Global well-posedness and energy decay for a one dimensional porous-elastic system subject to a neutral delay

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Abstract

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GLOBAL WELL-POSEDNESS AND ENERGY DECAY FOR A ONE DIMENSIONAL POROUS-ELASTIC SYSTEM SUBJECT TO A NEUTRAL DELAY

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Abstract. In this article, we consider a one-dimensional porous-elastic system with porous-viscosity and a distributed delay of neutral type. First, we prove the global existence and uniqueness of the solution by using the Faedo–Galerkin approximations along with some energy estimates. Then, based on the energy method and by constructing a suitable Lyapunov functional as well as under an appropriate assumptions on the kernel of neutral delay term, we show that despite of the destructive nature of delays in general, the damping mechanism considered provockes an exponential decay of the solution for the case of equal speed of wave propagation. In the case of lack of exponential stability, we show that the solution decays polynomially.

1. Introduction

In 1972, Goodman and Cowin [14] have given an extension of the classical elasticity theory to porous media by introducing the concept of a continuum theory of granular materials with interstitial voids into the theory of elastic solids with voids. In addition, Nunziato and Cowin [22] have presented a nonlinear theory for the behavior of porous solids in which the skeletal or matrix material is elastic and the interstices are void of material. In this theory the bulk density is written as the product of two fields, the matrix material density field and the volume fraction field. Furthermore, this representation introduces an additional degree of kinematic freedom. The intended applications of the theory of elastic materials with voids are to geological materials like rocks and soils and to manufactured porous materials.

In [23], Quintanilla gave the first investigation concerning the study of asymptotic behavior of the solutions for a one-dimensional porous-elastic system where he proved that the damping through porous-viscosity is not strong enough to provoke an exponential decay. In [1, 2], Apalara showed that the same system considered in [23] is exponentially stable for the case of equal speeds of wave propagation. In [4], Casas and Quintanilla studied the one-dimensional porous-elastic system in the presence of the usual thermal effect with microtemperature damping and they used the semi-group approach to prove the exponential stability of the solutions irrespective of the speeds of wave propagations. In [5], Casas and Quintanilla proved that the combination of porous-viscosity and thermal effects provokes an exponential stability of the solutions. In [18], Magańa and Quintanilla showed that viscoelasticity damping and temperature produced slow decay in time and when

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the viscoelasticity is coupled with porous damping or with microtemperatures, the system decays in an exponential way.

Delay effect arises in many applications depending not only on the present state but also on some past occurrences and it has attracted lots of attentions from researchers in diverse fields of human endeavor such as mathematics, engineering, science, and economics. The presence of delay may be a source of instability of systems which are uniformly asymptotically stable in the absence of delay unless additional control terms have been used (see [6, 7, 13, 20, 21, 28]. Also, the introducing of this complementary control may lead to ill-posedness as shown in many works such as ([7, 24]) and the references therein. In addition to the well-known discrete delays, there are several others and we are interested here in the neutral delay where the delay is occurring in the second (highest) derivative, for more details, see previous studies ([8]-[11, 16, 27]) and the references therein.

Among the investigations that have been realized concerning the asymptotic behavior with neutral delay, we cite the work of Tatar [26] where he considered the following damped wave equation with neutral delay

$$\begin{cases} u_{tt} = u_{xx} - u_t - \displaystyle\int_0^t h\left(t-s\right) u_{tt}\left(s\right) ds, \ x \in (0,1)\,, \ t > 0, \\ u\left(0,t\right) = u\left(1,t\right) = 0, \ t \geq 0, \\ u\left(x,0\right) = u_0\left(x\right), \ u_t\left(x,0\right) = u_1\left(x\right), \ x \in (0,1)\,, \end{cases}$$

and he showed that the solution decays in exponential manner under some conditions on the kernel of distributed neutral delay.

In [25] Seghour et al. studied the following thermoelastic laminated system with neutral delay

$$\begin{cases} \rho w_{tt} + G\left(\psi - w_x\right)_x + A w_t = 0, \ x \in (0,1)\,, \ t > 0, \\ I_\rho\left(3s_{tt} - \psi_{tt}\right) - G\left(\psi - w_x\right) - \left(3s - \psi\right) + \mu\theta_x = 0, \ x \in (0,1)\,, \ t > 0, \\ 3I_\rho\left(s_t + \displaystyle\int_0^t h\left(t-r\right) s_t\left(r\right) dr\right)_t + 3G\left(\psi - w_x\right) + 4\gamma s - 3s_{xx} = 0, \ x \in (0,1)\,, \ t > 0, \\ \theta_t - \kappa\theta_{xx} + \mu\left(3s - \psi\right)_{tx} = 0, \ x \in (0,1)\,, \ t > 0, \end{cases}$$

with boundary conditions

$$\begin{cases} \psi\left(0,t\right) = s\left(0,t\right) = \theta_x\left(0,t\right) = w_x\left(0,t\right) = 0, \ t \geq 0, \\ \theta\left(1,t\right) = w\left(1,t\right) = s_x\left(1,t\right) = \psi_x\left(1,t\right) = 0, \ t \geq 0, \end{cases}$$

and initial data

$$\begin{cases} \left(w, \psi, s, \theta\right)\left(x,0\right) = \left(w_0, \psi_0, s_0, \theta_0\right), \ x \in (0,1)\,, \\ \left(w_t, \psi_t, s_t\right)\left(x,0\right) = \left(w_1, \psi_1, s_1\right), \ x \in (0,1)\,, \end{cases}$$

and they showed that the dissipation given by the combination of neutral delay with the heat effect and the frictional damping stabilize exponentially the system in the case of equal wave speeds. In the opposite one, and with an additional assumption on the kernel, they proved a polynomial stability.

In [15], Kerbal and Tatar considered the following neutrally retarded viscoelastic Timoshenko system

$$\begin{cases} \varphi_{tt} = \left(\varphi_x + \psi\right)_x, \\ \left(\psi_t + \displaystyle\int_0^t k\left(t-s\right) \psi_t\left(s\right) ds\right)_t = \psi_{xx} - \displaystyle\int_0^t g\left(t-s\right) \psi_{xx}\left(s\right) ds - \left(\varphi_x + \psi\right), \end{cases}$$

for $x \in (0,1)$, $t > 0$ with initial and boundary conditions

$$\begin{cases} \varphi(x,0) = \varphi_0(x), \ \varphi_t(x,0) = \varphi_1(x), \ x \in (0,1), \\ \psi(x,0) = \psi_0(x), \ \psi_t(x,0) = \psi_1(x), \ x \in (0,1), \\ \varphi(0,t) = \varphi(1,t) = \psi(0,t) = \psi(1,t) = 0, \ t \geq 0, \end{cases}$$

and under certain conditions on the kernel, they proved that the neutral delay does not prevent the system from being stabilized by the viscoelastic term.

Motivated by the previous works, in this paper, we consider the following porous-elastic system with porous-viscosity subject to a distributed delay of neutral type

$$\begin{cases} \rho u_{tt} - \mu u_{xx} - b\phi_x = 0, \ x \in (0,1), \ t > 0, \\ J\left(\phi_t + \int_0^t k(t-s)\phi_t(s)\,ds\right)_t - \delta\phi_{xx} + bu_x + \xi\phi + \mu_1\phi_t = 0, \ x \in (0,1), \ t > 0, \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in (0,1), \\ \phi(x,0) = \phi_0(x), \ \phi_t(x,0) = \phi_1(x), \ x \in (0,1), \\ u_x(0,t) = u_x(1,t) = \phi(0,t) = \phi(1,t) = 0, \ t > 0, \end{cases} \tag{1.1}$$

where the functions u and ϕ represent respectively the displacement of the solid elastic material and the volume fraction . The parameter ρ designate the mass density and J equals to the product of the mass density by the equilibrated inertia The coefficients μ, δ, ξ, μ_1 are positive constants represent the constitutive parameters defining the coupling among the different components of the materials such that

$$\mu\xi > b^2. \tag{1.2}$$

where b is a real number different from zero. The initial data u_0, u_1, ϕ_0, ϕ_1 belongs to the suitable functional space and the integral represents the neutral delay term where k is the relaxation function that specified in the preliminaries. The system (1.1) was constructed by considering the following basic evolution equations of the one-dimensional porous materials theory

$$\rho u_{tt} = T_x, \ J\left(\phi_t + \int_0^t k(t-s)\phi_t(s)\,ds\right)_t = H_x + D, \tag{1.3}$$

where T, H and D represent respectively the stress tensor, the equilibrated stress vector and the equilibrated body force. Consequently, to get the system (1.1) we take the constitutive equations T, H and D at this form

$$\begin{aligned} T &= \mu u_x + b\phi, \ H = \delta\phi_x, \\ D &= -bu_x - \xi\phi - \mu_1\phi_t, \end{aligned} \tag{1.4}$$

and by combination (1.4) in (1.3), we obtain (1.1).

The main goal of this paper is to prove a global well-posedness of the problem by using the Faedo-Galerkin method. Moreover, based on the multipliers method and under some assumptions on the kernel of neutral delay, we show that despite of the destructive nature of delays in general, the porous-viscosity given an exponential behavior for the case of equal speeds of wave propagation, that is

$$\chi = \frac{\mu}{\rho} - \frac{\delta}{J} = 0. \tag{1.5}$$

In the opposite one, we establish an polynomial stability result.

