Propagation of radius of analyticity for solutions to a fourth order nonlinear Schr\"odinger equation

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Abstract

We prove that the uniform radius of spatial analyticity $\sigma(t)$ of solution at time t to the one-dimensional fourth order nonlinear Schr\"odinger equation $$ i\partial_tu-\partial_x^4u =|u |^2u $$ cannot decay faster than $1/ \sqrt{t}$ for large t, given initial data that is analytic with fixed radius σ_0. The main ingredients in the proof are a modified Gevrey space, a method of approximate conservation law and a Strichartz estimate for free wave associated with the equation.

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PROPAGATION OF RADIUS OF ANALYTICITY FOR SOLUTIONS TO A FOURTH ORDER NONLINEAR SCHRÖDINGER EQUATION

TEGEGNE GETACHEW, BIRILEW BELAYNEH, AND ACHENEF TESFAHUN

ABSTRACT. We prove that the uniform radius of spatial analyticity $\sigma(t)$ of solution at time t to the one-dimensional fourth order nonlinear Schrödinger equation

$$i\partial_t u - \partial_x^4 u = |u|^2 u$$

cannot decay faster than $1/\sqrt{t}$ for large t, given initial data that is analytic with fixed radius σ_0. The main ingredients in the proof are a modified Gevrey space, a method of approximate conservation law and a Strichartz estimate for free wave associated with the equation.

1. INTRODUCTION

We consider the Cauchy problem for one-dimensional fourth order cubic nonlinear Schrödinger equation,

$$\begin{cases} i\partial_t u - \partial_x^4 u = |u|^2 u & (x,t) \in \mathbb{R}\times\mathbb{R}, \\ u(0) = f, \end{cases} \tag{4NLS}$$

where u is a complex-valued function. This equation was studied in the context of stability of solitons in magnetic materials (for more physical background, see [12, 13]), and has been extensively studied in recent years; see for instance [2, 16, 18–20, 23, 24]. The mass and energy,

$$M(t) = \frac{1}{2}\int_{\mathbb{R}} |u|^2\,dx,$$
$$E(t) = \frac{1}{2}\int_{\mathbb{R}} |\partial_x^2 u|^2 + \frac{1}{2}|u|^4\,dx$$

are conserved by the flow of (4NLS).

Low regularity well-posedness for (4NLS) was recently studied by Seong [28]. The author proved that (4NLS) is locally and globally well-posed for initial data $f \in H^s(\mathbb{R}), s \geqslant -1/2$. The author also showed that the Cauchy problem is midly ill-posed in the sense that the solution map fails to be locally uniformly continuous on $H^s(\mathbb{R}), s < -\frac{1}{2}$. Well-posedness and long-time behavior of solution for the higher dimensional version of (4NLS) was also studied in [23].

The main concern of this paper is to study the property of spatial analyticity of the solution $u(x,t)$ to (4NLS), given a real analytic initial data $f(x)$ with uniform

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radius of analyticity σ_0, so that there is a holomorphic extension to a complex strip

$$S_{\sigma_0} = \{x + iy \in \mathbb{C} : |y| < \sigma_0\}.$$

Information about the domain of analyticity of a solution to a PDE can be used to gain a quantitative understanding of the structure of the equation, and to obtain insight into underlying physical processes. It is classical since the work of Kato and Masuda [14] that, for solutions of nonlinear dispersive PDEs with analytic initial data, the radius of analyticity, $\sigma(t)$, of the solution might decrease with t. Bourgain [3] used a simple argument in the context of Kadomtsev Petviashvili equation to show that $\sigma(t)$ decays exponentially in t.

Rapid progress has been made lately in obtaining an algebraic decay rate of the radius, i.e., $\sigma(t) \sim t^{-\theta}$ for some $\theta \geqslant 1$, to various nonlinear dispersive PDEs, see eg., [1,10,25–27,29–31]. The method used in these papers was first introduced by Selberg and Tesfahun [27] in the context of the Dirac-Klein-Gordon equations, which is based on an approximate conservation laws and Bourgan's Fourier restriction method. For earlier studies concerning properties of spatial analyticity of solutions for a large class of nonlinear partial differential equations, see eg., [3,6–9,11,14,17,21,22].

The radius of analyticity of a function can be related to decay properties of its Fourier transform. It is therefore natural to take initial data in Gevrey space $G^{\sigma,s}$ defined by the norm

$$\|f\|_{G^{\sigma,s}(\mathbb{R})} = \left\| \exp(\sigma|\xi|)\langle\xi\rangle^s \widehat{f} \right\|_{L^2_\xi(\mathbb{R})} \quad (\sigma \geqslant 0),$$

where $\langle\xi\rangle = \sqrt{1+\xi^2}$. For $\sigma = 0$, this space coincides with the Sobolev space $H^s(\mathbb{R})$, with norm

$$\|f\|_{H^s(\mathbb{R})} = \|\langle\xi\rangle^s \hat{f}\|_{L^2_\xi(\mathbb{R})},$$

while for $\sigma > 0$, any function in $G^{\sigma,s}(\mathbb{R})$ has a radius of analyticity of at least σ at each point $x \in \mathbb{R}$. This fact is contained in the the Paley–Wiener Theorem, whose proof can be found in [15] in the case $s = 0$; the general case follows from a simple modification.

