Novel results on the multi-parameters Mittag-Leffler function

Yu-Mei Pan[1] and Jian-Gen Liu[2]

[1]China University of Mining and Technology - Xuzhou Campus
[2]China University of Mining and Technology

July 18, 2022

Abstract

In this article, the multi-parameters Mittag-Leffler function is studied in detail. As a consequence, a series of novel results such as the integral representation, series representation and Mellin transform to the above function, are obtained. Especially, we associate the multi-parameters Mittag- Leffler function with two special functions which are the generalized Wright hypergeometric and the Fox's-H functions. Meanwhile, some interesting integral operators and derivative operators of this function, are also discussed.

Novel results on the multi-parameters Mittag-Leffler function

Yu-Mei Pan[1], Jian-Gen Liu[1⋆,2]

1. School of Mathematics, China University of Mining and Technology, Xuzhou 221116, Jiangsu, People's Republic of China

2. State Key Laboratory for Geomechanics and Deep Underground Engineering, China University of Mining and Technology, Xuzhou 221116, Jiangsu, People's Republic of China.

Abstract: In this article, the multi-parameters Mittag-Leffler function is studied in detail. As a consequence, a series of novel results such as the integral representation, series representation and Mellin transform to the above function, are obtained. Especially, we associate the multi-parameters Mittag- Leffler function with two special functions which are the generalized Wright hypergeometric and the Fox's-H functions. Meanwhile, some interesting integral operators and derivative operators of this function, are also discussed.

Key words: Multi-parameters Mittag-Leffler function; Special functions; Riemann-Liouville integral

MSC: 00A05; 33E12

1. Introductions

Fractional calculus is one of the important branches of modern pure mathematics and applied mathematics. However, the special functions play an essential role in constructing differential general fractional derivatives and building relations and identities between other fractional derivatives [1,2]. Among them, the Mittag-Leffler function is an important function in the field of fractional calculus, which was first introduced by the Swedish mathematician M.G. Mittag Leffler at the beginning of 20-th century to deal with the divergent series [3]. The Mittag-Leffler function is generally regarded as a generalized form of the exponential function. Usually, it naturally occurs as the solution of fractional differential equations or fractional integral equations [4]. Subsequently, the generalized form of the Mittag-Leffler function and its applications have aroused wide interest of many scholars from all around the world and given a series of novel results about them (see Ref. [1,2,5,6]).

Let us recall the Mittag-Leffler function in detail. In 1903, the Mittag-Leffler function of one parameter was defined as [7]

$$E_{\alpha}(z)=\sum_{n=0}^{\infty}\frac{z^{n}}{\Gamma(\alpha n+1)},(\alpha,z\in\mathbb{C},\Re(\alpha)>0),$$

⋆Corresponding author. E-mail:ljgzr557@126.com(J.G. Liu)

where $\Gamma(\cdot)$ denotes the Euler's gamma function. When $\alpha = 1$, it can be reduced into the classical exponential function [1].

In 1905, Wiman introduced the two parameters Mittag-Leffler function [8], defined as

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, (\alpha, \beta, z \in \mathbb{C}, \Re(\alpha) > 0).$$

And then in 1971, T.R. Prabhakar gave the three parameters Mittag-Leffler function also known as the Prabhakar function [9], which was defined by

$$E_{\alpha,\beta}^{\gamma}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, (\alpha, \beta, \gamma, z \in \mathbb{C}, \Re(\alpha) > 0),$$

this function plays a fundamental role in the description of the anomalous dielectric properties in disordered materials and heterogeneous systems manifesting simultaneous nonlocality and nonlinearity [10].

Next, Shukla and Prajapati introduced another extension form of the Mittag-Leffler function in Ref.[11] (see also Srivastava and Tomovski [12]), which was defined as

$$E_{\alpha,\beta}^{\delta,q}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!},$$

where $q, \alpha, \beta, \delta, z \in \mathbb{C}$ and $(\delta)_{nq} = \frac{\Gamma(\delta+nq)}{\Gamma(\delta)}$ is the generalized Pochhammer symbol [13].