This paper is organized as follows. In Section 2, we introduce some assumptions and transformations needed in the next sections to prove the main result. In Section

3,we prove the existence and uniqueness of the solution. In Section 4, we show the decay of the energy. In Section 5 and 6, we use the energy method to prove the exponential and polynomial stability result.

2. Preliminaries

In this section we present our assumptions on both kernels and introduce the energy functional and some other functional.

We use the standard Lebesgue space $L^2(0,1)$ and the Sobolev space $H_0^1(0,1)$ with their usual scalar products and norms. Let define the space $\mathcal{H}$ as

$$\mathcal{H} = H_*^1(0,1) \times L_*^2(0,1) \times H_0^1(0,1) \times L^2(0,1),$$

where $H_*^1(0,1) = H^1(0,1) \cap L_*^2(0,1)$ such that

$$L_*^2(0,1) = \left\{ f \in L^2(0,1) : \int_0^1 f(x)\,dx = 0 \right\}.$$

$$H_*^2(0,1) = \left\{ \psi \in H^2(0,1) : \psi_x(0) = \psi_x(1) = 0 \right\}.$$

To simplify the calculations, we are obliged to announce this Lemma which are usable in the following sections.

(H1) The kernel k is a nonnegative continuously differentiable and summable function satisfying

$$k'(t) \leq 0, \ \forall t \geq 0, \ \bar{k} = \int_0^\infty k(s)ds.$$

(H2) $\exp(\varsigma t)\, k(t) \in L^1(\mathbb{R}_+)$ for some $\varsigma > 0$.

Note that if $\int_0^{+\infty} e^{\varsigma s} k(s)ds < \infty$ and $\lim\limits_{t \longrightarrow \infty} \exp(\varsigma t)\, k(t) < \infty$, then

$$\int_0^{+\infty} e^{\varsigma s} |k'(s)|\, ds = -\int_0^{+\infty} e^{\varsigma s} k'(s)ds = -e^{\varsigma s} k(s) \mid_0^\infty + \varsigma \int_0^{+\infty} e^{\varsigma s} k(s)ds < \infty.$$

Lemma 1 ([25])**.** *For any function $\psi \in C^1\left([0,\infty); L^2(0,1)\right)$ and any $k \in C^1([0,\infty))$, we have the following identity*

$$\begin{aligned} &\int_0^1 \psi(t) \left(\int_0^t k(t-s)\, \psi_t(s)ds \right) dx \\ &= -\frac{1}{2} (k' \Box \psi)(t) + \frac{1}{2} \frac{d}{dt} \int_0^1 \left(\int_0^t k(t-s)\, \psi^2(s)ds \right) dx \\ &+ \frac{k(t)}{2} \int_0^1 \psi^2 dx - k(t) \int_0^1 \psi(0)\psi(t)\, dx, \end{aligned}$$

where

$$(k \Box \psi) = \int_0^t k(t-s) \left(\int_0^1 (\psi(t) - \psi(s))^2\, dx \right) ds, \ t \geq 0.$$

Theorem 1 ([3])**.** *Let $B_0 \subset B_1 \subset B_2$ be three Banach spaces. We assume that the embedding of B_1 in B_2 is continuous and that the embedding of B_0 in B_1 is compact. Let p, r such that $1 \leq p, r \leq +\infty$. For $T > 0$, we define*

$$E_{p,r} = \left\{ v \in L^p(0,T;B_0) \qquad \frac{dv}{dt} \in L^r(0,T;B_2) \right\}.$$

i) If $p < +\infty$, the embedding of $E_{p,r}$ in $L^p(0,T;B_1)$ is compact.
ii) If $p = +\infty$ and $r > 1$, the embedding of $E_{p,r}$ in $C^0(0,T;B_1)$ is compact.

In view of the boundary conditions, our system can have solutions (uniform in the variable x), which do not decay. To avoid such case and also to be able to use Poincaré's inequality for u , we perform the following transformation

From $(1.1)_1$, we observe that

$$\int_0^1 u_{tt} dx = 0.$$

If we take $\upsilon(t) = \int_0^1 u dx$, we observe that $\upsilon(0) = \int_0^1 u_0 dx$ and $\upsilon'(0) = \int_0^1 u_1 dx$. Moreover, υ is a solution of the following initial value problem

$$\begin{cases} \upsilon''(t) = 0, \ \forall t \geq 0, \\ \upsilon(0) = \int_0^1 u_0 dx, \quad \upsilon'(0) = \int_0^1 u_1 dx. \end{cases}$$

The solution of the problem is given by

$$\upsilon(t) = \int_0^1 u(x,t)\, dx = t \int_0^1 u_1(x)\, dx + \int_0^1 u_0(x)\, dx$$

Consequently, if we let

$$\bar{u}(x,t) = u(x,t) - t\int_0^1 u_1(x)\, dx - \int_0^1 u_0(x)\, dx,$$

we have

$$\int_0^1 \bar{u}(x,t)\, dx = 0, \ \forall t \geq 0.$$

In what follows, we will work with $\bar{u}$ but, for convenience, we write u instead of $\bar{u}$ with initial data given as

$$\bar{u}_0(x) = u_0(x) - \int_0^1 u_0(x)\, dx, \ \bar{u}_1(x) = u_1(x) - \int_0^1 u_1(x)\, dx.$$

3. Global well-posedness

In this section, we will prove the global existence and the uniqueness of the solution of problem (1.1) by using the classical Faedo-Galerkin approximations along with some priori estimates. The well-posedness of (1.1) is given by the following theorem.

Theorem 2. *Assume that* ***(H1)-(H2),*** (1.2) *hold, and the initial data*

$$\begin{aligned} &(u_0, u_1) \in H_*^1(0,1) \times L_*^2(0,1), \\ &(\phi_0, \phi_1) \in H_0^1(0,1) \times L^2(0,1), \end{aligned} \tag{3.1}$$

problem (1.1) *has a unique global strong solution*

$$\begin{aligned} u &\in C\left(\mathbb{R}_+, H_*^2(0,1) \cap H_*^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, H_*^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, L^2(0,1)\right), \\ \phi &\in C\left(\mathbb{R}_+, H^2(0,1) \cap H_0^1(0,1)\right) \cap C^1\left(\mathbb{R}_+, H_0^1(0,1)\right) \cap C^2\left(\mathbb{R}_+, L^2(0,1)\right). \end{aligned} \tag{3.2}$$

In addition, the solution (u, ϕ) depends continuously on the initial data.

Proof. We divide the proof into three steps: we first construct Faedo–Galerkin approximations, then thanks to a priori estimates we look to prove that $t_n = T$ for $n \in \mathbb{N}$. Finally, we pass to the limit.

Step 1: Faedo–Galerkin approximations.

We construct approximations of the solution (u, ϕ) by the Faedo-Galerkin method as follows (see Refs [12] and [19]): For every $n \geq 1$, let $W_n = span\{e_1, e_2, ..., e_n\}$ be a Hilbert basis (orthonormal basis) of $H_*^2(0,1) \cap H_*^1(0,1)$ and $L_*^2(0,1)$. Also, we denote by $\Gamma_n = span\{\sigma_1, \sigma_2, ..., \sigma_n\}$ a Hilbertian basis of $H^2(0,1) \cap H_0^1(0,1)$ and $L^2(0,1)$. For given initial data

$$(u_0, u_1) \in H_*^1(0,1) \times L_*^2(0,1),$$
$$(\phi_0, \phi_1) \in H_0^1(0,1) \times L^2(0,1),$$

we seek functions $y_j^n, h_j^n \in C^2([0,T])$, such that the approximations

$$\begin{cases} u^n(x,t) = \sum_{j=1}^{j=n} y_j^n(t) e_j(x), \\ \phi^n(x,t) = \sum_{j=1}^{j=n} h_j^n(t) \sigma_j(x), \end{cases} \tag{3.3}$$

check the following approximate problem

$$\begin{cases} \rho u_{tt}^n, -\mu u_{xx}^n - b\phi_x^n = 0, \\ J\phi_{tt}^n + J\left(\int_0^t k(t-s)\phi_t^n(s)ds\right)_t \\ -\delta\phi_{xx}^n + bu_x^n + \xi\phi^n + \mu_1\phi_t^n = 0, \end{cases} \tag{3.4}$$

with the initial data

$$\begin{cases} u^n(x,0) = u_0^n(x), \ u_t^n(x,0) = u_1^n(x), \\ \phi^n(x,0) = \phi_0^n(x), \ \phi_t^n(x,0) = \phi_1^n(x), \end{cases} \tag{3.5}$$

which satisfies

$$\begin{cases} u_0^n = \sum_{j=1}^n \left\{\int_0^1 u_0 e_j dx\right\} e_j \underset{n\to\infty}{\longrightarrow} u_0 \text{ strongly in } H_*^1(0,1), \\ u_1^n = \sum_{j=1}^n \left\{\int_0^1 u_1 e_j dx\right\} e_j \underset{n\to\infty}{\longrightarrow} u_1 \text{ strongly in } L_*^2(0,1), \\ \phi_0^n = \sum_{j=1}^n \left\{\int_0^1 \phi_0 \sigma_j dx\right\} \sigma_j \underset{n\to\infty}{\longrightarrow} \phi_0 \text{ strongly in } H_0^1(0,1), \\ \phi_1^n = \sum_{j=1}^n \left\{\int_0^1 \phi_1 \sigma_j dx\right\} \sigma_j \underset{n\to\infty}{\longrightarrow} \phi_1 \text{ strongly in } L^2(0,1). \end{cases} \tag{3.6}$$

