Bifurcation for an overdetermined problem in the complement of a ball in $\mathbb{R}^N$

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Abstract

We investigate the existence of a family nontrivial exterior domain $\tilde{\Omega}\subset \mathbb{R}^{N}$ $\left(N\geq2, N\neq3\right)$, bifurcating from the complement of a ball such that \begin{equation} \Delta u=0\,\, \text{in}\,\,\tilde{\Omega}, \,\, u=u_0,\,\,\partial_\nu u=\gamma H+C_0\,\,\text{on}\,\,\partial\tilde{\Omega},\,\,\,\lim _{r \rightarrow+\infty} u=0\,\, \text{or}\,\,+\infty\nonumber \end{equation} has a positive solution, where the Neumann condition is non-constant with H is mean curvature and γ, C_0 are constants. This result gives a negative answer to the Berestycki-Caffarelli-Nirenberg conjecture on overdetermined elliptic problems in the complement of the ball.

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Bifurcation for an overdetermined problem in the complement of a ball in $\mathbb{R}^N$ *

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Abstract

We investigate the existence of a family nontrivial exterior domain $\tilde{\Omega} \subset \mathbb{R}^N$ $(N \geq 2, N \neq 3)$, bifurcating from the complement of a ball such that

$$\Delta u = 0 \text{ in } \tilde{\Omega},\ u = u_0,\ \partial_\nu u = \gamma H + C_0 \text{ on } \partial\tilde{\Omega},\ \lim_{r\to+\infty} u = 0 \text{ or } +\infty$$

has a positive solution, where the Neumann condition is non-constant with H is mean curvature and γ, C_0 are constants. This result gives a negative answer to the Berestycki-Caffarelli-Nirenberg conjecture on overdetermined elliptic problems in the complement of the ball.

Keywords: Overdetermined problem; Bifurcation; Neumann boundary
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1 Introduction

Consider the following overdetermined problem

$$\begin{cases} \Delta u + f(u) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \\ \partial_\nu u = \text{const} & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where ν is the unit outer normal on $\partial\Omega$. In 1971, J. Serrin [11] proved that, if $\Omega \subset \mathbb{R}^N$ was a bounded domain of class C^2 such that problem (1.1) with $f(s) \equiv 1$ admitted a solution, Ω was a ball. Problem (1.1) arises, for example, in fluid dynamics [12] and the linear theory of torsion of a solid straight bar of cross section [13]. Serrin's proof was

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inspired by Aleksandrov's reflection principle [1] for constant mean curvature hypersurfaces. The reflection principle was also used by Gidas, Ni and Nirenberg [6] to derive radial symmetry results for positive solutions of semilinear elliptic equations. After that paper the reflection principle was called the moving plane method.

There are many papers deals with the classifying of domains such that problem (1.1) has a bounded positive solution. In particular, Berestycki, Caffarelli and Nirenberg [2] **conjectured** that if $f : [0, +\infty) \to \mathbb{R}$ is a Lipschitz function on a domain Ω in $\mathbb{R}^N$ such that $\mathbb{R}^N \setminus \overline{\Omega}$ is connected, then the existence of a bounded positive solution to problem (1.1) implies that Ω is a ball, or a half-space, or the complement of a ball, or a generalized cylinder $B^k \times \mathbb{R}^{N-k}$, where B^k is a round ball in $\mathbb{R}^k$.

Recently, Filippo Morabito [10] obtained nontrivial exterior domains in $\mathbb{R}^3$ such that problem (1.1) with $f(s) \equiv 0$ and non-constant Neumann condition has nontrivial solutions. Morabito's work supports the solution of this overdetermined problem, so it is of great significance to the counter example of BCN's conjecture. Concretely, consider the overdetermined problems with non-constant Neumann condition involving the mean curvature on an exterior domain

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N \backslash \Omega, \\ u = u_0 & \text{on } \partial\Omega, \\ \frac{\partial u}{\partial \nu} = \gamma H + C & \text{on } \partial\Omega, \\ \lim_{|x| \to +\infty} u = 0 \text{ or } +\infty, \end{cases} \tag{1.2}$$

where H is the mean curvature and ν is the unit outward normal about $\partial\Omega$, u_0, C, γ are constants. Ω is the ball of radius R_0 centred at the origin in $\mathbb{R}^N$ $(N \geq 2,\ N \neq 3)$. When $N \geq 4$, $\lim_{|x| \to +\infty} u = 0$. When $N = 2$, $\lim_{|x| \to +\infty} u = +\infty$. We obtain the existence of a family of bifurcation branches of domain is small perturbations of $\mathbb{R}^N \backslash \Omega$ which support solution of problem below

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N \backslash \tilde{\Omega}, \\ u = u_0 & \text{on } \partial\tilde{\Omega}, \\ \frac{\partial u}{\partial \nu} - \gamma H = C_0 & \text{on } \partial\tilde{\Omega}, \\ \lim_{r \to +\infty} u = \text{ or } +\infty. \end{cases} \tag{1.3}$$

This problem relates to electrodynamics [4, 9]. The aim of this paper is to extend the corresponding results of [10] into the general dimensional $N \geq 2, \neq 3$. When $N \geq 4$, under the spherical coordinates, there are $n - 1$ angle variable, while the free boundary transformation of [10] relies only on the one angle variable. To overcome this difficulty, we introduce G invariant group into the high dimensional problem. There are also some difficulties with Green's function for $N \geq 4$ when we solve poisson's equation. We use the Fourier expansion about spherical harmonics functions to overcome these. The method we will used is some different for the cases of $N \geq 4$ and $N = 2$. So we deal with the two cases separately.

We first give some notations about Laplace operator in the spherical coordinates, orthogonal group and work spaces. If we use the spherical coordinates (r,φ), where $\varphi=(\varphi_1,\varphi_2,\cdots,\varphi_{N-1})$, from Appendix we see that the equation $\Delta u=0$ can be written as

$$\Delta u=\frac{\partial^2 u}{\partial r^2}+\frac{N-1}{r}\frac{\partial u}{\partial r}+\sum_{k=1}^{N-1}\frac{\partial^2 u}{\partial\varphi_k^2}\frac{1}{r^2p_k^2}+\sum_{k=1}^{N-2}\frac{\partial u}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{r^2p_k^2}=0,$$

where $p_k=\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{k-1}$. Let $\mathbb{S}^{N-1}$ be the unit sphere in $\mathbb{R}^N$ with the spherical coordinates metric.

Let $O(N-1)$ be the group consisting of $(N-1)\times(N-1)$ orthogonal matrices, and let G be the subgroup of $O(N-1)$ consisting of those elements which fix $e_{N-1}=(1,0,\ldots,0)$. We say a function v defined on $\mathbb{S}^{N-1}$ is G-invariant if $v(O\varphi)=v(\varphi)$ for $\varphi\in\mathbb{S}^{N-1}$ and $O\in G$. It is clear that a G-invariant function v on $\mathbb{S}^{N-1}$ can be written as $v(\varphi)=\widetilde{v}(\varphi_1)$ for some function $\widetilde{v}$ defined on $[0,\pi]$, where φ_1 denotes the first coordinate of φ. In the following, given a G-invariant function v on $\mathbb{S}^{N-1}$, we will always use $\widetilde{v}$ to denote the corresponding function defined on $[0,\pi]$ such that $v(\varphi)=\widetilde{v}(\varphi_1)$.

Let

$$\begin{aligned}X^{m+2,\alpha}:=\ &\Big\{f(\varphi)\in C^{m+2,\alpha}\left([0,\pi]^{N-2}\times[0,2\pi]\right),\ f\text{ is }G-\text{invariant}\\ &\pi-\text{periodic in}\,\varphi_1,\varphi_2,\cdots,\varphi_{N-2}\text{ and }2\pi-\text{periodic in},\varphi_{N-1}\Big\},\end{aligned}$$

with $m\geqslant 1$. $X_{G,1}^{m+2,\alpha}=\overline{\left\{h\widetilde{Y}_l(\varphi_1),\ l\geqslant 0,\ \text{all } h\in\mathbb{R}\right\}}\subseteq X_G^{m+2,\alpha}$,

$$\tilde{X}_G^{m+2,\alpha}:=\left\{f\in C_G^{m+2,\alpha}\left(\mathbb{R}^N\backslash\Omega\right),\text{such that }f(r,\varphi)\in X_G^{m+2,\alpha},\forall r\in[R_0,+\infty)\right\}$$

and

$$\tilde{X}_{G,1}^{m+2,\alpha}:=\left\{f\in C_G^{m+2,\alpha}\left(\mathbb{R}^N\backslash\Omega\right),\text{such that }f(r,\varphi)\in X_{G,1}^{m+2,\alpha},\forall r\in[R_0,+\infty)\right\}.$$

The rest of this paper is arranged as follows. In Section 2, we study the case of $N\geq 4$. In Section 3 is devoted to the case of $N=2$.

