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Analysis and numerical simulation of the time fractional diffusion equation by using mimetic finite differences

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Abstract

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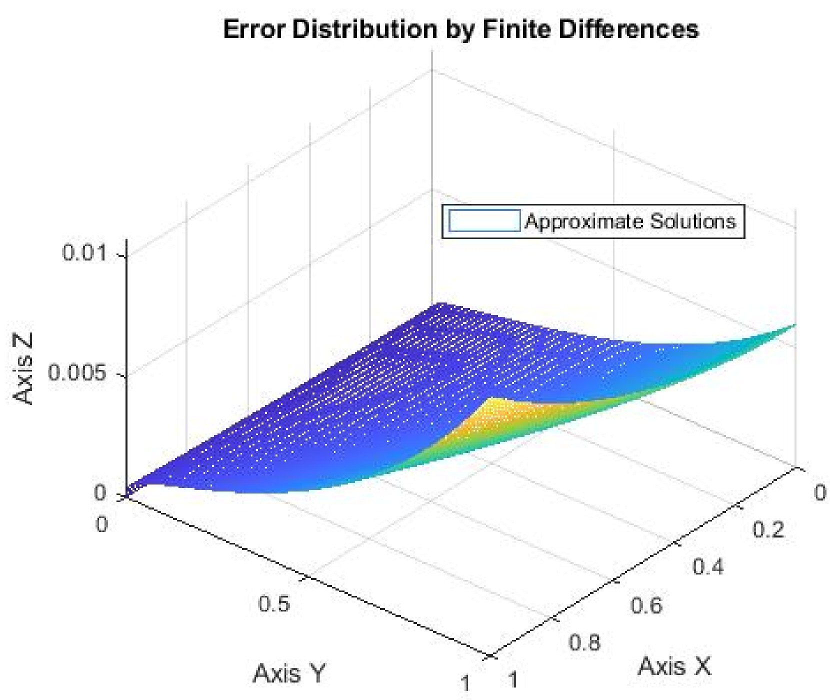

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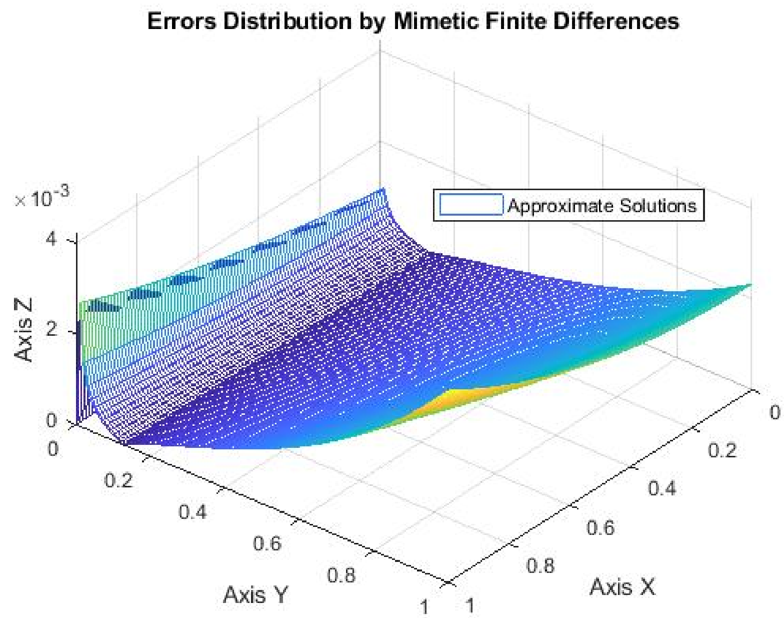

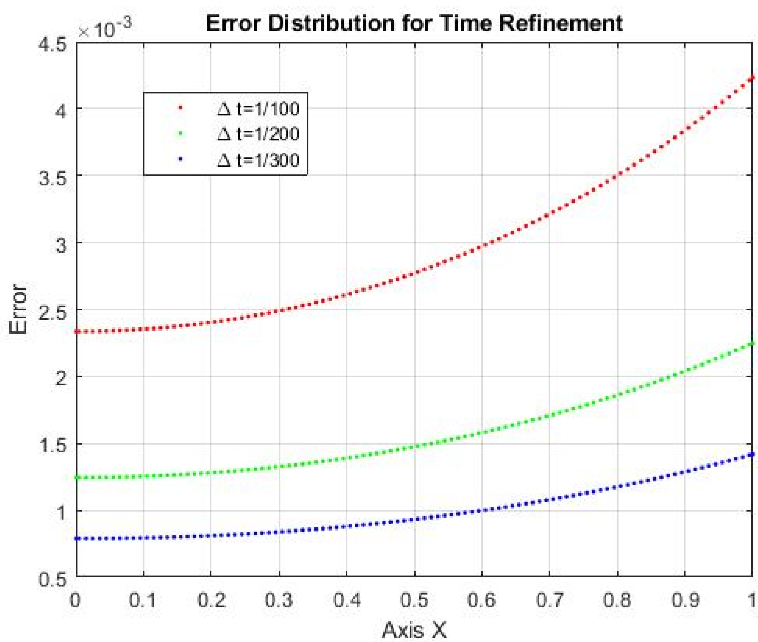

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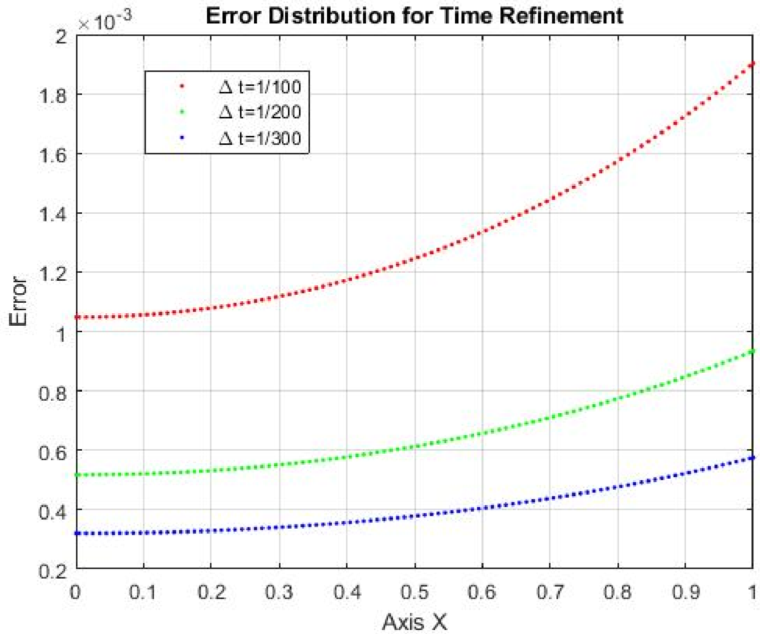

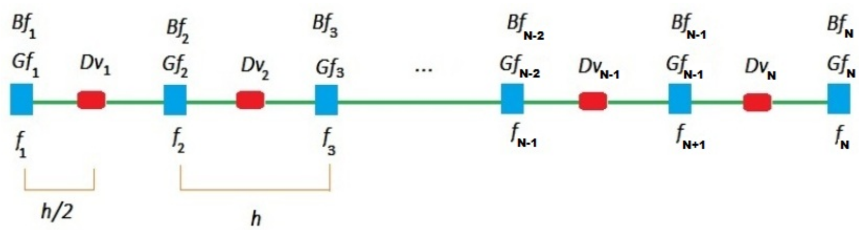

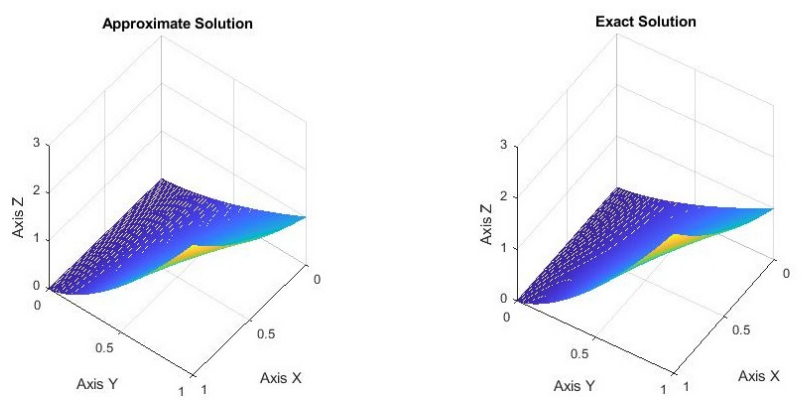

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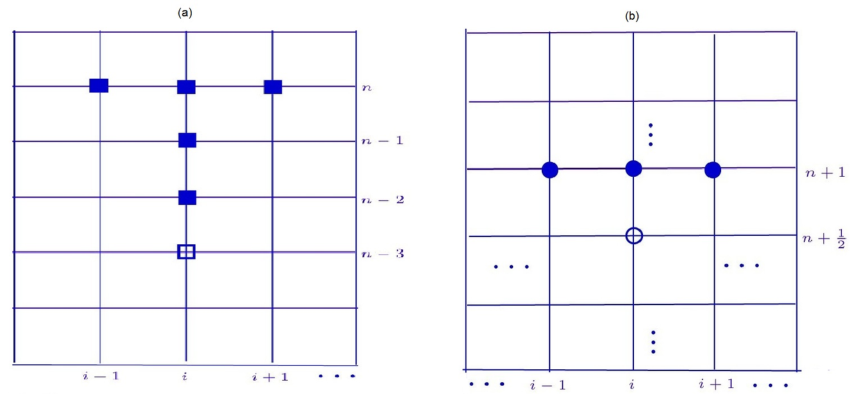

