mathbb{R}^+, X_0)$  of Eq. (1.1) which satisfies the integral equation*

$$x(t) = \mathcal{U}_B(t, s)x(s) + \lim_{\xi \rightarrow \infty} \int_s^t \mathcal{U}_B(t, h)\xi R(\xi, A)f(h)dh \quad \text{for } t \geq s \geq 0.$$

Moreover,  $\lim_{\xi \rightarrow \infty} \int_s^t \mathcal{U}_B(t, h)\xi R(\xi, A)f(h)dh \in X_0$  exists uniformly for  $t \geq s$  in compact sets in  $\mathbb{R}$ .

### 3. MAIN RESULTS

**3.1. Asymptotic periodic functions and their spectral characterization.** We begin this subsection by recalling the concept of asymptotic periodic functions on the half line. It is noted that our definition of asymptotic periodicity is slightly different from the concept used in many previous works, and period 1 is not a restriction, but just for the reader's convenience. All results can be easily stated for the general case of period.

**Definition 3.1.1** ([4]). A function  $f \in BC(\mathbb{R}^+, X)$  is said to be asymptotic 1-periodic if

$$\lim_{t \rightarrow \infty} (f(t+1) - f(t)) = 0.$$

**Remark 3.1.2** ([4]). In [12] the authors considered the concept of asymptotic periodicity of functions in the sense of our Definition 3.1.1. However, there is an error when it is shown that this concept is equivalent to the following definition of asymptotic

1-periodicity that is widely used in the literature:  $f$  is asymptotic 1-periodic if and only if

$$f(t) = p(t) + q(t), \quad (3.1)$$

where  $p, q$  are continuous functions such that  $p$  is 1-periodic and  $\lim_{t \rightarrow \infty} q(t) = 0$ . A counter-example and some sufficient conditions for the asymptotic periodicity in the sense of our Definition 3.1.1 are given in [13]. Below, we present a simpler counter-example.

**Example 3.1.3.** Let  $f \in BC(\mathbb{R}^+, \mathbb{R})$  defined as  $f(t) := \sin \sqrt{t}, t \in \mathbb{R}^+$ . Since  $\lim_{t \rightarrow \infty} (\sqrt{t+1} - \sqrt{t}) = 0$  we have

$$\lim_{t \rightarrow \infty} (f(t+1) - f(t)) = \lim_{t \rightarrow \infty} (\sin \sqrt{t+1} - \sin \sqrt{t}) = 0.$$

If  $f(t) = p(t) + q(t)$ , where  $p$  is continuous and 1-periodic,  $\lim_{t \rightarrow \infty} q(t) = 0$ , then

$$f(n) = p(1) + q(n), \quad n \in \mathbb{N}.$$

Therefore, the following limit exists:

$$\lim_{n \rightarrow \infty} f(n) = \lim_{n \rightarrow \infty} \sin \sqrt{n} = p(1) \in \mathbb{R}.$$

Substituting  $n = k^2$  into the formula gives

$$\lim_{k \rightarrow \infty} \sin k = p(1) \in \mathbb{R}$$

An elementary argument shows that this is impossible. In fact, if that is true, then

$$\lim_{k \rightarrow \infty} (\sin(k+2) - \sin(k)) = 0.$$

Consequently,

$$\lim_{k \rightarrow \infty} 2 \sin(1) \cos(k+1) = 0.$$

It follows that  $\lim_{k \rightarrow \infty} \cos k = 0$ . Arguing in the same way we can show that  $\lim_{k \rightarrow \infty} \sin k = 0$ .

Therefore,

$$1 = \lim_{k \rightarrow \infty} (\cos^2 k + \sin^2 k) = 0.$$

This is a contradiction that shows that  $\lim_{k \rightarrow \infty} \sin k$  does not exist, and  $f$  cannot be expressed as (3.1).