2 The case of $N\geq 4$

Clearly, $u_0\left(\frac{R_0}{r}\right)^{N-2}:=u_*$ is a solution of problem (1.2). Calculate the value of the constant C can be seen as follows. Since

$$\frac{\partial u_*}{\partial\nu}_{|\partial\Omega}=\frac{\partial u_*}{\partial r}_{|r=R_0}=\frac{\partial}{\partial r}\left[u_0\left(\frac{R_0}{r}\right)^{N-2}\right]_{|r=R_0}=-\frac{(N-2)\,u_0}{R_0}$$

and $H=-1/R_0$, one has that

$$C=\frac{\partial u}{\partial\nu}_{|\partial\Omega}-\gamma H_{|\partial\Omega}=\frac{\gamma-(N-2)\,u_0}{R_0}:=C_0.$$

The domain $\tilde{\Omega}$ satisfies the following structure

$$\Omega_x := \{r \leqslant R_0 + x(\varphi_1)\}$$

with $x(\varphi_1) = \varepsilon \widetilde{Y}_l(\varphi_1) + \sum_{k=2}^{m+1} \varepsilon^k \Lambda_{lk}(\varphi_1) + O(\varepsilon^{m+2})$, where ε is small, $Y_l(\varphi) = \widetilde{Y}_l(\varphi_1)$ $\left(\text{with } \left\|\widetilde{Y}_l\right\|_{L^2} = 1\right)$ denotes the real-valued spherical harmonic functions. The value of the corresponding parameter γ are

$$\gamma = \gamma_l + \sum_{k=1}^{m} \varepsilon^k \gamma_{lk} + O\left(\varepsilon^{m+1}\right),$$

where $l \in \mathbb{N}, l \neq 1$,

$$\gamma_l := \frac{2u_0\left[(N-2)l + (N-2)^2 - (N-1)\right]}{l^2 + (N-2)l - (N-1)},$$

and γ_{lk} is the coefficient of higher order terms.

The main result is the following theorem:

Theorem 2.1. *Suppose $N \geq 4$ and $l \in \mathbb{N}$, $l \neq 1$. For each value of γ_l, there exists a C^∞ bifurcation branch of solutions to the free boundary problem (1.3) with free boundary $\partial\tilde{\Omega}$ in $C^{m+2,\alpha}$ of the form*

$$\left\{r = R_0 + \varepsilon \widetilde{Y}_l(\varphi_1) + \sum_{k=2}^{m+1} \varepsilon^k \Lambda_{lk}(\varphi_1) + O\left(\varepsilon^{m+2}\right)\right\},$$

with

$$\gamma(\varepsilon) = \gamma_l + \sum_{k=1}^{m} \varepsilon^k \gamma_{lk} + O\left(\varepsilon^{m+1}\right)$$

and $m \geqslant 1$.

Solving (1.3) is equal to solving problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^N \backslash \Omega_x, \\ u = u_0 & \text{on } \partial\Omega_x, \\ \lim_{r\to+\infty} u = 0, \end{cases} \tag{2.1}$$

and

$$F(x,\gamma) := \frac{\partial u}{\partial \nu}_{|\partial\Omega_x} - \gamma H_{|\partial\Omega_x} + \frac{(N-2)u_0}{R_0} - \frac{\gamma}{R_0} = 0. \tag{2.2}$$

Since u_* satisfies the above equation when $x \equiv 0$, $F(0,\gamma) = 0$ for any γ. Therefore, finding nontrivial domains emanating from $B^c_{R_0}$ such that the problem (1.3) has a bounded positive solution is equivalent to study the the nontrivial solutions of $F(x,\gamma) = 0$.

We divide our arguments into two steps. In the next subsection we first study the existence of (2.1). Then, by the Crandall-Rabinowitz Bifurcation Theorem, we prove Theorem 2.1 by studying the operator equation $F(x,\gamma) = 0$.

2.1 A Dirichlet problem

In condition $\Omega_x = \{r \leqslant R_0 + x(\varphi),\, x(\varphi) = \varepsilon S(\varphi)\}$, applying the transformation

$$u(r,\varphi) = u_0 \left(\frac{R_0}{r}\right)^{N-2} + \bar{u}(r,\varphi) \tag{2.3}$$

to the problem of (2.1), we can get

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \mathbb{R}^N \setminus \Omega_x, \\ \bar{u} = u_0 \frac{(R_0+\varepsilon S(\varphi))^{N-2} - R_0^{N-2}}{(R_0+\varepsilon S(\varphi))^{N-2}} & \text{on } \partial\Omega_x, \\ \lim_{r\to+\infty} \bar{u} = 0. \end{cases} \tag{2.4}$$

In order to go from Ω_x to the fixed domain of the ball of radius R_0, we take the Hanzawa transformation W:

$$\begin{cases} r = \tilde{r} + \chi(R_0 - \tilde{r})\,\varepsilon S(\varphi), \\ \varphi = \tilde{\varphi}, \end{cases} \tag{2.5}$$

where χ is a cutoff function identically equal to 1 when $|y| \leq 1/4a$ and identically equal to 0 when $|y| \geq 3/4a$ with $\left|\frac{\mathrm{d}^k\gamma}{\mathrm{d}t^k}\right| \leq Ca^{-k}, a > 0$ small.

To get $\frac{\partial\tilde{r}}{\partial r}$ and $\frac{\partial^2\tilde{r}}{\partial r^2}$, we take the derivative of (2.5) with respect to r, so we have

$$\frac{\partial\tilde{r}}{\partial r} - \frac{\mathrm{d}\chi(R_0-\tilde{r})}{\mathrm{d}\tilde{r}}\frac{\partial\tilde{r}}{\partial r}\varepsilon S(\varphi) = 1.$$

It's not difficult to deduce that

$$\begin{aligned} \frac{\partial\tilde{r}}{\partial r} &= \frac{1}{1-\varepsilon\frac{\mathrm{d}\chi(R_0-\tilde{r})}{\mathrm{d}\tilde{r}}S}, \\ \frac{\partial^2\tilde{r}}{\partial r^2} &= -\frac{\chi''(R_0-\tilde{r})\,\varepsilon S}{(1-\varepsilon\chi'(R_0-\tilde{r})\,S)^3}, \end{aligned}$$

where χ' is the first order derivative of χ and χ'' is the second order derivative of χ. These two are evaluated at $t = R_0 - \tilde{r}$. To get $\frac{\partial\tilde{r}}{\partial\varphi_k}$ and $\frac{\partial^2\tilde{r}}{\partial\varphi_k^2}$, we take the derivative of (2.5) with respect to $\varphi_k = \tilde{\varphi}_k$ $(1 \leq k \leq N-1)$, so we have

$$\frac{\partial\tilde{r}}{\partial\varphi_k} - \frac{\mathrm{d}\chi(R_0-\tilde{r})}{\mathrm{d}\tilde{r}}\frac{\partial\tilde{r}}{\partial\varphi_k}\varepsilon S(\varphi) + \varepsilon\chi(R_0-\tilde{r})\frac{\partial S}{\partial\tilde{\varphi}_k} = 0.$$

It's not difficult to deduce that

$$\begin{aligned} \frac{\partial\tilde{r}}{\partial\varphi_k} &= -\frac{\chi(R_0-\tilde{r})\,\varepsilon\frac{\partial S}{\partial\tilde{\varphi}_k}}{1-\varepsilon\frac{\mathrm{d}\chi(R_0-\tilde{r})}{\mathrm{d}\tilde{r}}S}, \\ \frac{\partial^2\tilde{r}}{\partial\varphi_k^2} &= -\frac{\chi''(R_0-\tilde{r})\,\chi^2(R_0-\tilde{r})\,\varepsilon^3 S\left(\frac{\partial S}{\partial\tilde{\varphi}_k}\right)^2}{(1-\varepsilon\chi'(R_0-\tilde{r})\,S)^3} - \frac{2\varepsilon^2\chi'(R_0-\tilde{r})\,\chi\left(\frac{\partial S}{\partial\tilde{\varphi}_k}\right)^2}{(1-\varepsilon\chi'(R_0-\tilde{r})\,S)^2} \\ &\quad -\frac{\varepsilon\chi(R_0-\tilde{r})\frac{\partial^2 S}{\partial\tilde{\varphi}_k^2}}{(1-\varepsilon\chi'(R_0-\tilde{r})\,S)}. \end{aligned}$$

Letting

$$\tilde{u}\left(\tilde{r},\tilde{\varphi}\right):=\bar{u}\circ W,$$

then one has that

$$\begin{aligned}
\frac{\partial u}{\partial r} &= \frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial\tilde{r}}{\partial r},\\
\frac{\partial^2 u}{\partial r^2} &= \frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}\left(\frac{\partial\tilde{r}}{\partial r}\right)^2+\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial^2\tilde{r}}{\partial r^2},\\
\frac{\partial u}{\partial\varphi_k} &= \frac{\partial\tilde{u}}{\partial\tilde{\varphi}_k}+\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial\tilde{r}}{\partial\varphi_k},\\
\frac{\partial^2 u}{\partial\varphi_k^2} &= \frac{\partial^2\tilde{u}}{\partial\tilde{\varphi}_k^2}+2\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{\varphi}_k}\frac{\partial\tilde{r}}{\partial\varphi_k}+\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial^2\tilde{r}}{\partial\varphi_k^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}\left(\frac{\partial\tilde{r}}{\partial\varphi_k}\right)^2.
\end{aligned}$$

We compute the Laplace operator in coordinates $(\widetilde{r},\widetilde{\varphi})$ as follows:

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial r^2}+\frac{N-1}{r}\frac{\partial u}{\partial r}++\sum_{k=1}^{N-1}\frac{\partial^2 u}{\partial\varphi_k^2}\frac{1}{r^2p_k^2}+\sum_{k=1}^{N-2}\frac{\partial u}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{r^2p_k^2}\\
&= \frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}\left(\frac{\partial\tilde{r}}{\partial r}\right)^2+\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial^2\tilde{r}}{\partial r^2}+\frac{N-1}{r}\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial\tilde{r}}{\partial r}\\
&\quad+\sum_{k=1}^{N-1}\left[\frac{\partial^2\tilde{u}}{\partial\tilde{\varphi}_k^2}+2\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{\varphi}_k}\frac{\partial\tilde{r}}{\partial\varphi_k}+\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial^2\tilde{r}}{\partial\varphi_k^2}+\frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}\left(\frac{\partial\tilde{r}}{\partial\varphi_k}\right)^2\right]\frac{1}{r^2\tilde{p}_k^2}\\
&\quad+\sum_{k=1}^{N-2}\left[\frac{\partial\tilde{u}}{\partial\tilde{\varphi}_k}+\frac{\partial\tilde{u}}{\partial\tilde{r}}\frac{\partial\tilde{r}}{\partial\varphi_k}\right]\frac{(N-k-1)\cot\tilde{\varphi}_k}{r^2\tilde{p}_k^2}\\
&= \frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}\left[\left(\frac{\partial\tilde{r}}{\partial r}\right)^2+\sum_{k=1}^{N-1}\left(\frac{\partial\tilde{r}}{\partial\varphi_k}\right)^2\frac{1}{r^2\tilde{p}_k^2}\right]\\
&\quad+\frac{\partial\tilde{u}}{\partial\tilde{r}}\left[\frac{\partial^2\tilde{r}}{\partial r^2}+\frac{N-1}{r}\frac{\partial\tilde{r}}{\partial r}+\sum_{k=1}^{N-1}\frac{\partial^2\tilde{r}}{\partial\varphi_k^2}\frac{1}{r^2\tilde{p}_k^2}+\sum_{k=1}^{N-2}\frac{\partial\tilde{r}}{\partial\varphi_k}\frac{(N-k-1)\cot\tilde{\varphi}_k}{r^2\tilde{p}_k^2}\right]\\
&\quad+\sum_{k=1}^{N-1}\frac{\partial^2\tilde{u}}{\partial\tilde{\varphi}_k^2}\left[\frac{1}{r^2\tilde{p}_k^2}\right]+\sum_{k=1}^{N-1}\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{\varphi}_k}\left[2\frac{\partial\tilde{r}}{\partial\varphi_k}\frac{1}{r^2\tilde{p}_k^2}\right]+\sum_{k=1}^{N-2}\frac{\partial\tilde{u}}{\partial\tilde{\varphi}_k}\left[\frac{(N-k-1)\cot\tilde{\varphi}_k}{r^2\tilde{p}_k^2}\right]\\
&= \frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}\left[1+\varepsilon\left(2\chi'S\right)\right]+\frac{\partial\tilde{u}}{\partial\tilde{r}}\left[\varepsilon\left(-\chi''S\right)+(N-1)\left(\frac{1}{\tilde{r}}+\varepsilon\frac{\tilde{r}\chi'S-\chi S}{\tilde{r}^2}\right)\right.\\
&\quad\left.+\sum_{k=1}^{N-1}\varepsilon\left(-\chi\frac{\partial^2 S}{\partial\tilde{\varphi}_k^2}\frac{1}{\tilde{r}^2\tilde{p}_k^2}\right)+\sum_{k=1}^{N-2}\varepsilon\left(-\chi\frac{\partial S}{\partial\tilde{\varphi}_k}\frac{(N-k-1)\cot\tilde{\varphi}_k}{\tilde{r}^2\tilde{p}_k^2}\right)\right]\\
&\quad+\sum_{k=1}^{N-1}\frac{\partial^2\tilde{u}}{\partial\tilde{\varphi}_k^2}\left[\frac{1}{\tilde{r}^2\tilde{p}_k^2}+\varepsilon\frac{-2\chi S}{\tilde{r}^3\tilde{p}_k^2}\right]+\sum_{k=1}^{N-1}\frac{\partial^2\tilde{u}}{\partial\tilde{r}\partial\tilde{\varphi}_k}\left[\varepsilon\left(-2\chi\frac{\partial S}{\partial\tilde{\varphi}_k}\frac{1}{\tilde{r}^2\tilde{p}_k^2}\right)\right]\\
&\quad+\sum_{k=1}^{N-2}\frac{\partial\tilde{u}}{\partial\tilde{\varphi}_k}\left[\frac{(N-k-1)\cot\tilde{\varphi}_k}{\tilde{r}^2\tilde{p}_k^2}+\varepsilon\left(\frac{-2\chi S}{\tilde{r}^3}\times\frac{(N-k-1)\cot\tilde{\varphi}_k}{\tilde{r}^2\tilde{p}_k^2}\right)\right]+O\left(\varepsilon^2\right),
\end{aligned}$$

so we can get

$$\Delta=\tilde{\Delta}+\varepsilon\left(A_1+A_2+A_3+A_4\right)+\varepsilon^2B,$$

where

$$\begin{aligned}\tilde{\Delta}\tilde{u}\left(\tilde{r},\tilde{\varphi}\right) &= \frac{\partial^2\tilde{u}}{\partial\tilde{r}^2}+\frac{N-1}{\tilde{r}}\frac{\partial\tilde{u}}{\partial\tilde{r}}+\sum_{k=1}^{N-1}\frac{\partial^2\tilde{u}}{\partial\tilde{\varphi}_k^2}\frac{1}{\tilde{r}^2\tilde{p}_k^2}+\sum_{k=1}^{N-2}\frac{\partial\tilde{u}}{\partial\tilde{\varphi}_k}\frac{\left(N-k-1\right)\cot\tilde{\varphi}_k}{\tilde{r}^2\tilde{p}_k^2},\\ \tilde{p}_k &= \sin\tilde{\varphi}_1\sin\tilde{\varphi}_2\cdots\sin\tilde{\varphi}_{k-1},\end{aligned}$$

and

$$\begin{aligned}A_1 &:= S\left[2\chi'\frac{\partial^2}{\partial\tilde{r}^2}+\chi'\frac{N-1}{\tilde{r}}\frac{\partial}{\partial\tilde{r}}-\chi''\frac{\partial}{\partial\tilde{r}}-\chi\frac{N-1}{\tilde{r}^2}\frac{\partial}{\partial\tilde{r}}\right],\\ A_2 &:= -2\frac{\chi S}{\tilde{r}^3}\left[\sum_{k=1}^{N-1}\frac{\partial^2}{\partial\tilde{\varphi}_k^2}\frac{1}{\tilde{p}_k^2}+\sum_{k=1}^{N-2}\frac{\partial}{\partial\tilde{\varphi}_k}\frac{\left(N-k-1\right)\cot\tilde{\varphi}_k}{\tilde{p}_k^2}\right],\\ A_3 &:= -\frac{\chi}{\tilde{r}^2}\left[\sum_{k=1}^{N-1}\frac{\partial^2 S}{\partial\tilde{\varphi}_k^2}\frac{1}{\tilde{p}_k^2}+\sum_{k=1}^{N-2}\frac{\partial S}{\partial\tilde{\varphi}_k}\frac{\left(N-k-1\right)\cot\tilde{\varphi}_k}{\tilde{p}_k^2}\right]\frac{\partial}{\partial\tilde{r}},\\ A_4 &:= -\frac{2\chi}{\tilde{r}^2}\left[\sum_{k=1}^{N-1}\frac{\partial S}{\partial\tilde{\varphi}_k}\frac{1}{\tilde{p}_k^2}\right]\frac{\partial^2}{\partial\tilde{r}\partial\tilde{\varphi}_k}.\end{aligned}$$

B changes with the variable ε respectively, which is the series expands higher-order terms of Laplace operator.

