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Regularity Criterion for the Liquid Crystal Flow in $\mathbb{R}^3$}

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July 11, 2022

Abstract

We obtain that a weak solution (u,d) to the Liquid Crystal system is strong, if any two components of u and ∇d satisfy Ladyzhenskaya-Prodi-Serrin's condition, that is $$\|u^1\|_{L^{r,s}}+\|u^2\|_{L^{r,s}}+\|\nabla d\|_{L^{r,s}}\leq \infty, \qquad\frac{3}{r}+\frac{2}{s}\leq1.$$ We also prove that the velocity u is bounded locally if any two comonents of u and ∇d belong to $L^{6,\infty}$ and u belongs to $L^{2+\delta}$ ($\delta>0$).

Regularity Criterion for the Liquid Crystal Flow in $\mathbb{R}^3$

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Abstract We obtain that a weak solution (u, d) to the Liquid Crystal system is strong, if any two components of u and ∇d satisfy
Ladyzhenskaya-Prodi-Serrin's condition, that is

$$\|u^1\|_{L^{r,s}} + \|u^2\|_{L^{r,s}} + \|\nabla d\|_{L^{r,s}} \leq \infty, \qquad \frac{3}{r} + \frac{2}{s} \leq 1.$$

We also prove that the velocity u is bounded locally if any two comonents of u and ∇d belong to $L^{6,\infty}$ and u belongs to $L^{2+\delta}$ $(\delta > 0)$.

Keywords Regularity criterion; Liquid Crystal flows

1. Introduction

In this paper, we will consider the regularity of the weak solutions of the incompressible Nematic liquid flows:

$$\begin{cases} \partial_t u - \mu\Delta u + u \cdot \nabla u + \nabla p = -\lambda\nabla \cdot (\nabla d \odot \nabla d), & \text{in } \Omega \times \mathbb{R}_+, \\ \partial_t d + u \cdot \nabla d = \gamma(\Delta d - f(d)), & \text{in } \Omega \times \mathbb{R}_+, \\ \text{div} u = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ u(x, 0) = u_0, \quad d(x, 0) = d_0, & \text{on } \Omega. \end{cases} \tag{1.1}$$

where μ, λ, γ are positive constants, $u = u(t, x) \in \mathbb{R}^3$ is the velocity field, $p = p(t, x) \in \mathbb{R}$ is the pressure field, $d = d(t, x) \in \mathbb{R}^3$ is the macroscopic/continuum molecule orientation of the nematic liquid crystal flow. $u_0(x) : \mathbb{R}^3 \to \mathbb{R}^3$ is the initial velocity field with $\nabla \cdot u_0 = 0$ and d_0 is the initial director field with $|d_0| \leq 1$. The notation $\nabla d \odot \nabla d$ denotes the 3×3 matrix whose (i, j)th entry is

given by $\partial_i d \cdot \partial_j d\,(1 \le i, j \le 3)$. The Ginzburg-Landau approximate function, $f(d) = \frac{1}{\epsilon^2}(|d|^2 - 1)d$ is smooth and bounded function, where $\epsilon > 0$ is a constant.

The system (1.1) is a simplified representation of the Ericksen-Leslie system [3],[7], however, it still remains most of essential features of the Ericksen-Leslie system. When $d \equiv 0$, the Liquid Crystal flow reduces to the Navier Stoke equation:

$$\begin{cases} \partial_t u - \mu\Delta u + u \cdot \nabla u + \nabla p = 0, \\ \mathrm{div} u = 0, \end{cases} \tag{1.2}$$

with $u(x, 0) = u_0$, $u|_{\partial\Omega} = 0$. In the classical literatures, Leray [6] and Hopf [5] established the existence weak solution to the Navier Stokes system (1.2). However, the smoothness of the Leray-Hopf weak solution is still a challenging open problem. Serrin [11] proved that if the weak solution to the Navier Stokes equation (1.2) satisfies the well-know Ladyzhenskaya-Prodi-Serrin type condition

$$u \in L^q(0, T; L^p(\mathbb{R})^3), \quad \text{with } \frac{2}{q} + \frac{3}{p} \le 1, \quad p > 1,$$

then the weak solution u is regular in $\mathbb{R}^3 \times (0, T)$. Subsequently, whenever the boundary data of pressure p is bounded, Choe [1] proved the L^∞ regularity of u up to boundary under assumption

$$u \in L^{r,s}, \qquad \text{with } \frac{2}{s} + \frac{3}{r} \le 1, \quad r > 3,$$
$$\text{or } u \in L^{3,\infty} \quad \text{with } \|u\|_{L^{3,\infty}} \le \epsilon_0$$

here, ϵ_0 is a small positive constant. Later, for $p = 3, q = \infty$ (i.e.; $u \in L^\infty(0, T; L^3(\mathbb{R}^3))$), Escauriaza-Seregin-Sverák [4] established the regularity of the Navier Stokes equation (1.2) by using backward uniqueness for the parabolic operator.