Further, Salim and Faraj presented in [15] by the following expression of the form

$$E_{\rho,\sigma,\tau}^{\delta,q,r}(z) = \sum_{n=0}^{\infty} \frac{(\delta)_{qn}}{\Gamma(\rho n + \sigma)} \frac{z^n}{(\tau)_{rn}}, \tag{1}$$

where $z, \rho, \sigma, \delta, \tau \in \mathbb{C}$, $\min\{\Re(\rho), \Re(\sigma), \Re(\delta), \Re(\tau)\} > 0$, $q, r > 0$ and $q \leq \Re(\rho) + r$. Meanwhile, its corresponding integral operator was defined as

$$\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,q,r} f\right)(x) = \int_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,q,r}\left(w(x-t)^{\rho}\right) f(t)\, dt. \tag{2}$$

Later in 2014, Özarslan extended the Mittag-Leffler function [17] as follows:

$$E_{\alpha,\beta}^{\gamma,c}(z;p) = \sum_{n=0}^{\infty} \frac{B_p(\gamma+n, c-\gamma)}{B(\gamma, c-\gamma)} \frac{(c)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!}, (p \geq 0; \Re(c) > \Re(\gamma) > 0),$$

where

$$B_p(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1} e^{-\frac{p}{t(1-t)}}\, dt, (\min\{\Re(x), \Re(y), \Re(p)\} > 0), \tag{3}$$

is the extended Euler's Beta function [18].

In 2017, Rahman et al. presented in [19] the function of the form

$$E_{\rho,\sigma}^{\delta,q,c}(z;p)=\sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{z^n}{n!},\tag{4}$$

where $z,\rho,\sigma,\delta,c\in\mathbb{C};\Re(\rho),\Re(\delta),\Re(c)>0;p\geq 0,q>0$ with B_p as an extension of the beta function. Its the corresponding integral operator was defined by [19]

$$\left(\varepsilon_{a+,\rho,\sigma}^{w,\delta,q,c}f\right)(x;p)=\int_a^x(x-t)^{\sigma-1}E_{\rho,\sigma,}^{\delta,q,c}(w(x-t)^{\rho};p)\,f(t)\,dt.\tag{5}$$

The another recent extension was defined by Andrić et al. in Ref.[20]

$$E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)=\sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{z^n}{(\tau)_{nr}},\tag{6}$$

where $\rho,\sigma,\tau,\delta,c\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau),\Re(\delta),\Re(c)>0$, $p\geq 0,r>0$, and $0<q\leq r+\Re(\rho)$.

The purpose of this paper is to derive some novel results for the multi-parameters Mittag-Leffler function. Besides, the relationship between the multi-parameters Mittag-Leffler function and the generalized Wright hypergeometric and the Fox's-H functions, are also found. To the best knowledge of the authors, these good results have not appeared elsewhere.

2. Preliminary

Before giving novel results for the multi-parameters Mittag-Leffler function, let us introduce the relevant basic knowledge. Throughout this investigation, we need the following well-known facts and rules to study the various properties and relations formulas of the function $E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)$.

Definition 1 [20] Let $\rho,\sigma,\tau,\delta,c,w\in\mathbb{C}$, $\Re(\rho),\Re(\sigma),\Re(\tau),\Re(\delta),\Re(c)>0$, with $p\geq 0,r>0$, and $0<q\leq r+\Re(\rho)$. Let $f\in L_1[a,b]$ and $x\in[a,b]$. Then the generalized fractional integral operator $\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c}f$ is defined by the following expression

$$\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c}f\right)(x;p)=\int_a^x(x-t)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,r,q,c}(w(x-t)^{\rho};p)\,f(t)\,dt.\tag{7}$$

Definition 2 [21] The left Riemann-Liouville fractional integral ${}^{RL}I_{c+}^{\alpha}$ and derivative ${}^{RL}D_{c+}^{\alpha}$ of order α of a function $f(x)\in C[a,b]$ are defined as

$${}^{RL}I_{c+}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_c^x(x-t)^{\alpha-1}f(t)\,dt,(\Re(\alpha)>0)\tag{8}$$

and

$$^{RL}D_{c+}^{\alpha}f(x)=\frac{d^n}{dx^n}\left(^{RL}I_{c+}^{n-\alpha}f(x)\right),(\Re(\alpha)\geq 0,n=[\Re(\alpha)]+1), \tag{9}$$

where $[x]$ denotes the greatest integer in the real number x. In the above definition, we note that, the λ order fraction integral of a power function $(t-a)^{\mu-1}$ can be written as follows [21]

$$I_{a+}^{\lambda}\left[(t-a)^{\mu-1}\right](x)=\frac{\Gamma(\mu)}{\Gamma(\lambda+\mu)}(x-a)^{\lambda+\mu-1},(\lambda,\mu\in\mathbb{C},\Re(\lambda),\Re(\mu)>0). \tag{10}$$