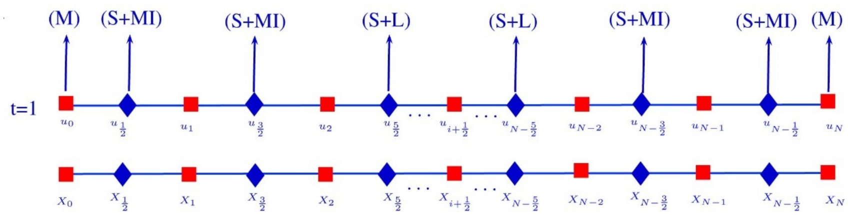

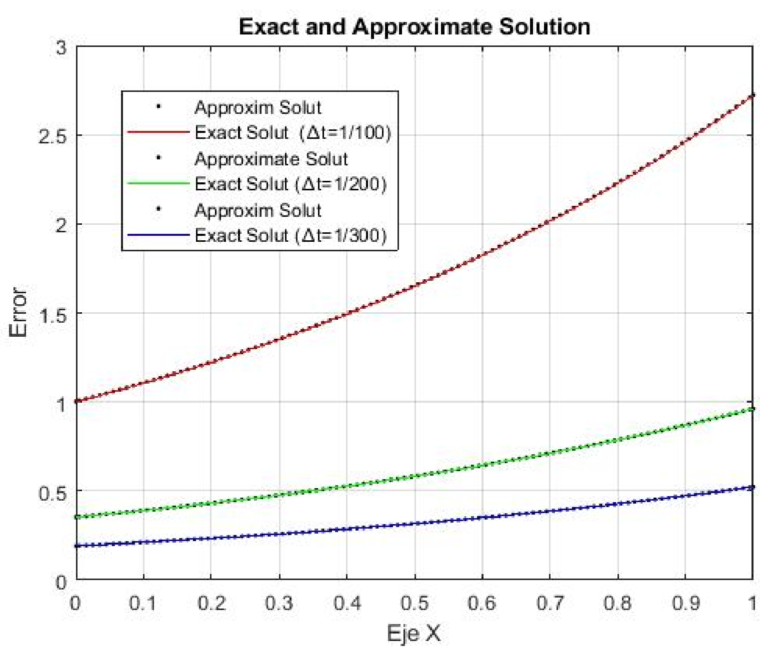

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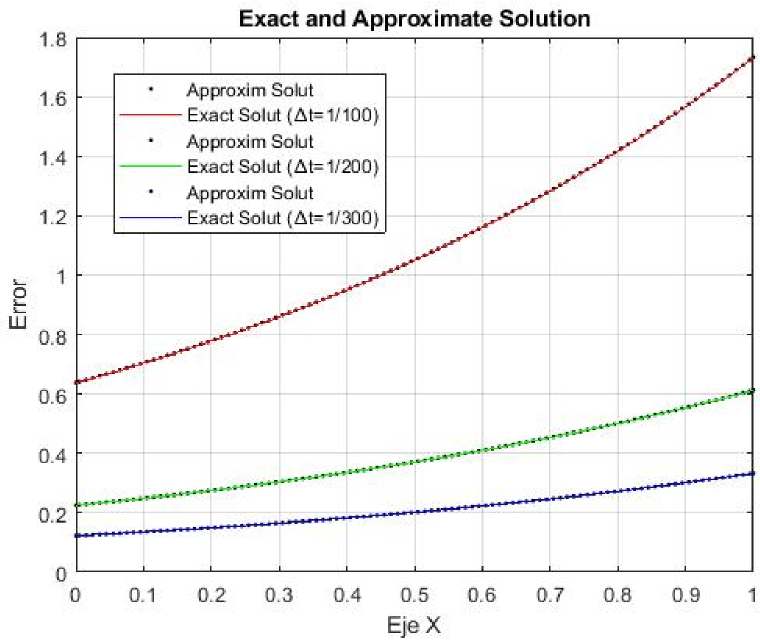

ANALYSIS AND NUMERICAL SIMULATION OF THE TIME FRACTIONAL DIFFUSION EQUATION BY USING MIMETIC FINITE DIFFERENCES

MARDO GONZALES HERRERA, SHWETA SRIVASTAVA*, AND CÉSAR E. TORRES LEDESMA

Abstract. This paper is devoted to the numerical treatment of time fractional diffusion equation with mixed boundary conditions. A new scheme based on the combination of the implicit finite difference method for Caputo derivative in time and the mimetic finite difference in space is considered for solving this problem. The stability analysis of the proposed scheme is given by using Von-Neumann method. The numerical results are provided to demonstrate the effectiveness of the proposed method as compared with other finite difference methods.

1. Introduction

Fractional partial differential equations (FPDEs) plays an important role in various fields of science and engineering, which has received increasing attention during the past 20 years. The various applications of fractional PDEs verified experimentally started to accelerate, see [7, 9, 13, 15, 17, 3]. Mainardi [12] proposed the fractional version of the time diffusion equation, which is obtained from the classical diffusion equation by replacing the first-order time derivative with a fractional derivative of order $\alpha \in (0,1)$. Later Gorenflo et al. [8] showed that this equation is derived by considering continuous time random walk problems, which are in general non-Markovian processes. In the last years, a number of numerical methods have been developed to solve the time fractional diffusion equation with Dirichlet boundary conditions. Liu et al.[10] used a first-order finite difference scheme in both time and space directions and derived the stability estimates for this equation. Yuste [18] presented a difference scheme based on the weighted average methods for ordinary (non-fractional) diffusion equations. In [11], a finite difference/spectral method based on a finite difference scheme in time and Legendre spectral methods in space is designed.

There is considerable literature on the development and applications of numerical methods for the time fractional diffusion equation with Dirichlet/Neumann boundary conditions. However, to the best of author's knowledge, we did not find any paper dealing with the time-fractional diffusion equation with mixed boundary conditions. We believe that it is very important to develop a numerical method to solve this kind of equation with mixed boundary conditions. Therefore in this paper, we propose a new numerical method to solve the time fractional diffusion equation with mixed boundary conditions. This new numerical method is a combination of the implicit finite difference method to approximate the time fractional derivative and the mimetic finite difference method to approximate the spatial variable. More precisely, we are going to study the following fractional problem

$$c\varrho\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \lambda\frac{\partial^2 u(x,t)}{\partial x^2} + g(x,t)\ , \quad (x,t)\in D=[0,L]\times[0,T], \tag{1.1}$$

where $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is the Caputo fractional derivative of order $\alpha\in(0,1)$ defined as

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{\partial u(x,s)}{\partial s}(t-s)^{-\alpha}ds, \qquad 0\le t\le T. \tag{1.2}$$

Consider the initial condition associated with equation (1.1),

$$u(x,0) = f(x)\ , \qquad x\in[0,L], \tag{1.3}$$

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and the Neumann-Robin type boundary condition

$$\left\{\begin{array}{rcl} -\lambda\frac{\partial u(0,t)}{\partial x} & = & q(t) \qquad , \; t\in[0,T] \\ -\lambda\frac{\partial u(L,t)}{\partial x} & = & h(t)\Big(u(L,t)-u^{\infty}\Big), \end{array}\right. \tag{1.4}$$

where: c is the specific heat, λ : is the thermal conductivity coefficient, ϱ is the density, h is the heat transfer coefficient, u^{∞} is the environmental temperature and q be the heat flux.

2. Preliminary Results

The basic idea of the numerical scheme for the time fractional diffusion equation is combining the implicit finite difference method to discretize the temporal variable [11] and the mimetic finite difference scheme to discretize the spatial variable [4]. The values of the functions u and g in the mesh points are denoted by $u_i^n = u(x_i, t_n)$ and $g_i = g(x_i)$, respectively.

2.1. **Discretization in the time: An implicit finite difference scheme.** We follow the ideas of [11] to discretize the Caputo fractional partial derivative with respect to the time. Let us discretize the time interval as $t_k := k\Delta t,\ k = 0,1,2,\cdots,K$, where $\Delta t := \frac{T}{K}$ is the time step. Then from the quadrature formula as in [3], for all $0\le k\le K-1$, we have

$$\frac{\partial^{\alpha}u(x,t_{k+1})}{\partial t^{\alpha}} = \frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\frac{u(x,t_{j+1})-u(x,t_j)}{\Delta t}\int_{t_j}^{t_{j+1}}\frac{ds}{(t_{k+1}-s)^{\alpha}} + r_{\Delta t}^{k+1}, \tag{2.1}$$

where

$$r_{\Delta t}^{k+1} \le c_u\left[\frac{1}{\Gamma(1-\alpha)}\sum_{j=0}^{k}\int_{t_j}^{t_{j+1}}\frac{t_{j+1}+t_j-2s}{(t_{k+1}-s)^{\alpha}}ds + O\left(\Delta t^2\right)\right], \tag{2.2}$$

is the truncation error given in [11] and c_u is a constant which depends only of u.
Let

$$S(k) = (k+1)^{1-\alpha} + 2\Big((k)^{1-\alpha} + (k-1)^{1-\alpha} + (k-2)^{1-\alpha} + \cdots + (1)^{1-\alpha}\Big) - \frac{2}{2-\alpha}(k-1)^{1-\alpha}.$$

Now we are going to show that $|S(k)|$ is bounded for all $\alpha\in[0,1]$, and $\forall\ k\ge 1$, in the following lemma. Just for the reader's convenience, we give the details of the proof here.