Paley-Wiener Theorem. *Let $\sigma > 0$ and $s \in \mathbb{R}$, then the following are equivalent*

(a) $f \in G^{\sigma,s}(\mathbb{R})$,

(b) f *is the restriction to* $\mathbb{R}$ *of a function* F *which is holomorphic in the strip*

$$S_\sigma = \{x + iy \in \mathbb{C} : \ |y| < \sigma\}.$$

Moreover, the function F *satisfies the estimates*

$$\sup_{|y|<\sigma} \|F(\cdot + iy)\|_{H^s(\mathbb{R})} < \infty.$$

By using the Grevey space, the energy method and the method of approximate conservation law it is not difficult to derive a linear decay rate, $\sigma(t) \sim 1/t$, for the radius of analyticity solution to (4NLS). In the present paper, we derive the decay rate $\sigma(t) \sim 1/\sqrt{t}$, by using a modified Gevrey space that was introduced recently in [4,5], a Strichartz estimate and the method of approximate conservation law. The modified Gevrey space, denoted $H^{\sigma,s}(\mathbb{R})$, is obtained from the Gevrey space

$G^{\sigma,s}(\mathbb{R})$ by replacing the exponential weight $\exp(\sigma|\xi|)$ with the hyperbolic weight $\cosh(\sigma|\xi|)$, i.e.,

$$\|f\|_{H^{\sigma,s}(\mathbb{R})} = \left\|\cosh(\sigma|\xi|)\langle\xi\rangle^{s}\widehat{f}\right\|_{L^2_\xi(\mathbb{R})} \qquad (\sigma \geqslant 0).$$

Since $\frac{1}{2}\exp(\sigma|\xi|) \leqslant \cosh(\sigma|\xi|) \leqslant \exp(\sigma|\xi|)$, this norm is equivalent with the $G^{\sigma,s}(\mathbb{R})$–norm, i.e.,

$$\|f\|_{H^{\sigma,s}(\mathbb{R})} \sim \|f\|_{G^{\sigma,s}(\mathbb{R})}. \tag{1}$$

Therefore, the statement of Paley-Wiener Theorem still holds for functions in $H^{\sigma,s}(\mathbb{R})$.

The simple estimate

$$\frac{1-\exp(-\sigma|\xi|)}{|\xi|} \leqslant \sigma$$

can be used in the $G^{\sigma,s}$-set up to derive a linear decay rate, $\sigma(t) \sim 1/t$. In comparison, the decay rate $\sigma(t) \sim 1/\sqrt{t}$ obtained in the $H^{\sigma,s}$-set up of this paper stems from the σ^2-factor of the following estimate:

$$\frac{1-[\cosh(\sigma|\xi|)]^{-1}}{|\xi|^2} \leqslant \sigma^2.$$

We state our main result as follows.

Theorem 1 (Asymptotic lower bound for σ)**.** *Let $f \in H^{\sigma_0,2}(\mathbb{R})$ for some $\sigma_0 > 0$. Then the* [1]*global solution u of* (4NLS) *satisfies*

$$u(t) \in H^{\sigma,2}(\mathbb{R}) \quad \textit{for all} \quad t > 0,$$

with the radius of analyticity σ satisfying the asymptotic lower bound

$$\sigma := \sigma(t) \geqslant c/\sqrt{t} \quad \textit{as} \quad t \to \infty,$$

where $c > 0$ is constant depending on the initial data norm $\|f\|_{H^{\sigma_0,2}(\mathbb{R})}$.

So it follows from Theorem 1 that the solution $u(x,t)$ at any time t is analytic in the strip $S_{\sigma(t)}$ (due to (1) and the Paley-Wiener Theorem) with radius decaying at the rate $\sigma(t) \sim 1/\sqrt{t}$.

Remark 1. It is possible to show that the statement of Theorem 1 holds true for the general fourth order semilinear nonlinear Schrödinger equation (pNLS),

$$i\partial_t u - \partial_x^4 u = |u|^p u,$$

for any even integer $p \geqslant 2$. However, we do not pursue this issue here.

Notation. For any positive numbers a and b, the notation $a \lesssim b$ stands for $a \leqslant cb$, where c is a positive constant that may change from line to line. Moreover, we denote $a \sim b$ when $a \lesssim b$ and $b \lesssim a$.

The rest of the sections are organized as follows. In Section 2 we introduce function spaces, recall some linear estimates and prove local well-posedness of

[1]As a consequence of the embedding $H^{\sigma_0,2}(\mathbb{R}) \hookrightarrow H^2(\mathbb{R})$ and the existing well-posedness theory in $H^2(\mathbb{R})$ (see [28]), the Cauchy problem (4NLS) has a unique, smooth solution for all time, given initial data $f \in H^{\sigma_0,2}$.

(4NLS) for initial data $f \in H^{\sigma,2}$. In Section 3, an approximate conservation law for a modified mass + energy functional associated with $u_\sigma = \cosh(\sigma|D|)u$ is derived. Theorem 1 is proved in Section 4 by combining the results from Sections 2 and 3. Finally, Section 5 is dedicated for proving a key nonlinear estimate that is crucial in the proof of the approximate conservation law.