Through 3.4, we get

$$\begin{cases} \rho\langle u_{tt}^n, e_k\rangle_{L^2(0,1)} - \mu\langle u_{xx}^n, e_k\rangle_{L^2(0,1)} - b\langle\phi_x^n, e_k\rangle_{L^2(0,1)} = 0, \\ J\langle\phi_{tt}^n, \sigma_k\rangle_{L^2(0,1)} + J\left\langle\left(\int_0^t k(t-s)\phi_t^n(s)ds\right)_t, \sigma_k\right\rangle_{L^2(0,1)} \\ -\delta\langle\phi_{xx}^n, \sigma_k\rangle_{L^2(0,1)} + b\langle u_x^n, \sigma_k\rangle_{L^2(0,1)} + \xi\langle\phi^n, \sigma_k\rangle_{L^2(0,1)} + \mu_1\langle\phi_t^n, \sigma_k\rangle_{L^2(0,1)} = 0, \end{cases} \tag{3.7}$$

with (u_0^n, u_1^n) and (ϕ_0^n, ϕ_1^n) are chosen, respectively, in W_n and Γ_n. According to the standard ordinary differential equations theory, the finite dimensional problem (3.7) has a solution $\left(y_j^n, h_j^n\right)_{j=1,..,n} \in C^2([0,t_n])^2$. Then, thanks to a priori estimates that follow imply that in fact $t_n = T$, $\forall T > 0$.

Step 2: Energy estimates

A priori estimate I.

For every $n \geq 1$, we use integration by parts in (3.7), we get

$$\begin{cases} \rho\int_0^1 u_{tt}^n e_k dx + \mu\int_0^1 u_x^n e_{kx} dx - b\int_0^1 \phi_x^n e_k dx = 0, \\ J\int_0^1 \phi_{tt}^n \sigma_k dx + J\int_0^1 \sigma_k\left(\int_0^t k(t-s)\phi_t^n(s)ds\right)_t dx \\ +\delta\int_0^1 \phi_x^n \sigma_{kx} dx + b\int_0^1 u_x^n \sigma_k dx + \xi\int_0^1 \phi^n \sigma_k dx \\ +\mu_1\int_0^1 \phi_t^n \sigma_k dx = 0, \qquad \forall k = 1, .., n. \end{cases} \tag{3.8}$$

Multiplying $(3.8)_1$ and $(3.8)_2$, respectively, by $(y_k^n)_t$ and $(h_k^n)_t$, then, by using integration by parts, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left\{\rho\left(u_t^n\right)^2+\mu\left(u_x^n\right)^2+J\left(\phi_t^n\right)^2+2bu_x^n\phi^n+\xi\left(\phi^n\right)^2+\delta\left(\phi_x^n\right)^2\right\}dx\\&+J\int_0^1\phi_t^n\left(\int_0^t k\left(t-s\right)\phi_t^n(s)ds\right)_t dx+\mu_1\int_0^1\left(\phi_t^n\right)^2dx=0.\end{aligned}\tag{3.9}$$

Note that

$$\left(\int_0^t k\left(t-s\right)\phi_t^n(s)ds\right)_t=k\left(t\right)\phi_t^n(0)+\int_0^t k\left(t-s\right)\phi_{tt}^n(s)ds.$$

Then

$$\begin{aligned}&J\int_0^1\phi_t^n\left(\int_0^t k\left(t-s\right)\phi_t^n(s)ds\right)_t dx\\&=Jk\left(t\right)\int_0^1\phi_t^n\phi_t^n(0)dx+J\int_0^1\phi_t^n\left(\int_0^t k\left(t-s\right)\phi_{tt}^n(s)ds\right)dx.\end{aligned}$$

By using Lemma (1), we get

$$\begin{aligned}&J\int_0^1\phi_t^n\left(\int_0^t k\left(t-s\right)\phi_t^n(s)ds\right)_t dx\\&=Jk\left(t\right)\int_0^1\phi_t^n\phi_t^n(0)dx-\frac{J}{2}\left(k'\Box\phi_t^n\right)(t)\\&+\frac{J}{2}\frac{d}{dt}\int_0^1\left(\int_0^t k\left(t-s\right)\left(\phi_t^n\right)^2(s)ds\right)dx\\&+J\frac{k(t)}{2}\int_0^1\left(\phi_t^n\right)^2dx-Jk(t)\int_0^1\phi_t^n(0)\phi_t^n dx.\end{aligned}$$

So, (3.9) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1\left\{\rho\left(u_t^n\right)^2+\mu\left(u_x^n\right)^2+J\left(\phi_t^n\right)^2+2bu_x^n\phi^n+\xi\left(\phi^n\right)^2+\delta\left(\phi_x^n\right)^2\right\}dx\\&+\frac{J}{2}\frac{d}{dt}\int_0^1\left(\int_0^t k\left(t-s\right)\left(\phi_t^n\right)^2(s)ds\right)dx=\frac{J}{2}\left(k'\Box\phi_t^n\right)(t)\\&-\left(J\frac{k(t)}{2}+\mu_1\right)\int_0^1\left(\phi_t^n\right)^2dx\ \le\ 0.\end{aligned}\tag{3.10}$$

Now integrating (3.10), we obtain

$$\begin{aligned}&\frac{1}{2}\int_0^1\left\{\rho\left(u_t^n\right)^2+\mu\left(u_x^n\right)^2+J\left(\phi_t^n\right)^2+2bu_x^n\phi^n+\xi\left(\phi^n\right)^2+\delta\left(\phi_x^n\right)^2\right\}dx\\&+\frac{J}{2}\int_0^1\left(\int_0^t k\left(t-s\right)\left(\phi_t^n\right)^2(s)ds\right)dx\\&\le\frac{1}{2}\int_0^1\left\{\rho\left(u_1^n\right)^2+J\left(\phi_1^n\right)^2+\xi\left(\phi_0^n\right)^2+\left[\mu\left(u_x^n\right)^2+2bu_x^n\phi^n+\delta\left(\phi_x^n\right)^2\right](x,0)\right\}dx.\end{aligned}$$

Hence, the previous inequality takes the following form

$$E^n\left(t\right)\le E^n\left(0\right),$$

where

$$E^n(t) = \frac{1}{2}\int_0^1 \left\{\rho\left(u_t^n\right)^2 + \mu\left(u_x^n\right)^2 + J\left(\phi_t^n\right)^2 + 2bu_x^n\phi^n + \xi\left(\phi^n\right)^2 + \delta\left(\phi_x^n\right)^2\right\}dx$$
$$+\frac{J}{2}\int_0^1\left(\int_0^t k(t-s)\left(\phi_t^n\right)^2(s)ds\right)dx. \tag{3.11}$$

Note that

$$\mu\left(u_x^n\right)^2 + 2bu_x^n\phi^n + \xi\left(\phi^n\right)^2 = \frac{1}{2}\left[\mu\left(u_x^n + \frac{b}{\mu}\phi^n\right)^2 + \xi\left(\phi^n + \frac{b}{\xi}u_x^n\right)^2\right.$$
$$\left.+\left(\mu - \frac{b^2}{\xi}\right)\left(u_x^n\right)^2 + \left(\xi - \frac{b^2}{\mu}\right)\left(\phi^n\right)^2\right],$$

and because $\mu\xi > b^2$, we deduce that

$$\mu\left(u_x^n\right)^2 + 2bu_x^n\phi^n + \xi\left(\phi^n\right)^2 > \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)\left(u_x^n\right)^2 + \left(\xi - \frac{b^2}{\mu}\right)\left(\phi^n\right)^2\right].$$

Consequently, $E^n(t)$ is non-negative.
In view of the hypotheses on the function k, we deduce

$$0 \leq E^n(t) \leq E^n(0).$$

Now, since the sequences $\left(u_0^n\right)_{n\in\mathbb{N}}, \left(u_1^n\right)_{n\in\mathbb{N}}, \left(\phi_0^n\right)_{n\in\mathbb{N}}, \left(\phi_1^n\right)_{n\in\mathbb{N}}$, converge (see (3.6)), using (**H1**) and (**H2**), we can find a positive constant C independent of n such that

$$E^n(t) \leq C. \tag{3.12}$$

Then $t_n = T$, for all $T > 0$.

A priori estimate II

Through (3.3), also as $\left(y_j^n, h_j^n\right)_{j=1,..,n} \in \left(C^2[0,T]\right)^2$ and

$$\left(e_j\right)_{j\geq 1} \subset H_*^2(0,1) \cap H_*^1(0,1) \subset H^1(0,L) \hookrightarrow C(0,L),$$
$$\left(\sigma_j\right)_{j\geq 1} \subset H^2(0,1) \cap H_0^1(0,1) \subset H^1(0,L) \hookrightarrow C(0,L),$$

we have

$$\left\{\begin{array}{l} u^n \in C^2\left(0,T;\ H_*^2(0,1) \cap H_*^1(0,1)\right), \\ \phi^n \in C^2\left(0,T;\ H^2(0,1) \cap H_0^1(0,1)\right), \end{array}\right. \tag{3.13}$$

and from (3.13), we get

$$\int_0^1 \left(u_{xx}^n\right)^2 + \left(\phi_{xx}^n\right)^2 dx < \infty,\ \forall t \in [0,T]. \tag{3.14}$$

Step 3 : The limit process.