So by the transformation W, problem (2.4) becomes

$$\begin{cases}\tilde{\Delta}\tilde{u}\left(\tilde{r},\tilde{\varphi}\right)=\varepsilon J\left(\tilde{r},\tilde{\varphi}\right) & \text{in } \mathbb{R}^N\setminus\Omega,\\ \tilde{u}\left(R_0,\tilde{\varphi}\right)=g\left(\tilde{\varphi}\right) & \text{on } \partial\Omega,\\ \lim_{\tilde{r}\to+\infty}\tilde{u}\left(\tilde{r},\tilde{\varphi}\right)=0,\end{cases}\tag{2.6}$$

with

$$J\left(\tilde{r},\tilde{\varphi}\right)=-\sum_{i=1}^{4}A_i\tilde{u}\left(\tilde{r},\tilde{\varphi}\right)-\varepsilon B\tilde{u}\left(\tilde{r},\tilde{\varphi}\right),$$

and

$$\begin{aligned}g\left(\tilde{\varphi}\right) &= u_0\frac{\left(R_0+\varepsilon S\left(\tilde{\varphi}\right)\right)^{N-2}-R_0^{N-2}}{\left(R_0+\varepsilon S\left(\tilde{\varphi}\right)\right)^{N-2}}\\ &= u_0\left[C_{N-2}^1\frac{\varepsilon S}{R_0}+C_{N-2}^2\frac{\varepsilon^2S^2}{R_0^2}+O\left(\varepsilon^3\right)\right]\left(1-\varepsilon\frac{S}{R_0}+O\left(\varepsilon^2\right)\right)^{N-2}\\ &= u_0\left[\left(N-2\right)\varepsilon\frac{S}{R_0}+C_{N-2}^2\frac{\varepsilon^2S^2}{R_0^2}-\left(N-2\right)^2\varepsilon^2\frac{S^2}{R_0^2}+O\left(\varepsilon^3\right)\right].\end{aligned}\tag{2.7}$$

The Poisson equation of (2.6) contains the solution $\tilde{u}$ in integral form ([7])

$$\tilde{u}=-\int_{\partial\Omega}\frac{\partial G}{\partial\nu}gd\sigma''-\int_{\mathbb{R}^N\setminus\Omega}G\varepsilon Jdvol'',\tag{2.8}$$

where $G=G\left(\tilde{r},\tilde{\varphi},r'',\varphi''\right)$ is the Green function for (2.6) in $\mathbb{R}^N\setminus\Omega$. $\tilde{x}=\left(\tilde{r},\tilde{\varphi}\right)$, $x''=\left(r'',\varphi''\right)$ are points on $\mathbb{R}^N\setminus\Omega$. So we can use the simpler notation $G\left(\tilde{x},x''\right)$. G has the representation:

$$G\left(\tilde{x},x''\right)=\frac{1}{N\left(N-2\right)\omega_N}\frac{1}{\left|\tilde{x}-x''\right|^{N-2}}-\frac{1}{N\left(N-2\right)\omega_N}\left(\frac{R_0}{\left|\tilde{x}\right|}\right)^{N-2}\frac{1}{r_1^{N-2}},$$

where

$$r_1=\sqrt{\left(\frac{R_0^2}{|\tilde{x}|}\right)^2+|x''|^2-2R_0^2\frac{|x''|}{|\tilde{x}|}\cos\zeta},$$

with $\cos\zeta=\frac{x''\cdot\tilde{x}}{\|x''\|\|\tilde{x}\|}$ and ω_N is the surface area of an N-dimensional unit sphere. Similar to [10] by Banach's Fixed Point Theorem, the existence and uniqueness of solution to problem (2.8) can be proved.

From (2.7) and (2.8) we immediately get $\tilde{u}=O(\varepsilon)$, and then we get $\sum_{i=1}^{4}A_i\tilde{u}\left(\tilde{r},\tilde{\varphi}\right)=O\left(\varepsilon\right)$, so

$$\tilde{u}=-\int_{\partial\Omega}\frac{\partial G}{\partial\nu}gd\sigma''+O\left(\varepsilon^2\right). \tag{2.9}$$

Since $\tilde{u}\in\tilde{X}_G^{m+2,\alpha}$, then $\frac{\partial u}{\partial\nu}|_{\partial\Omega_x}\in\tilde{X}_G^{m+1,\alpha}$, so the F of (2.2) satisfies not only $F:X_G^{m+2,\alpha}\times\mathbb{R}\to X_G^{m,\alpha}$ but also $F:X_{G,1}^{m+2,\alpha}\times\mathbb{R}\to X_{G,1}^{m,\alpha}$ when $x=x\left(\varphi_1\right)=\varepsilon S\left(\varphi_1\right)$.

By the [8, Proposition 3.2.11], we have the real-valued spherical harmonics form under the G invariant group

$$g_1\left(\tilde{\varphi}\right)=\sum_{l=0}^{+\infty}a_l\widetilde{Y}_l\left(\tilde{\varphi}_1\right),\ \ \forall g_1\in X_G^{m+2,\alpha},\ a_l=\int_0^{\pi}\widetilde{g}_1\left(\tilde{\varphi}_1\right)\widetilde{Y}_l\left(\tilde{\varphi}_1\right)\mathrm{d}\tilde{\varphi}_1.$$

According to [8, Proposition 3.3.13], the first term of (2.9) can be written as the real-valued spherical harmonics expansion

$$-\int_{\partial\Omega}\frac{\partial G}{\partial\nu}g\left(\tilde{\varphi}\right)\mathrm{d}\sigma''=\sum_{l=0}^{\infty}\left(\frac{R_0}{\tilde{r}}\right)^{l+N-2}b_l\widetilde{Y}_l\left(\tilde{\varphi}_1\right),\ b_l=\int_0^{\pi}\widetilde{g}\left(\tilde{\varphi}_1\right)\widetilde{Y}_l\left(\tilde{\varphi}_1\right)\mathrm{d}\tilde{\varphi}_1,\ g\in X_{G,1}^{m+2,\alpha}.$$

If you compute $\frac{\partial\tilde{u}}{\partial\tilde{r}}|_{\tilde{r}=R_0}$ and substitute (2.7) into

$$\frac{\partial\tilde{u}}{\partial\tilde{r}}|_{\tilde{r}=R_0}=-\sum_{l=0}^{\infty}\frac{l+N-2}{R_0}b_l\widetilde{Y}_l\left(\tilde{\varphi}_1\right)+O\left(\varepsilon^2\right),$$

and you get

$$\begin{aligned}\frac{\partial\tilde{u}}{\partial\tilde{r}}|_{\tilde{r}=R_0}&=-u_0\frac{(N-2)\,\varepsilon}{R_0}\sum_{l=0}^{\infty}\left(l+N-2\right)a_l\widetilde{Y}_l\left(\tilde{\varphi}_1\right)+O\left(\varepsilon^2\right),\\ a_l&=\int_0^{\pi}\widetilde{S}\left(\tilde{\varphi}_1\right)\widetilde{Y}_l\left(\tilde{\varphi}_1\right)\mathrm{d}\tilde{\varphi}_1.\end{aligned}$$

Let $\nu=\left(n_r,n_{\varphi_1},n_{\varphi_2},\cdots,n_{\varphi_{N-1}}\right)$ denote the unit outer normal vector on $\partial\Omega_{\varepsilon S}$, so

$$\begin{aligned}\nu&=\left(n_r,n_{\varphi_1},n_{\varphi_2},\cdots,n_{\varphi_{N-1}}\right),\\ &=\left(n_{\tilde{r}}\frac{\partial\tilde{r}}{\partial r},n_{\tilde{\varphi}_1}\frac{\partial\tilde{\varphi}_1}{\partial\varphi_1},n_{\tilde{\varphi}_2}\frac{\partial\tilde{\varphi}_2}{\partial\varphi_2},\cdots,n_{\tilde{\varphi}_{N-1}}\frac{\partial\tilde{\varphi}_{N-1}}{\partial\varphi_{N-1}}\right),\\ &=\left(n_{\tilde{r}}\frac{1}{1-\varepsilon\chi'S},n_{\tilde{\varphi}_1}\frac{\varepsilon\chi S}{1-\varepsilon\chi'S},n_{\tilde{\varphi}_2}\frac{\varepsilon\chi S}{1-\varepsilon\chi'S},\cdots,n_{\tilde{\varphi}_{N-1}}\frac{\varepsilon\chi S}{1-\varepsilon\chi'S}\right),\\ &=\left(n_{\tilde{r}},0,0,\cdots,0\right)+\varepsilon\left(\chi'S,\chi S,\cdots,\chi S\right).\end{aligned}$$

Using the formulas of (2.3) and $\frac{\partial \tilde{r}}{\partial r}, \frac{\partial \tilde{r}}{\partial \varphi_k}, \nu$, the values of $\frac{\partial u}{\partial \nu}|_{\partial\Omega_{\varepsilon S}}$ is computed in follow

$$\begin{aligned}
\frac{\partial u}{\partial \nu}|_{\partial\Omega_{\varepsilon S}} &= \frac{\partial}{\partial r}\left[u_0\left(\frac{R_0}{r}\right)^{N-2}\right]_{|r=R_0+\varepsilon S} + \frac{\partial \tilde{u}}{\partial \tilde{r}}|_{\tilde{r}=R_0} + O\left(\varepsilon^2\right) \\
&= -u_0\frac{(N-2)\,R_0^{N-2}}{(R_0+\varepsilon S)^{N-1}} - u_0\frac{(N-2)\,\varepsilon}{R_0}\sum_{l=0}^{\infty}(l+N-2)\,a_l\widetilde{Y}_l\left(\tilde{\varphi}_1\right) + O\left(\varepsilon^2\right) \\
&= -\frac{(N-2)\,u_0}{R_0}\left[1-(N-1)\,\frac{\varepsilon S}{R_0} + O\left(\varepsilon^2\right)\right] - \\
&\quad u_0\frac{(N-2)\,\varepsilon}{R_0}\sum_{l=0}^{\infty}(l+N-2)\,a_l\widetilde{Y}_l\left(\tilde{\varphi}_1\right) + O\left(\varepsilon^2\right).
\end{aligned}$$