Definition 3 [22,24] Let $0<\mu<1$, $0\leq v\leq 1$, and $f\in L_1(a,b)$. The left Hilfer fractional derivative $D_{a+}^{\mu,v}$ with respect to x is defined as follows

$$\left(D_{a+}^{\mu,v}f\right)(x)=\left(I_{a+}^{v(1-\mu)}\frac{d}{dx}\left(I_{a+}^{(1-v)(1-\mu)}f\right)\right)(x). \tag{11}$$

Eq.(11) yields the classical Riemann-Liouville fraction derivative operator D_{a+}^{μ} when $v=0$. Moreover, in its special case when $v=1$, the definition (11) would reduce to the familiar Caputo fractional derivative operator [12](see, for details, [23,p.90 et seq.]). According to Eq.(11), the following result holds true for the fractional derivative operator $D_{a+}^{\mu,v}$ [10]

$$\left(D_{a+}^{\mu,v}(t-a)^{\lambda-1}\right)(x)=\frac{\Gamma(\lambda)}{\Gamma(\lambda-\mu)}(x-a)^{\lambda-\mu-1},(x>a;0<\mu<1;0\leq v\leq 1;\Re(\lambda)>0). \tag{12}$$

The generalized Wright hypergeometric function is defined as[23]

$${}_p\Psi_q(z)={}_p\Psi_q\left[\begin{array}{cc}(a_i,A_i)_{1,p} & \\ & ,z\\ (b_j,B_j)_{1,q} & \end{array}\right]\equiv\sum_{n=0}^{\infty}\frac{\Pi_{i=1}^{p}\Gamma(a_i+A_in)}{\Pi_{j=1}^{q}\Gamma(b_j+B_jn)}\frac{z^n}{n!}$$

where $z\in\mathbb{C};a_i,b_j\in\mathbb{C};A_i,B_j\in\mathbb{R};(i=1,2,...,p;j=1,2,...,q)$.

Definition 4 [23] The relationship between the Wright generalized hypergeometric and Fox's-H function is defined as

$${}_p\Psi_q\left[\begin{array}{cccc}(a_1,A_1) & \cdots & (a_p,A_p) & \\ & & & ,z\\ (b_1,B_1) & \cdots & (b_q,B_q) & \end{array}\right]=H_{p,q+1}^{1,p}\left[-z\left|\begin{array}{c}(1-a_1,A_1),\cdots,(1-a_p,A_p)\\ (0,1),(1-b_1,B_1),\cdots,(1-b_q,B_q)\end{array}\right.\right], \tag{13}$$

where $a_j,A_j\in\mathbb{C};\Re(A_j)>0;(j=1,\ldots,p)$.

The Mellin transform of the function $f(z)$ is defined as[24]

$$\mathrm{M}\{f(z);s\}=F(s)=\int_0^{\infty}z^{s-1}f(z)\,dz,(\Re(s)>0). \tag{14}$$

3. Main results

In this section, we present the main results of this paper.

Theorem 1 If the extended Mittag-Leffler function is defined by (6), then

$$E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) = \frac{1}{B(\delta, c-\delta)} \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}} E_{\rho,\sigma,r}^{c,\tau,q}(t^q z)\, dt, \tag{15}$$

where $E_{\rho,\sigma,r}^{c,\tau,q}(z)$ is defined in (1).