2. Local well-posedness theory

2.1. **Function spaces and linear estimates.** The Bourgain space, $X^{s,b}$, associated with (4NLS) is defined to be the closure of the Schwartz space $\mathcal{S}(\mathbb{R}\times\mathbb{R})$ under the norm

$$\|u\|_{X^{s,b}} = \|\langle\xi\rangle^s \langle\tau+\xi^4\rangle^b \widetilde{u}(\tau,\xi)\|_{L^2_{\tau,\xi}(\mathbb{R}\times\mathbb{R})},$$

where $\widetilde{u}$ denotes the space-time Fourier transform given by

$$\widetilde{u}(\xi,\tau) = \mathcal{F}_{x,t}[u](\xi,\tau) = \int_{\mathbb{R}^{1+1}} e^{-i(t\tau+x\xi)} u(x,t)\,dxdt.$$

The spatial Fourier transform $\widehat{f}$ is defined by

$$\widehat{f}(\xi) = \mathcal{F}_x[f](\xi) = \int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx.$$

The restriction to time slab $(0,T)\times\mathbb{R}$ of the Bourgain space, denoted $X^{s,b}_T$, is a Banach space when equipped with the norm

$$\|u\|_{X^{s,b}_T} = \inf\{\|v\|_{X^{s,b}} : v = u \text{ on } (0,T)\times\mathbb{R}\}.$$

We have the embedding $X_T^{s,\frac12+} \subset C([0,T],H^s)$. In particular,

$$\|u\|_{L^\infty_T H^s} \lesssim \|u\|_{X_T^{s,\frac12+}}, \tag{2}$$

where we used the notation $L^q_T X := L^q_t X([0,T]\times\mathbb{R})$ and $a+ := a+\varepsilon$ for sufficiently small $\varepsilon > 0$.

Now consider the Cauchy problem for the linear fourth order Schrödinger equation

$$\begin{cases} i\partial_t u - \partial_x^4 u = F, \\ u(0) = f \end{cases}$$

whose solution is given by Duhamel's formula

$$u(t) = e^{-it\partial_x^4} f - i\int_0^t e^{-i(t-s)\partial_x^4} F(s)\,ds. \tag{3}$$

It is standard that the following energy inequality holds:

$$\|u\|_{X_T^{s,\frac12+}} \lesssim \|f\|_{H^s} + \|F\|_{X_T^{s,-\frac12+}} \qquad (T>0). \tag{4}$$

Local well-posedness of (4NLS) can easily be proved using energy inequality, Sobolev embedding and a standard contraction argument in the [2]Gevrey-Bourgain type space, $X^{\sigma,s,b}$, whose norm is defined by

$$\|u\|_{X^{\sigma,s,b}} = \left\| e^{\sigma|D|}u \right\|_{X^{s,b}}.$$

In the case $\sigma = 0$, this space coincides with the Bourgain space $X^{s,b}$. The restrictions of $X^{\sigma,s,b}$ to a time slab $(0,T)\times\mathbb{R}$, denoted $X_T^{\sigma,s,b}$, is defined in a similar way as above.

Now applying $\cosh(\sigma|D|)$ to (3) and using (4) we obtain

$$\|u\|_{X_T^{\sigma,s,\frac{1}{2}+}} \lesssim \|f\|_{H^{\sigma,s}} + \|F\|_{X_T^{\sigma,s,-\frac{1}{2}+}}. \tag{5}$$

2.2. **Local well-posedness of (4NLS) in $H^{2,\sigma}$.** Consider the integral formulation of (4NLS):

$$u(t) = e^{-it\partial_x^4}f - i\int_0^t e^{-i(t-s)\partial_x^4}(|u|^2u)(s)\,ds. \tag{6}$$

We claim that the following nonlinear estimate holds:

$$\left\| |u|^2u \right\|_{X_T^{\sigma,2,-\frac{1}{2}+}} \leqslant cT^{\frac{1}{2}} \|u\|^3_{X_T^{\sigma,2,\frac{1}{2}+}}. \tag{7}$$

Then (5) combined with a standard contraction argument in the space $X_T^{\sigma,2,\frac{1}{2}+}$ implies existence of a unique solution $u \in X_T^{\sigma,2,\frac{1}{2}+}$ for (4NLS) with existence time $T \sim \|f\|^{-4}_{H^{\sigma,2}}$. Moreover,

$$\|u\|_{X_T^{\sigma,2,\frac{1}{2}+}} \leqslant c\,\|f\|_{H^{\sigma,2}}. \tag{8}$$

It remains to prove the estimate (7). Since $u_\sigma = \cosh(\sigma|D|)u$, (7) reduces to

$$\left\| \langle D\rangle^2 \cosh(\sigma|D|)\left[|\operatorname{sech}(\sigma|D|)u_\sigma|^2 \operatorname{sech}(\sigma|D|)u_\sigma \right] \right\|_{L_T^2L_x^2} \leqslant cT^{\frac{1}{2}}\|u_\sigma\|^3_{X_T^{2,\frac{1}{2}+}}. \tag{9}$$

By Plancherel,

$$\begin{aligned}
\text{LHS (9)} &= \left\| \langle\xi\rangle^2\cosh(\sigma|\xi|)\mathcal{F}_x\left[|\operatorname{sech}(\sigma|D|)u_\sigma|^2 \operatorname{sech}(\sigma|D|)u_\sigma \right](\xi) \right\|_{L_T^2L_\xi^2} \\
&= \left\| \int_{\xi=\xi_1-\xi_2+\xi_3} \langle\xi\rangle^2\cosh(\sigma|\xi|)\prod_{j=1}^3 \operatorname{sech}(\sigma|\xi_j|)\widehat{u_\sigma}(\xi_1,t)\overline{\widehat{u_\sigma}(\xi_2,t)}\widehat{u_\sigma}(\xi_3,t)\;d\xi_1d\xi_2d\xi_3 \right\|_{L_T^2L_\xi^2} \\
&\leqslant 8\left\| \int_{\xi=\xi_1-\xi_2+\xi_3} \langle\xi\rangle^2|\widehat{u_\sigma}(\xi_1,t)||\overline{\widehat{u_\sigma}(\xi_2,t)}||\widehat{u_\sigma}(\xi_3,t)|\;d\xi_1d\xi_2d\xi_3 \right\|_{L_T^2L_\xi^2}
\end{aligned}$$

where to obtain the third line we used the rough estimate

$$\cosh(\sigma|\xi|)\prod_{j=1}^3 \operatorname{sech}(\sigma|\xi_j|) \leqslant 8,$$

[2]In fact, a contraction argument in the space $L_T^\infty H^{\sigma,2}$ combined with a simple Sobolev embedding can be used to prove local well-posedness of (4NLS) for initial data $f \in H^{\sigma,2}$. However, the space $X_T^{\sigma,2,\frac{1}{2}+}$, which captures the dispersive nature of the equation, is needed in the proof of the approximate conservation law (see Section 5).

which in turn follows from the triangle inequality $|\xi| \leqslant \sum_{j=1}^{3} |\xi_j|$.