By the fact of $\partial\Omega_x = \{r = R_0 + x(\varphi) = R_0 + \varepsilon S(\varphi)\}$, we can calculate the value of the Fréchet derivative of $\frac{\partial u}{\partial \nu}|_{\partial\Omega_x}$ with respect to a perturbation $x(\varphi)$ of the domain Ω at $x = 0$ below

$$\left[D_x\left(\frac{\partial u}{\partial \nu}|_{\partial\Omega_x}\right)\right]x = x\frac{(N-2)\,(N-1)\,u_0}{R_0^2} - \frac{(N-2)\,u_0}{R_0^2}\sum_{l=0}^{\infty}(l+N-2)\,a_l\left(x\right)\widetilde{Y}_l\left(\tilde{\varphi}_1\right),$$

with

$$a_l\left(x\right) = \int_0^{\pi}\widetilde{x}\left(\tilde{\varphi}_1\right)\widetilde{Y}_l\left(\tilde{\varphi}_1\right)\mathrm{d}\tilde{\varphi}_1.$$

2.2 Proof of Theorem 2.1

We prove of Theorem 2.1 by solve (2.2), that is $F(x,\gamma) = 0$, on $\partial\Omega_x$, with

$$F(x,\gamma) = \frac{\partial u}{\partial \nu}|_{\partial\Omega_x} - \gamma H_{|\partial\Omega_x} + \frac{(N-2)\,u_0}{R_0} - \frac{\gamma}{R_0}.$$

From the (8.1) of [5] we have

$$H_{|\partial\Omega_x} = -\frac{1}{R_0} + \frac{1}{N-1}Qx + O\left(\|x\|_{C^2}^2\right),$$

where

$$Qx = \Delta_{\partial\Omega}x + |II|^2x = \frac{1}{R_0^2}\left[\sum_{k=1}^{N-1}\frac{\partial^2 x}{\partial\varphi_k^2}\frac{1}{p_k^2} + \sum_{k=1}^{N-2}\frac{\partial x}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{p_k^2}\right] + \frac{N-1}{R_0^2}x.$$

Here $\Delta_{\partial\Omega}$ stands for the Laplace-Beltrami operator of the surface $\partial\Omega$ and $|II|^2$ denotes the square of the norm of the second fundamental form which is $\sum_{i=1}^{N-1}k_i^2$ with k_i being the principal curvature. The principal curvatures are all equal to $-R_0^{-1}$ about a sphere of radius R_0 in $\mathbb{R}^N$.

The Fréchet derivative $F_x(0,\gamma) := D_x F(0,\gamma)$ can be taken in follow:

$$\begin{aligned}
[F_x(0,\gamma)]\,x &= -\frac{\gamma}{(N-1)R_0^2}\left[\sum_{k=1}^{N-1}\frac{\partial^2 x}{\partial\varphi_k^2}\frac{1}{p_k^2}+\sum_{k=1}^{N-2}\frac{\partial x}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{p_k^2}\right]-\frac{\gamma}{R_0^2}x \\
&\quad +\left[D_x\left(\frac{\partial u}{\partial\nu \mid \partial\Omega_x}\right)\right]x \\
&= -\frac{\gamma}{(N-1)R_0^2}\left[\sum_{k=1}^{N-1}\frac{\partial^2 x}{\partial\varphi_k^2}\frac{1}{p_k^2}+\sum_{k=1}^{N-2}\frac{\partial x}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{p_k^2}\right]-\frac{\gamma}{R_0^2}x \\
&\quad +x\frac{(N-1)u_0}{R_0^2}-\frac{(N-2)u_0}{R_0^2}\sum_{l=0}^{\infty}(l+N-2)\,a_l(x)\,\widetilde{Y}_l(\varphi_1).
\end{aligned}$$

If $x=\widetilde{Y}_l(\varphi_1)$, then

$$\begin{aligned}
QY_l=\Delta_{\partial\Omega}\widetilde{Y}_l+|II|^2\widetilde{Y}_l &= \frac{1}{R_0^2}\left[-l(l+N-2)\widetilde{Y}_l+(N-1)\widetilde{Y}_l\right] \\
&= \frac{\widetilde{Y}_l}{R_0^2}\left[-l^2-(N-2)l+N-1\right].
\end{aligned}$$

Consequently

$$[F_x(0,\gamma)]\,\widetilde{Y}_l(\varphi)=\left[-\tfrac{\gamma}{N-1}\tfrac{-l^2-(N-2)l+N-1}{R_0^2}+\tfrac{(N-1)u_0}{R_0^2}-\tfrac{(N-2)u_0}{R_0^2}(l+N-2)\right]\widetilde{Y}_l(\varphi).$$

The value of γ such that the operator $[F_x(0,\gamma)]$ is degenerate is what we got for γ_l. Namely

$$-\frac{\gamma}{N-1}\frac{-l^2-(N-2)l+N-1}{R_0^2}+\frac{(N-1)u_0}{R_0^2}-\frac{(N-2)u_0}{R_0^2}(l+N-2)=0,$$

then

$$\gamma_l := \frac{(N-1)u_0\left[(N-2)l+(N-2)^2-(N-1)\right]}{l^2+(N-2)l-(N-1)}. \tag{2.10}$$

The denominator of (2.10) is not 0, which is $l\neq 1$ and $l\neq -(N-1)$. Because $l\geq 0$, so γ_l is meaningful with $l\in\mathbb{N}, l\neq 1$. The Crandall-Rabinowitz Theorem [3] can be use on it.

The real Banach spaces $X=X_{G,1}^{m+2,\alpha}$ and $Y=X_1^{m,\alpha}$ are given. The operator $F(x,\gamma)$: $X\times\mathbb{R}\to Y$ and $A_\gamma := F_x(0,\gamma)$: $X\to Y$. If we set $\gamma=\gamma_l$ then

$$\ker A_{\gamma_l}=\left\{\widetilde{Y}_l\right\},\quad \operatorname{Im} A_{\gamma_l}=Y\ominus\left\{\widetilde{Y}_l\right\},$$

so that

$$\dim(\ker A_{\gamma_l})=1=\operatorname{codim}(\operatorname{Im} A_{\gamma_l}).$$

Observe that

$$[F_{\gamma x}(0,\gamma_l)]\,x=-\frac{1}{(N-1)R_0^2}\left[\sum_{k=1}^{N-1}\frac{\partial^2 x}{\partial\varphi_k^2}\frac{1}{p_k^2}+\sum_{k=1}^{N-2}\frac{\partial x}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{p_k^2}\right]-\frac{1}{R_0^2}x.$$

If $x = \widetilde{Y}_l(\varphi_1)$ then

$$[F_{\gamma x}(0,\gamma_l)]\,\widetilde{Y}_l(\varphi_1) = -\frac{1}{(N-1)R_0^2}\left[-l(l+N-2)+N-1\right]\widetilde{Y}_l(\varphi_1) \notin \mathrm{Im}(A_{\gamma_l}).$$

The curve $t \to (x(t), \gamma(t))$ satisfies $(x(0), \gamma(0)) = (0, \gamma_l)$ and $x'(0) = \widetilde{Y}_l(\varphi_1)$, By the Crandall-Rabinowitz Theorem $(0, \gamma_l)$ is a bifurcation point of the equation $F(x, \gamma) = 0$. The boundary $\partial\tilde{\Omega}$ has the form

$$\left\{r = R_0 + \varepsilon\widetilde{Y}_l(\varphi_1) + o(\varepsilon)\right\},$$

with ε small enough and $\gamma(\varepsilon) = \gamma_l + o(1)$. □

3 The case of $N = 2$

For $N = 2$, by polar coordinates, we have that

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2}.$$

Clearly, $u = u_0 + \ln\left(\frac{r}{R_0}\right) := u^*$ is the solution of

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2\backslash\Omega, \\ u = u_0 & \text{on } \partial\Omega, \\ \lim_{|x|\to+\infty} u = +\infty. \end{cases}$$

We see that

$$\frac{\partial u^*}{\partial \nu}_{|\partial\Omega} = \frac{1}{R_0}.$$

Taking

$$C_0 = \frac{1}{R_0} + \frac{\gamma}{2R_0},$$

then u^* is the trivial solution of the following overdetermined problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2\backslash\Omega \\ u = u_0 & \text{on } \partial\Omega \\ \frac{\partial u}{\partial \nu} - \gamma H = C_0 & \text{on } \partial\Omega \\ \lim_{r\to+\infty} u = +\infty. \end{cases}$$

We study this problem on perturbation domain

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2\backslash\tilde{\Omega} \\ u = u_0 & \text{on } \partial\tilde{\Omega} \\ \frac{\partial u}{\partial \nu} - \gamma H = C_0 & \text{on } \partial\tilde{\Omega} \\ \lim_{r\to+\infty} u = +\infty. \end{cases} \tag{3.1}$$

The domain $\tilde{\Omega}$ satisfies the following structure:

$$\Omega_x := \{r \leqslant R_0 + x(\varphi)\}$$

with $x(\varphi) = \varepsilon \cos(l\varphi) + \sum_{k=2}^{m+1} \varepsilon^k \Lambda_{lk}(\varphi) + O(\varepsilon^{m+2})$, where ε is small, $\Lambda_{lk}(\varphi, z)$ denote the coefficients of the power series expansion of $x(\varphi)$ with respect to the variable ε. The value of the corresponding parameter γ are

$$\gamma = \gamma_l + \sum_{k=1}^{m} \varepsilon^k \gamma_{lk} + O\left(\varepsilon^{m+1}\right)$$

with $\gamma_l := -2/(1+l)$, $l \neq -1$.