Proof. Using equation (3) in equation (6), one gets

$$E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) = \sum_{n=0}^{\infty} \left[\int_0^1 t^{\delta+nq-1}(1-t)^{c-\delta-1} \right] \frac{1}{B(\delta, c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{z^n}{(\tau)_{nr}}. \tag{16}$$

Interchanging the order of summation and integration in equation (16), which is guaranteed under the assumptions given in the statement of the conditions in (6), we obtain

$$E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) = \frac{1}{B(\delta, c-\delta)} \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}} \sum_{n=0}^{\infty} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{(t^q z)^n}{(\tau)_{nr}} dt. \tag{17}$$

Plugging equation (1) into equation (17), the desired result is proofed. $\square$

Corollary 1 Taking $t = \frac{u}{1+u}$ in **Theorem 1**, we have

$$E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) = \frac{1}{B(\delta, c-\delta)} \int_0^{\infty} \frac{u^{\delta-1}}{(u+1)^c} e^{-\frac{p(1+u)^2}{u}} E_{\rho,\sigma,r}^{c,\tau,q}\left(\left(\frac{u}{1+u}\right)^q z\right) du.$$

Corollary 2 Taking $t = \sin^2\theta$ in **Theorem 1**, the following integral representation is yielded

$$E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) = \frac{2}{B(\delta, c-\delta)} \int_0^{\frac{\pi}{2}} \sin^{2\delta-1}\theta \cos^{2c-2\delta-1}\theta e^{-\frac{p}{\sin^2\theta\cos^2\theta}} E_{\rho,\sigma,r}^{c,\tau,q}\left(\sin^{2q} z\right) d\theta.$$

Theorem 2 If the conditions about parameters $\rho, \sigma, \tau, \delta, c, q, r, w$ in (6) is satisfied, for any interval $(a,b) \subset \mathbb{R}$ and any function $f \in L_1(a,b)$, then the generalized fractional integral operator $\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c} f$ can be written as

$$\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c} f\right)(x;p) = \sum_{n=0}^{\infty} \frac{B_p(\delta+nq, c-\delta)\, w^n}{B(\delta, c-\delta)} \frac{(c)_{nq}}{(\tau)_{nr}} \left({}^{RL}I_{a+}^{\rho n+\sigma} f\right)(x;p), \tag{18}$$

where the series on the right-hand is locally uniformly convergent.

Proof. The starting point is the series formula (6) for the multi parameters Mittag-Leffler function. This series is known to be locally uniformly convergent in $|z| < \frac{r^r \Re(\rho)^{\Re(\rho)}}{q^q}$, which converges for all z

provided that $q < r + \Re(\rho)$ [17]. Thus we can interchange the summation and integration in the formula (19), and work as follows

$$\begin{aligned}\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c}f\right)(x;p) &= \int\limits_a^x (x-t)^{\sigma-1} E_{\rho,\sigma,\tau}^{\delta,r,q,c}\left(w(x-t)^{\rho};p\right) f(t)\,dt \\ &= \int\limits_a^x (x-t)^{\sigma-1} \sum_{n=0}^{\infty} \frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{w^n (x-t)^{\rho n}}{(\tau)_{nr}} f(t)\,dt \\ &= \sum_{n=0}^{\infty} \frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)} \frac{(c)_{nq}}{(\tau)_{nr}} \frac{w^n}{\Gamma(\rho n+\sigma)} \int\limits_a^x (x-t)^{\rho n+\sigma-1} f(t)\,dt \\ &= \sum_{n=0}^{\infty} \frac{B_p(\delta+nq,c-\delta)\,w^n}{B(\delta,c-\delta)} \frac{(c)_{nq}}{(\tau)_{nr}} \left({}^{RL}I_{a+}^{\rho n+\sigma}f\right)(x). \quad \square\end{aligned} \tag{19}$$

Note 1 By using Eq.(18) defined in **Theorem 2**, then the proof of **Theorem 2.5** in [14] can skip two steps.

Meanwhile, we note that Eq.(18) is a generalization of the following fractional integral operator.

Proposition 1 Taking $p = 0$ in **Theorem 2**, Eq.(18) reduces to the generalized fractional integral operator $\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,q,r}f$ in Eq.(7). That is,

$$\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,q,r}f\right)(x) = \sum_{n=0}^{\infty} \frac{(\delta)_{nq}}{(\tau)_{nr}} \left({}^{RL}I_{a+}^{\rho n+\sigma}f\right)(x). \tag{20}$$