For the periodic domain, Choe [1] proved that if any two components of the velocity $u := (u^2, u^2, u^3)$ meet the Ladyzhenskaya-Prodi-Serrin condition, that is

$$u^1, u^2 \in L^{r,s}, \quad \frac{3}{r} + \frac{2}{s} \le 1, \quad r > 3,$$

then they concluded $u \in L^\infty(0,\infty;H^1) \cap L^2(0,\infty;H^2)$. They also established that if any two components of a suitable weak solution lie in $L^{6,\infty}_{loc}$, then the third component belongs to $L^{5,5}_{loc}$.

For system (1.1), Lin-Liu established the existence of global weak solution under assumption that $u_0 \in L^2$ and $d_0 \in H^1$. In [8], Lin-Liu proved regularity of the suitable weak solution under the CKN [2]. Furthermore, Liu-Min-Zhang [9] established a regularity criterion under the Serrin-type condition [11]. A weak solution $(u, \nabla d)$ to the incompressible liquid crystal system (1.1) is smooth in Q when the velocity u satisfies the following condition:

$$u \in L^{p,q}_{x,t}(Q) \quad \text{with } \frac{3}{p} + \frac{2}{q} = 1, \quad 3 < p < \infty.$$

Later, Liu-Ming-Zhang [10] obtained the endpoint case(i.e. $u, \nabla d \in L^\infty(0,T;L^3(\mathbb{R}^3))$) regularity for the Liquid Crystals system (1.1) in $\mathbb{R}^3$. In this paper, inspired by Choe-Bae [1], we shall extend the result by establishing the similar regularity criteria to the system (1.1).

Since precise values of the parameters μ, λ, γ and ϵ in the system (1.1) don't play a role in the paper, we simply assume that

$$\mu = \lambda = \gamma = \epsilon = 1.$$

Our main results state as follows:

Theorem 1.1. Assume that the initial value u_0 and d_0 are periodic, that is, $u_0(x+z) = u_0(x)$ and $d_0(x+z) = d_0(x)$ for any $x \in \mathbb{R}^3$, $t \in \mathbb{R}^+$, $z \in 0, 1$. Assume that a pair function $(u, \nabla d)$ is a weak solution to the system (1.1) for the periodic domain. If u^1, u^2, ∇d satisfy Ladyzhenskaya-Prodi-Serrin's condition:

$$\|u^1\|_{L^{r,s}((0,1)^3\times\mathbb{R}_+)} + \|u^2\|_{L^{r,s}((0,1)^3\times\mathbb{R}_+)} + \|\nabla d\|_{L^{r,s}((0,1)^3\times\mathbb{R}_+)} < +\infty, \quad \frac{3}{r} + \frac{2}{s} \le 1,\ r > 3.$$

Then we have

$$u \in L^\infty(\mathbb{R}_+;\mathbb{H}^1((0,1)^3)) \cap L^2(\mathbb{R}_+;\mathbb{H}^2((0,1)^3),$$
$$d \in L^\infty(\mathbb{R}_+;\mathbb{H}^2((0,1)^3)) \cap L^2(\mathbb{R}_+;\mathbb{H}^3((0,1)^3),$$

and therefore $(u, \nabla d)$ is a strong solution to the system (1.1). Furthermore, ∇u and Δd satisfy the following:

$$\begin{aligned}
&\int_{(0,1)^3} (|\nabla u|^2 + |\Delta d|^2) dx(T) \\
&\leq \exp[CT^{1-\frac{2r}{s(r-3)}}(\|u^1\|_{L^{r,s}} + \|u^2\|_{L^{r,s}}\| + \nabla d\|_{L^{r,s}})^{\frac{2r}{r-3}}] \\
&\quad \Big[\int_{(0,1)^3} |\nabla u_0|^2 dx + C(T)\Big].
\end{aligned}$$