Lemma 2.1. [11] *For all $\alpha\in[0,1]$ and for all $k\ge 1$, there is a positive constant $C\ge 0$, independent of α and k, such that $|S(k)|\le C$.*

Proof. For $\alpha = 0$ we have

$$\begin{aligned} S(k) &= (k+1) + 2\left[k + (k-1) + (k-2) + (k-3) + \cdots + 1\right] - (k+1)^2 \\ &= (k+1)(1-k-1) + 2\frac{k(k+1)}{2} = 0. \end{aligned} \tag{2.3}$$

Now for $\alpha\in(0,1]$, we claim that

$$S(k) = (k+1)^{1-\alpha} + 2\Big[(k)^{1-\alpha} + (k-1)^{1-\alpha} + (k-2)^{1-\alpha} + \cdots + (1)^{1-\alpha}\Big] - \left(\frac{2}{2-\alpha}\right)(k+1)^{2-\alpha} = \sum_{i=0}^{k} a_i, \tag{2.4}$$

where

$$a_i = (i+1)^{1-\alpha} + (i)^{1-\alpha} - \frac{2}{2-\alpha}\Big[(i+1)^{2-\alpha} - (i)^{2-\alpha}\Big].$$

Evaluating for each i, we get

$$\begin{aligned}
&i=0;\;\; a_0=(1)^{1-\alpha}-\frac{2}{2-\alpha}(1)^{2-\alpha}\\
&i=1;\;\; a_1=(2)^{1-\alpha}+(1)^{1-\alpha}-\frac{2}{2-\alpha}\left[(2)^{2-\alpha}-(1)^{2-\alpha}\right]\\
&\vdots\\
&i=k;\;\; a_k=(k+1)^{1-\alpha}+(k)^{1-\alpha}-\frac{2}{2-\alpha}\left[(k+1)^{2-\alpha}-(k)^{2-\alpha}\right].
\end{aligned}$$

Thus we have,

$$\begin{aligned}
\sum_{i=0}^{k}a_i &= a_0+a_1+a_2+a_3+\cdots+a_k\\
&=2\left[(1)^{1-\alpha}+(2)^{1-\alpha}+(3)^{1-\alpha}+(4)^{1-\alpha}+\cdots+(k-2)^{1-\alpha}+(k-1)^{1-\alpha}+(k)^{1-\alpha}\right]+(k+1)^{1-\alpha}\\
&\quad-\frac{2}{2-\alpha}(k+1)^{2-\alpha}=S(k).
\end{aligned}$$

According to (2.4), we will show that $\sum_{i=1}^{\infty}a_i$ is convergent. Therefore, it is enough to prove that $|a_i|\le\frac{1}{i^{1+\alpha}}$, for i large enough. Note that for $i\ge 2$, we have

$$|a_i|=(i)^{1-\alpha}\left|\left(1+\frac{1}{i}\right)^{1-\alpha}+1-\frac{2i}{2-\alpha}\left(\left(1+\frac{1}{i}\right)^{2-\alpha}-1\right)\right| \tag{2.5}$$

Let

$$\Theta_1=\left(1+\frac{1}{i}\right)^{1-\alpha}\qquad\text{and}\qquad\Theta_2=\left(1+\frac{1}{i}\right)^{2-\alpha}-1.$$

Thus by Newton's binomial theorem

$$\begin{aligned}
\Theta_1=&1+(1-\alpha).\frac{1}{i}+\frac{(1-\alpha)(-\alpha)}{2!}\left(\frac{1}{i^2}\right)+\frac{(1-\alpha)(-\alpha)(-\alpha-1)}{3!}\left(\frac{1}{i^3}\right)\\
&+\frac{(1-\alpha)(-\alpha)(-\alpha-1)(-\alpha-2)}{4!}\left(\frac{1}{i^4}\right)+\cdots
\end{aligned} \tag{2.6}$$

and

$$\begin{aligned}
\Theta_2=&1+\frac{(2-\alpha)}{1!}.\frac{1}{i}+\frac{(2-\alpha)(1-\alpha)}{2!}\frac{1}{i^2}+\frac{(2-\alpha)(1-\alpha)(-\alpha)}{3!}\frac{1}{i^3}\\
&+\frac{(2-\alpha)(1-\alpha)(-\alpha)(-\alpha-1)}{4!}\frac{1}{i^4}+\cdots
\end{aligned} \tag{2.7}$$

Combining (2.5), (2.6), (2.7) and taking $i=k$, we get

$$\begin{aligned}
|a_k|&=(k)^{1-\alpha}\left|\left(\frac{1}{2!}-\frac{2}{3!}\right)(1-\alpha)(-\alpha).\frac{1}{k^2}+\left(\frac{1}{3!}-\frac{2}{4!}\right)(1-\alpha)(-\alpha)(-\alpha-1)-\frac{1}{k^3}+\cdots\right|\\
&=(k)^{1-\alpha}\left|\frac{1}{3!}(1-\alpha)(-\alpha).\frac{1}{k^2}+\frac{2}{4!}(1-\alpha)(-\alpha)(-\alpha-1)-\frac{1}{k^3}+\cdots\right|.
\end{aligned}$$

Therefore

$$\begin{aligned}
|a_k|\le(k)^{1-\alpha}&\left|\frac{(1-\alpha)}{3!}\frac{1}{k^2}\right||-\alpha|+\left|\frac{2(1-\alpha)(-\alpha)}{4!}\frac{1}{k^3}\right||-\alpha-1|+\left|\frac{3(1-\alpha)}{5!}\frac{1}{k^4}\right||-\alpha||-\alpha-1||-\alpha-2|+\cdots\\
&\le\frac{(1-\alpha)(\alpha)}{3!}\frac{1}{k^{1+\alpha}}\left(1+\frac{1}{k}+\frac{1}{k^2}+\cdots\right)\le\frac{2(1-\alpha)(\alpha)}{3!}\frac{1}{k^{1+\alpha}}\le\frac{1}{k^{1+\alpha}}.
\end{aligned} \tag{2.8}$$

Hence we conclude that $|S(k)|$ is bounded. □

From Lemma 2.1 and since $\frac{1}{\Gamma(2-\alpha)} \leq 2$, for all $\alpha \in [0,1]$, we have

$$\left| \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{t_{j+1}+t_j-2s}{(t_{k+1}-s)^{\alpha}} ds \right| \leq 2\Delta t^{2-\alpha}. \tag{2.9}$$

Now by using (2.9), we get

$$\begin{aligned}
\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \int_{t_j}^{t_{j+1}} \frac{\partial u(x;s)}{\partial t}(t_{k+1}-s)^{-\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^{k} \left[\frac{u_i^{j+1}-u_i^j}{\Delta t} + O(\Delta t) \right] \int_{t_j}^{t_j+1} ((k+1)(\Delta t)-s)^{-\alpha} ds \\
&= \frac{1}{\Gamma(1-\alpha)} \left(\frac{1}{1-\alpha}\right) \left(\frac{1}{\Delta t^\alpha}\right) \sum_{j=0}^{k} \left\{ \left(u_i^{j+1}-u_i^j\right) \left[(k-j+1)^{1-\alpha}-(k-j)^{1-\alpha}\right] \right\} + \\
&\quad \left(\frac{1}{\Gamma(1-\alpha)}\right) \left(\frac{1}{1-\alpha}\right) \sum_{j=0}^{k} \left[(k-j+1)^{1-\alpha}-(k-j)^{1-\alpha}\right] O(\Delta t^{2-\alpha}).
\end{aligned} \tag{2.10}$$

Thus the approximation of the fractional derivative is given by

$$\frac{\partial^\alpha u_i}{\partial t^\alpha} = O(\alpha,\Delta t) \sum_{j=0}^{k} \omega(\alpha,j)(u_i^{k-j+1}-u_i^{k-j}), \tag{2.11}$$

where

$$O(\alpha,\Delta t) = \frac{1}{\Gamma(1-\alpha)(1-\alpha)\Delta t^\alpha}. \tag{2.12}$$

and

$$\omega(\alpha,j) = (j+1)^{1-\alpha} - (j)^{1-\alpha}, \quad \forall\, j = 0,1,2\cdots,k, \tag{2.13}$$

are positive for all $\alpha, j, \Delta t$.

$$\begin{aligned}
\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} &= O(\alpha,\Delta t) \sum_{j=0}^{k} (u_i^{k-j+1}-u_i^{k-j}) \left[(j+1)^{1-\alpha}-(j)^{1-\alpha}\right] \\
&\quad + \frac{1}{(1-\alpha)\Gamma(1-\alpha)} \sum_{j=0}^{k} \left[(j+1)^{1-\alpha}-(j)^{1-\alpha}\right] O(\Delta t^{2-\alpha}).
\end{aligned} \tag{2.14}$$

Therefore, the expression for fractional derivative becomes,

$$\frac{\partial^\alpha u(x,t_{k+1})}{\partial t^\alpha} = O(\alpha,\Delta t) \sum_{j=0}^{k} (u_i^{k-j+1}-u_i^{k-j}) \left[(j+1)^{1-\alpha}-(j)^{1-\alpha}\right] + O(\Delta t^{2-\alpha}).$$

Remark 2.2. The expression (2.11) provides the values of the time fractional derivative at $t=0$, which is not required by the implicit finite difference scheme.