From (3.12)-(3.14), we conclude that

$$\begin{aligned}
&\left(u^n\right)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; H_*^2(0,1) \cap H_*^1(0,1)\right),\\
&\left(u_t^n\right)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; L_*^2(0,1)\right),\\
&\left(\phi^n\right)_{n\in\mathbb{N}^*} \text{ is bounded in } L^\infty\left(0,T; H^2(0,1) \cap H_0^1(0,1)\right),\\
&\left(\phi_t^n\right)_{n\in\mathbb{N}^*} \text{ is bounded in } L^2\left(0,T; L^2(0,1)\right).
\end{aligned} \tag{3.15}$$

By using Aubin–Lions–Simon theorem (1), Since

The embedding of $H_*^1(0,1)$ in $L_*^2(0,1)$ is continuous.

The embedding of $H_*^2(0,1)\cap H_*^1(0,1)$ in $H_*^1(0,1)$ is compact.

The embedding of $H_0^1(0,1)$ in $L^2(0,1)$ is continuous.

The embedding of $H^2(0,1)\cap H_0^1(0,1)$ in $L^2(0,1)$ is compact.

Then, we get the embedding of $E_{\infty,\infty}$ in $C(0,T;H_*^1(0,1))$ is compact where

$$\begin{aligned} E_{\infty,\infty} = \big\{ u^n/\ u^n \in L^\infty\left(0,T;H_*^2(0,1)\cap H_*^1(0,1)\right), \\ \left. u_t^n = \frac{du^n}{dt} \in L^\infty\left(0,T;L_*^2(0,1)\right) \right\}, \end{aligned}$$

also, the embedding of $\tilde{E}_{\infty,\infty}$ in $C([0,T],H_0^1(0,1))$ is compact where

$$\begin{aligned} \tilde{E}_{\infty,\infty} = \big\{ \phi^n/\ \phi^n \in L^\infty\left(0,T;H^2(0,1)\cap H_0^1(0,1)\right), \\ \left. \phi_t^n = \frac{d\phi^n}{dt} \in L^\infty\left(0,T;L^2(0,1)\right) \right\}, \end{aligned}$$

by (3.15), we get $(u^n)_{n\in\mathbb{N}^*}$, $(\phi^n)_{n\in\mathbb{N}^*}$ bounded in $E_{\infty,\infty}$, $\tilde{E}_{\infty,\infty}$ respectively, then there exist $(u^m)_{m\geq 1}$ sub sequence of $(u^n)_{n\geq 1}$ and $(\phi^m)_{m\geq 1}$ sub sequence of $(\phi^n)_{n\geq 1}$ such that

$$u^m \overset{m\to\infty}{\longrightarrow} u \text{ strongly in } C(0,T;H_*^1(0,1)), \tag{3.16}$$

$$\phi^m \overset{m\to\infty}{\longrightarrow} \phi \text{ strongly in } C(0,T;H_0^1(0,1)), \tag{3.17}$$

by using (3.13), (3.16) and (3.17), we arrive at

$$\|u_t^m - u_t\|_X = \left\| \frac{d}{dt}u^m - u_t \right\|_X \overset{m\to\infty}{\longrightarrow} 0,$$

$$\|\phi_t^m - \phi_t\|_Y = \left\| \frac{d}{dt}\phi^m - \phi_t \right\|_Y \overset{m\to\infty}{\longrightarrow} 0,$$

where $X = C(0,T;\ H_*^1(0,1))$ and $Y = C(0,T;\ H_0^1(0,1))$, then we conclude that

$$u_t^m \overset{m\to\infty}{\longrightarrow} u_t \text{ strongly in } X = C(0,T;H_*^1(0,1)), \tag{3.18}$$

$$\phi_t^m \overset{m\to\infty}{\longrightarrow} \phi_t \text{ strongly in } Y = C(0,T;H_0^1(0,1)). \tag{3.19}$$

Again, by using (3.13), (3.18) and (3.19), we obtain

$$\|u_{tt}^m - u_{tt}\|_Z = \left\| \frac{d}{dt}u_t^m - u_{tt} \right\|_Z \overset{m\to\infty}{\longrightarrow} 0, \tag{3.20}$$

$$\|\phi_{tt}^m - \phi_{tt}\|_Z = \left\| \frac{d}{dt}\phi_t^m - \phi_{tt} \right\|_Z \overset{m\to\infty}{\longrightarrow} 0,$$

where $Z = C(0,T;\ L^2(0,1))$, then we deduce that

$$u_{tt}^m \overset{m\to\infty}{\longrightarrow} u_{tt} \text{ strongly in } C(0,T;L^2(0,1)), \tag{3.21}$$

and

$$\phi_{tt}^m \overset{m\to\infty}{\longrightarrow} \phi_{tt} \text{ strongly in } C(0,T;L^2(0,1)). \tag{3.22}$$

By passing to the limit in (3.8)-(3.6), then we get that, the problem (1.1) accepts a strong solution satisfies (3.2).

The proof now can be completed arguing as in [17, Théorème 3.1]

Continuous dependence and uniqueness
For uniqueness, let us assume that $\left(\Lambda^1, \Upsilon^1\right)$ and $\left(\Lambda^2, \Upsilon^2\right)$ are two global solutions of (1.1). Then, $(\chi, \Xi) = \left(\Lambda^1 - \Lambda^2, \Upsilon^1 - \Upsilon^2\right)$ satisfies

$$\begin{cases} \rho\chi_{tt} - \mu\chi_{xx} - b\Xi_x = 0, \ x \in (0,1)\,, \ t > 0, \\ J\left(\Xi_t + \displaystyle\int_0^t k\,(t-s)\,\Xi_t\,(s)\,ds\right)_t - \delta\Xi_{xx} + b\chi_x + \xi\Xi + \mu_1\Xi_t = 0, \ x \in (0,1)\,, \ t > 0, \\ \chi\,(x,0) = \chi_t\,(x,0) = \Xi\,(x,0) = \Xi_t\,(x,0) = 0, \ x \in (0,1)\,, \\ \chi_x\,(0,t) = \chi_x\,(1,t) = \Xi\,(0,t) = \Xi\,(1,t) = 0, \ t > 0, \end{cases} \tag{3.23}$$

Multiplying $(3.23)_1$ by χ_t , $(3.23)_2$ by Ξ_t, integrating the results over $(0,1)$, and summing them up, we obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho\chi_t^2 + \mu\chi_x^2 + J\Xi_t^2 + 2b\chi_x\Xi + \xi\Xi^2 + \delta\Xi_x^2\right) dx \\ &+ \frac{J}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right) dx \\ &= \frac{J}{2}\left(k'\Box\Xi_t\right)(t) - \left(J\frac{k\,(t)}{2} + \mu_1\right)\int_0^1 \Xi_t^2 dx \le 0. \end{aligned} \tag{3.24}$$

Then, a simple integration over $(0,t)$ and combining by the initial data on(χ, Ξ) , gives

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left(\rho\chi_t^2 + \mu\chi_x^2 + J\Xi_t^2 + 2b\chi_x\Xi + \xi\Xi^2 + \delta\Xi_x^2\right) dx \\ &+ \frac{J}{2}\int_0^1 \left(\int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right) dx \ \le \ 0, \end{aligned}$$

which implies that $(\chi, \Xi) = (0,0)$. So, the problem (1.1) has a unique global solution.
Now, by integrating (3.24), using Young's inequality and the positivity of energy, we get

$$\begin{aligned} E\,(t) \le E\,(0) &+ \frac{1}{2}\int_0^t \left[\int_0^1 \left(\rho\chi_t^2 + \mu\chi_x^2 + J\Xi_t^2 + 2b\chi_x\Xi + \xi\Xi^2 + \delta\Xi_x^2\right) dx\right. \\ &\left. + \frac{J}{2}\int_0^1 \left(\int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right) dx\right] d\tau \\ &\le E\,(0) + \frac{1}{2}\int_0^t \left[\int_0^1 \left(\rho\chi_t^2 + (\mu + |b|)\,\chi_x^2 + J\Xi_t^2 + (\xi + |b|)\,\Xi^2 + \delta\Xi_x^2\right) dx\right. \\ &\left. + \frac{J}{2}\int_0^1 \left(\int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right) dx\right] d\tau \\ &\le E\,(0) + \varsigma_1\int_0^t \left[\int_0^1 \left(\chi_t^2 + \chi_x^2 + \Xi_t^2 + \Xi^2 + \Xi_x^2\right) dx\right. \\ &\left. + \int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right] d\tau \end{aligned} \tag{3.25}$$

On the other hand, we have

$$E(t) > \varsigma_2\int_0^1 \left(\chi_t^2 + \chi_x^2 + \Xi_t^2 + \Xi^2 + \Xi_x^2 + \int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right) dx.$$

Applying Gronwall's inequality to (3.25), we obtain

$$\int_0^1 \left(\chi_t^2 + \chi_x^2 + \Xi_t^2 + \Xi^2 + \Xi_x^2 + \int_0^t k\,(t-s)\,\Xi_t^2(s)ds\right) dx \leq e^{\varsigma_3 t} E\,(0)\,.$$

This shows that solution of problem (1.1) depends continuously on the initial data. This ends the proof of Theorem (2). $\square$

4. Stability result

In this section, we use the energy method to study the asymptotic behavior of solutions of the system (1.1). First, we state and prove the following lemma.