Theorem 1.2. Set $u = (u^1, u^2, u^3)$ and $Q = \Omega \times \mathbb{R}_+$. Let (u, d, p) be a suitable weak solution of (1.1). Assume that any two of the three components u^1, u^2, u^3 of u and ∇d lie in $L^{6,\infty}_{loc}(Q)$. Let $u \in L^{2+\delta}_{loc}(Q)$ with $\delta > 0$. Then we have

$$u \in L^{\infty,\infty}_{loc}(Q).$$

2. Regularity Criterion for the Periodic Domain

In this section, we shall introduce a regularity criterion for periodic domain and prove Theorem 1.1. Throughout this section, we suppose that u_0 is weakly divergence free, u_0 and d_0 satisfy periodicity in space:

$$u_0(x+z) = u_0(x), \quad d_0(x+z) = d_0(x) \quad \text{for any } x \in \mathbb{R}^3 \quad \text{and } z \in \{0,1\}^3.$$

Let $u_0 \in \mathbb{H}^1(\mathbb{R}^3/N^3)$, $d_0 \in \mathbb{H}^2(\mathbb{R}^3/N^3)$, $u, \nabla d \in L^\infty(\mathbb{R}_+; L^2(\mathbb{R}^3/N^3)) \cap L^2(\mathbb{R}_+; \mathbb{H}^1(\mathbb{R}^3/N^3))$, $|d_0| = 1$ and (u, d) is a weak solution to the Liquid Crystal system (1.1), where N is the set of integers.

The proof of Theorem 1.1. Set $\Omega = (0,1)^3$. First, we give the needed a priori estimates in the following. Testing $(1.1)_1$ and $(1.1)_3$ by u and $-\Delta d + f(d)$, integrating over $\mathbb{R}^3$ and summing them up, we have

$$\frac{1}{2}\frac{d}{dt}\int_\Omega (|u|^2 + |\nabla d|^2 + \frac{1}{2}(|d|^2 - 1)^2) dx + \int_\Omega (|\nabla u|^2 + |\Delta d - f(d)|^2) dx = 0. \tag{2.1}$$

In the above we have used the facts that $\nabla \cdot u =$ and

$$\langle u \cdot \nabla u, u\rangle = \langle \nabla p, u\rangle = \langle u \cdot \nabla d, f(d)\rangle = \left\langle u, \nabla \frac{|\nabla d|^2}{2}\right\rangle = 0.$$

Then we have

$$\|u\|_{L^\infty(0,T;L^2)} + \|u\|_{L^2(0,T;\mathbb{H}^1)} \le C. \tag{2.2}$$

Multiplying $(1.1)_3$ by d and integrating by parts, we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |d|^2 dx + \int_\Omega (|\nabla d|^2 + |d|^4) dx &= \int_\Omega (|d|^2 + (d\cdot\nabla)u\cdot d) dx \\ &\le \frac{1}{2}\int_\Omega |d|^4 dx + \int_\Omega (|d|^2 + \frac{1}{2}|\nabla u|^2) dx. \end{aligned} \tag{2.3}$$

Using (2.1) (2.2) (2.3) and Gronwall's inequality, we get

$$\|d\|_{L^\infty(0,T;\mathbb{H}^1)} + \|d\|_{L^2(0,T;\mathbb{H}^2)} \le C. \tag{2.4}$$

We apply Δ to $(1.1)_3$ with respect to the spatial variables and multiple it by Δd. Integrating by parts, one can obtains

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Delta d\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2 &= -\sum_{i,k=1}^3 \int_\Omega \Delta u^i \partial_i d^k \Delta d^k dx \\ &\quad - 2\sum_{i,k=1}^3 \int_\Omega \nabla u^i \partial_i \nabla d^k \Delta d^k - \int_\Omega \Delta f(d)\Delta d dx. \end{aligned} \tag{2.5}$$

Multiplying both side of $(1.1)_1$ by Δu, using the incompressibility and integrating over Ω, we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\nabla u\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 &= \int_\Omega (u\cdot\nabla u)\cdot\Delta u dx \\ &\quad + \sum_{i,j,k=1}^3 \int_\Omega \partial_{ij} d^k \partial_j d^k \Delta u^i dx + \sum_{i,j,k=1}^3 \int_\Omega \partial_i d^k \partial_{ii} d^k \Delta u^i dx \\ &= \int_\Omega (u\cdot\nabla u)\cdot\Delta u dx + \sum_{i,j,k=1}^3 \int_\Omega \partial_i d^k \partial_{ii} d^k \Delta u^i dx. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2) \\ \le &\int_\Omega (u\cdot\nabla u)\cdot\Delta u dx + 2\sum_{i,k=1}^3 \int_\Omega u^i \partial_i \nabla d^k \nabla\Delta d^k - \int_\Omega \Delta f(d)\Delta d dx \\ :=& I_1 + I_2 + I_3. \end{aligned}$$