The main result in this section is the following theorem.

Theorem 3.1. *Suppose $N = 2$ and $l \in \mathbb{N}$, $l \neq 1$. For each value of γ_l, there exists a C^∞ bifurcation branch of solutions to the free boundary problem (3.1) with free boundary $\partial\tilde{\Omega}$ in $C^{m+2,\alpha}$ of the form*

$$\left\{ r = R_0 + \varepsilon \cos(l\varphi) + \sum_{k=2}^{m+1} \varepsilon^k \Lambda_{lk}(\varphi) + O\left(\varepsilon^{m+2}\right) \right\},$$

with

$$\gamma = \gamma_l + \sum_{k=1}^{m} \varepsilon^k \gamma_{lk} + O\left(\varepsilon^{m+1}\right)$$

and $m \geqslant 1$.

Solving (3.1) is equal to solving problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathbb{R}^2 \backslash \Omega_x, \\ u = u_0 & \text{on } \partial\Omega_x, \\ \lim_{r \to +\infty} u = +\infty, \end{cases} \tag{3.2}$$

and

$$F(x, \gamma) := \frac{\partial u}{\partial \nu}_{|\partial\Omega_x} - \gamma H_{|\partial\Omega_x} - \frac{1}{R_0} - \frac{\gamma}{2R_0} = 0.$$

Since u_* satisfies the above equation when $x \equiv 0$, $F(0, \gamma) = 0$ for any γ. Therefore, finding nontrivial domains emanating from $B^c_{R_0}$ such that the problem (3.1) has a positive solution is equivalent to study the the nontrivial solutions of $F(x, \gamma) = 0$.

In condition $\Omega_x = \{r \leqslant R_0 + x(\varphi),\ x(\varphi) = \varepsilon S(\varphi)\}$, applying the transformation

$$u(r, \varphi) = u^* + \bar{u}(r, \varphi),$$

to the problem of (3.2), we can get

$$\begin{cases} \Delta \bar{u} = 0 & \text{in } \mathbb{R}^2 \setminus \Omega_x \\ \bar{u} = \ln \frac{R_0}{R_0 + \varepsilon S(\varphi)} & \text{on } \partial\Omega_x, \\ \lim_{r \to +\infty} \bar{u} < +\infty. \end{cases} \tag{3.3}$$

In order to go from Ω_x to the fixed domain of the ball of radius R_0, we take the Hanzawa transformation W

$$\begin{cases} r = \tilde{r} + \chi\left(R_0 - \tilde{r}\right)\varepsilon S\left(\varphi\right), \\ \varphi = \tilde{\varphi}. \end{cases}$$

The value of $\frac{\partial \tilde{r}}{\partial r}$, $\frac{\partial^2 \tilde{r}}{\partial r^2}$, $\frac{\partial \tilde{r}}{\partial \varphi}$ and $\frac{\partial^2 \tilde{r}}{\partial \varphi^2}$ as same as the value in section 2. Taking the Hanzawa transformation on the $\bar{u}$, we get $\tilde{u}$.

We compute the Laplace operator in coordinates $(\tilde{r}, \tilde{\varphi})$ as follows

$$\begin{aligned}
\Delta u &= \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} \\
&= \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2}\left(\frac{\partial \tilde{r}}{\partial r}\right)^2 + \frac{\partial \tilde{u}}{\partial \tilde{r}}\frac{\partial^2 \tilde{r}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{u}}{\partial \tilde{r}}\frac{\partial \tilde{r}}{\partial r} \\
&\quad + \frac{1}{r^2}\left[\frac{\partial^2 \tilde{u}}{\partial \tilde{\varphi}^2} + 2\frac{\partial^2 \tilde{u}}{\partial \tilde{r}\partial \tilde{\varphi}}\frac{\partial \tilde{r}}{\partial \varphi} + \frac{\partial \tilde{u}}{\partial \tilde{r}}\frac{\partial^2 \tilde{r}}{\partial \varphi^2} + \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2}\left(\frac{\partial \tilde{r}}{\partial \varphi}\right)^2\right] \\
&= \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2}\left[\left(\frac{\partial \tilde{r}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial \tilde{r}}{\partial \varphi}\right)^2\right] + \frac{\partial \tilde{u}}{\partial \tilde{r}}\left[\frac{\partial^2 \tilde{r}}{\partial r^2} + \frac{1}{r}\frac{\partial \tilde{r}}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \tilde{r}}{\partial \varphi^2}\right] \\
&\quad + \frac{\partial^2 \tilde{u}}{\partial \tilde{\varphi}^2}\left[\frac{1}{r^2}\right] + \frac{\partial^2 \tilde{u}}{\partial \tilde{r}\partial \tilde{\varphi}}\left[2\frac{\partial \tilde{r}}{\partial \varphi}\frac{1}{r^2}\right] \\
&= \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2}\left[1 + \varepsilon\left(2\chi' S\right)\right] + \frac{\partial \tilde{u}}{\partial \tilde{r}}\left[\varepsilon\left(-\chi'' S\right) + \left(\frac{1}{\tilde{r}} + \varepsilon\frac{\tilde{r}\chi' S - \chi S}{\tilde{r}^2}\right) + \varepsilon\left(-\chi\frac{\partial^2 S}{\partial \tilde{\varphi}^2}\frac{1}{\tilde{r}^2}\right)\right] \\
&\quad + \frac{\partial^2 \tilde{u}}{\partial \tilde{\varphi}^2}\left[\frac{1}{\tilde{r}^2} + \varepsilon\frac{-2\chi S}{\tilde{r}^3}\right] + \frac{\partial^2 \tilde{u}}{\partial \tilde{r}\partial \tilde{\varphi}}\left[\varepsilon\left(-2\chi\frac{\partial S}{\partial \tilde{\varphi}}\frac{1}{\tilde{r}^2}\right)\right] + O\left(\varepsilon^2\right),
\end{aligned}$$

so we can get

$$\Delta = \tilde{\Delta} + \varepsilon\left(A_1 + A_2 + A_3 + A_4\right) + \varepsilon^2 B,$$

where

$$\tilde{\Delta}\tilde{u}\left(\tilde{r}, \tilde{\varphi}\right) = \frac{\partial^2 \tilde{u}}{\partial \tilde{r}^2} + \frac{1}{\tilde{r}}\frac{\partial \tilde{u}}{\partial \tilde{r}} + \frac{1}{\tilde{r}^2}\frac{\partial^2 \tilde{u}}{\partial \tilde{\varphi}^2},$$

and

$$\begin{aligned}
A_1 &:= S\left[2\chi'\frac{\partial^2}{\partial \tilde{r}^2} + \chi'\frac{1}{\tilde{r}}\frac{\partial}{\partial \tilde{r}} - \chi''\frac{\partial}{\partial \tilde{r}} - \chi\frac{1}{\tilde{r}^2}\frac{\partial}{\partial \tilde{r}}\right] \\
A_2 &:= -2\frac{\chi S}{\tilde{r}^3}\frac{\partial^2}{\partial \tilde{\varphi}^2}, \\
A_3 &:= -\frac{\chi}{\tilde{r}^2}\frac{\partial^2 S}{\partial \tilde{\varphi}^2}\frac{\partial}{\partial \tilde{r}}, \\
A_4 &:= -\frac{2\chi}{\tilde{r}^2}\frac{\partial S}{\partial \tilde{\varphi}}\frac{\partial^2}{\partial \tilde{r}\partial \tilde{\varphi}}.
\end{aligned}$$

By the transformation W, problem (3.3) becomes

$$\begin{cases} \tilde{\Delta}\tilde{u}\left(\tilde{r}, \tilde{\varphi}\right) = \varepsilon J\left(\tilde{r}, \tilde{\varphi}\right) & \text{in } \mathbb{R}^2 \setminus \Omega \\ \tilde{u}\left(R_0, \tilde{\varphi}\right) = g\left(\tilde{\varphi}\right) & \text{on } \partial\Omega, \\ \lim_{\tilde{r}\to+\infty} \tilde{u}\left(\tilde{r}, \tilde{\varphi}\right) < +\infty, \end{cases} \tag{3.4}$$

with

$$J(\tilde{r},\tilde{\varphi}) = -\sum_{i=1}^{4} A_i \tilde{u}(\tilde{r},\tilde{\varphi}) - \varepsilon B \tilde{u}(\tilde{r},\tilde{\varphi}),$$

and

$$g(\tilde{\varphi}) = \ln \frac{R_0}{R_0 + \varepsilon S(\tilde{\varphi})} = -\frac{\varepsilon S}{R_0} + \frac{1}{2}\frac{\varepsilon^2 S^2}{R_0^2} + O\left(\varepsilon^3\right).$$

Suppose S is even, 2π-periodic. Consequently

$$S(\tilde{\varphi}) = \sum_{l=0}^{+\infty} a_l \cos(l\tilde{\varphi})$$

with

$$a_l := \frac{1}{c_l}\int_0^{2\pi} S(\alpha)\cos(l\tilde{\varphi})\, d\tilde{\varphi},$$

with $c_{il} = \pi$ if $l > 0$, or $c_l = 2\pi$ if $l = 0$.