By symmetry we may assume $|\xi_1| \leqslant |\xi_2| \leqslant |\xi_3|$; this implies $|\xi| \leqslant 3|\xi_3|$. Then, denoting $v_\sigma = \mathcal{F}^{-1}\left[|\widehat{u_\sigma}|\right]$, we have by Plancherel, Hölder, Sobolev embedding and (2),

$$\begin{aligned}
\text{LHS} &\leqslant c \left\| \int_{\xi=\xi_1-\xi_2+\xi_3} \widehat{v_\sigma}(\xi_1,t)\overline{\widehat{v_\sigma}(\xi_2,t)}\langle\xi_3\rangle^2\widehat{v_\sigma}(\xi_3,t)\; d\xi_1 d\xi_2 d\xi_3 \right\|_{L^2_T L^2_\xi} \\
&\leqslant c \left\| \mathcal{F}_x\left[v_\sigma \cdot \overline{v_\sigma} \cdot \langle D\rangle^2 v_\sigma \right](\xi) \right\|_{L^2_T L^2_\xi} \\
&= c \left\| v_\sigma \cdot \overline{v_\sigma} \cdot \langle D\rangle^2 v_\sigma \right\|_{L^2_T L^2_x} \\
&\leqslant cT^{\frac12} \|v_\sigma\|^2_{L^\infty_T L^\infty_x} \left\| \langle D\rangle^2 v_\sigma \right\|_{L^\infty_T L^2_x} \\
&\leqslant cT^{\frac12} \|v_\sigma\|^2_{L^\infty_T H^2} \|v_\sigma\|_{L^\infty_T H^2} \\
&\leqslant cT^{\frac12} \|u_\sigma\|^3_{X_T^{2,\frac12+}}
\end{aligned}$$

which proves (9).

3. Approximate conservation law

We derive an approximate mass + energy conservation for

$$u_\sigma := \cosh(\sigma|D|)u,$$

where $D = -i\partial_x$ and u is a solution to (4NLS) (hence $u = \operatorname{sech}(\sigma|D|)u_\sigma$). To do this, we define a modified mass + energy functional associated with u_σ by

$$\mathcal{E}_\sigma(t) := \frac{1}{2}\int_{\mathbb{R}} |u_\sigma|^2 + |\partial_x^2 u_\sigma|^2 + \frac{1}{2}|u_\sigma|^4 \, dx. \tag{10}$$

Since $u = u_0$, and hence $\mathcal{E}_0(t) = M(t) + E(t)$, we have $\mathcal{E}_0(t) = \mathcal{E}_0(0)$ for all t. However, this fails to hold when $\sigma > 0$. In what follows we will nevertheless prove the approximate conservation

$$\sup_{0\leqslant t\leqslant T} \mathcal{E}_\sigma(t) = \mathcal{E}_\sigma(0) + \sigma^2 \mathcal{O}\left([1+\mathcal{E}_\sigma(0)]^3 \right) \tag{11}$$

for T as in the local existence theory of the proceeding Section. Thus, in the limit as $\sigma \to 0$, we recover the conservation $\mathcal{E}_0(t) = \mathcal{E}_0(0)$ for $0 \leqslant t \leqslant T$.

The rest of the section is dedicated for the proof of (11). Applying the operator $\cosh(\sigma|D|)$ to (4NLS) we obtain

$$i\partial_t u_\sigma - \partial_x^4 u_\sigma = |u_\sigma|^2 u_\sigma + N(u_\sigma), \tag{12}$$

where

$$N(u_\sigma) = -|u_\sigma|^2 u_\sigma + \cosh(\sigma|D|)\left[|\operatorname{sech}(\sigma|D|)u_\sigma|^2 \operatorname{sech}(\sigma|D|)u_\sigma \right]. \tag{13}$$

Differentiating $\mathcal{E}_\sigma(t)$, and then using (12)–(13) and integration by parts, we obtain

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}_\sigma(t) &= \operatorname{Re}\int_{\mathbb{R}} \overline{u_\sigma}\partial_t u_\sigma + \partial_x^2\overline{u_\sigma}\partial_x^2\partial_t u_\sigma + |u_\sigma|^2\overline{u_\sigma}\partial_t u_\sigma dx \\
&= \operatorname{Re}\int_{\mathbb{R}} \partial_t u_\sigma \left(\overline{u_\sigma + \partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma}\right) dx \\
&= -\operatorname{Re} i\int_{\mathbb{R}} \left(\partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma + N(u_\sigma)\right)\left(\overline{u_\sigma + \partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma}\right) dx \\
&= \operatorname{Im}\int_{\mathbb{R}} |\partial_x^2 u_\sigma|^2 + |u_\sigma|^4 + |\partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma|^2 + N(u_\sigma)\left(\overline{u_\sigma + \partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma}\right) dx \\
&= \operatorname{Im}\int_{\mathbb{R}} N(u_\sigma)\left(\overline{u_\sigma + \partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma}\right) dx.
\end{aligned}$$