Proposition 2 Taking $\tau = r = 1$ **Theorem 2**, Eq.(18) reduces to the generalized fractional integral operator $\varepsilon_{a+,\rho,\sigma}^{w,\delta,q,c}f$ in Eq.(5), which can be written as

$$\left(\varepsilon_{a+,\rho,\sigma}^{w,\delta,q,c}f\right)(x;p) = \sum_{n=0}^{\infty} \frac{B_p(\delta+nq,c-\delta)\,w^n}{B(\delta,c-\delta)} \frac{(c)_{nq}}{n!} \left({}^{RL}I_{a+}^{\rho n+\sigma}f\right)(x;p). \tag{21}$$

Theorem 3 The-multi parameters Mittag-Leffler fractional integral operator (7) with the parameters satisfying conditions in (6), interacts naturally with Riemann-Liouville differintegral operators in the following ways. For any $f \in L^1$ function, and any $\mu \in \mathbb{C}$ such that $\Re(\mu) + \Re(\sigma) > 0$, we have

$${}^{RL}I_{a+}^{\mu}\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c}f\right)(x;p) = \left(\varepsilon_{a+,\rho,\sigma+\mu,\tau}^{w,\delta,r,q,c}f\right)(x;p). \tag{22}$$

Further, if $\Re(\mu) > 0$, then

$${}^{RL}I_{a+}^{\mu}\left(\varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c}f\right)(x;p) = \varepsilon_{a+,\rho,\sigma,\tau}^{w,\delta,r,q,c}\left({}^{RL}I_{a+}^{\mu}f\right)(x;p). \tag{23}$$

Proof. For the first identity, by the series formula (18) it is enough to show that

$${}^{RL}I_{a+}^{\mu}\,{}^{RL}I_{a+}^{\rho n+\sigma}f = {}^{RL}I_{a+}^{\rho n+\sigma+\mu}f, (n \geq 0), \tag{24}$$

which is clearly true by the semigroup property for Riemann-Liouville fractional integral. For the second identity, by (18) it is enough to show that

$$ {}^{RL}I_{a+}^{\mu}\, {}^{RL}I_{a+}^{\rho n+\sigma} f = {}^{RL}I_{a+}^{\rho n+\sigma}\, {}^{RL}I_{a+}^{\mu} f, (n \geq 0)\,, \tag{25} $$

which again is clearly true, by the basic properties of Riemann-Liouville differintegrals. □

Note 2 Equations (22) and (23) are generalization of the other differintegral operators in [9,13].

Theorem 4 The expression for the multi-parameters Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)$ in terms of the generalized Wright hypergeometric function as follows

$$ E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) = \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}}\, {}_3\Psi_4 \left[\begin{matrix} (c,q) & (\tau,0) & (1,1) & \\ (\delta,0) & (c-\delta,0) & (\sigma,\rho) & (\tau,r) \end{matrix}, zt^q \right] dt. \tag{26} $$

Proof. By using equation (15), we can directly get

$$ \begin{aligned} E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) &= \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}} \sum_{n=0}^{\infty} \frac{1}{B(\delta,c-\delta)} \frac{(c)_{nq}}{\Gamma(\rho n+\sigma)} \frac{(t^q z)^n}{(\tau)_{nr}} dt \\ &= \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}} \sum_{n=0}^{\infty} \frac{\Gamma(c)\Gamma(n+1)}{\Gamma(\delta)\Gamma(c-\delta)} \frac{\Gamma(c+nq)\Gamma(\tau)}{\Gamma(c)\Gamma(\rho n+\sigma)} \frac{(t^q z)^n}{\Gamma(\tau+nr)\, n!} dt \\ &= \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}}\, {}_3\Psi_4 \left[\begin{matrix} (c,q) & (\tau,0) & (1,1) & \\ (\delta,0) & (c-\delta,0) & (\sigma,\rho) & (\tau,r) \end{matrix}, zt^q \right] dt. \quad \square \end{aligned} \tag{27} $$

Theorem 5 The representation for the multi-parameters Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)$ in terms of the Fox's H-function as follows

$$ \begin{aligned} &E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p) \\ &= \int_0^1 t^{\delta-1}(1-t)^{c-\delta-1} e^{-\frac{p}{t(1-t)}} H_{3,5}^{1,3} \left[-z \middle| \begin{matrix} (1-c,q),(1-\tau,0),(0,1) \\ (0,1),(1-\delta,0),(1-c+\delta,0),(1-\sigma,\rho),(1-\tau,r) \end{matrix} \right]. \end{aligned} \tag{28} $$