Lemma 2. *Let* (u, ϕ) *be a solution of system* (1.1)*. Then the energy associated to the system* (1.1) *is defined by*

$$\begin{aligned} E(t) &= \frac{1}{2}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right) dx \\ &+ \frac{J}{2}\int_0^1 \left(\int_0^t k\,(t-s)\,\phi_t^2(s)ds\right) dx, \end{aligned} \tag{4.1}$$

satisfies

$$E'(t) \leq \frac{J}{2}\,(k'\Box\phi_t)\,(t) - \mu_1 \int_0^1 \phi_t^2 dx. \tag{4.2}$$

Proof. Multiplying $(1.1)_1, (1.1)_2$ by u_t, ϕ_t and integrating over $(0,1)$ and summing them up, we obtain

$$\begin{aligned} &\frac{d}{2dt}\int_0^1 \left(\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right) dx \\ &+ J\int_0^1 \left[\phi_t \left(\int_0^t k\,(t-s)\,\phi_t(s)ds\right)_t\right] dx \;=\; -\mu_1 \int_0^1 \phi_t^2 dx. \end{aligned} \tag{4.3}$$

Note that

$$\left(\int_0^t k\,(t-s)\,\phi_t(s)ds\right)_t = k\,(t)\,\phi_t(0) + \int_0^t k\,(t-s)\,\phi_{tt}(s)ds.$$

Then

$$\begin{aligned} &J\int_0^1 \left[\phi_t \left(\int_0^t k\,(t-s)\,\phi_t(s)ds\right)_t\right] dx \\ &= J\int_0^1 \phi_t \left(k\,(t)\,\phi_t(0) + \int_0^t k\,(t-s)\,\phi_{tt}(s)ds\right) dx \\ &= Jk(t)\int_0^1 \phi_t(0)\phi_t dx + J\int_0^1 \phi_t \left(\int_0^t k\,(t-s)\,\phi_{tt}(s)ds\right) dx. \end{aligned}$$

By applying the result in Lemma(1), we obtain

$$
\begin{aligned}
& J\int_0^1\left[\left(\int_0^t k\left(t-s\right)\phi_t(s)ds\right)_t\phi_t\right]dx \\
& = Jk(t)\int_0^1 \phi_t(0)\phi_t dx - \frac{J}{2}\left(k'\Box\phi_t\right)(t) \\
& + \frac{J}{2}\frac{d}{dt}\int_0^1\left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right)dx \\
& + J\frac{k(t)}{2}\int_0^1 \phi_t^2 dx - Jk(t)\int_0^1 \phi_t(0)\phi_t dx.
\end{aligned}
\tag{4.4}
$$

Inserting (4.4) in (4.3), we have

$$
\begin{aligned}
& \frac{d}{2dt}\int_0^1\left[\rho u_t^2 + \mu u_x^2 + J\phi_t^2 + 2bu_x\phi + \xi\phi^2 + \delta\phi_x^2\right. \\
& \left. + J\left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right)\right]dx \\
& = \frac{J}{2}\left(k'\Box\phi_t\right)(t) - J\frac{k(t)}{2}\int_0^1 \phi_t^2 dx - \mu_1\int_0^1 \phi_t^2 dx \\
& \leq \frac{J}{2}\left(k'\Box\phi_t\right)(t) - \mu_1\int_0^1 \phi_t^2 dx.
\end{aligned}
$$

□

Remark 1. *The energy $E(t)$ defined by (4.1) is non-negative. In fact,*

$$
\begin{aligned}
\mu u_x^2 + 2bu_x\phi + \xi\phi^2 = \frac{1}{2}\Bigg[& \mu\left(u_x + \frac{b}{\mu}\phi\right)^2 + \xi\left(\phi + \frac{b}{\xi}u_x\right)^2 \\
& + \left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2\Bigg],
\end{aligned}
$$

since $\mu\xi > b^2$, we deduce that

$$
\mu u_x^2 + 2bu_x\phi + \xi\phi^2 > \frac{1}{2}\left[\left(\mu - \frac{b^2}{\xi}\right)u_x^2 + \left(\xi - \frac{b^2}{\mu}\right)\phi^2\right].
$$

Consequently,

$$
\begin{aligned}
E(t) > \frac{1}{2}\int_0^1 & \left\{\rho u_t^2 + J\phi_t^2 + \mu_1 u_x^2 + \delta\phi_x^2 + \xi_1\phi^2\right. , \\
& \left. + J\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right\}dx,
\end{aligned}
$$

where $\xi_1 = \frac{1}{2}\left(\xi - \frac{b^2}{\mu}\right)$ and $\mu_1 = \frac{1}{2}\left(\mu - \frac{b^2}{\xi}\right)$. Then $E(t)$ is non-negative.

4.1. **Exponential stability.** In this subsection, we establish an exponential decay result of solutions the problem (1.1) in the case when (1.5) holds. For that, we need the following lemmas to achieve our goal.

Lemma 3. *Let* (u, ϕ) *be a solution of system* (1.1)*. Then, the functional*

$$F_1(t) = J\int_0^1 \phi\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)dx + \frac{b\rho}{\mu}\int_0^1 \phi\left(\int_0^x u_t(y)\,dy\right)dx + \frac{\mu_1}{2}\int_0^1 \phi^2 dx,$$

satisfies for any $\varepsilon_0 > 0$,

$$F_1'(t) \le -\delta\int_0^1 \phi_x^2 dx - 2\xi_1\int_0^1 \phi^2 dx + \left[\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right]\int_0^1 \phi_t^2 dx + \varepsilon_0\int_0^1 u_t^2 dx + \frac{J\bar{k}}{2}\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx. \tag{4.5}$$

Proof. By differentiating $F_1(t)$ and integrating by parts, we obtain

$$F_1'(t) = J\int_0^1 \phi_t^2 dx + J\int_0^1 \phi_t\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx - \delta\int_0^1 \phi_x^2 dx - b\int_0^1 u_x\phi dx - 2\xi_1\int_0^1 \phi^2 dx + b\int_0^1 u_x\phi dx + \frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t(y)\,dy\right)dx. \tag{4.6}$$

Using Young's and Cauchy-Schwarz inequalities, we obtain

$$J\int_0^1 \phi_t\left(\int_0^t k(t-s)\,\phi_t(s)ds\right)dx \le \frac{J}{2}\int_0^1 \phi_t^2 dx + \frac{J\bar{k}}{2}\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx. \tag{4.7}$$

Using Young's inequality, we get

$$\frac{b\rho}{\mu}\int_0^1 \phi_t\left(\int_0^x u_t(y)\,dy\right)dx \le \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\int_0^1 \phi_t^2 dx + \varepsilon_0\int_0^1 u_t^2 dx. \tag{4.8}$$

Inserting (4.7) and (4.8) into (4.6), we obtain (4.5). □

Lemma 4. *Let* (u, ϕ) *be a solution of system* (1.1)*. Then, the functional*

$$F_2(t) = \frac{\delta\rho b}{\mu J}\int_0^1 \phi_x u_t dx + b\int_0^1\left(\phi_t + \int_0^t k(t-s)\,\phi_t(s)ds\right)u_x dx,$$

satisfies, for any $\varepsilon_1 > 0$,

$$F_2'(t) \le -\frac{b^2}{4J}\int_0^1 u_x^2 dx + C_{\varepsilon_1}\int_0^1 \phi_x^2 dx + \varepsilon_1(2 + k(0))\int_0^1 u_t^2 dx + \frac{b^2 k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx + \frac{b^2 k(0)}{4\varepsilon_1}\int_0^1\left(\int_0^t |k'(t-s)|\,\phi_x^2(s)ds\right)dx + \frac{\mu_1^2}{J}\int_0^1 \phi_t^2 dx + \frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx} dx, \tag{4.9}$$

where $C_{\varepsilon_1} = \dfrac{\delta b^2}{\mu J} + \dfrac{b^2 k^2(0)}{4\varepsilon_1} + \dfrac{\xi^2}{2J}$.