Consequently, integrating over the time interval $[0, s]$, where $s \leqslant T$, we get

$$\mathcal{E}_\sigma(s) = \mathcal{E}_\sigma(0) + \mathcal{R}_\sigma(s), \tag{14}$$

where

$$\mathcal{R}_\sigma(s) = \operatorname{Im}\int_0^s\int_{\mathbb{R}} N(u_\sigma)\left(\overline{u_\sigma + \partial_x^4 u_\sigma + |u_\sigma|^2 u_\sigma}\right) dxdt. \tag{15}$$

The quantity $\mathcal{R}_\sigma(s)$ satisfies the estimate (the proof is given in the last section)

$$\sup_{0\leqslant s\leqslant T} |\mathcal{R}_\sigma(s)| \leqslant c\sigma^2\left(1 + \|u_\sigma\|^2_{X_T^{2,\frac{1}{2}+}}\right)^3 \tag{16}$$

for all $u_\sigma \in X_T^{2,\frac{1}{2}+}$, where c depends on T.

By (8), we have

$$\|u_\sigma\|_{X_T^{2,\frac{1}{2}+}} = \|u\|_{X_T^{2,\sigma,\frac{1}{2}+}} \leqslant c\,\|f\|_{H^{\sigma,2}} = c\,\|u_\sigma(0)\|_{H^2}. \tag{17}$$

Now, since

$$\begin{aligned}
\mathcal{E}_\sigma(0) &= \frac{1}{2}\int_{\mathbb{R}}\left(|u_\sigma(x,0)|^2 + |\partial_x^2 u_\sigma(x,0)|^2 + \frac{1}{2}|u_\sigma(x,0)|^4\right) dx \\
&\gtrsim \|u_\sigma(\cdot,0)\|^2_{H^2}
\end{aligned}$$

it follows from (17) that

$$\|u_\sigma\|_{X_T^{2,\frac{1}{2}+}} \lesssim \sqrt{\mathcal{E}_\sigma(0)}. \tag{18}$$

Finally, using (18) in (16) we obtain the desired estimate (11).

4. Proof of Theorem 1

Suppose that $u(\cdot,0) = f \in H^{\sigma_0,2}(\mathbb{R})$ for some $\sigma_0 > 0$. This implies $u_{\sigma_0}(\cdot,0) = \cosh(\sigma_0|D)|)f \in H^2$, and hence

$$\mathcal{E}_{\sigma_0}(0) \lesssim \|u_{\sigma_0}(\cdot,0)\|^2_{H^2(\mathbb{R})} + \|u_{\sigma_0}(\cdot,0)\|^4_{H^2(\mathbb{R})} < \infty,$$

where we also used the Sobolev embedding $H^2(\mathbb{R}) \subset L^4_x(\mathbb{R})$.

Now following the argument in [27] (see also [25]) we can construct a solution on $[0, T_0]$ for arbitrarily large time T_0. This is achieved by applying the approximate conservation (11), so as to repeat the local result on successive short time intervals of size T to reach T_0, by adjusting the strip width parameter $\sigma \in (0, \sigma_0]$ of the solution according to the size of T_0.

To achieve this first note that by (11),

$$\begin{aligned} \sup_{0\leqslant t\leqslant \delta} \mathcal{E}_\sigma(t) &\leqslant \mathcal{E}_\sigma(0) + c\sigma^2 \left[1+\mathcal{E}_\sigma(0)\right]^3 \\ &\leqslant \mathcal{E}_{\sigma_0}(0) + c\sigma^2 \left[1+\mathcal{E}_\sigma(0)\right]^3, \end{aligned} \tag{19}$$

for some $\delta \in (0, T]$. Here to get the second line we used the fact that $\mathcal{E}_\sigma(0) \leqslant \mathcal{E}_{\sigma_0}(0)$ which holds for $\sigma \leqslant \sigma_0$ as $\cosh x$ is increasing for $x \geqslant 0$. Thus,

$$\sup_{0\leqslant t\leqslant \delta} \mathcal{E}_\sigma(t) \leqslant 2\mathcal{E}_{\sigma_0}(0) \tag{20}$$

provided that

$$c\sigma^2 \left[1+\mathcal{E}_{\sigma_0}(0)\right]^3 \leqslant \mathcal{E}_{\sigma_0}(0). \tag{21}$$

Next, we apply the local theory with initial time $t = \delta$ and time-step size T to extend the solution from $[0, \tau]$ to $[\tau, \tau + T]$. By (11) and (20) we obtain

$$\sup_{\delta\leqslant t\leqslant \delta+T} \mathcal{E}_\sigma(t) \leqslant \mathcal{E}_\sigma(\delta) + c\sigma^2 \left[1+2\mathcal{E}_\sigma(0)\right]^3. \tag{22}$$

Proceeding this way we can cover all time intervals $[0, T]$, $[T, 2T]$, $[2T, 3T]$, etc., and then apply induction (see e.g, [4]) to establish

$$\sup_{0\leqslant t\leqslant T_0} \mathcal{E}_\sigma(t) \leqslant 2\mathcal{E}_{\sigma_0}(0) \quad \text{for} \quad \sigma \geqslant c/\sqrt{T_0}, \tag{23}$$

where $c > 0$ depends on $\mathcal{E}_{\sigma_0}(0)$. This would in turn imply

$$\sup_{0\leqslant t\leqslant T_0} \|u(t)\|_{H^{\sigma,2}(\mathbb{R})} < \infty \quad \text{for} \quad \sigma \geqslant c/\sqrt{T_0}$$

which proves Theorem 1.