Similar to [10] by Banach's fixed point theorem, the existence and uniqueness of solutions of problem (3.4) is proved. Then, as that of [10, Theorem 3.2], we get

$$\tilde{u}(\tilde{r},\tilde{\varphi}) = -\frac{\varepsilon}{R_0}\sum_{l=1}^{+\infty} a_l \left(\frac{R_0}{\tilde{r}}\right)^l \cos(l\tilde{\varphi}) + O\left(\varepsilon^2\right).$$

Clearly

$$\frac{\partial \tilde{u}}{\partial \tilde{r}}|^{\tilde{r}=R_0} = \frac{\partial \tilde{u}_1}{\partial \tilde{r}} + O\left(\varepsilon^2\right) = \frac{\varepsilon}{R_0}\sum_{l=1}^{+\infty} a_l \frac{1}{R_0}\cos(l\tilde{\varphi}) + O\left(\varepsilon^2\right).$$

Using the formulas of $\frac{\partial \tilde{u}}{\partial \tilde{r}}|^{\tilde{r}=R_0}$ and $\frac{\partial \tilde{r}}{\partial r}, \frac{\partial \tilde{r}}{\partial \varphi}, \nu$, The values of $\frac{\partial u}{\partial \nu}|^{\partial\Omega_{\varepsilon S}}$ is computed in follow

$$\begin{aligned}
\frac{\partial u}{\partial \nu}|^{\partial\Omega_{\varepsilon S}} &= \frac{\partial}{\partial r}\left(\ln\frac{r}{R_0}\right)_{|r=R_0} + \varepsilon S + \frac{\partial \tilde{u}}{\partial \tilde{r}}|^{\tilde{r}=R_0} + O\left(\varepsilon^2\right) \\
&= \frac{1}{R_0} - \varepsilon\frac{S}{R_0^2} + \frac{\varepsilon}{R_0}\sum_{l=1}^{+\infty} a_l \frac{1}{R_0} l\cos(l\varphi) + O\left(\varepsilon^2\right).
\end{aligned}$$

By the fact of $\partial\Omega_x = \{r = R_0 + x(\varphi) = R_0 + \varepsilon S(\varphi)\}$, we can calculate the value of the Fréchet derivative of $\frac{\partial u}{\partial \nu}|^{\partial\Omega_x}$ with respect to a perturbation $x(\varphi)$ of the domain Ω at $x = 0$ below

$$\left[D_x\left(\frac{\partial u}{\partial \nu}|\partial\Omega_x\right)\right]x = -\frac{x}{R_0^2} + \frac{1}{R_0^2}\sum_{l=1}^{+\infty} a_l(x)\, l\cos(l\varphi),$$

with

$$a_l := \frac{1}{c_l}\int_0^{2\pi} x(\alpha)\cos(l\varphi)\, d\varphi,$$

with $c_l = \pi$ if $l > 0$, or $c_l = 2\pi$ if $l = 0$.

Proof of Theorem 3.1. The Fréchet derivative $F_x(0,\gamma) := D_x F(0,\gamma)$ can be taken in follow:

$$\begin{aligned}
[F_x(0,\gamma)]\,x &= -\frac{\gamma}{2}\left(\frac{1}{R_0^2}\frac{\partial^2 x}{\partial\varphi^2}+\frac{1}{R_0^2}x\right)+\left[D_x\left(\frac{\partial u}{\partial\nu}|\partial\Omega_x\right)\right]x \\
&= -\frac{\gamma}{2}\left(\frac{1}{R_0^2}\frac{\partial^2 x}{\partial\varphi^2}+\frac{1}{R_0^2}x\right)-\frac{x}{R_0^2}+\frac{1}{R_0^2}\sum_{l=1}^{+\infty} l\cos(l\varphi)\,a_l(x).
\end{aligned}$$

If $x=\cos(l\varphi)$ then $Jx=\left(-\frac{l^2}{R_0^2}+\frac{1}{R_0^2}\right)\cos(l\varphi)$ and $a_m(x)=\delta(l-m)$. Consequently

$$[F_x(0,\gamma)]\cos(l\varphi)=\left[-\frac{\gamma}{2}\left(-\frac{l^2}{R_0^2}+\frac{1}{R_0^2}\right)-\frac{1}{R_0^2}+\frac{l}{R_0^2}\right]\cos(l\varphi).$$

Then as that of [10, Theorem 3.2] we can obtain the desired conclusions. □

4 Appendix

The expression of spherical coordinates of the N dimensional Laplace equation may already exist, but we have not found relevant literature, we give our calculation process here for safety.

If we use the Cartesian coordinates $(x_1, x_2, \cdots, x_N)$, the equation $\Delta u = 0$ can be written as

$$\Delta u=\frac{\partial^2 u}{\partial x_1^2}+\frac{\partial^2 u}{\partial x_2^2}+\cdots+\frac{\partial^2 u}{\partial x_N^2}=0.$$

The relationship between the N dimension spherical coordinates $(r,\varphi_1,\varphi_2,\cdots,\varphi_{N-1})$ and the cartesian coordinates $(x_1, x_2, \cdots, x_N)$ is

$$\begin{cases}
x_1 &= r\cos\varphi_1 \\
x_2 &= r\sin\varphi_1\cos\varphi_2 \\
x_3 &= r\sin\varphi_1\sin\varphi_2\cos\varphi_3 \\
\cdots & \\
x_{N-1} &= r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{N-2}\cos\varphi_{N-1} \\
x_N &= r\sin\varphi_1\sin\varphi_2\cdots\sin\varphi_{N-2}\sin\varphi_{N-1},
\end{cases}$$

So we have

$$
\begin{aligned}
\Delta u &= \sum_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2} \\
&= \sum_{i=1}^{N} \partial \frac{\frac{\partial u}{\partial r}\frac{\partial r}{\partial x_i} + \sum_{k=1}^{N-1} \frac{\partial u}{\partial \varphi_k}\frac{\partial \varphi_k}{\partial x_i}}{\partial x_i} \\
&= \sum_{i=1}^{N} \frac{\partial\left(\frac{\partial u}{\partial r}\right)}{\partial x_i}\frac{\partial r}{\partial x_i} + \sum_{i=1}^{N} \frac{\partial u}{\partial r}\frac{\partial\left(\frac{\partial r}{\partial x_i}\right)}{\partial x_i} + \sum_{i=1}^{N}\sum_{k=1}^{N-1}\left[\frac{\partial\left(\frac{\partial u}{\partial \varphi_k}\right)}{\partial x_i}\frac{\partial \varphi_k}{\partial x_i} + \frac{\partial u}{\partial \varphi_k}\frac{\partial\left(\frac{\partial \varphi_k}{\partial x_i}\right)}{\partial x_i}\right] \\
&= \sum_{i=1}^{N} \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x_i}\right)^2 + \sum_{i=1}^{N}\sum_{k=1}^{N-1} \frac{\partial^2 u}{\partial r \partial \varphi_k}\frac{\partial r}{\partial x_i}\frac{\partial \varphi_k}{\partial x_i} + \sum_{i=1}^{N} \frac{\partial u}{\partial r}\frac{\partial^2 r}{\partial x_i^2} + \\
&\quad \sum_{i=1}^{N}\sum_{k,j=1}^{N-1} \frac{\partial^2 u}{\partial \varphi_k \partial \varphi_j}\frac{\partial \varphi_k}{\partial x_i}\frac{\partial \varphi_j}{\partial x_i} + \sum_{i=1}^{N}\sum_{k=1}^{N-1} \frac{\partial^2 u}{\partial \varphi_k \partial r}\frac{\partial \varphi_k}{\partial x_i}\frac{\partial \varphi_k}{\partial x_i} + \sum_{i=1}^{N}\sum_{k=1}^{N-1} \frac{\partial u}{\partial \varphi_k}\frac{\partial^2 \varphi_k}{\partial x_i^2} \\
&= \frac{\partial^2 u}{\partial r^2}\left[\sum_{i=1}^{N}\left(\frac{\partial r}{\partial x_i}\right)^2\right] + \frac{\partial u}{\partial r}\left[\sum_{i=1}^{N}\frac{\partial^2 r}{\partial x_i^2}\right] + \sum_{k=1}^{N-1}\frac{\partial^2 u}{\partial \varphi_k^2}\left[\sum_{i=1}^{N}\left(\frac{\partial \varphi_k}{\partial x_i}\right)^2\right] + \\
&\quad \sum_{k=1}^{N-1}\frac{\partial u}{\partial \varphi_k}\left[\sum_{i=1}^{N}\frac{\partial^2 \varphi_k}{\partial x_i^2}\right] + 2\sum_{k=1}^{N-1}\frac{\partial^2 u}{\partial r \partial \varphi_k}\left[\sum_{i=1}^{N}\frac{\partial r}{\partial x_i}\frac{\partial \varphi_k}{\partial x_i}\right] + \\
&\quad 2\sum_{1=k<j=2}^{N-1}\frac{\partial^2 u}{\partial \varphi_k \partial \varphi_j}\left[\sum_{i=1}^{N}\frac{\partial \varphi_k}{\partial x_i}\frac{\partial \varphi_j}{\partial x_i}\right],
\end{aligned}
$$

and then we need to calculate the coefficients on these six terms. The first step is to compute $\frac{\partial r}{\partial x_i}$, $\frac{\partial^2 r}{\partial x_i^2}$, $\frac{\partial \varphi_k}{\partial x_i}$ and $\frac{\partial^2 \varphi_k}{\partial x_i^2}$.