Proof. By differentiating $F_2(t)$, and integrating by parts, we obtain

$$F_2'(t) = \frac{\rho b}{\mu}\chi \int_0^1 \phi_t u_{tx} dx - \frac{b^2}{J}\int_0^1 u_x^2 dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx$$
$$+ b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\phi_t(s)ds\right)dx - \frac{b\mu_1}{J}\int_0^1 \phi_t u_x dx. \tag{4.10}$$

Integrating by parts with respect to t the last term of (4.10),we have

$$\begin{aligned} & b\int_0^1 u_{tx}\left(\int_0^t k(t-s)\phi_t(s)ds\right)dx \\ &= b\int_0^1 u_{tx}\left[k(0)\phi(t) - k(t)\phi(0) + \int_0^t k'(t-s)\phi(s)ds\right]dx \\ &= -bk(0)\int_0^1 u_t\phi_x dx + bk(t)\int_0^1 u_t\phi_x(0)dx \\ &- b\int_0^1 u_t\left(\int_0^t k'(t-s)\phi_x(s)ds\right)dx. \end{aligned}$$

Then, (4.10) becomes

$$\begin{aligned} F_2'(t) &= \frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx}dx - \frac{b^2}{J}\int_0^1 u_x^2 dx + \frac{\delta b^2}{\mu J}\int_0^1 \phi_x^2 dx \\ &- bk(0)\int_0^1 u_t\phi_x dx + bk(t)\int_0^1 u_t\phi_x(0)dx - \frac{b\xi}{J}\int_0^1 \phi u_x dx \\ &- b\int_0^1 u_t\left(\int_0^t k'(t-s)\phi_x(s)ds\right)dx - \frac{b\mu_1}{J}\int_0^1 \phi_t u_x dx. \end{aligned} \tag{4.11}$$

By using Young's inequality, we arrive at

$$-bk(0)\int_0^1 u_t\phi_x dx \le \varepsilon_1\int_0^1 u_t^2 dx + \frac{b^2k^2(0)}{4\varepsilon_1}\int_0^1 \phi_x^2 dx, \tag{4.12}$$

$$-\frac{b\mu_1}{J}\int_0^1 \phi_t u_x dx \le \frac{b^2}{4J}\int_0^1 u_x^2 dx + \frac{\mu_1^2}{J}\int_0^1 \phi_t^2 dx \tag{4.13}$$

and

$$\begin{aligned} +bk(t)\int_0^1 u_t\phi_x(0)dx &\le \varepsilon_1 k(t)\int_0^1 u_t^2 dx + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx \\ &\le \varepsilon_1 k(0)\int_0^1 u_t^2 dx + \frac{b^2k(t)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx, \end{aligned} \tag{4.14}$$

Young's and Cauchy-Schwarz inequalities leads to

$$\begin{aligned} &- b\int_0^1 u_t\left(\int_0^t k'(t-s)\phi_x(s)ds\right)dx \\ &\le \varepsilon_1\int_0^1 u_t^2 dx + \frac{b^2k(0)}{4\varepsilon_1}\int_0^1\left(\int_0^t |k'(t-s)|\phi_x^2(s)ds\right)dx. \end{aligned} \tag{4.15}$$

By using Young's and Poincaré inequalities, we have

$$-\frac{b\xi}{J}\int_0^1 \phi u_x dx \le \frac{b^2}{4J}\int_0^1 u_x^2 dx + \frac{\xi^2}{J}\int_0^1 \phi_x^2 dx. \tag{4.16}$$

By substituting (4.12)-(4.16) in (4.11) and taking into account that $\chi = 0$, we get (4.9). □

Lemma 5. *Let* (u, ϕ) *be a solution of system* (1.1)*. Then, the functional*

$$F_3(t) = -\int_0^1 u u_t dx,$$

satisfies,

$$F_3'(t) \leq -\rho \int_0^1 u_t^2 dx + \frac{b^2}{2\mu} \int_0^1 \phi_x^2 dx + \frac{3\mu}{2} \int_0^1 u_x^2 dx. \tag{4.17}$$

Proof. Differentiating $F_2(t)$ and integrating by parts, we obtain

$$F_3'(t) = -\rho \int_0^1 u_t^2 dx + \mu \int_0^1 u_x^2 dx - b \int_0^1 u \phi_x dx,$$

Young's and Poincarré inequalities give (4.17). □

Lemma 6. *(*[25]*)Let* (u, ϕ) *be a solution of system* (1.1)*. Then, the functionals*

$$F_4(t) = e^{-\varsigma t} \int_0^1 \left(\int_0^t e^{\varsigma s} \tilde{H}_1(t-s) \phi_t^2(s) ds \right) dx,$$
$$F_5(t) = e^{-\tau t} \int_0^1 \left(\int_0^t e^{\tau s} \tilde{H}_2(t-s) \phi_x^2(s) ds \right) dx,$$

satisfy, $\forall t \geq 0$,

$$F_4'(t) = -\varsigma F_4(t) + \tilde{H}_1(0) \int_0^1 \phi_t^2 dx - \int_0^1 \left(\int_0^t k(t-s) \phi_t^2(s) ds \right) dx, \tag{4.18}$$

$$F_5'(t) = -\tau F_5(t) + \tilde{H}_2(0) \int_0^1 \phi_x^2 dx - \int_0^1 \left(\int_0^t |k'(t-s)| \phi_x^2(s) ds \right) dx, \tag{4.19}$$

where $\tilde{H}_1(t) = \int_t^\infty e^{\varsigma s} |k(s)| ds$ *and* $\tilde{H}_2(t) = \int_t^\infty e^{\tau s} |k'(s)| ds$.

Now, we define the Lyapunov functional $\mathcal{L}(t)$ by

$$\mathcal{L}(t) = NE(t) + N_1 F_1(t) + N_2 F_2(t) + F_3(t) + N_3 F_4(t) + N_4 F_5(t), \tag{4.20}$$

where N, N_1, N_2, N_3 and N_4 are positive constants.

Theorem 3. *Let* (u, ϕ) *be a solution of* (1.1)*. Then, there exist two positive constants* κ_1 *and* κ_2 *such that the Lyapunov functional* (4.20) *satisfies*

$$\kappa_1 (E(t) + F_4(t) + F_5(t)) \leq \mathcal{L}(t) \leq \kappa_2 (E(t) + F_4(t) + F_5(t)), \ \forall t \geq 0, \tag{4.21}$$

and

$$\mathcal{L}'(t) \leq -\beta_1 (E(t) + F_4(t) + F_5(t)) + C_2 k(t) + N_2 \frac{\rho b}{\mu} \chi \int_0^1 \phi_t u_{tx} dx, \ \beta_1 > 0. \tag{4.22}$$

Proof. From (4.20), we have

$$\begin{aligned}
&\left|\mathcal{L}(t)-NE\left(t\right)-N_3F_4\left(t\right)-N_4F_5\left(t\right)\right| \\
&\leq N_1J\int_0^1\left|\phi\right|.\left|\phi_t+\int_0^t k\left(t-s\right)\phi_t(s)ds\right|dx+N_1\frac{\mu_1}{2}\int_0^1\phi^2dx \\
&+N_1\frac{\left|b\right|\rho}{\mu}\int_0^1\left|\phi\right|\left(\int_0^x\left|u_t\left(y\right)\right|dy\right)dx+N_2\frac{\delta\rho\left|b\right|}{\mu J}\int_0^1\left|\phi_x\right|\left|u_t\right|dx \\
&+N_2\left|b\right|\int_0^1\left|u_x\right|\left|\phi_t+\int_0^t k\left(t-s\right)\phi_t(s)ds\right|dx+\rho\int_0^1\left|u\right|\left|u_t\right|dx.
\end{aligned}$$

By using Young's ,Cauchy-Schwarz and poincarré inequalities ,we obtain

$$\left|\mathcal{L}(t)-NE\left(t\right)-N_3F_4\left(t\right)-N_4F_5\left(t\right)\right|\leq\lambda_1E(t).$$

Therefore,

$$\left(N-\lambda_1\right)E\left(t\right)+N_3F_4\left(t\right)+N_4F_5\left(t\right)\leq\mathcal{L}(t)\leq\left(N+\lambda_1\right)E\left(t\right)+N_3F_4\left(t\right)+N_4F_5\left(t\right),$$

by choosing N (depending on N_1, N_2, N_3, N_4) sufficiently large we obtain (4.21) with

$$\begin{aligned}
\kappa_1&=\min\left\{N-\lambda_1,N_3,N_4\right\}, \\
\kappa_2&=\max\left\{N+\lambda_1,N_3,N_4\right\}.
\end{aligned}$$

Now, by differentiating $\mathcal{L}\left(t\right)$, exploiting (4.2), (4.5), (4.9), (4.17), (4.18), (4.19) and setting $\varepsilon_0=\dfrac{\rho}{4N_1}$, $\varepsilon_1=\dfrac{\rho}{4N_2\left(2+k(0)\right)}$, we get

$$\begin{aligned}
\mathcal{L}'(t)\leq&-\left[N\mu_1-N_1\left(\frac{3J}{2}+\frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right)-N_3\tilde{H}_1\left(0\right)-N_2\frac{\mu_1^2}{J}\right]\int_0^1\phi_t^2dx \\
&+\frac{NJ}{2}\left(k'\square\phi_t\right)\left(t\right)-\frac{\rho}{2}\int_0^1u_t^2dx-2N_1\xi_1\int_0^1\phi^2dx \\
&-\left[\delta N_1-N_2C_{\varepsilon_1}-\frac{b^2}{2\mu}-N_4\tilde{H}_2\left(0\right)\right]\int_0^1\phi_x^2dx \\
&-\left(\frac{b^2}{4J}N_2-\frac{3\mu}{2}\right)\int_0^1u_x^2dx-\varsigma N_3F_4\left(t\right)-\tau N_4F_5\left(t\right) \\
&-\left(N_3-\frac{J\bar{k}}{2}N_1\right)\int_0^1\left(\int_0^t k\left(t-s\right)\phi_t^2(s)ds\right)dx \\
&-\left[N_4-\frac{N_2^2b^2k(0)\left(2+k(0)\right)}{\rho}\right]\int_0^1\left(\int_0^t\left|k'\left(t-s\right)\right|\phi_x^2(s)ds\right)dx \\
&+\frac{b^2N_2^2k(t)\left(2+k(0)\right)}{\rho}\int_0^1\phi_{0x}^2dx+N_2\frac{\rho b}{\mu}\chi\int_0^1\phi_tu_{tx}dx.
\end{aligned}$$