5. Proof of estimate (16)

We can write

$$\mathcal{R}_\sigma(s) = \underbrace{\operatorname{Im} \int_0^s \int_{\mathbb{R}} N(u_\sigma) \left(1+\partial_x^4\right) \overline{u_\sigma} dxdt}_{:=\mathcal{R}_\sigma^{(1)}(s)} + \underbrace{\operatorname{Im} \int_0^s \int_{\mathbb{R}} N(u_\sigma) |u_\sigma|^2 \overline{u_\sigma} dxdt}_{:=\mathcal{R}_\sigma^{(2)}(s)},$$

where

$$N(u_\sigma) = -|u_\sigma|^2 u_\sigma + \cosh(\sigma|D|) \left\{ |\operatorname{sech}(\sigma|D|) u_\sigma|^2 \operatorname{sech}(\sigma|D|) u_\sigma \right\}.$$

So (16) reduces to proving the nonlinear estimates

$$\sup_{0\leqslant s\leqslant T} |\mathcal{R}_\sigma^{(1)}(s)| \lesssim \sigma^2 \|u_\sigma\|^4_{X_T^{2,b}}, \tag{24}$$

$$\sup_{0\leqslant s\leqslant T} |\mathcal{R}_\sigma^{(2)}(s)| \lesssim \sigma^2 \|u_\sigma\|^6_{X_T^{2,b}}. \tag{25}$$

To prove (24) and (25) we need the following estimate from [4, Lemma 3]:

$$\xi = \sum_{j=1}^3 \xi_j \quad \Rightarrow \quad \left| 1 - \cosh|\xi| \prod_{j=1}^3 \operatorname{sech}|\xi_j| \right| \leqslant 8 \sum_{j\neq k=1}^3 |\xi_j||\xi_k|. \tag{26}$$

In addition, the proof of (24) shall make use of the following space-time estimate from [28, Lemma 3.2]:

$$\|\langle D\rangle^{\frac{1}{2}} u\|_{L_t^4 L_x^\infty(\mathbb{R}\times\mathbb{R})} \lesssim \|u\|_{X^{0,\frac{1}{2}+}} \tag{27}$$

This estimate is deduced from the Strichartz estimate for free wave,

$$\|\langle D\rangle^{\frac{1}{2}} e^{it\partial_x^4} f\|_{L_t^4 L_x^\infty(\mathbb{R}\times\mathbb{R})} \lesssim \|f\|_{L_x^2(\mathbb{R})}, \tag{28}$$

and the standard transference principle. To be more precise (28) is proved in [28] with $\langle D\rangle$ replaced by $|D|$. However, the proof can be easily modified to deduce that (28) still holds.

5.1. **Proof of (24).** By Plancherel and (26),

$$\begin{aligned}
\left|\mathcal{R}_\sigma^{(1)}(s)\right| &= \left|\int_0^s \int_{\mathbb{R}} \widehat{N(u_\sigma)}(\xi,t)\cdot\left(1+\xi^4\right)\overline{\widehat{u_\sigma}(\xi,t)}d\xi dt\right| \\
&= \left|\int_0^s \int_{\mathbb{R}^4} \left(1+\xi^4\right)\left(1-\cosh(\sigma|\xi|)\prod_{j=1}^3 \operatorname{sech}(\sigma|\xi_j|)\right)\widehat{u_\sigma}(\xi_1,t)\widehat{u_\sigma}(\xi_2,t)\overline{\widehat{u_\sigma}(\xi_3,t)\widehat{u_\sigma}(\xi,t)}d\mu(\xi)dt\right|, \\
&\leqslant 8\sigma^2\int_0^T\int_{\mathbb{R}^4}\langle\xi\rangle^4\left(\sum_{j\neq k=1}^3|\xi_j||\xi_k|\right)|\widehat{u_\sigma}(\xi_1,t)||\widehat{u_\sigma}(\xi_2,t)||\widehat{u_\sigma}(\xi_3,t)||\widehat{u_\sigma}(\xi,t)|d\mu(\xi)dt,
\end{aligned}$$

where $d\mu(\xi)$ is a measure $d\mu(\xi) = \delta(\xi-\xi_1-\xi_2+\xi_3)d\xi_1 d\xi_2 d\xi_3 d\xi$. This measure imposes the condition $\xi = \xi_1+\xi_2-\xi_3$

By symmetry of our argument, we may assume $|\xi_1| \leqslant |\xi_2| \leqslant |\xi_3|$, and hence $|\xi| \leqslant 3|\xi_3|$. Then, denoting $v_\sigma = \mathcal{F}_x^{-1}(|\widehat{u_\sigma}|)$, we have by Plancherel, Hölder, (2) and (27),