2.2. **Discretization of the spatial variable using the mimetic finite difference scheme.** Now we are going to use a discrete version of divergence theorem to determine the discrete gradient. The divergence theorem says that

$$\int_\Omega \nabla.\overrightarrow{v} f \; dV + \int_\Omega \overrightarrow{v} \nabla f \; dV = \int_{\partial\Omega} f \overrightarrow{v} \overrightarrow{n} \; dS. \tag{2.15}$$

where Ω is a domain and $\partial\Omega$ is the boundary of the domain, $\overrightarrow{n}$ is the exterior normal, f is a scalar function defined over the boundary $\partial\Omega$.

Let f, g be the scalar fields and $\overrightarrow{v}$, $\overrightarrow{w}$ be the vector fields, the appropriate inner product of the continuum are given by

$$\langle f, g\rangle = \int_{\Omega} fg\ dV, \qquad \langle \overrightarrow{v}, \overrightarrow{w}\rangle = \int_{\Omega} \overrightarrow{v}\overrightarrow{w}\ dV. \tag{2.16}$$

Thus the equation (2.15) can be written as

$$\langle \nabla.\overrightarrow{v}, f\rangle + \langle \overrightarrow{v}, \nabla f\rangle = \int_{\partial\Omega} f\overrightarrow{v}\overrightarrow{n}\ dS. \tag{2.17}$$

The divergence theorem in one dimension becomes an integration by parts, thus we have

$$\int_0^1 \frac{dv}{dx} f\ dx + \int_0^1 v\frac{df}{dx}\ dx = v(1)f(1) - v(0)f(0). \tag{2.18}$$

A discrete form of the conservation law needs to be constructed in order to satisfy the local conservation in each cell interval so that the law of global conservation is fulfilled throughout the investigated interval.

Definition 2.3 ([4], [5])**.** Let $\overrightarrow{v} : R \rightarrow R^{N+1}$, be a discrete vector function defined on the nodes of the one-dimensional mesh, such that $v(t) = (v_0(t), v_1(t), \cdots, v_N(t))$, $\forall\ t \in R$. $D_v \in R^N$ represents the approximation in the centers of the cells $\nabla\overrightarrow{v}$, the divergence in the centers of the cell are defined as:

$$\begin{aligned} D_v \subset R^{N+1} &\rightarrow R^N \\ (D_v)_{i+\frac{1}{2}} &= \frac{(v_{i+1} - v_i)}{h} \qquad \text{for } i = 0, 1, 2, \cdots, n-1. \end{aligned}$$

The approximation of the divergence in the centers of the cells coincide with the central difference scheme, which is expressed as the matrix $D_{(N)\times(N+1)}$

$$D = \frac{1}{h}\begin{pmatrix} -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 1 \end{pmatrix}_{(N)\times(N+1)}$$

Definition 2.4 ([4], [5])**.** Let $f = (f_0,\ f_{1/2},\ f_{3/2}, \cdots,\ f_{N-1/2},\ f_N)^T \in R^{N+2}$ be a discrete function defined in the center of the cell and in the domain border of a one-dimensional mesh. Further, let $Gf \in R^{N+1}$ represents the approximations at the nodes ∇f. The gradient $Gf \subset R^{N+2} \rightarrow R^{N+1}$ defined in a one-dimensional mesh at the boundary points, has the form

$$\begin{aligned} (Gf)_0 &= \frac{-\frac{8}{3}f_0 + 3f_{\frac{1}{2}} - \frac{1}{3}f_{\frac{3}{2}}}{h}, \\ (Gf)_N &= \frac{\frac{8}{3}f_N - 3f_{N-\frac{1}{2}} + \frac{1}{3}f_{N-\frac{3}{2}}}{h}. \end{aligned}$$

The gradient Gf at the interior points coincides with the central difference scheme, that is

$$(Gf)_i = \frac{f_{i+\frac{1}{2}} - f_{i-\frac{1}{2}}}{h} \quad \text{for } i = 1, 2, \cdots n-1,$$

matrix $G_{(N+1)\times(N+2)}$ is expressed as,

$$G = \frac{1}{h}\begin{pmatrix} \frac{-8}{3} & 3 & \frac{-1}{3} & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & -1 & 1 & 0 & \cdots & \cdots & \cdots & 0 & 0 \\ 0 & 0 & -1 & 1 & \cdots & \cdots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & -1 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \cdots & 0 & -1 & 1 & 0 \\ 0 & \cdots & \cdots & \cdots & \cdots & 0 & \frac{1}{3} & -3 & \frac{8}{3} \end{pmatrix}_{(N+1)\times(N+2)}$$

Observation 2.5. *Now we have the matrix operators, the conditions of the discrete operators gradient* (G) *and divergence* (D) *will be reformulated.*

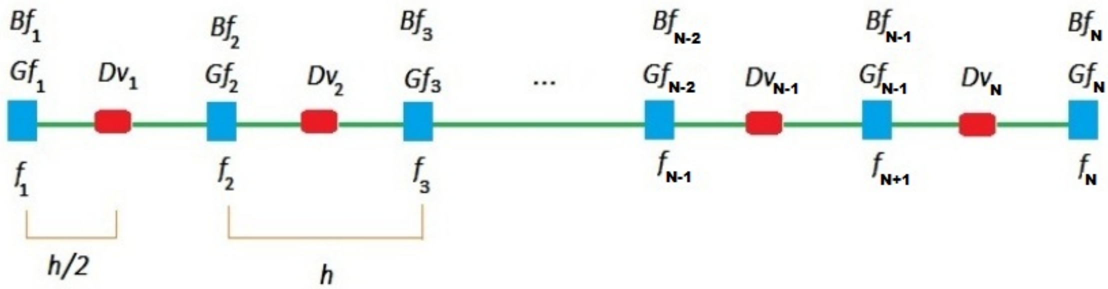

FIGURE 1. One-dimensional Uniform Staggered Mesh

where (D) indicates the approach for the divergence operator applied to the vector function in the cell centers and (G) denotes the approximation for the gradient operator applied to the scalar function in the nodes of the cells and the border.

In figure (1), the cell with uniform spacing and length $h = \frac{1}{N}$, over the interval $[0,1]$ is considered. The investigated interval is divided into n sub-intervals, each node has coordinate $x_i = (i \times h)$, for $0 \le i \le N$.

Each cell has a central point, means the interval $[x_i, x_{i+1}]$ includes the center of coordinates $x_{i+1/2}$.

The discrete form of the gradient operator (G) inside the domain and in the border are given by (see [4, 5])

$$(Gu)_i = \frac{u_{i+\frac{1}{2}} - u_{i-\frac{1}{2}}}{h} \quad 1 \le i \le N, \tag{2.19}$$

$$(Gu)_0 = \frac{-8}{3h}u_0 + \frac{3}{h}u_{\frac{1}{2}} - \frac{1}{3h}u_{\frac{3}{2}}, \tag{2.20}$$

$$(Gu)_N = \frac{8}{3h}u_N - 3u_{N-\frac{1}{2}} + \frac{1}{3h}u_{N-\frac{3}{2}}. \tag{2.21}$$

The mimetic discretization of the divergence (D) in the centers of each cell is given by

$$(D_v)_{i+\frac{1}{2}} = \frac{v_{i+1} - v_i}{h} \quad 0 \le i \le N. \tag{2.22}$$

Definition 2.6 ([4], [5])**.** Let $v : R \to R^{N+1}$ be a discrete vector function defined on the nodes of the one-dimensional mesh such that $v(t) = (v_0(t), v_1(t), v_2(t), \ldots, v_N(t))\,, \forall\ t \in R$. Further, let $D_v \in R^N$ represents the discrete approximation of the divergence (D) in the centers of the cells, the divergence ($\widehat{D}$) is defined as :

$$\begin{aligned} \widehat{D}_v : R^{N+1} &\rightarrow R^{N+2} \\ \left(\widehat{D}_v\right)_{i+\frac{1}{2}} &= \frac{(v_{i+1} - v_i)}{h}, \quad \forall \text{ i} = 0, 1, \ldots, n-1; \\ (D_v)_0 &= (D_v)_N = 0. \end{aligned}$$

The Matrix operator $\widehat{D}$ is expressed as :

$$\widehat{D} = \frac{1}{h} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ 0 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}_{(N+2)\times(N+1)}$$