Proof. Substituting equation (13) into equation (27), we can directly get the result. □

Theorem 6 The Mellin transform of the multi-parameters Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)$ is given by

$$ \left\{ E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s \right\} = \frac{\Gamma(s)\Gamma(q)\Gamma(c+s-\delta)}{\Gamma(\delta)\Gamma(c-\delta)} \cdot {}_3\Psi_3 \left[\begin{matrix} (c,r) & (1,1) & (\delta+s,q) \\ (q,\tau) & (\sigma,\rho) & (c+2s,q) \end{matrix}, z \right], \tag{29} $$

where ${}_3\Psi_3$ is the generalized Wright hypergeometric function.

Proof. Taking the Mellin transform of the multi-parameters Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)$, we have

$$\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}=\int_0^\infty p^{s-1}E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)\,dp. \tag{30}$$

Inserting equation (6) into equation (30), we get

$$\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}=\frac{1}{B(\delta,c-\delta)}\int_0^\infty p^{s-1}\left(\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}e^{-\frac{p}{t(1-t)}}E_{\rho,\sigma,r}^{c,\tau,q}(t^q z)\,dt\right)dp. \tag{31}$$

Interchanging the order of integrals in equation (31), which is valid because of the conditions in the statement of the equation(6), yields

$$\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}=\frac{1}{B(\delta,c-\delta)}\int_0^1 t^{\delta-1}(1-t)^{c-\delta-1}E_{\rho,\sigma,r}^{c,\tau,q}(t^q z)\left(\int_0^\infty p^{s-1}e^{-\frac{p}{t(1-t)}}\,dp\right)dt. \tag{32}$$

Now taking $u=\frac{p}{t(1-t)}$ in equation (32) and using the fact that $\Gamma(s)=\int\limits_0^\infty u^{s-1}e^{-u}du$, one obtains

$$\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}=\frac{\Gamma(s)}{B(\delta,c-\delta)}\int_0^1 t^{\delta+s-1}(1-t)^{c+s-\delta-1}E_{\rho,\sigma,r}^{c,\tau,q}(t^q z)\,dt. \tag{33}$$

Interchanging the order of summation and integration, we have

$$\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}=\frac{\Gamma(s)}{B(\delta,c-\delta)}\sum_{n=0}^\infty\frac{(c)_{nr}}{(q)_{n\tau}}\frac{z^n}{\Gamma(\rho n+\sigma)}\int_0^1 t^{qn+\delta+s-1}(1-t)^{c+s-\delta-1}dt \tag{34}$$

Next, by considering the Beta function in equation (34), the following result was obtained

$$\begin{aligned}\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}&=\frac{\Gamma(s)}{B(\delta,c-\delta)}\sum_{n=0}^\infty\frac{(c)_{nr}}{(q)_{n\tau}}\frac{z^nB(qn+\delta+s,c+s-\delta)}{\Gamma(\rho n+\sigma)}\\&=\sum_{n=0}^\infty\frac{\Gamma(s)\Gamma(q)}{\Gamma(\delta)}\frac{\Gamma(c+nr)}{\Gamma(c-\delta)}\frac{\Gamma(1+n)}{\Gamma(q+n\tau)}\frac{\Gamma(\delta+s+nq)}{\Gamma(\rho n+\sigma)}\frac{\Gamma(c+s-\delta)}{\Gamma(c+2s+nq)}\frac{z^n}{n!}\\&=\frac{\Gamma(s)\Gamma(q)}{\Gamma(\delta)}\frac{\Gamma(c+s-\delta)}{\Gamma(c-\delta)}\cdot{}_3\Psi_3\left[\begin{array}{ccc}(c,r)&(1,1)&(\delta+s,q)\\(q,\tau)&(\rho,\sigma)&(c+2s,q)\end{array},z\right].\quad\square\end{aligned} \tag{35}$$

Corollary 3 Taking $s=1$ in **Theorem 6**, we obtain

$$\begin{aligned}\mathrm{M}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p);s\right\}&=\int_0^\infty E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)dp\\&=\frac{(c-\delta)\Gamma(q)}{\Gamma(s)}\cdot{}_3\Psi_3\left[\begin{array}{ccc}(c,r)&(1,1)&(\delta+1,q)\\(q,\tau)&(\rho,\sigma)&(c+2,q)\end{array},z\right].\end{aligned} \tag{36}$$