$$\begin{aligned}
\left|\mathcal{R}_\sigma^{(1)}(s)\right| &\leqslant c\sigma^2\int_0^T\int_{\mathbb{R}^4}\langle\xi\rangle^4|\xi_2||\xi_3|\widehat{v_\sigma}(\xi_1,t)\widehat{v_\sigma}(\xi_2,t)\overline{\widehat{v_\sigma}(\xi_3,t)\widehat{v_\sigma}(\xi,t)}d\mu(\xi)dt \\
&\leqslant c\sigma^2\int_0^T\int_{\mathbb{R}^4}\widehat{v_\sigma}(\xi_1,t)\cdot\langle\xi_2\rangle\widehat{v_\sigma}(\xi_2,t)\cdot\overline{\langle\xi_3\rangle^{\frac{5}{2}}\widehat{v_\sigma}(\xi_3,t)}\cdot\overline{\langle\xi\rangle^{\frac{5}{2}}\widehat{v_\sigma}(\xi,t)}d\mu(\xi)dt \\
&= c\sigma^2\int_0^T\int_{\mathbb{R}}\mathcal{F}_x\left[v_\sigma\cdot\langle D\rangle v_\sigma\cdot\overline{\langle D\rangle^{\frac{5}{2}}v_\sigma}\right](\xi,t)\cdot\overline{F_x\left[\langle D\rangle^{\frac{5}{2}}v_\sigma\right](\xi,t)}d\xi dt \\
&= c\sigma^2\int_0^T\int_{\mathbb{R}} v_\sigma\cdot\langle D\rangle v_\sigma\cdot\overline{\langle D\rangle^{\frac{5}{2}}v_\sigma}\cdot\overline{\langle D\rangle^{\frac{5}{2}}v_\sigma}\,dxdt \\
&\leqslant c\sigma^2T^{\frac{1}{2}}\|v_\sigma\|_{L_T^\infty L_x^2}\|\langle D\rangle v_\sigma\|_{L_T^\infty L_x^2}\|\langle D\rangle^{\frac{5}{2}}v_\sigma\|_{L_T^4 L_x^\infty}^2 \\
&\leqslant c\sigma^2T^{\frac{1}{2}}\|v_\sigma\|_{X_T^{1,\frac{1}{2}+}}^2\|\langle D\rangle^2 v_\sigma\|_{X_T^{0,\frac{1}{2}+}}^2 \\
&\leqslant c\sigma^2T^{\frac{1}{2}}\|u_\sigma\|_{X_T^{2,\frac{1}{2}+}}^4.
\end{aligned}$$

This proves (24).

5.2. **Proof of** (25)**.** By Plancherel and (26),

$$
\begin{aligned}
\left|\mathcal{R}_\sigma^{(3)}(s)\right| &= \Big|\int_0^s\int_{\mathbb{R}} \mathcal{F}_x\left[N(u_\sigma)\right](\xi,t)\cdot\overline{\mathcal{F}_x\left[|u_\sigma|^2u_\sigma\right](\xi,t)}d\xi dt\Big| \\
&= \Big|\int_0^s\int_{\mathbb{R}^6}\left(1-\cosh(\sigma|\xi|)\prod_{j=1}^3\operatorname{sech}(\sigma|\xi_j|)\right)\prod_{j=1}^2\widehat{u_\sigma}(\xi_j,t)\overline{\widehat{u_\sigma}(\xi_3,t)}\cdot\prod_{j=4}^5\overline{\widehat{u_\sigma}(\xi_j,t)}\widehat{u_\sigma}(\xi_6,t)d\nu(\xi)dt\Big|, \\
&\leqslant 8\sigma^2\int_0^T\int_{\mathbb{R}^6}\left(\sum_{j\neq k=1}^3|\xi_j||\xi_k|\right)\prod_{j=1}^6|\widehat{u_\sigma}(\xi_j,t)|d\nu(\xi)dt,
\end{aligned}
$$

where $d\nu(\xi)$ is the measure

$$
d\nu(\xi)=\delta\begin{pmatrix}\xi-\xi_1-\xi_2+\xi_3\\ \xi-\xi_4-\xi_5+\xi_6\end{pmatrix}\prod_{j=1}^6 d\xi_j.
$$

This measure impose the conditions $\xi=\xi_1+\xi_2-\xi_3=\xi_4+\xi_5-\xi_6$.

Again, assuming $|\xi_1|\leqslant|\xi_2|\leqslant|\xi_3|$ by symmetry, we have

$$
\begin{aligned}
\left|\mathcal{R}_\sigma^{(3)}(s)\right| &\leqslant c\sigma^2\int_0^T\int_{\mathbb{R}^6}|\xi_2||\xi_3|\prod_{j=1}^6|\widehat{u_\sigma}(\xi_j,t)|d\nu(\xi)dt \\
&= c\sigma^2\int_0^T\int_{\mathbb{R}}\mathcal{F}_x\left[v_\sigma|D|v_\sigma\overline{|D|v_\sigma}\right](\xi,t)\cdot\overline{\mathcal{F}_x\left[|v_\sigma|^2v_\sigma\right](\xi,t)}d\xi dt \\
&= c\sigma^2\int_0^T\int_{\mathbb{R}}v_\sigma|D|v_\sigma\overline{|D|v_\sigma}\cdot|v_\sigma|^2\overline{v}_\sigma dxdt \\
&\leqslant c\sigma^2T\|v_\sigma\|^4_{L^\infty_TL^\infty_x}\||D|v_\sigma\|^2_{L^\infty_TL^2_x} \\
&\leqslant c\sigma^2T\|v_\sigma\|^6_{L^\infty_TH^2} \\
&\leqslant c\sigma^2T\|u_\sigma\|^6_{X_T^{2,\frac12+}}
\end{aligned}
$$

which proves (25).

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References

[1] B. Belayneh, E. Tegegn and A. Tesfahun, *Lower bound on the radius of analyticity of solution for fifth order KdV-BBM Equation.* Nonlinear Differ. Equ. Appl. NoDEA, 29 (6) (2022).

[2] M. Ben-Artzi, H. Koch, J. Saut *Dispersion estimates for fourth order Schrödinger equations,* , C. R. Acad. Sci. Paris Sér. I Math. 330 (2000), no. 2, 87–92.