For the first term I_1, by using $\nabla \cdot u = 0$, we have

$$I_1 = -\int_\Omega \nabla(u \cdot \nabla u) \cdot \nabla u dx = -\sum_{i,j,k=1}^{3} \int_\Omega u_k^i u_i^j u_k^j dx. \tag{2.7}$$

When $i \neq 3$, integrating by apart, we obtain

$$\begin{aligned}
\int_\Omega u_k^i u_i^j u_k^j dx &= -\int_\Omega u^i (u_i^j u_k^j)_k dx \leq \epsilon \int_\Omega |\nabla^2 u|^2 dx + \frac{C}{\epsilon} \int_\Omega |u^i|^2 |\nabla u|^2 dx \\
&\leq \epsilon \int_\Omega |\nabla^2 u|^2 dx + C\Big(\int_\Omega |u^i|^r dx\Big)^{\frac{2}{r}} \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{r-3}{r}} \Big(\int_\Omega |\nabla u|^6 dx\Big)^{\frac{1}{r}} \\
&\leq \epsilon \int_\Omega |\nabla^2 u|^2 dx + C\Big(\int_\Omega |u^i|^r dx\Big)^{\frac{2}{r}} \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{r-3}{r}} \Big(\int_\Omega |\nabla^2 u|^2 dx\Big)^{\frac{3}{r}} \\
&\leq \epsilon \int_\Omega |\nabla^2 u|^2 dx + \frac{C}{\epsilon}\Big(\int_\Omega |u^i|^r dx\Big)^{\frac{2}{r-3}} \Big(\int_\Omega |\nabla u|^2 dx\Big).
\end{aligned} \tag{2.8}$$

When $i = 3, j \neq 3$, we have

$$\int_\Omega u_k^3 u_3^j u_k^j dx \leq \epsilon \int_\Omega |\nabla^2 u|^2 dx + \frac{C}{\epsilon}\Big(\int_\Omega |u^j|^r dx\Big)^{\frac{2}{r-3}} \Big(\int_\Omega |\nabla u|^2 dx\Big). \tag{2.9}$$

When $i = 3,\ j = 3$, since $\nabla \cdot u = 0$, we have

$$\begin{aligned}
\int_\Omega u_k^3 u_3^3 u_k^3 dx &= -\int_\Omega u_k^3 (u_1^1 + u_2^2) u_k^3 dx \\
&\leq \epsilon \int_\Omega |\nabla^2 u|^2 dx + \frac{C}{\epsilon}\Big(\int_\Omega (|u^1|^r + |u^2|^r) dx\Big)^{\frac{2}{r-3}} \Big(\int_\Omega |\nabla u|^2 dx\Big).
\end{aligned} \tag{2.10}$$

For I_2, by the similar estimate, we get

$$\begin{aligned}
I_2 = \int_\Omega \nabla u : \nabla^2 d \Delta d dx &= -\int_\Omega \partial_j \Delta d \partial_k d \partial_k u^j dx \\
&\leq \epsilon \int_\Omega |\nabla \Delta d|^2 dx + \frac{C}{\epsilon} \int_\Omega |\nabla d|^2 |\nabla u|^2 dx \\
&\leq \epsilon \int_\Omega |\nabla \Delta d|^2 dx + C\Big(\int_\Omega |\nabla d|^r dx\Big)^{\frac{2}{r}} \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{r-3}{r}} \Big(\int_\Omega |\nabla u|^6 dx\Big)^{\frac{1}{r}} \\
&\leq \epsilon \int_\Omega |\nabla \Delta d|^2 dx + C\Big(\int_\Omega |\nabla d|^r dx\Big)^{\frac{2}{r}} \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{r-3}{r}} \Big(\int_\Omega |\nabla^2 u|^2 dx\Big)^{\frac{3}{r}} \\
&\leq \epsilon \int_\Omega (|\nabla \Delta d|^2 + |\nabla^2 u|^2) dx + \frac{C}{\epsilon}\Big(\int_\Omega |\nabla d|^r dx\Big)^{\frac{2}{r-3}} \Big(\int_\Omega |\nabla u|^2 dx\Big).
\end{aligned} \tag{2.11}$$