2.3. **Discrete divergence theorem (Castillo-Grone approach** [4, 5]**).** In general, the Castillo-Grone approach given in (2.15) expresses the law of conservation of the equation (2.17). It can be represented in the form of a weighted internal product of discrete vector and scalar functions on a stepped grid as,

$$\langle \widehat{D}v, f \rangle_Q + \langle v, Gf \rangle_P = \langle Bv, f \rangle_I \tag{2.23}$$

where D, G and B are the discrete versions of their corresponding continuums: divergence ($\nabla.$), gradient (∇), and the border operator ($\frac{\partial}{\partial \vec{n}}$). The $\langle \ , \ \rangle$ represents the generalization of the internal product with weights Q, P and I. Using the identity (2.23), a relation is obtained for the border operator

$$B = Q\widehat{D} + G^t P, \tag{2.24}$$

where $[Q]$, $[P]$ and $[I]$ are positive definite matrices of order $(N+2) \times (N+2)$, $(N+1) \times (N+1)$ and $(N+2) \times (N+2)$ respectively, which are used to determine the form of $\widehat{D}$ and G. Thus the matrix B is given as

$$B = \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{1}{8} & -\frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ -\frac{1}{8} & \frac{1}{8} & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & 0 & \cdots & 0 & -\frac{1}{8} & \frac{1}{8} \\ 0 & 0 & 0 & \cdots & 0 & \frac{1}{8} & -\frac{1}{8} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 1 \end{pmatrix}_{(N+2)\times(N+2)}$$

The discrete second order operators introduced by Castillo-Grone are given in [4], [5] and [6].
Therefore the mimetic operators gradient (G), divergence ($\widehat{D}$) are of second order, both inside of the domain as well as at the border, in a uniform refined mesh of a one-dimensional domain.
Thus (2.15) can be written as

$$\langle \nabla.\vec{v}, f \rangle + \langle \vec{v}, \nabla f \rangle = \int_{\partial\Omega} f \vec{v} \vec{n} \, dS. \tag{2.25}$$

3. Numerical Scheme

3.1. Incorporation of discrete temporal and spatial scheme for fractional order diffusion equation. From (1.1), we have

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{a\partial^2 u(x,t)}{\partial x^2} + \overline{g}(x,t), \tag{3.1}$$

where $a = \frac{\lambda}{c\varrho}$ is the thermal diffusion coefficient and $\overline{g}(x,t) = \frac{g(x,t)}{c\varrho}$. Using implicit finite difference relation for fractional order time derivative and mimetic discretization for the spatial variable, we obtain

$$O(\alpha,k)\sum_{j=0}^{k}\omega(\alpha,j)\left(u_i^{k-j+1} - u_i^{k-j}\right) = a[D][G]u_i^k + \overline{g}_i^k,$$

Now, $\forall\ k \geq 1$ and $i = 0$, we have

$$\begin{aligned} O(\alpha,k)\,\omega(\alpha,0)\left(u_0^{k+1} - u_0^k\right) + O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_0^{k-j+1} - u_0^{k-j}\right) &= a[D][G]u_0^{k+1} + \overline{g}_0^{k+1}, \\ O(\alpha,k)\left(u_0^{k+1} - u_0^k\right) + O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_0^{k-j+1} - u_0^{k-j}\right) &= a[D][G]u_0^{k+1} + \overline{g}_0^{k+1}. \end{aligned}$$

(3.2)

Using the approximations for Neumann and Robin boundary conditions (1.4), we get

$$\frac{-\lambda\partial u(0,t)}{\partial x} = q(t) \implies -\lambda[B][G]u_i^{k+1} = q^{k+1}, \qquad \forall t \in [0,T] \tag{3.3}$$

$$\begin{aligned} \frac{-\lambda\partial u(L,t)}{\partial x} &= h(t)u(L,t) - h(t)u^\infty \\ h(t)\,u(L,t) + \lambda\frac{\partial u}{\partial x}(L,t) &= h(t)\,u^\infty \implies h(t)\,u^{k+1} + \beta[B][G]u^{k+1} = h(t)\,u^\infty \end{aligned} \tag{3.4}$$

substituting (3.3) in (3.2), equation becomes

$$\begin{aligned} &O(\alpha,k)\left(u_0^{k+1} - u_0^k\right) + O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_0^{k-j+1} - u_0^{k-j}\right) = a[D][G]u_0^{k+1} + \overline{g}_0^{k+1} \\ &\beta[B][G]u_0^{k+1} = q^{k+1} \qquad , \qquad \beta = -\lambda \\ &O(\alpha,k)\left(u_0^{k+1} - u_0^k\right) + O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_0^{k-j+1} - u_0^{k-j}\right) = \left(a[D][G] - \beta[B][G]\right)u_0^{k+1} + \overline{g}_0^{k+1} + q^{k+1} \end{aligned}$$

$$\left(O(\alpha,k) - a[D][G] + \beta[B][G]\right)u_0^{k+1} = O(\alpha,k)\,u_0^k - O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_0^{k-j+1} - u_0^{k-j}\right) + \overline{g}_0^{k+1} + q^{k+1} \tag{3.5}$$

Further, $\forall\ k \geq 1$ and $i = 1, 2, \ldots, N-1$, we have

$$O(\alpha,k)\left(u_i^{k+1} - u_i^k\right) + O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_i^{k-j+1} - u_i^{k-j}\right) = a[D][G]u_i^{k+1} + \overline{g}_i^{k+1}$$

$$\left(O(\alpha,k) - a[D][G]\right)u_i^{k+1} = O(\alpha,k)\,u_i^k - O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_i^{k-j+1} - u_i^{k-j}\right) + \overline{g}_i^{k+1} \tag{3.6}$$

And, $\forall\ k \geq 1$ and $i = N$, we have

$$O(\alpha,k)\left(u_N^{k+1}-u_N^{k}\right)+O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_N^{k-j+1}-u_N^{k-j}\right)=a[D][G]\left(u_N^{k+1}\right)+\overline{g}_N^{k+1}$$
$$h(t)\,u_N^{k+1}+\beta[B][G]u_N^{k+1}=h(t)\,u^{\infty}\qquad,\quad \beta=-\lambda.$$

Thus, we obtain

$$O(\alpha,k)\,u_N^{k+1}-O(\alpha,k)\,u_N^{k}+O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_N^{k-j+1}-u_N^{k-j}\right)=\left(a[D][G]-h(t)-\beta[B][G]\right)u_N^{k+1}$$
$$+h(t)\,u^{\infty}+\overline{g}_N^{k+1}$$

$$\left(O(\alpha,k)-a[D][G]+h(t)+\beta[B][G]\right)u_N^{k+1}=O(\alpha,k)\,u_N^{k}-O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(u_N^{k-j+1}-u_N^{k-j}\right)$$
$$+h(t)\,u^{\infty}+\overline{g}_N^{k+1}. \tag{3.7}$$

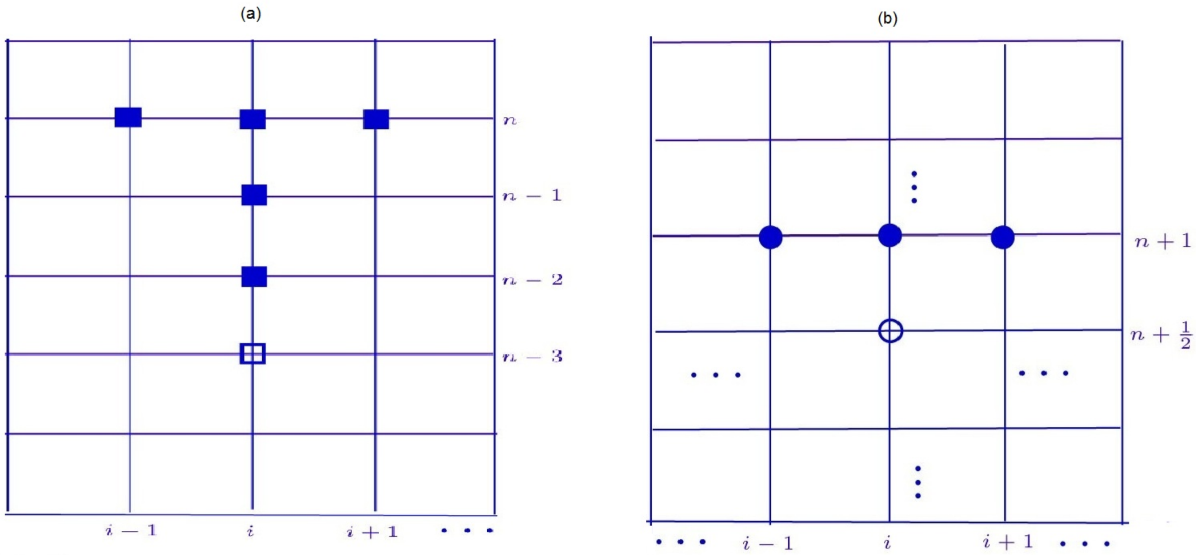

FIGURE 2. (a) Mesh for the Fractional Order Derivative, (b) Mesh for the Integer Order Derivative

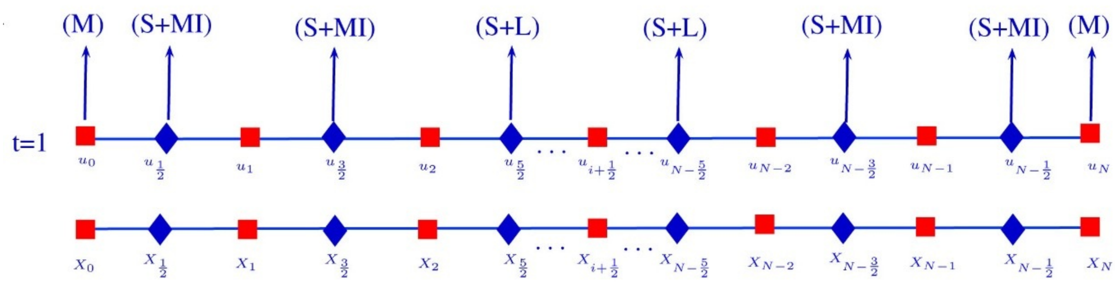

FIGURE 3. One-dimensional mesh for mimetic operators: B,G,D and, S operators

From the graph it can be seen that:

$$\begin{aligned} L &= D \cdot G \\ S &= O(\alpha, k) \\ M &= A + B \cdot G \\ MI &= A + \cdot M \cdot B \cdot G - D \cdot G \end{aligned}$$

where A is a matrix of order $(N+2) \times (N+2)$ which has nonzero entries on its diagonal, which correspond to the nodes of the boundary. The values associated with the inputs are not null.
Then $O(\alpha, \Delta t) = [T]_{(N+2)\times(N+2)}$ is a matrix of order $(N+2) \times (N+2)$.