We select our parameters appropriately as follows
First, we choose N_2 large enough such that

$$\frac{b^2}{4J}N_2-\frac{3\mu}{2}>0.$$

We pick N_4 large such that

$$N_4-\frac{N_2^2b^2k(0)\left(2+k(0)\right)}{\rho}>0.$$

We select N_1 large enough such that :

$$\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4 \tilde{H}_2(0) > 0.$$

We choose N_3 large such that

$$N_3 - \frac{J\bar{k}}{2} N_1 > 0.$$

Finally, we take N large enough (even larger so that (4.21) remains valid) such that

$$N\mu_1 - N_1\left(\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3\tilde{H}_1(0) - N_2\frac{\mu_1^2}{J} > 0.$$

All these choices leads to

$$\begin{aligned}\mathcal{L}'(t) \leq{}& -\alpha_1\int_0^1\left(\phi_t^2+\phi_x^2+u_t^2+u_x^2+\phi^2\right)dx - \int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\ &+\alpha_2 k(t)\int_0^1 \phi_{0x}^2 dx - \varsigma N_3 F_4(t) - N_4\tau F_5(t) + N_2\frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_{tx}dx, \qquad (4.23)\end{aligned}$$

where $\alpha_1, \alpha_2 > 0$.
On the other hand, from Eq. (4.1) and by using Young's inequality, we obtain

$$\begin{aligned}E(t) \leq{}& \frac{1}{2}\int_0^1\left(\rho u_t^2 + J\phi_t^2 + (\mu+|b|)\,u_x^2 + \delta\phi_x^2 + (\xi+|b|)\,\phi^2\right)dx \\ &+\frac{J}{2}\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\ \leq{}& \varrho_1\left(\int_0^1\left(u_t^2+\phi_t^2+u_x^2+\phi_x^2+\phi^2\right)dx\right. \\ &\left.+\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx\right),\ \ \varrho_1 \ > \ 0,\end{aligned}$$

which implies that

$$-\int_0^1\left(u_t^2+\phi_t^2+u_x^2+\phi_x^2+\phi^2\right)dx - \int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \leq -\varrho_2 E(t),\ \varrho_2 > 0. \qquad (4.24)$$

The combination of (4.23) and (4.24) gives (4.22) with $C_2 = \alpha_2\int_0^1 \phi_{0x}^2 dx$. $\square$

We are now ready to state and prove the following exponential stability result

Lemma 7. *Let (u,ϕ) be a solution of* (1.1) *and assume that* (1.2), ***(H1)-H(2)*** *hold and $\chi = 0$. Then, there exist two positive constants τ_1 and τ_2 such that*

$$E(t) \leq \tau_2 e^{-\tau_1 t},\ \forall t \geq 0. \qquad (4.25)$$

Proof. By using (4.22) and the right side of (4.21), we get

$$\mathcal{L}'(t) \leq -C_1\mathcal{L}(t) + C_2 k(t), \qquad (4.26)$$

where $C_1 = \dfrac{\beta_1}{\kappa_2} > 0$.
Multiplying (4.26) by $\exp(C_1 t)$, we obtain

$$\frac{d}{dt}\left(\mathcal{L}(t)\exp(C_1 t)\right) \leq C_2\exp(C_1 t)\,k(t). \qquad (4.27)$$

Integrating over $(0,T)$ the inequation (4.27) and choosing C_1 smaller than ς, we have

$$\mathcal{L}(T)\exp\left(C_1T\right) \leq \mathcal{L}(0) + C_2\int_0^T \exp\left(\varsigma t\right)k(t)dt$$
$$\leq \mathcal{L}(0) + C_2\int_0^\infty \exp\left(\varsigma t\right)k(t)dt.$$

Thanks to the hypothesis **(H2)**, we can write

$$\mathcal{L}(T) \leq C_3\exp\left(-C_1T\right),\ C_3 > 0,$$

which yields the serial result (4.25), using the fact that $F_4(t), F_5(t)$ are positive and the other side of the equivalence relation (4.21) again. The proof is complete. □

4.2. **Polynomial stability.** Here, we prove a polynomial decay result of solutions of the problem (1.1) when (1.5) does not holds by assuming that the function k verifies the same hypotheses **(H1)-(H2)** and the additional assumption

- **(H3)** $-\omega k(t) \leq k'(t) \leq 0$, where ω is a positive constant.

In order to establish the desired result of this subsection, we need to use the second-order energy $E_2(t)$ which has been calculate by using the multiplier technique as in the case of $E(t)$. For that, by differentiating $(1.1)_1$ and $(1.1)_2$ with respect to time, we obtain the following new system

$$\begin{cases} \rho u_{ttt} = \mu u_{xxt} + b\phi_{xt},\ x \in (0,1),\ t > 0, \\ J\phi_{ttt} + J\left(\displaystyle\int_0^t k(t-s)\phi_t(s)\,ds\right)_{tt} = \delta\phi_{xxt} - bu_{xt} - \xi\phi_t + \mu_1\phi_{tt},\ x \in (0,1),\ t > 0, \end{cases} \tag{4.28}$$

with boundary conditions

$$u_{xt}(0,t) = u_{xt}(1,t) = \phi_t(0,t) = \phi_t(1,t) = 0,\ t \geq 0,$$

and initial data

$$\begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ u_{tt}(x,0) = u_2(x),\ x \in (0,1), \\ \phi(x,0) = \phi_0(x),\ \phi_t(x,0) = \phi_1(x),\ \phi_{tt}(x,0) = \phi_2(x),\ x \in (0,1). \end{cases}$$

Note that

$$\begin{aligned} &\left(\int_0^t k(t-s)\phi_t(s)\,ds\right)_{tt} \\ &= \left(\int_0^t k(s)\phi_t(t-s)\,ds\right)_{tt} \\ &= \left(\int_0^t k(s)\phi_{tt}(t-s)\,ds + k(t)\phi_t(0)\right)_t \\ &= \int_0^t k(t-s)\phi_{ttt}(s)\,ds + k(t)\phi_{tt}(0) + k'(t)\phi_t(0). \end{aligned}$$

Then, the system (4.28) can be rewritten as follows

$$\begin{cases} \rho u_{ttt} = \mu u_{xxt} + b\phi_{xt},\ x \in (0,1),\ t > 0 \\ J\phi_{ttt} + J\displaystyle\int_0^t k(t-s)\phi_{ttt}(s)\,ds + Jk(t)\phi_2 + Jk'(t)\phi_1 \\ = \delta\phi_{xxt} - bu_{xt} - \xi\phi_t - \mu_1\phi_{tt},\ x \in (0,1),\ t > 0, \end{cases} \tag{4.29}$$

where $\phi_2 = \phi_{tt}(0)$ and $\phi_1 = \phi_t(0)$ are depend on x.

Lemma 8. *The second-order energy* $E_2(t)$ *associated to the system* (1.1) *is defined by*

$$E_2(t) = \frac{1}{2}\int_0^1 \left(\rho u_{tt}^2 + J\phi_{tt}^2 + \xi\phi^2 + \delta\phi_{xt}^2 + \mu u_{xt}^2 + 2b\phi_t u_{tx}\right) dx + \frac{J}{2}\int_0^1 \left(\int_0^t k(t-s)\,\phi_{tt}^2(s)\,ds\right) dx, \tag{4.30}$$

satisfies

$$E_2'(t) \le -Jk'(t)\int_0^1 \phi_1\phi_{tt}dx - \mu_1\int_0^1 \phi_{tt}^2 dx + \frac{J}{2}\left(k'\Box\phi_{tt}\right)(t), \tag{4.31}$$

and

$$E_2(t) \le l, \ \forall t \ge 0. \tag{4.32}$$

Proof. By multiplying $(4.29)_1$ by u_{tt}, $(4.29)_2$ by ϕ_{tt}, integrating over $(0,1)$ and summing up, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_0^1 \left(\rho u_{tt}^2 + J\phi_{tt}^2 + \xi\phi^2 + \delta\phi_{xt}^2 + \mu u_{xt}^2 + 2b\phi_t u_{tx}\right) dx \\ &+ Jk(t)\int_0^1 \phi_{tt}\phi_2 dx + Jk'(t)\int_0^1 \phi_{tt}\phi_1 dx \\ &+ J\int_0^1 \phi_{tt}\left(\int_0^t k(t-s)\,\phi_{ttt}(s)ds\right) dx = -\mu_1\int_0^1 \phi_{tt}^2 dx.\end{aligned} \tag{4.33}$$

By using again the result in lemma (1) to estimate the last term of (4.33), we get

$$\begin{aligned}&J\int_0^1 \phi_{tt}\left(\int_0^t k(t-s)\,\phi_{ttt}(s)ds\right) dx \\ &= \frac{J}{2}\frac{d}{dt}\int_0^1 \left(\int_0^t k(t-s)\,\phi_{tt}^2(s)\,ds\right) dx - Jk(t)\int_0^1 \phi_2\phi_{tt}dx \\ &+ \frac{Jk(t)}{2}\int_0^1 \phi_{tt}^2 dx - \frac{J}{2}\left(k'\Box\phi_{tt}\right)(t).\end{aligned} \tag{4.34}$$

By using the positivity of $k(t)$ and the combination of (4.33) with (4.34), we have (4.30) and (4.31).