Owing to $f(d) = (|d|^2 - 1)d$ and using Young's, Sobolev's and Hölder's inequality, we have

$$I_3 = -\int_\Omega \nabla f(d) \nabla \Delta d dx \leq 3 \sum_{i=1}^{3} \int_\Omega \partial_i d \partial_i \Delta d |d|^2 dx - \sum_{i=1}^{3} \int_\Omega \partial_i d \partial_i \Delta d dx$$

$$
\begin{aligned}
&\le \epsilon \int_\Omega |\nabla\Delta d|^2 dx + C\int_\Omega |\nabla d|^2 dx + C\int_\Omega |\nabla d|^2 |d|^4 dx \\
&\le \epsilon \int_\Omega |\nabla\Delta d|^2 dx + C\int_\Omega |\nabla d|^2 dx + C\Big(\int_\Omega |\nabla d|^6 dx\Big)^{\frac{1}{3}}\Big(\int_\Omega |d|^6 dx\Big)^{\frac{2}{3}} \\
&\le \epsilon \int_\Omega |\nabla\Delta d|^2 dx + C\int_\Omega |\nabla d|^2 dx + C\int_\Omega |\nabla^2 d|^2 dx\Big(\int_\Omega |\nabla d|^2 dx\Big)^2.
\end{aligned}
\tag{2.12}
$$

By (2.7)-(2.12), we get

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}(\|\nabla u\|_{L^2}^2 + \|\Delta d\|_{L^2}^2) + (\|\Delta u\|_{L^2}^2 + \|\nabla\Delta d\|_{L^2}^2) \\
&\qquad \le C\Big(\Big(\int_\Omega |u^1|^r\Big)^{\frac{2}{r-3}} + \Big(\int_\Omega |u^2|^r\Big)^{\frac{2}{r-3}} + \Big(\int_\Omega |\nabla d|^r\Big)^{\frac{2}{r-3}}\Big)\int_\Omega |\nabla u|^2 dx \\
&\qquad\quad + C\int_\Omega |\nabla d|^2 dx + C\int_\Omega |\nabla^2 d|^2 dx\Big(\int_\Omega |\nabla d|^2 dx\Big)^2.
\end{aligned}
$$

According to Gronwall's inequality, one obtains

$$
\begin{aligned}
&\int_\Omega (|\nabla u|^2 + |\Delta d|^2) dx(t) \le \\
&e^{\int_0^t \left(\int_\Omega \left(\sum_{i=1,2} \|u^i|^r + |\nabla d|^r\right) dx\right)^{\frac{2}{r-3}} d\tau} \times \Big(\int_\Omega |\nabla u_0|^2 dx \\
&\qquad + C\int_0^t \int_\Omega |\Delta d|^2 dx d\tau \sup_{0<\tau<t} \Big(\int_\Omega |\nabla d|^2 dx(\tau)\Big)^2\Big).
\end{aligned}
$$

For u^1, u^2, $\nabla d \in L^{r,s}$, with $\frac{3}{r} + \frac{2}{s} \le 1$, $r > 3$, then from Hölder's inequality, we get

$$
\int_0^T \Big(\int_\Omega |u^1|^r + |u^2|^r + |\nabla d|^r dx\Big)^{\frac{2}{r-3}} dt \le T^{1-\frac{1}{s}\frac{2r}{r-3}} (\|u^1\|_{L^{r,s}} + \|u^2\|_{L^{r,s}} + \|\nabla d\|_{L^{r,s}})^{\frac{2r}{r-3}},
$$

where, if $\frac{3}{r} + \frac{2}{s} \le 1$, then $1 - \frac{1}{s}\frac{2r}{r-3} \ge 0$. By using (2.3), we get

$$
\int_\Omega (|\nabla u|^2 + |\Delta d|^2) dx(T) \le \exp\Big(CT^{1-\frac{1}{s}\frac{2r}{r-3}}\left(\|u^1\|_{L^{r,s}} + \|u^2\|_{L^{r,s}} + \|\nabla d\|_{L^{r,s}}\right)^{\frac{2r}{r-3}}\Big)\Big[\int_\Omega |\nabla u_0|^2 dx + C(T)\Big].
$$

3. Interior regularity

In this section, we will study a suitable weak solution $(u, \nabla d)$ to the Liquid Crystal system (1.1) with $u_0(x) \in L^2$, $d_0(x) \in \mathbb{H}^1$, and $\nabla \cdot u_0 = 0$ in a weak sense. We also assume a so called no slip conditions $(u|_{\partial\Omega=0}, d|_{\partial\Omega=0})$.