4. Stability analysis

In this section, we prove an unconditional stable estimates of the proposed scheme with mixed boundary conditions. The expanded form of equation(3.5) for the left most boundary for $k \geq 1$ and $i = 0$ is given by

$$\begin{aligned} &\left(O(\alpha, k) - \frac{8\beta}{3h}\right) u_0^{k+1} + \frac{3}{h} u_{1/2}^{k+1} - \frac{1}{3h} u_{3/2}^{k+1} \\ &\qquad = O(\alpha, k)\, u_0^k - O(\alpha, k) \sum_{j=1}^{k} \omega(\alpha, j) \left(u_0^{k-j+1} - u_0^{k-j}\right) + \overline{g}_0^{k+1} + q^{k+1} \end{aligned} \tag{4.1}$$

Further, for the inner node solution, $\forall\ k \geq 1$ and $i = 1, 2, \ldots, N-1$, we have from (3.6)

$$\begin{aligned} &-\frac{a}{h^2}\, u_{i-1/2}^{k+1} + \left(O(\alpha, k) + \frac{2a}{h^2}\right) u_{i+1/2}^{k+1} - \frac{a}{h^2}\, u_{i+3/2}^{k+1} \\ &\qquad = O(\alpha, k)\, u_{i+1/2}^k - O(\alpha, k) \sum_{j=1}^{k} \omega(\alpha, j) \left(u_{i+1/2}^{k-j+1} - u_{i+1/2}^{k-j}\right) + \overline{g}_{i+1/2}^{k+1} \end{aligned} \tag{4.2}$$

Now, we look for the right boundary, so (3.7) in the expanded form $\forall\ k \geq 1$ and $i = N$ is given by

$$\begin{aligned} &(O(\alpha, k)) u_N^{k+1} + \left(h_k + \frac{8\lambda}{3h}\right) u_N^{k+1} + \left(\frac{\lambda}{3h}\right) u_{N-3/2}^{k+1} - \left(\frac{3\lambda}{h}\right) u_{N-1/2}^{k+1} \\ &\qquad = O(\alpha, k)\, u_N^k - O(\alpha, k) \sum_{j=1}^{k} \omega(\alpha, j) \left(u_N^{k-j+1} - u_N^{k-j}\right) + h_k u^\infty + \overline{g}_N^{k+1}. \end{aligned} \tag{4.3}$$

Theorem 4.1. *The fully discrete scheme defined by (4.1), (4.2), and (4.3) for the solution of (1.1) with $0 < \alpha < 1$ on finite domain is unconditionally stable.*

Proof. Let the solution of above fully discrete equation is given by $u_j^k = \xi_k e^{(iwjh)}$, where $i = \sqrt{-1}$. First we consider the stability of left boundary nodes for which the solution in given by the equation(4.1).
Let $u_j^k = \xi_k e^{(iwjh)}$ be the form of solution then substituting it in (4.1) we obtain,

$$\begin{aligned} &\left(O(\alpha, k) - \frac{8\beta}{3h}\right) \xi_{k+1} + \frac{3}{h} \xi_{k+1}\, e^{(iwh/2)} - \frac{1}{3h} \xi_{k+1}\, e^{(3iwh/2)} \\ &\qquad = O(\alpha, k)\, \xi_k - O(\alpha, k) \sum_{j=1}^{k} \omega(\alpha, j) \left(\xi_{k-j+1} - \xi_{k-j}\right) + \overline{g}_0^{k+1} \end{aligned}$$

After simplifying, we get

$$\left(O\left(\alpha,k\right)-\frac{8\beta}{3h}\right)\xi_{k+1}+\frac{1}{3h}\xi_{k+1}\ \{9e^{(iwh/2)}-e^{(3iwh/2)}\}$$
$$=O\left(\alpha,k\right)\xi_k-O\left(\alpha,k\right)\sum_{j=1}^{k}\omega\left(\alpha,j\right)\left(\xi_{k-j+1}-\xi_{k-j}\right)+\overline{g}_0^{k+1}$$

$$\left(O\left(\alpha,k\right)-\frac{8\beta}{3h}+\frac{1}{3h}\Big\{8e^{(iwh/2)}-2ie^{(iwh)}\sin(wh/2)\Big\}\right)\xi_{k+1}$$
$$=O\left(\alpha,k\right)\xi_k-O\left(\alpha,k\right)\sum_{j=1}^{k}\omega\left(\alpha,j\right)\left(\xi_{k-j+1}-\xi_{k-j}\right)+\overline{g}_0^{k+1}$$

$$\xi_{k+1}\ =\frac{\xi_k+\sum_{j=1}^{k}\omega\left(\alpha,j\right)\left(\xi_{k-j}-\xi_{k-j+1}\right)+\overline{g}_0^{k+1}}{\left(1+\frac{8e^{(iwh/2)}-2ie^{(iwh)}\sin(wh/2)-8\beta}{3h\ O(\alpha,k)}\right)}$$

Here we can see that denominator is always greater than 1 as i.e.

$$\frac{8e^{(iwh/2)}-2ie^{(iwh)}\sin(wh/2)-8\beta}{3h\ O\left(\alpha,k\right)}=\frac{8e^{(iwh/2)}-2ie^{(iwh)}+8\lambda}{3h\ O\left(\alpha,k\right)}\geq 0$$

which is always true. Hence, we observe from the above equation that

$$\xi_1\leq\xi_0 \tag{4.4}$$

and

$$\xi_{k+1}\leq\xi_k+\sum_{j=1}^{k}\omega\left(\alpha,j\right)\left(\xi_{k-j}-\xi_{k-j+1}\right)+\overline{g}_0^{k+1},\ \forall k\geq 2. \tag{4.5}$$

Therefore for $k=1$, the above inequality implies

$$\xi_2\leq\xi_1+\omega(\alpha,2)(\xi_0-\xi_1)+\overline{g}_0^2$$

and using the result $\xi_1\leq\xi_0$ and the positiveness of the coefficients $w(\alpha,j)$ (2.13), it follows that $\xi_2\leq\xi_1$. Repeating the process, we have

$$\xi_{k+1}\leq\xi_k\leq\xi_{k-1}\leq...\leq\xi_1\leq\xi_0.$$

Thus for the left boundary nodes, we can see that the stability estimates holds.
Now we look for stability estimates at the inner points of the domain, in (4.2) after simplifying we get,

$$-\frac{a}{h^2}\ e^{-iwh/2}\xi_{k+1}+\left(O\left(\alpha,k\right)+\frac{2a}{h^2}\right)e^{iwh/2}\xi_{k+1}-\frac{a}{h^2}\ e^{3iwh/2}\xi_{k+1}$$
$$=O\left(\alpha,k\right)e^{iwh/2}\xi_k-O\left(\alpha,k\right)\sum_{j=1}^{k}\omega\left(\alpha,j\right)e^{iwh/2}\left(\xi_{k-j+1}-\xi_{k-j}\right)+\overline{g}_{i+1/2}^{k+1}$$

$$\xi_{k+1}\left(\frac{2ai}{h^2}\sin(wh/2)+O\left(\alpha,k\right)e^{iwh/2}-\frac{2ai}{h^2}\sin(wh/2)\ e^{iwh}\right)$$
$$=\left(O\left(\alpha,k\right)\xi_k-O\left(\alpha,k\right)\sum_{j=1}^{k}\omega\left(\alpha,j\right)\left(\xi_{k-j+1}-\xi_{k-j}\right)\right)e^{iwh/2}+\overline{g}_{i+1/2}^{k+1}$$

$$\xi_{k+1}=\frac{\left(\xi_k+\sum_{j=1}^{k}\omega\left(\alpha,j\right)\left(\xi_{k-j+1}-\xi_{k-j}\right)\right)e^{iwh/2}+\overline{g}_{i+1/2}^{k+1}/O\left(\alpha,k\right)}{\left(e^{iwh/2}+\frac{2ai}{h^2\ O(\alpha,k)}\sin(wh/2)\ (1-e^{iwh})\right)}$$