Theorem 7 For the multi-parameters Mittag-Leffler function $E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)$, the following differentiation formula holds

$$\frac{d^n}{dp^n}\left\{E_{\rho,\sigma,\tau}^{\delta,r,q,c}(z;p)\right\} = (-1)^n \frac{\Gamma(\delta-n)\Gamma(c-\delta-n)\Gamma(c)}{\Gamma(\delta)\Gamma(c-\delta)\Gamma(c-2n)} E_{\rho,\sigma,\tau}^{\delta-n,r,q,c-2n}(z;p) \tag{37}$$

Proof. Taking the p-derivative n times in equation (6), we can directly get the result. $\square$

Theorem 8 Let $x > a\,(a \in \mathbb{R}^+ := [0,+\infty))\,, 0 < \mu < 1, 0 \le v \le 1, w \in \mathbb{C}$ and all parameters meet the conditions in (6). The following expressions hold

$$\left(I_{a+}^{\lambda}\left[(t-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,r,q,c}(w(t-a)^{\rho};p)\right]\right)(x) = (x-a)^{\sigma+\lambda-1}E_{\rho,\sigma+\lambda,\tau}^{\delta,r,q,c}(w(x-a)^{\rho};p)\,, \tag{38}$$

$$\left(D_{a+}^{\lambda}\left[(t-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,r,q,c}(w(t-a)^{\rho};p)\right]\right)(x) = (x-a)^{\sigma-\lambda-1}E_{\rho,\sigma-\lambda,\tau}^{\delta,r,q,c}(w(x-a)^{\rho};p) \tag{39}$$

and

$$\left(D_{a+}^{\mu,v}\left[(t-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,r,q,c}(w(t-a)^{\rho};p)\right]\right)(x) = (x-a)^{\sigma-\mu-1}E_{\rho,\sigma-\mu,\tau}^{\delta,r,q,c}(w(x-a)^{\rho};p)\,. \tag{40}$$

Proof. By applying equations (6) and (12) on the left-hand of the first identity, it is straightforward to get this result.

For the second identity, we use a similar discussion.

For the last identity, indeed, we have

$$\begin{aligned}
&\left(D_{a+}^{\mu,v}\left[(t-a)^{\sigma-1}E_{\rho,\sigma,\tau}^{\delta,r,q,c}(w(t-a)^{\rho};p)\right]\right)(x) = \\
&= \left(D_{a+}^{\mu,v}\left[\sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n(t-a)^{\rho n+\sigma-1}}{(\tau)_{nr}}\right]\right)(x) \\
&= \sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{nr}}\left(D_{a+}^{\mu,v}\left[(t-a)^{\rho n+\sigma-1}\right]\right)(x) \\
&= \sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{w^n}{(\tau)_{nr}}\frac{\Gamma(\rho n+\sigma)}{\Gamma(\rho n+\sigma-\mu)}(x-a)^{\rho n+\sigma-\mu-1} \\
&= (x-a)^{\sigma-\mu-1}\sum_{n=0}^{\infty}\frac{B_p(\delta+nq,c-\delta)}{B(\delta,c-\delta)}\frac{(c)_{nq}}{\Gamma(\rho n+\sigma)}\frac{\Gamma(\rho n+\sigma)}{(\tau)_{nr}}\frac{[w(x-a)^{\rho}]^n}{\Gamma(\rho n+\sigma-\mu)} \\
&= (x-a)^{\sigma-\mu-1}E_{\rho,\sigma-\mu,\tau}^{\delta,r,q,c}(w(x-a)^{\rho};p)\,. \quad \square
\end{aligned}$$

4. Concluding remarks

In this paper, we have studied the multi-parameters Mittag-Leffler function, together with the corresponding fractional integral operator that contains this function as its kernel. Meanwhile, we have obtained a new series expression for this transform, in terms of the classical Riemann-Liouville fractional integrals. Furthermore, we have showed the Mellin transform of the multi-parameters Mittag-Leffler function in terms of the generalized Wright hypergeometric and Fox's-H functions. Finally, the fractional integrals and fractional derivatives of the multi-parameters Mittag-Leffler function were also described.