**Proposition 3.1.4** ([4]). *The following assertions are valid:*

- i) Let  $x \in BC(\mathbb{R}^+, X_0)$ . Then,  $\sigma(x) = \emptyset$  if and only if  $x \in C_0(\mathbb{R}^+, X_0)$ ;
- ii) Let  $p \in \mathbb{R}$  and  $x \in BC(\mathbb{R}^+, X_0)$ . Then,  $\sigma(x) \subset \{e^{ip}\}$  if and only if

$$\lim_{t \rightarrow \infty} (x(t+1) - e^{ip}x(t)) = 0.$$

**Lemma 3.1.5** ([4]). *Assume that  $Q(t), t \in \mathbb{R}^+$  is a family of bounded linear operators in  $X_0$  that satisfies*

- (1) *The function  $\mathbb{R}^+ \times X_0 \ni (t, x) \mapsto Q(t)x \in X_0$  is continuous,*
- (2)  *$Q(t+1) = Q(t)$  for all  $t \in \mathbb{R}^+$ ,*
- (3)  *$\sup_{0 \leq t \leq 1} \|Q(t)\| < \infty$ .*

Then, for each  $x(\cdot) \in BC(\mathbb{R}^+, X_0)$  we have

$$\sigma(Qx(\cdot)) \subset \sigma(x(\cdot)),$$

where  $Q$  denotes the operator in  $BC(\mathbb{R}^+, X_0)$  defined as

$$[Qx(\cdot)](t) := Q(t)x(t), \quad t \in \mathbb{R}^+.$$

### 3.2. Asymptotic periodic solution.

**Definition 3.2.1.** A function  $x(\cdot) \in BC(\mathbb{R}^+, X_0)$  is said to be an *asymptotic mild solution* of Eq.(1.1) if there exists a function  $\epsilon(\cdot) \in C_0(\mathbb{R}^+, X)$  such that

$$x(t) = \mathcal{U}_B(t, s)x(s) + \lim_{\xi \rightarrow \infty} \int_s^t \mathcal{U}_B(t, h)\xi R(\xi, A)[f(h) + \epsilon(h)]dh, \quad (3.2)$$

for all  $t \geq s \geq 0$ .

Now, for  $T$  is an operator in a Banach space  $X_0$ , we denote  $\sigma_\Gamma(T) := \sigma(T) \cap \Gamma$ . We also recall the following well known result on the spectrum of the "monodromy" operators

$$P(t) := \mathcal{U}_B(t+1, t)$$

for each  $t \geq 0$ . When  $t = 1$  we denote  $P := P(1)$ . In particular,  $P = \mathcal{U}_B(1, 0)$  if  $(\mathcal{U}_B(t, s))_{t \geq s \geq 0}$  is a 1-periodic process. Let us denote by  $\mathcal{P}$  the operator of multiplication  $u \mapsto \mathcal{P}u$  defined as

$$\mathcal{P}u(t) = P(t)u(t). \quad (3.3)$$

**Lemma 3.2.2.** Let  $(\mathcal{U}_B(t, s))_{t \geq s \geq 0}$  be a 1-periodic process in  $X_0$ . Then, for each  $t \geq 0$

$$\sigma(P(t)) \setminus \{0\} = \sigma(P) \setminus \{0\}.$$

*Proof.* See [3, Lemma 7.2.2, p. 197]. □

The unique existence of asymptotic mild solution of Eq.(1.1) is implied from Theorem 2.3.3, by in fact that  $f \in BC(\mathbb{R}^+, X) \subset L_{loc}^1 BC(\mathbb{R}^+, X)$ . Now we prove the relation between the spectral of asymptotic mild solution  $x$  with spectral of  $P$  and  $f$ .

**Lemma 3.2.3.** Let  $x(\cdot) \in BC(\mathbb{R}^+, X_0)$  be an asymptotic mild solution of Eq. (1.1) and  $f \in BC(\mathbb{R}^+, X)$ . Then,

$$\sigma(x) \subset \sigma_\Gamma(P) \cup \sigma(f). \quad (3.4)$$

*Proof.* By the definition of asymptotic mild solutions there is a function  $\epsilon(\cdot) \in C_0(\mathbb{R}^+, X)$  such that, for each  $t \in \mathbb{R}^+$

$$x(t+1) = \mathcal{U}_B(t+1, t)x(t) + \lim_{\xi \rightarrow \infty} \int_t^{t+1} \mathcal{U}_B(t+1, h)\xi R(\xi, A)(f(h) + \epsilon(h))dh. \quad (3.5)$$

For  $\xi > \omega$  we set  $f_\xi = \xi R(\xi, A)f$ . Note that  $\sigma(f_\xi) \subset \sigma(f)$  and  $f_\xi \in BC(\mathbb{R}^+, X_0)$ .