We will use the following spaces:

$$
H_0^1(\Omega) = \text{closure of } C_0^\infty(\Omega, \mathbb{R}^3) \text{ in the norm } \Big(\int_\Omega |\nabla u|^2 dx\Big)^{\frac{1}{2}},
$$

$$
\begin{aligned}
H^{-1}(\Omega) &= \text{the dual space of } \mathbb{H}_0^1(\Omega),\\
\mathcal{V} &= C_0^\infty(\Omega, \mathbb{R}^3) \cap \{u : \nabla \cdot u = 0\},\\
H &= \text{closure of } \mathcal{V} \text{ in } L^2(\Omega),\\
V_1 &= \text{closure of } \mathcal{V} \text{ in } H^1(\Omega),\\
V_2 &= \text{closure of } \mathcal{V} \text{ in } H^2(\Omega),\\
Q &= \Omega \times \mathbb{R}_+, \quad Q_T = \Omega \times (0, T).
\end{aligned}
$$

Given $N > 0$, we take $\epsilon = T/N$. Thus, there exist sequences u_N, p_N and d_N such that

$$
u_N \in C(0, T; H) \cap L^2(0, T; V_1), \quad d_N \in C(0, T; V_1) \cap L^2(0, T; V_2),
$$

$$
\partial_t u_N + \Psi_\epsilon(u_N) \cdot \nabla u_N - \Delta u_N + \nabla p_N = -\nabla \cdot (\nabla d_N \odot \nabla d_N), \tag{3.1}
$$

$$
\partial_t d_N + u_N \cdot \nabla d_N = \Delta d_N - f(d_N), \tag{3.2}
$$

$$
u_N(0) = u_0, \quad \nabla \cdot u_N = 0, \quad d_N(0) = d_0, \quad |d_0| = 1.
$$

and u_N, p_N and d_N converge to limits u_*, p_* and d_* such that

$$
u_N \to u_* \begin{cases} \text{strong in } L^2(Q_T), \\ \text{weakly in } L^2(0, T; V_1), \\ \text{weak-star in } L^\infty(0, T; H). \end{cases}
$$

$$
d_N \to d_* \begin{cases} \text{strong in } L^2(0, T; H), \\ \text{weakly in } L^2(0, T; V_2), \\ \text{weak-star in } L^\infty(0, T; V_1). \end{cases}
$$

$$
p_N \to p_* \quad \text{weakly in } L^{5/3}(Q_T).
$$

The triplet function (u_*, p_*, d_*) is a weak solution of the Liquid Crystal system (3.1). Now, we define the "retarded mollifier" $\Psi_\epsilon(u)$ which appears in (3.2). Suppose $\psi \in C^\infty$ and

$$
\psi \geq 1 \quad \text{and} \quad \int\int \psi dx dt = 1,
$$

$$\text{supp}\psi \subset \{(x,t) : |x|^2 < t, 1 < t < 2\}.$$

For $v \in L^2(0,T;V_1)$, let $\tilde{v} : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^3$ be

$$\tilde{v} = \begin{cases} v(x,t) & \text{if } (x,t) \in Q_T, \\ 0, & \text{otherwise.} \end{cases}$$

Set

$$\Psi_\epsilon(v) = (x,t) = \delta^{-4} \int_{-\infty}^{-\infty} \int_{\mathbb{R}^3} \psi(\frac{y}{\epsilon}, \frac{\tau}{\epsilon}) \tilde{v}(x-y, t-\tau) dy d\tau.$$

Note that the value of $\Psi_\epsilon(v)$ at times t depend only on the values of v at times $\tau \in (t-2\epsilon, t-\epsilon)$. From the Appendix in CKN [2], and references therein, for $v \in L^\infty(0,T;H) \cap L^2(0,T;V_1)$, we have

$$\nabla \cdot \Psi_\epsilon(v) = 0,$$

$$\sup_{0 \le t \le T} \int_\Omega |\Psi_\epsilon(v)|^2(x,t)| dx \le \text{ess} \sup_{0<t<T} \int_\Omega |v|^2 dx.$$