[3] J. Bourgain, *On the Cauchy problem for the Kadomtsev Petviashvili equation*, Geom. Funct. Anal. 3 (4) (1993) 315-341.

[4] T.T. Dufera, S. Mebrate, and A. Tesfahun, *On the persistence of spatial analyticity for the Beam equation*, J. Math. Anal. and Appl., 126001 (2022).

[5] ______, *Improved Lower Bound for the Radius of Analyticity of Solutions to the fifth order KdV-BBM model*, Preprint (2022).

[6] A. B. Ferrari and E. S. Titi, *Gevrey regularity for nonlinear analytic parabolic equations*, Comm. Partial Differential Equations **23** (1998), no. 1-2, 1–16.

[7] C. Foias, R. Temam, *Gevrey class regularity for the solutions of the Navier-Stokes equations*, J. Funct. Anal. 87 (1989) 359–369.
[8] P. Gérard, Y. Guo, and E. S. Titi, *On the radius of analyticity of solutions to the cubic Szegő equation*, Ann. Inst. H. Poincaré Anal. Non Linéaire **32** (2015), no. 1, 97–108.
[9] H. Hannah, A. A. Himonas, and G. Petronilho, *Gevrey regularity of the periodic gKdV equation*, J. Differential Equations **250** (2011), no. 5, 2581–2600.
[10] A. A. Himonas, K. Henrik, and Selberg S., *On persistence of spatial analyticity for the dispersion-generalized periodic kdv equation*, Nonlinear Analysis: Real World Applications **38** (2017), 35–48.
[11] A. A. Himonas and G. Petronilho, *Analytic well-posedness of periodic gKdV*, J. Differential Equations **253** (2012), no. 11, 3101–3112.
[12] V. Karpman *Lyapunov approach to the soliton stability in highly dispersive systems. I. Fourth order nonlinear Schrödinger equations* , Phys. Lett. A 215 (1996), no. 5-6, 254–256.
[13] V. Karpman, A. Shagalov *Stability of solitons described by nonlinear Schrödingertype equations with higher-order dispersion*, Phys. D 144 (2000), no. 1-2, 194–210.
[14] T. Kato T, K. Masuda, *Nonlinear evolution equations and analyticity I*, Ann. Inst. H. Poincare Anal. Non Lineaire, 3 (1986) 455–467.
[15] Y. Katznelson, *An introduction to harmonic analysis*, corrected ed., Dover Publications, Inc., New York, 1976.
[16] C. Kwak, *Periodic fourth-order cubic NLS: Local well-posedness and Non-squeezing property*, J. Math. Anal. Appl. 461 (2018), no. 2, 1327–1364.
[17] C. D. Levermore and M. Oliver, *Analyticity of solutions for a generalized Euler equation*, J. Differential Equations **133** (1997), no. 2, 321–339.
[18] T. Oh, P. Sosoe, N. Tzvetkov, *An optimal regularity result on the quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation*, J.Éc. polytech. Math. 5 (2018), 793–841.
[19] T. Oh, N. Tzvetkov, *Quasi-invariant Gaussian measures for the cubic fourth order nonlinear Schrödinger equation*, Probab. Theory Related Fields 169 (2017), 1121–1168.
[20] T. Oh, Y. Wang, *Global well-posedness of the periodic cubic fourth order NLS in negative Sobolev spaces*, Forum Math. Sigma 6 (2018), e5, 80 pp.
[21] M. Oliver and E. S. Titi, *On the domain of analyticity of solutions of second order analytic nonlinear differential equations*, J. Differential Equations **174** (2001), no. 1, 55–74.
[22] S. Panizzi, *On the domain of analyticity of solutions to semilinear Klein-Gordon equations*, Nonlinear Anal. **75** (2012), no. 5, 2841–2850.
[23] B. Pausader, *The cubic fourth-order Schrödinger equation*, J. Funct. Anal. 256 (2009), no. 8, 2473–2517.
[24] J. Segata *Modified wave operators for the fourth-order non-linear Schrödinger-type equation with cubic non-linearity*, Math. Methods Appl. Sci. 29 (2006), no. 15, 1785–1800. J. Funct. Anal. 256 (2009), no. 8, 2473–2517.
[25] S. Selberg and D. O. da Silva, *Lower bounds on the radius of spatial analyticity for the KdV equation*, Ann. Henri Poincaré (2016). doi:10.1007/s00023-016-0498-1.
[26] S. Selberg and A. Tesfahun, *On the radius of spatial analyticity for the quartic generalized KdV equation*, Annales Henri Poincaré 18, 3553–3564 (2017)
[27] ———, *On the radius of spatial analyticity for the 1d Dirac-Klein-Gordon equations*, Journal of Differential Equations **259** (2015), 4732–4744.
[28] K. Seong, *Well-Posedness and Ill-Posedness for the Fourth order cubic nonlinear Schrödinger equation in negative Sobolev spaces*, J. Math. Anal. and Appl., 125342 (2021).
[29] A. Tesfahun, *On the radius of spatial analyticity for cubic nonlinear Schrodinger equation*, J. Differential Equations 263 (2017) 7496–7512.
[30] ———, *Remark on the persistence of spatial analyticity for cubic nonlinear Schrödinger equation on the circle.* Nonlinear Differ. Equ. Appl. NoDEA, (2019) 26:12
[31] ———, *Asymptotic lower bound for the radius of spatial analyticity to solutions of KdV equation*, Comm. Contemp. Math. 21, 08, 1850061 (2019).

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