Let us denote

$$F_\xi(t) := \int_t^{t+1} \mathcal{U}_B(t+1, h)f_\xi(h)dh.$$

Observe that the operator taking  $f_\xi$  to  $F_\xi$  commutes with  $S$ , and it is a bounded linear operator from  $BC(\mathbb{R}^+, X_0)$  into itself, so by Lemma 3.1.5,

$$\sigma(F_\xi) \subset \sigma(f_\xi).$$

Moreover,  $F_\xi \in BC(\mathbb{R}^+, X_0)$  and

$$F_\xi(t) \Rightarrow F(t) := \lim_{\xi \rightarrow \infty} \int_t^{t+1} \mathcal{U}_B(t+1, h) f_\xi(h) dh \in X_0.$$

which shows that

$$\sigma(F) \subset \sigma(F_\xi) \subset \sigma(f_\xi) \subset \sigma(f).$$

Also, if we denote

$$\varepsilon(t) = \lim_{\xi \rightarrow \infty} \int_t^{t+1} \mathcal{U}_B(t+1, h) \xi R(\xi, A) \epsilon(h) dh$$

then  $\varepsilon(\cdot) \in C_0(\mathbb{R}^+, X_0)$ . Hence, for the function

$$w(t) := \lim_{\xi \rightarrow \infty} \int_t^{t+1} \mathcal{U}_B(t+1, h) \xi R(\xi, A) (f(h) + \epsilon(h)) dh = F(t) + \varepsilon(t)$$

we have

$$\sigma(w) = \sigma(F) \subset \sigma(f).$$

The periodicity of the evolution process  $(\mathcal{U}_B(t, s))_{t \geq s}$  yields that  $P(t)$  is 1-periodic, so it commutes with the translation  $S$ . Therefore, (3.5) gives

$$\bar{S}\bar{x} = \bar{\mathcal{P}}\bar{x} + \bar{F}.$$

Let  $0 \neq \lambda_0 \notin \sigma_\Gamma(P) \cup \sigma(f)$  and let  $V$  be a fixed small open neighborhood of  $\lambda_0$  such that

$$V \cap (\sigma_\Gamma(P) \cup \sigma(f)) = \emptyset.$$

Using the identity

$$R(\lambda, \bar{S})\bar{S}\bar{x} = \lambda R(\lambda, \bar{S})\bar{x} - \bar{x}, \quad \text{for } \lambda \in V : |\lambda| \neq 1$$

we have

$$R(\lambda, \bar{S})(\bar{\mathcal{P}}\bar{x} + \bar{F}) = R(\lambda, \bar{S})\bar{S}\bar{x} = \lambda R(\lambda, \bar{S})\bar{x} - \bar{x}.$$

Together with the fact that  $R(\lambda, \bar{S})\bar{\mathcal{P}}\bar{x} = \bar{\mathcal{P}}R(\lambda, \bar{S})\bar{x}$  we obtain

$$\begin{aligned} \bar{x} + R(\lambda, \bar{S})\bar{F} &= \lambda R(\lambda, \bar{S})\bar{x} - \bar{\mathcal{P}}R(\lambda, \bar{S})\bar{x} \\ &= (\lambda - \bar{\mathcal{P}})R(\lambda, \bar{S})\bar{x}. \end{aligned}$$

Since  $\lambda \in V$  the operator  $\lambda - \bar{\mathcal{P}}$  is invertible and its inverse is determined by  $R(\lambda, \bar{\mathcal{P}})$ . Therefore, for all  $\lambda \in V$  such that  $|\lambda| \neq 1$  we have

$$R(\lambda, \bar{S})\bar{x} = R(\lambda, \bar{\mathcal{P}})(\bar{x} + R(\lambda, \bar{S})\bar{F}).$$

Since  $R(\lambda, \bar{\mathcal{P}})\bar{x}$  is analytic in  $V$  and  $R(\lambda, \bar{S})\bar{F}$  is analytically extendable in a neighborhood of  $\lambda_0$ , the complex function  $R(\lambda, \bar{S})\bar{x}$  is analytically extendable to a neighborhood of  $\lambda_0$ . That is  $\lambda_0 \notin \sigma(x)$ . This proves (3.4), completing the proof of the lemma.  $\square$