Since the denominator is always greater than 1, and following the same procedure as we did for the left most boundary, it can be concluded that $\xi_{k+1} \leq \xi_k$ and the solution is stable at the inner nodes.
Furthermore we look for the stability of solution for the right most boundary. Assuming the form of solution, the equation (4.3) becomes

$$(O(\alpha,k))\xi_{k+1}e^{iwNh} + \left(h_k + \frac{8\lambda}{3h}\right)\xi_{k+1}e^{iwNh} + \left(\frac{\lambda}{3h}\right)\xi_{k+1}e^{iw(N-3/2)h} - \left(\frac{3\lambda}{h}\right)\xi_{k+1}e^{iw(N-1/2)h}$$
$$= O(\alpha,k)\,\xi_k e^{iwNh} - O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(\xi_{k-j+1} - \xi_{k-j}\right)e^{iwNh} + h_k u^{\infty} + \overline{g}_N^{k+1}.$$

$$\Big((O(\alpha,k)) + h_k + \frac{\lambda}{3h}\left(16\sin^2(wh/4) + 2i\sin(wh/2)(4 - e^{-iwh})\right)\Big)\xi_{k+1}$$
$$= O(\alpha,k)\,\xi_k - O(\alpha,k)\sum_{j=1}^{k}\omega(\alpha,j)\left(\xi_{k-j+1} - \xi_{k-j}\right) + h_k u^{\infty}$$

Taking the upper bound for left hand coefficient of ξ_{k+1} and simplifying, we get

$$\xi_{k+1} \leq \frac{\xi_k + \sum_{j=1}^{k}\omega(\alpha,j)\left(\xi_{k-j} - \xi_{k-j+1}\right) + h_k u^{\infty}/O(\alpha,k)}{\left(1 + \frac{3h^2+26\lambda}{3h\ O(\alpha,k)}\right)}$$

We can see that $\left(1 + \frac{3h^2+26\lambda}{3h\ O(\alpha,k)}\right) \geq 1$ for all α, k, ω, and h_k, therefore it follows that the solution is stable for the right most boundary also. Hence from the above estimates, we can see that the solution is stable in all three cases means $\xi_{k+1} \leq \xi_k$. Thus we can conclude that the solution is stable throughout the domain.

□

5. Numerical Results

In this section, numerical tests are considered to show the numerical performance of the proposed formulation. From (1.1) and (3.1), the fractional partial differential equation in time with $\alpha = 0.5$ is given as

$$\frac{\partial^{0.5}u(x,t)}{\partial t^{0.5}} = \frac{\partial^2 u(x,t)}{\partial x^2} + e^x t^{0.5}\left(-t + \frac{\sqrt{t}\,\Gamma(2+\alpha)}{\Gamma(\frac{3}{2}+\alpha)}\right), \tag{5.1}$$

with domain

$$D = \{(x,t) : x,t \in [0,1], \tag{5.2}$$

the initial condition

$$u(x,0) = 0, \tag{5.3}$$

and Neumann and Robin type boundary conditions

$$\left.\begin{aligned} -\frac{\partial u}{\partial x}(0,t) &= -t^{1+0.5} \\ u(1,t) + \frac{\partial u}{\partial x}(1,t) &= 0 \end{aligned}\right\} \tag{5.4}$$

The exact solution is represented by the function: $u(x,t) = e^x t^{1+0.5}$.
The mimetic scheme used for the spatial discretization is expressed as

$$\begin{array}{rcl}\dfrac{\partial^2 u(x,t)}{\partial x^2} & \approx & D.(KG)u(x,t) \;\;.\;\; g(x,t) = g_i(x,t) \\ h(t) & \approx & A \\ \dfrac{\partial u}{\partial x} & \approx & B.G \\ q(t) & \approx & q_i(t).\end{array}$$

This can be written mimetically as

$$MI = \left[\widehat{A} + BG + \widehat{D}.(G)\right] u(x,t),$$

where $\widehat{A}$ is a diagonal matrix of order $(N+2) \times (N+2)$ that has non-null entries. The elements corresponding to the boundary conditions are

$$\widehat{A}(1,1) = 0 \qquad \widehat{A}(N+2, N+2) = -1.$$

The purpose of this example is to study the influence of the grid size Δx and time-step length Δt on the numerical solution obtained with Mimetic finite difference scheme. We evaluate the error between the exact and approximate solution with the maximum norm. The following Table shows the Error in the maximum norm for the implicit finite difference (I.F.D) and the mimetic finite difference method (M.F.D) at time $t = 0.75$ and $t = 1$. It can be seen from the Table (1) that the error decreases when the time step length Δt is reduced. Though the approximation error is optimal when the grid size reduces, there is a significant difference in error distribution between Mimetic and Implicit discretizations. These errors are higher in the case of Implicit finite difference method, see Table 1 compare to Mimetic finite difference scheme. This behavior is expected as Mimetic finite difference method is of second order in time, while the Implicit scheme is of first order only, which supports the proposed numerical scheme for fractional order PDEs.

Table 1. Variation of the approximation error for Implicit finite difference and Mimetic finite difference scheme with different time refinement level Δt.

Grid($\Delta x \times \Delta t$)	$E\,(\Delta x, \Delta t)$			
	$t = 0.75$		$t = 1$	
	IFD	MFD	IFD	MFD
$\frac{1}{100} \times \frac{1}{100}$	0.00414288	0.00190269	0.01080850	0.00423427
$\frac{1}{100} \times \frac{1}{200}$	0.00146142	$9.34982209e^{-04}$	0.00384188	0.00224898
$\frac{1}{100} \times \frac{1}{300}$	$7.73882266e^{-04}$	$5.76161563e^{-04}$	0.00205081	0.00141923

It can be seen from the Table (2) that the error slowly decreases, when the grid size (Δx) is reduced using the Mimetic finite difference scheme (M.F.D). However, the same does not happen with respect to the Implicit finite difference scheme (I.F.D). Here, the error increases due to round off errors, when we reduce the grid size (Δx). This shows that even by varying the grid size (Δx), the (M.F.D) scheme shows some advantage over the (I.F.D) method. This verifies the efficiency of the proposed Mimetic finite difference method over the Implicit finite difference scheme.

Next, the distribution of error obtained with the Mimetic finite difference scheme for different time step length Δt and different time instances are depicted in the following figures. As expected, the distribution of error reduces when the time step is reduced, see Figure 4(a) at time $t = 0.75$. In addition, we can observe that distribution of error reduced by half with each successive refinement. Similar is the case of distribution error at time $t = 1$, see Figure 5(a). This behavior supports the Mimetic finite difference over Implicit finite difference method.

Furthermore, the next Figures presents the Approximate and exact solution with different time step length Δt. This shows that the approximation error between the exact solution and the discrete one improves with the

TABLE 2. Variation of the approximation error for Implicit finite difference and Mimetic finite difference scheme with different space grid refinement (Δx) level

Grid$(\Delta x \times \Delta t)$	$E(\Delta x, \Delta t)$			
	$t = 0.75$		$t = 1$	
	IFD	MFD	IFD	MFD
$\frac{1}{100} \times \frac{1}{100}$	0.0041428757	0.0019026870	0.010808523	0.0042342705
$\frac{1}{200} \times \frac{1}{100}$	0.004195998	0.0019008641	0.0109263755	0.0042286135
$\frac{1}{300} \times \frac{1}{100}$	0.004205837	0.0019006521	0.0109482022	0.0042278331

further refinement of the time step length, see Figure 4(b) at time $t = 0.75$, and Figure 5(b) at time $t = 1$. This comparative study is performed in order to see the effects of Mimetic finite difference method.

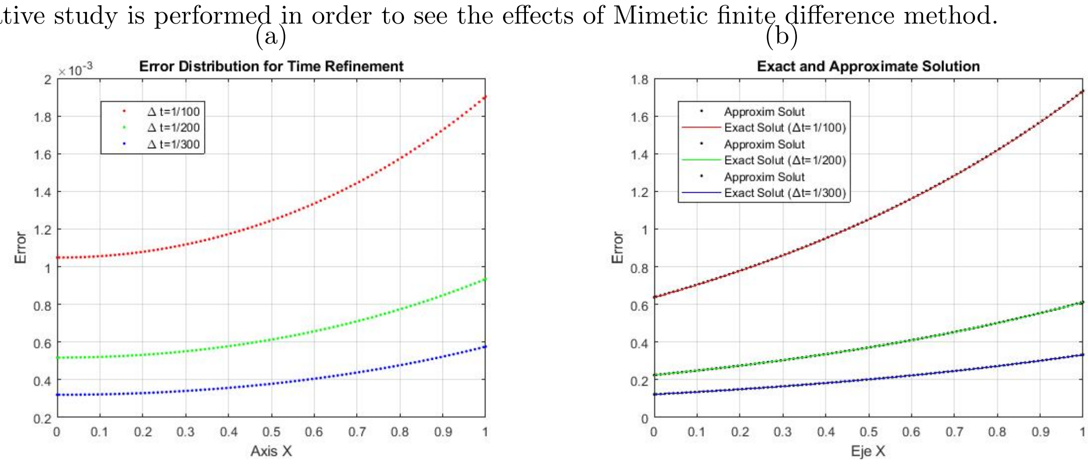

FIGURE 4. Approximation error obtained with Mimetic finite difference method with grid size $\frac{1}{100} \times \frac{1}{100}$, $\frac{1}{100} \times \frac{1}{200}$, $\frac{1}{100} \times \frac{1}{300}$ at time $t = 0.75$ (a) Distribution of error, (b) comparison of exact and approximate solution.