For $v \in L^s(\Omega)$, it is clear that

$$\int_\Omega |\Psi_\epsilon(v)|^s dx \le C \int_\Omega |v|^s dx.$$

Set $u_N := (u_N^1, u_N^2, u_N^3)$ and $\Psi_\epsilon(v) := (\Psi_\epsilon^1(v), \Psi_\epsilon^2(v), \Psi_\epsilon^3(v))$. In what following, we shall prove that if u_N^1, u_N^2, $\nabla d_N \in L^{5,5}_{loc}(Q)$, then $u_N^3 \in L^{5,5}_{loc}(Q)$ and u_N is locallly bounded. Thus, the suitable weak solution (u_*, d_*) is also bounded.

In the following, we shall introduce some Lemmas which will be used in the proof of the main Theorem of this section.

Lemma 3.1 ([8]Theroem 2.5). Suppose $\Omega \subset \mathbb{R}^3$ be a smooth bounded domain. Let u_0 and d_0 smooth enough. Let $s, p \in (1,\infty)$ with $n \le \frac{2}{s} + \frac{n}{p}$, $\frac{1}{p} + \frac{1}{n} < 1$ and $\frac{1}{q} = \frac{1}{p} + \frac{1}{n}$, such that

$$u \in L^\infty(0,T;L^2) \cap L^2(0,T;\mathbb{H}^1),$$

$$d \in L^\infty(0,T;\mathbb{H}^1) \cap L^2(0,T;\mathbb{H}^2),$$

Then there exists a $P \in L^s(0,T;L^p(\Omega))$ with $\nabla P \in L^s(0,T;L^q(\Omega))$ which satisfies (1.1).

Lemma 3.2 ([8]Theroem 2.6). There exists constants ϵ and $C > 0$, such that if the triplet function (u, d, p) is a suitable weak solutions of (1.1) on Q_1 with the following inequality

$$\int\int_{Q_1} (|u|^3 + |\nabla d|^3 + |p|^{\frac{3}{2}})dxdt < \epsilon,$$

Then

$$|u(x,t)| + |\nabla d(x,t)| \leq C$$

for almost every point $(x,t) \in Q_{\frac{1}{2}}$.

Lemma 3.3. Suppose $u \in L^\infty(0,T;L^2(\Omega)) \cap L^2(0,T;\mathbb{H}^1(\Omega))$, where $\Omega \subset \mathbb{R}^3$, then

$$u \in L^{q,r}_{x,t}(\Omega \times (0,T)), \qquad \frac{1}{r} + \frac{3}{2q} = \frac{3}{4}.$$

Moreover, if $u|_{\partial\Omega} = 0$ or $\fint_\Omega u(x,t)dx = 0$, thus the following inequality valids:

$$\|u\|_{L^{q,r}(Q_T)} \leq C \sup_{0<t<T} \|u\|^{1-\frac{2}{r}}_{L^2(\Omega)} \|\nabla u\|^{\frac{2}{r}}_{L^2(Q_T)}.$$

The main result in this section is following:

Theorem 3.4. Suppose that any two of the three component u_N^1, u_N^2, u_N^3, of u_N, belong to $L^{6,\infty}_{loc}(Q)$, and $\nabla d_N \in L^{6,\infty}_{loc}(Q)$. Let $u_N \in L^\infty(0,T;L^{2+\delta}) \cap L^2(0,T;H^1)$, for some $0 < \delta < 1$. Let u_N^1, u_N^2 and ∇d_N are uniformly bounded in $L^{6,\infty}_{loc}(Q)$. Then

$$u_N \in L^{\infty,\infty}_{loc}(Q).$$

Moreover, $(u_*, \nabla d_*) \in L^{\infty,\infty}_{loc}(Q)$.