Acknowledgements

The authors are grateful to the anonymous referees for their useful comments and suggestions.

References

[1] Yang X J. Theory and Applications of Special Functions for Scientists and Engineers. Springer Nature, New York, USA, 2021.

[2] Yang X J. An introduction to hypergeometric, supertrigonometric, and superhyperbolic functions. Academic Press, 2021.

[3] Mittag-Leffler G M. Sur la nouvelle fonction $E_\alpha(x)$. CR Acad. Sci. Paris, 1903, 137(2): 554-558.

[4] Mehrez K, Sitnik S M. Functional inequalities for the Mittag-Leffler functions. Results in Mathematics, 2017, 72(1): 703-714.

[5] R. Gorenflo, A. A. Kilbas, F. Mainardi, S. Rogosin, Mittag-Leffler functions. Theory and Applications. Springer Monographs in Mathematics, Springer, Berlin, 2014.

[6] Gorenflo R, Kilbas A A, Mainardi F, et al. Mittag-Leffler functions, related topics and applications. New York, NY, USA: Springer, 2020.

[7] F. Mainardi, R. Gorenflo, Time-fractional derivatives in relaxation processes: a tutorial survey. Fract. Calc. Appl. Anal. 2007, 10(3), 269-308.

[8] Wiman A. uber den Fundamentalsatz in der Teorie der Funktionen $Ea(x)$. Acta Mathematica, 1905, 29: 191-201.

[9] T.R. Prabhakar, A singular integral equation with a generalized Mittag-Leffler function in the kernel, Yokohama Math. J., 1971(19), 7-15.

[10] Garra R, Garrappa R. The Prabhakar or three parameter Mittag-Leffler function: Theory and application. Communications in Nonlinear Science and Numerical Simulation, 2018, 56: 314-329.

[11] Shukla A K, Prajapati J C. On a generalization of Mittag-Leffler function and its properties. Journal of mathematical analysis and applications, 2007, 336(2): 797-811.

[12] Srivastava H M, Tomovski . Fractional calculus with an integral operator containing a generalized Mittag-Leffler function in the kernel. Applied Mathematics and Computation, 2009, 211(1): 198-210.

[13] Diaz R, Pariguan E. On hypergeometric functions and Pochhammer k-symbol. arXiv preprint math/0405596, 2004.

[14] Salim T O. Some properties relating to the generalized Mittag-Leffler function. Adv. Appl. Math. Anal, 2009, 4(1): 21-30.

[15] Salim T O, Faraj A W. A generalization of Mittag-Leffler function and integral operator associated with fractional calculus. J. Fract. Calc. Appl, 2012, 3(5): 1-13.
[16] Khan and Ahmed, On some properties of the generalized Mittag-Leffler function. SpringerPlus, 2013 2:33
[17] Özarslan M A, Ymaz B. The extended Mittag-Leffler function and its properties. Journal of Inequalities and Applications, 2014, 2014(1): 1-10.
[18] Chaudhry, M.A., Qadir, A., Rafique, M., Zubair, S.M.: Extension of Euler's beta function. J. Comput. Appl. Math. 78(1),19-32 (1997).
[19] Rahman G, Baleanu D, Qurashi M A, et al. The extended Mittag-Leffler function via fractional calculus. J. Nonlinear Sci. Appl, 2017, 10(8): 4244-4253.
[20] Andrić M, Farid G, Pečarić J. A further extension of Mittag-Leffler function. Fractional Calculus and Applied Analysis, 2018, 21(5): 1377-1395.
[21] Kilbas A A, Marichev O I, Samko S G. Fractional integrals and derivatives (theory and applications), 1993.
[22] R. Hilfer (Ed.), Applications of Fractional Calculus in Physics, World Scientific Publishing Company, Singapore, New Jersey, London and Hong Kong, 2000.
[23] A.A. Kilbas, H.M. Srivastava, J.J. Trujillo, Theory and Applications of Fractional Differential Equations, North-Holland Mathematical Studies, vol. 204, Elsevier, 2006.
[24] Yang X J, Gao F, Yang J. General fractional derivatives with applications in viscoelasticity. Academic Press, 2020.