According to

$$r^2 = x_1^2 + x_2^2 + \cdots + x_N^2,$$

there is

$$
\begin{aligned}
2r\frac{\partial r}{\partial x_i} &= 2x_i, \\
\frac{\partial r}{\partial x_i} &= \frac{x_i}{r}, \\
\frac{\partial^2 r}{\partial x_i^2} &= \frac{r - x_i\frac{x_i}{r}}{r^2} = \frac{r^2 - x_i^2}{r^3},
\end{aligned}
$$

so we have that

$$
\begin{aligned}
\sum_{i=1}^{N}\left(\frac{\partial r}{\partial x_i}\right)^2 &= \sum_{i=1}^{N}\left(\frac{x_i}{r}\right)^2 = 1, \\
\sum_{i=1}^{N}\frac{\partial^2 r}{\partial x_i^2} &= \sum_{i=1}^{N}\frac{r^2 - x_i^2}{r^3} = \frac{Nr^2 - r^2}{r^3} = \frac{N-1}{r}.
\end{aligned}
$$

According to

$$\frac{x_N^2 + x_{N-1}^2 + \cdots + x_{k+1}^2}{x_k^2} = \tan^2\varphi_k,$$

there is

$$
\begin{aligned}
\frac{2x_i}{x_k^2} &= 2\tan\varphi_k \sec^2\varphi_k \frac{\partial\varphi_k}{\partial x_i} \quad (k+1 \le i \le N)\,, \\
\frac{\partial\varphi_k}{\partial x_i} &= \frac{x_i}{x_k^2 \tan\varphi_k \sec^2\varphi_k} \quad (k+1 \le i \le N)\,, \\
\frac{-2x_k\left(x_N^2 + x_{N-1}^2 + \cdots + x_{k+1}^2\right)}{x_k^4} &= 2\tan\varphi_k \sec^2\varphi_k \frac{\partial\varphi_k}{\partial x_k} \quad (i = k)\,, \\
\frac{\partial\varphi_k}{\partial x_k} &= -\frac{x_N^2 + x_{N-1}^2 + \cdots + x_{k+1}^2}{x_k^3 \tan\varphi_k \sec^2\varphi_k} = -\frac{\tan\varphi_k}{x_k \sec^2\varphi_k} \quad (i = k)\,, \\
\frac{\partial\varphi_k}{\partial x_i} &= 0 \quad (1 \le i \le k-1)\,.
\end{aligned}
$$

Hence, we find that

$$
\begin{aligned}
\frac{\partial^2\varphi_k}{\partial x_i^2} &= \frac{x_k^2 \tan\varphi_k \sec^2\varphi_k - x_i x_k^2 \left[\sec^2\varphi_k \sec^2\varphi_k + \tan\varphi_k 2\sec\varphi_k \tan\varphi_k \sec\varphi_k\right] \frac{\partial\varphi_k}{\partial x_i}}{\left(x_k^2 \tan\varphi_k \sec^2\varphi_k\right)^2} \\
&= \frac{x_k^2 \tan^2\varphi_k \sec^2\varphi_k - x_i^2 \left[\sec^2\varphi_k + 2\tan^2\varphi_k\right]}{\left(x_k^2 \tan\varphi_k \sec^2\varphi_k\right)^2 \tan\varphi_k} \quad (k+1 \le i \le N)\,,
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial^2\varphi_k}{\partial x_k^2} &= \frac{-\sec^2\varphi_k \frac{\partial\varphi_k}{\partial x_k} x_k \sec^2\varphi_k + \tan\varphi_k \left[\sec^2\varphi_k + x_k 2\sec\varphi_k \tan\varphi_k \sec\varphi_k \frac{\partial\varphi_k}{\partial x_i}\right]}{\left(x_k^3 \sec^2\varphi_k\right)^2} \\
&= \frac{\tan\varphi_k \sec^2\varphi_k + \tan\varphi_k \sec^2\varphi_k - 2\tan^3\varphi_k}{\left(x_k^3 \sec^2\varphi_k\right)^2} \\
&= \frac{2\tan\varphi_k \sec^2\varphi_k - 2\tan^3\varphi_k}{\left(x_k^3 \sec^2\varphi_k\right)^2} \\
&= \frac{2\tan\varphi_k}{\left(x_k^3 \sec^2\varphi_k\right)^2}
\end{aligned}
$$

and

$$
\frac{\partial^2\varphi_k}{\partial x_i^2} = 0 \quad (1 \le i \le k-1)\,.
$$

The coefficient on the third term is

$$
\begin{aligned}
\sum_{i=1}^{N} \left(\frac{\partial\varphi_k}{\partial x_i}\right)^2 &= \sum_{i=k}^{N} \left(\frac{\partial\varphi_k}{\partial x_i}\right)^2 \\
&= \frac{x_N^2 + x_{N-1}^2 + \cdots + x_{k+1}^2 + x_k^2 \tan^4\varphi_k}{\left(x_k^2 \tan\varphi_k \sec^2\varphi_k\right)^2} \\
&= \frac{x_k^2 \tan^2\varphi_k + x_k^2 \tan^4\varphi_k}{\left(x_k^2 \tan\varphi_k \sec^2\varphi_k\right)^2} = \frac{x_k^2 \tan^2\varphi_k \left(1 + \tan^2\varphi_k\right)}{\left(x_k^2 \tan\varphi_k \sec^2\varphi_k\right)^2} \\
&= \frac{x_k^2 \tan^2\varphi_k \sec^2\varphi_k}{\left(x_k^2 \tan\varphi_k \sec^2\varphi_k\right)^2} = \frac{1}{x_k^2 \sec^2\varphi_k} \\
&= \frac{1}{r^2 p_k^2}\,.
\end{aligned}
$$

The coefficient on the fourth term is

$$\begin{aligned}
\sum_{i=1}^{N}\frac{\partial^2\varphi_k}{\partial x_i^2} &= \sum_{i=k}^{N}\frac{\partial^2\varphi_k}{\partial x_i^2}\\
&= \frac{(N-k)\,x_k^2\tan^2\varphi_k\sec^2\varphi_k-\left(x_{k+1}^2+\cdots+x_N^2\right)\left[\sec^2\varphi_k+2\tan^2\varphi_k\right]+2x_k^2\tan^4\varphi_k}{x_k^4\tan^3\varphi_k\sec^4\varphi_k}\\
&= \frac{(N-k)\,x_k^2\tan^2\varphi_k\sec^2\varphi_k-x_k^2\tan^2\varphi_k\left[\sec^2\varphi_k+2\tan^2\varphi_k\right]+2x_k^2\tan^4\varphi_k}{x_k^4\tan^3\varphi_k\sec^4\varphi_k}\\
&= \frac{(N-k-1)\,x_k^2\tan^2\varphi_k\sec^2\varphi_k}{x_k^4\tan^3\varphi_k\sec^4\varphi_k}=\frac{N-k-1}{x_k^2\tan\varphi_k\sec^2\varphi_k}\\
&= \frac{N-k-1}{r^2p_k^2}\cot\varphi_k.
\end{aligned}$$

The coefficient on the fifth term is

$$\begin{aligned}
\sum_{i=1}^{N}\frac{\partial r}{\partial x_i}\frac{\partial\varphi_k}{\partial x_i} &= \sum_{i=1}^{N}\frac{\partial r}{\partial x_i}\frac{\partial\varphi_k}{\partial x_i}\\
&= \frac{x_N^2+x_{N-1}^2+\cdots+x_{k+1}^2-x_k^2\tan^2\varphi_k}{rx_k^2\tan\varphi_k\sec^2\varphi_k}\\
&= 0.
\end{aligned}$$

The coefficient on the sixth term is

$$\begin{aligned}
\sum_{i=1}^{N}\frac{\partial\varphi_k}{\partial x_i}\frac{\partial\varphi_j}{\partial x_i} &= \frac{\left(x_N^2+x_{N-1}^2+\cdots+x_{j+1}^2\right)-x_j^2\tan^2\varphi_j}{x_k^2\tan\varphi_k\sec^2\varphi_k\times x_j^2\tan\varphi_j\sec^2\varphi_j}\\
&= 0.
\end{aligned}$$

And then finally we have the form of Laplace equation in N dimensional spherical coordinates is

$$\Delta u=\frac{\partial^2u}{\partial r^2}+\frac{N-1}{r}\frac{\partial u}{\partial r}+\sum_{k=1}^{N-1}\frac{\partial^2u}{\partial\varphi_k^2}\frac{1}{r^2p_k^2}+\sum_{k=1}^{N-2}\frac{\partial u}{\partial\varphi_k}\frac{(N-k-1)\cot\varphi_k}{r^2p_k^2}=0.$$

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