For the time fractional partial differential equation given in (5.1) together with the conditions (5.2),(5.3), (5.4), and the grid size $n = m = 100$, the maximum error using (I.F.D) is equal to $E(\Delta x; \Delta t) = 0.0108085$, while the maximum error using (M.F.D) is equal to $E(\Delta x; \Delta t) = 0.00423427$. Moreover, the distribution of the total errors of the approximate solution using Mimetic finite difference (M.F.D) and Implicit finite difference (I.F.D) are presented in Figure 6 (a) and (b) respectively. Further the approximate and exact solutions are depicted in Figure 7.

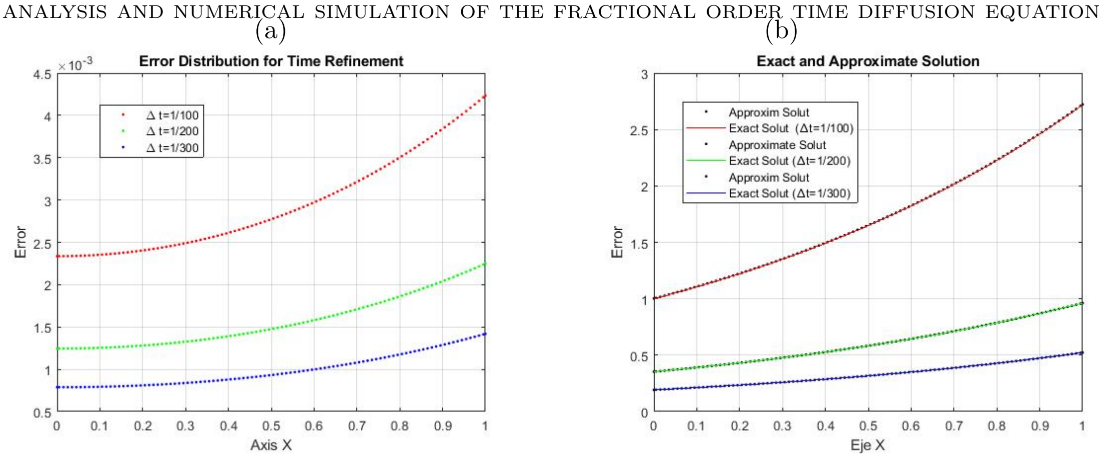

FIGURE 5. Approximation error obtained with Mimetic finite difference method with grid size $\frac{1}{100} \times \frac{1}{100}$, $\frac{1}{100} \times \frac{1}{200}$, $\frac{1}{100} \times \frac{1}{300}$ at time $t = 1$ (a) Distribution of error, (b) comparison of exact and approximate solution.

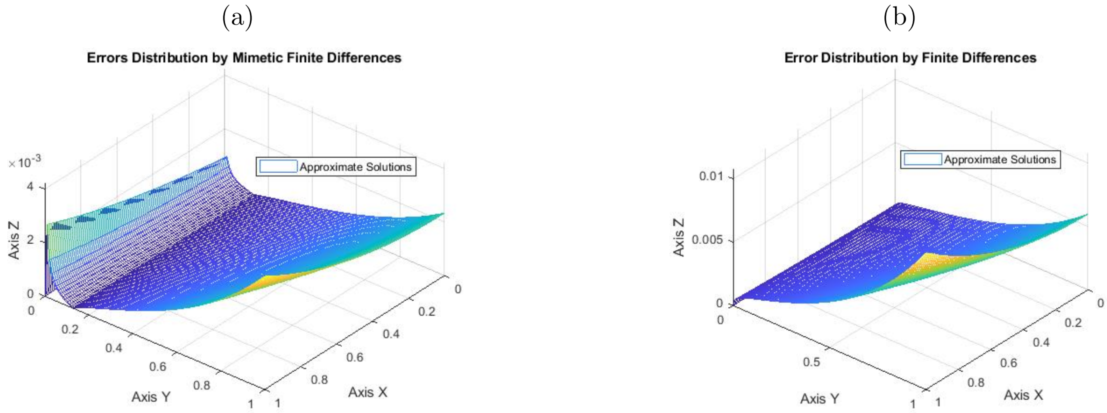

FIGURE 6. Distribution of errors of the approximate solution for the grid size $n = m = 100$.

6. Conclusions

In this paper, we present the mimetic finite difference scheme to solve the fractional order diffusion equation with mixed boundary conditions, characterized mainly by not using ghost points at the boundary of the domain. The result of this method presents an approximation order (Δx^2) for the spatial variable, while the implicit finite difference method for the time variable shows an approximation order $(\Delta t^{2-\alpha})$.
It is concluded that the approximation error between the exact and approximate solution using the mimetic finite difference method is much better than the error using the approximation method of implicit finite difference, given in [3].

7. 2-D Fractional Time Diffusion Equation Mimetic Code

```
Tmax = 1.;  % intervalo  de tiempo
format long
x0 = 0.;
xn = 1.;
```

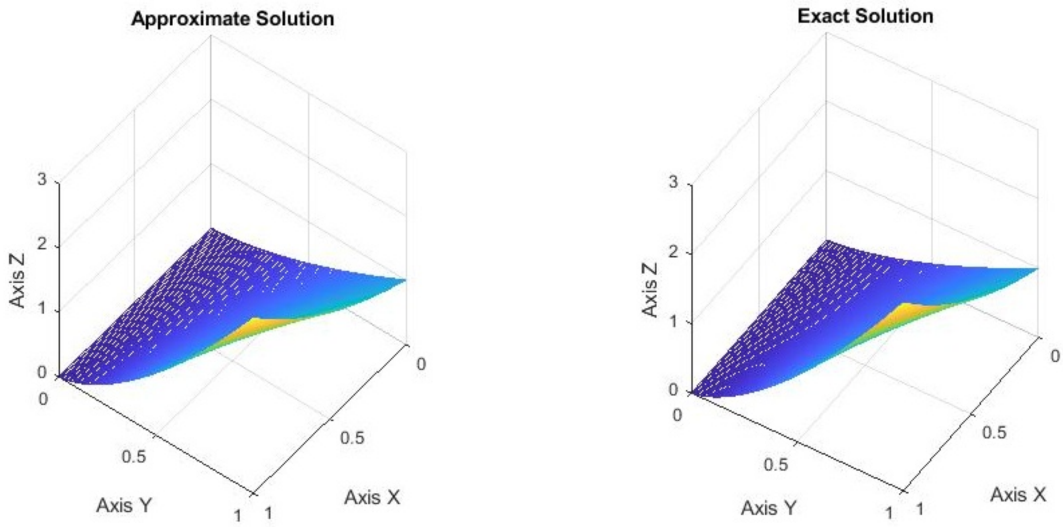

FIGURE 7. Approximate and exact solutions for a mesh $n = m = 100$.

```


n = 100;
m = 100;
k = Tmax/(m);
h = (xn-x0)/(n);

x = zeros(n+2,1);
t = zeros(m+1,1);
w = zeros(n+2,m+1);

x(1) = x0;
x(n+2) = xn;

for i = 2:n+1
   x(i) = h*((i-1)-1/2);
end

for j=1:m+1;
     t(j) = k*((j-1));
end



alpha = 0.5;

alfa0 = 0;
mu0 = 1;

alfa1 = -1;
mu1 = 1;

Mu = MatrixMu(n,mu0,mu1);
B = normalB(n);
G = gradiente(n)/h;
D = divergencia(n)/h;

%valores en la frontera


```

```
for i=1:n+2
   w(i,1)= 0;
end

A = matrixA(n,alfa0,alfa1);

omega = zeros(m+1,1);
sigma = 1/(gamma(2- alpha)*(k^alpha));
for j = 1:m+1
    omega(j) = (j)^(1-alpha) - (j-1)^(1-alpha);
end

Id = eye(n+2);
Id(1,1) = 0;
Id(n+2,n+2)= 0;

Fuente = zeros(n+2,m+1);
 for j = 1:m+1
   for i = 2:n+1
        Fuente(i,j) = exp(x(i))*(t(j)^alpha)*(-t(j) + (t(j)^(1/2))*(gamma(2+alpha)/ gamma(3/2+alpha)));
     end
end

for j = 1:m+1
   Fuente(1,j) = -t(j)^(1+alpha);
   Fuente(n+2,j)= 0;
end

MI = (A + Mu*B*G) - D*G;

for j=2:m
   suma = zeros(n+2,m+1);
     if j >= 2
        for jj=2:j
           suma(:,j) = suma(:,j) + omega(jj,1)*Id*(w(:,j-jj+2)-w(:,j-jj+1));
        end
    end
    w(:,j+1)= (sigma*Id + MI)\ (Fuente(:,j+1) + sigma*(Id*w(:,j)-suma(:,j)));
 end


u_real=zeros(n+2,m+1);

for j= 1:m+1
   for i = 1:n+2
        u_real(i,j) = exp(x(i))*t(j)^(1+alpha);
   end
end

 e = zeros(m+1,1);

 for j=1:m+1
    e(j) = norm(w(:,j)'-u_real(:,j)',inf);
 end


norm(e,'inf')
subplot(1,2,1)
[X T]=meshgrid(x,t);
mesh(X,T,w');
```

```
title('Approximate Solution')
xlabel('Axis X')
ylabel('Axis Y')
zlabel('Axis Z')
subplot(1,2,2)
[X T]=meshgrid(x,t);
mesh(X,T,u_real');
title('Exact Solution')
xlabel('Axis X')
ylabel('Axis Y')
zlabel('Axis Z')
% spy(w)
```

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