**Theorem 3.2.4.** *Let  $(\mathbf{M}_1)$  -  $(\mathbf{M}_3)$  are satisfied. Let  $\sigma_\Gamma(P) \subset \{1\}$  and  $x \in BC(\mathbb{R}^+, X_0)$  be an asymptotic mild solution of Eq. (1.1). Furthermore, let  $f \in BC(\mathbb{R}^+, X)$  in Eq. (1.1) be asymptotic 1-periodic. Then,  $x(\cdot)$  is asymptotic 1-periodic, i.e.,*

$$\lim_{t \rightarrow \infty} (x(t+1) - x(t)) = 0.$$



*Proof.* Since  $f$  is asymptotic 1-periodic,

$$\sigma(f) \subset \{1\}.$$

By Lemma 3.2.3,

$$\sigma(x) \subset \sigma_{\Gamma}(P) \cup \sigma(f) \subset \{1\}.$$

Then, by Proposition 3.1.4 we conclude that  $x(\cdot)$  is asymptotic 1-periodic.  $\square$

**3.3. Example.** To illustrate our results, we consider the following nondensely defined nonautonomous partial differential equation

$$\begin{cases} \frac{\partial x}{\partial t}(t, \zeta) = \frac{\partial^2}{\partial \zeta^2} x(t, \zeta) - b(t)x(t, \zeta) + g(\zeta) \cdot \sin \sqrt{t}, & \text{for } t \in \mathbb{R}_+ \text{ and } \zeta \in [0, \pi], \\ x(t, 0) = x(t, \pi) = 0, & \text{for } t \in \mathbb{R}_+, \end{cases} \quad (3.6)$$

where  $b(\cdot)$  is a 1-periodic function which satisfies  $0 < \bar{b} < b(\cdot)$  and  $g$  is  $L^2$  - integrable on  $[0, \pi]$ .

We set  $X := C([0, \pi], \mathbb{R})$ , the Banach space of continuous functions on  $[0, \pi]$ , equipped with the uniform norm topology, and we define  $A : \mathcal{D}(A) \subset X \rightarrow X$  by

$$\begin{cases} D(A) = \{z \in C^2([0, \pi], \mathbb{R}) : z(0) = z(\pi) = 0\}, \\ Az = z'' \end{cases}$$

We have  $(0, \infty) \subset \rho(A)$ ,

$$\|R(\lambda, A)\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0,$$

and

$$X_0 := \overline{\mathcal{D}(A)} = \{y \in C([0, \pi], \mathbb{R}) : y(0) = y(\pi) = 0\} \neq X.$$

Hence,  $(\mathbf{M}_1)$  is satisfied. We will use the fact that  $A$  generates a strongly continuous exponentially semigroup  $(T_0(t))_{t \geq 0}$  on  $X_0$  with

$$\|T_0(t)\| \leq e^{-t}, \quad \forall t \geq 0.$$

Moreover, as in [21, p. 414] the eigenvalues of  $A$  on  $i\mathbb{R}$  are determined from the set of solutions of the equations

$$\lambda - 1 = -n^2, \quad n = 1, 2, \dots$$

Obviously, there is only one root  $\lambda = 0$  that lies on  $i\mathbb{R}$ , so  $\sigma(A) \cap i\mathbb{R} = \{0\}$ . Since this semigroup is compact, the spectral mapping theorem yields that  $\sigma(T_0(1)) = e^{\sigma(A)} = \{1\}$ .

We now consider the family  $(B(t))_{t \geq 0}$  defined on  $X_0$  by  $B(t) = -b(t)I$ , for every  $t \geq 0$ . Since  $b(\cdot) \in L^1_{loc}(\mathbb{R}_+)$ ,  $t \mapsto B(t)x$  is strongly measurable. Hence,  $(\mathbf{M}_2)$  is satisfied. Clearly that  $B(\cdot)$  is 1-periodic so  $(\mathbf{M}_3)$  is fulfilled. We find that  $A + B(t)$  generates a unique 1-periodic strongly continuous evolutionary process  $(\mathcal{U}_B(t, s))_{t \geq s \geq 0}$  on  $X_0$  defined by

$$\mathcal{U}_B(t, s) = \exp \left( - \int_s^t b(\tau) d\tau \right) T_0(t - s).$$

For the monodromy operator  $P = \mathcal{U}_B(1, 0) = \exp \left( - \int_0^1 b(\tau) d\tau \right) T_0(1)$  we have

$$\sigma_{\Gamma}(P) = \{1\}.$$

Furthermore, if we assume that  $g \in X$ , then the function  $f(t) := \sin \sqrt{t} \cdot g(\cdot)$  is an asymptotic 1-periodic function taking values in  $X$  (see Example 3.1.3).

Therefore, by applying Theorem 3.2.4 we conclude that every asymptotic solution to Eq. (3.6) is asymptotic 1-periodic.

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