Now, by using the hypothesis **(H3)** and Young's inequality, we can write

$$-Jk'(t)\int_0^1 \phi_1\phi_{tt}dx \le J\delta_1\omega k(t)\int_0^1 \phi_{tt}^2 dx + \frac{J\omega k(t)}{4\delta_1}\int_0^1 \phi_1^2 dx, \tag{4.35}$$

letting $\delta_1 = \dfrac{1}{2\omega}$ and because $k'(t) \le 0$, then, (4.31) becomes

$$E_2'(t) \le \frac{J\omega^2 k(t)}{2}\int_0^1 \phi_1^2 dx = \zeta k(t),$$

where $\zeta = \dfrac{J\omega^2}{2}\displaystyle\int_0^1 \phi_1^2 dx > 0$. A simple integration over $(0,T)$ and by the hypothesis **(H1)**, we obtain (4.32). □

We introduce the following functional

$$\tilde{F}_2(t) = -\frac{\rho b}{\mu}\chi\int_0^1 \phi_t u_x dx,$$

that satisfies

$$\tilde{F}_2'(t) = -\frac{\rho b}{\mu}\chi \int_0^1 u_{tx}\phi_t dx - \frac{\rho b}{\mu}\chi \int_0^1 \phi_{tt} u_x dx.$$

By using Young's inequality, we get

$$-\frac{\rho b}{\mu}\chi \int_0^1 \phi_{tt} u_x dx \leq \frac{b^2}{8J}\int_0^1 u_x^2 dx + C_0 \int_0^1 \phi_{tt}^2 dx.$$

Then,

$$\tilde{F}_2'(t) \leq \frac{b^2}{8J}\int_0^1 u_x^2 dx + C_0 \int_0^1 \phi_{tt}^2 dx - \frac{\rho b}{\mu}\chi \int_0^1 u_{tx}\phi_t dx.$$

We define the following Lyapunov functional as follows

$$\begin{aligned} \mathcal{L}_1(t) &= N\left(E\left(t\right) + E_2(t)\right) + N_1 F_1\left(t\right) + N_2\left(F_2\left(t\right) + \tilde{F}_2(t)\right) + F_3\left(t\right) \\ &\quad + N_3 F_4\left(t\right) + N_4 F_5\left(t\right). \end{aligned} \tag{4.36}$$

The Lyapunov functional $\mathcal{L}_1$ defined by (4.36) is not equivalent to the energy functional E, but it is equivalent to $E + E_2 + F_4(t) + F_5(t)$. Indeed by using (4.36), Young's, Poincaré's and Cauchy-Schwarz inequalities, we have

$$\begin{aligned} &\left|\mathcal{L}_1(t) - N\left(E\left(t\right) + E_2(t)\right) - N_3 F_4\left(t\right) - N_4 F_5\left(t\right)\right| \\ &\leq \lambda_1 E\left(t\right) + \lambda_2 E_2(t) \\ &\leq \beta\left(E\left(t\right) + E_2(t)\right),\ \beta = \max\left(\lambda_1, \lambda_2\right), \end{aligned}$$

$$\begin{aligned} &\left(N - \beta\right)\left(E\left(t\right) + E_2(t)\right) + N_3 F_4\left(t\right) + N_4 F_5\left(t\right) \\ &\leq \mathcal{L}_1(t) \leq \left(N + \beta\right)\left(E\left(t\right) + E_2(t)\right) + N_3 F_4\left(t\right) + N_4 F_5\left(t\right). \end{aligned}$$

Now by choosing N sufficiently large, we obtain

$$\rho_1\left(E\left(t\right) + E_2(t) + F_4\left(t\right) + F_5\left(t\right)\right) \leq \mathcal{L}_1(t) \leq \rho_2\left(E\left(t\right) + E_2(t) + F_4\left(t\right) + F_5\left(t\right)\right),$$

where

$$\rho_1 = \min\left\{N - \beta, N_3, N_4\right\},\ \rho_2 = \max\left\{N + \beta, N_3, N_4\right\}$$

Therefore,

$$\mathcal{L}_1(t) \sim E + E_2 + F_4 + F_5.$$

Now, we are ready to state and prove the polynomial stability result

Lemma 9. *Let (u, ϕ) be a solution of* (1.1) *and assume that* (1.2), ***(H1)-H(3)*** *hold and $\chi \neq 0$. Then, there exits a positive constant C_3 such that*

$$E(t) \leq \frac{C_3}{t},\ t > 0.$$

Proof. First, note that when $\chi \neq 0$, we have

$$\begin{aligned} F_2'(t) + \tilde{F}_2'(t) &\leq -\frac{b^2}{8J}\int_0^1 u_x^2 dx + \left(\frac{\delta b^2}{\mu J} + \frac{b^2 k^2\left(0\right)}{4\varepsilon_1} + \frac{\xi^2}{2J}\right)\int_0^1 \phi_x^2 dx \\ &\quad + \varepsilon_1\left(2 + k\left(0\right)\right)\int_0^1 u_t^2 dx + \frac{b^2 k\left(t\right)}{4\varepsilon_1}\int_0^1 \phi_{0x}^2 dx + \frac{\mu_1^2}{J}\int_0^1 \phi_t^2 dx \\ &\quad + \frac{b^2 k\left(0\right)}{4\varepsilon_1}\int_0^1\left(\int_0^t k'\left(t-s\right)\phi_x^2(s) ds\right) dx + C_0\int_0^1 \phi_{tt}^2 dx. \end{aligned} \tag{4.37}$$

By differentiating $\mathcal{L}_1$ and using (4.2), (4.5), (4.37), (4.17), (4.18) and (4.19), we get

$$\begin{aligned}
\mathcal{L}_1'(t) \leq & -\left[N\mu_1 - N_1\left(\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3\tilde{H}_1(0) - N_2\frac{\mu_1^2}{J}\right]\int_0^1 \phi_t^2 dx \\
& + \frac{NJ}{2}\left(k'\Box\phi_t\right)(t) - \frac{\rho}{2}\int_0^1 u_t^2 dx - 2N_1\xi_1\int_0^1 \phi^2 dx \\
& - \left[\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\tilde{H}_2(0)\right]\int_0^1 \phi_x^2 dx \\
& - \left(\frac{b^2}{8J}N_2 - \frac{3\mu}{2}\right)\int_0^1 u_x^2 dx - \varsigma N_3 F_4(t) - \tau N_4 F_5(t) \\
& - \left(N_3 - \frac{J\bar{k}}{2}N_1\right)\int_0^1\left(\int_0^t k(t-s)\,\phi_t^2(s)ds\right)dx \\
& - \left[N_4 - \frac{N_2^2 b^2 k(0)\left(2+k(0)\right)}{\rho}\right]\int_0^1\left(\int_0^t \left|k'(t-s)\right|\phi_x^2(s)ds\right)dx \\
& + \frac{b^2 N_2^2 k(t)\left(2+k(0)\right)}{\rho}\int_0^1 \phi_{0x}^2 dx - \left(N\mu_1 - N_2 C_0\right)\int_0^1 \phi_{tt}^2 dx \\
& + \frac{JN}{2}\left(k'\Box\phi_{tt}\right)(t) + N\zeta k(t).
\end{aligned}$$

We select our parameters as follows First, we choose N_2 large enough such that

$$\frac{b^2}{8J}N_2 - \frac{3\mu}{2} > 0.$$

We pick N_4 large such that

$$N_4 - \frac{N_2^2 b^2 k(0)\left(2+k(0)\right)}{\rho} > 0.$$

We select N_1 large enough such that :

$$\delta N_1 - N_2 C_{\varepsilon_1} - \frac{b^2}{2\mu} - N_4\tilde{H}_2(0) > 0.$$

We choose N_3 large such that

$$N_3 - \frac{J\bar{k}}{2}N_1 > 0.$$

Finally, we take N large enough (even larger so that (4.21) remains valid) such that

$$\begin{cases} N\mu_1 - N_1\left(\frac{3J}{2} + \frac{b^2\rho^2}{4\mu^2\varepsilon_0}\right) - N_3\tilde{H}_1(0) - N_2\frac{\mu_1^2}{J} > 0, \\ \text{and} \\ N\mu_1 - N_2 C_0 > 0. \end{cases}$$

Which leads to

$$\mathcal{L}_1'(t) \leq -\omega_0 E(t) + \omega_1 k(t),$$

and by integrating over $(0,T)$, we get

$$\omega_0 E(T)T \leq -\mathcal{L}_1(T) + \mathcal{L}_1(0) + \omega_1 \int_0^T k(t)dt,$$
$$\leq \mathcal{L}_1(0) + \omega_1 \int_0^\infty k(t)dt = l.$$

So

$$E(T) \leq \frac{C_3}{T},$$

with

$$C_3 = \frac{l}{\omega_0}.$$

The proof is complete. □

Conclusion

In this paper, we studied the asymptotic behavior of the solution of porous-elastic system in the presence of neutral delay. Introducing a single damping mechanism given by this type of delay makes our problem different from those considered so far in the literature and under some assumptions imposed on the kernel of delay, we have been able to prove an explicit energy decay rate that depends of the wave speeds of propagation.

Conflict of interest

This work does not have any conflicts of interest, and there are no funders to report for this submission.

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