Proof. By using Sobolev's inequality and Hölder's inequality, we have

$$\begin{aligned}
\int\int |v|^5\eta^5 dxdt &\leq \int \Big(\int |v\eta|^9 dx\Big)^{\frac{1}{3}} \Big(|v\eta|^3 dx\Big)^{\frac{2}{3}} dt \\
&\leq C \int \Big(\int |\nabla |v\eta|^{\frac{3}{2}}|^2 dx\Big)\Big(|v\eta|^3 dx\Big)^{\frac{2}{3}} dt \\
&\leq C \sup_t \|v\eta\|^2_{L^3} \int\int |\nabla |v\eta|^{\frac{3}{2}}|^2 dxdt,
\end{aligned}$$

here η is a smooth cut-off function with compact support. If

$$\sup_t \int |v\eta|^3 dx + \int\int |\nabla |v\eta|^{\frac{3}{2}}|^2 dxdt \leq C, \tag{3.3}$$

then $v \in L^{5,5}_{loc}(Q)$. According to (3.3), we need find an energy estimate of u_N^3. In the following, we simplify the notation by $u = u_N$, $d = d_N$ and $p = p_N$. Denote $w := u^3$ and $U = \Psi_\epsilon(u)$, and consider

$$\begin{cases} \partial_t w - \Delta w + (U \cdot \nabla) w + \partial_{x_3} p = -\partial_{x_3} d \cdot \Delta d - \partial_{x_3}\left(\frac{|\nabla d|^2}{2}\right), \\ \nabla \cdot u = 0, \\ \partial_t d - \Delta d + u \cdot \nabla d = -f(d). \end{cases} \tag{3.4}$$

Taking divergence to both side of $(3.4)_3$, multiplying by test function $\eta^k |\nabla d| \nabla d$ and integrating over Ω, we obtain

$$\begin{aligned} &\frac{1}{3}\frac{d}{dt}\int_\Omega |\nabla d|^3 \eta^k dx - \frac{1}{3}\int_\Omega |\nabla d|^3 k \eta^{k-1} \eta_t dx + \int_\Omega \nabla^2 d : \nabla(\eta^k |\nabla d| \nabla d) dx \\ &= \int_\Omega u \cdot \nabla d \cdot \nabla(\eta^k |\nabla d| \nabla d) dx - \int_\Omega \nabla f(d) \eta^k |\nabla d| \nabla d dx. \end{aligned} \tag{3.5}$$

By using Hölder's inequality and Young's inequality, we can estimate the terms of the above equality.

$$\begin{aligned} &\int_\Omega \nabla^2 d : \nabla(\eta^k |\nabla d| \nabla d) dx \\ &= \int_\Omega \nabla^2 d [\eta^k |\nabla d| \nabla^2 d + k\eta^{k-1} \nabla\eta |\nabla d| \nabla d + \eta^k \nabla d \nabla |\nabla d|] dx \\ &= \int_\Omega |\nabla d||\nabla^2 d|^2 \eta^k + k\eta^{k-1} |\nabla d|^2 (\nabla |\nabla d|) \nabla\eta + |\nabla d||\nabla|\nabla d||^2 \eta^k dx \\ &\geq \frac{1}{2}\int_\Omega |\nabla d||\nabla^2 d|^2 \eta^k dx - C\int_\Omega |\nabla d|^3 |\nabla\eta|^2 \eta^{k-2} dx. \end{aligned} \tag{3.6}$$

Since $\nabla d \in L^{6,\infty}_{loc}$, then

$$\int_\Omega |\nabla d|^6 \eta^k dx \leq C. \tag{3.7}$$

When $i = 1, 2$, $u^i \in L^{6,\infty}_{loc}$, we get

$$\int_\Omega |u^i|^4 \eta^k dx \leq \left(\int_\Omega |u^i|^6 \eta^{\frac{3k}{2}}\right)^{\frac{2}{3}} |\Omega_\eta|^{\frac{1}{3}} \leq C. \tag{3.8}$$

When $i = 3$, by Hölder's inequality and Sobolev's inequality, we have

$$\int_\Omega |w|^4 \eta^k \leq \left(\int_\Omega |w|^9 \eta^{3k} dx\right)^{\frac{1}{3}} \left(\int_{\Omega_\eta} |w|^{\frac{3}{2}} dx\right)^{\frac{2}{3}} \leq C\|w\|_{L^2} |\Omega_\eta|^{\frac{1}{6}} \int_\Omega |\nabla(|w|^{\frac{3}{2}} \eta^{\frac{k}{2}})|^2 dx. \tag{3.9}$$

Combining (3.7)-(3.9), we have

$$
\begin{aligned}
&\int_\Omega u\cdot\nabla d\cdot\nabla(\eta^k|\nabla d|\nabla d)dx\\
\le&\int_\Omega 2|u||\nabla d|^2|\nabla^2 d|\eta^k+|u||\nabla d|^3k\eta^{k-1}|\nabla\eta|dx\\
\le&\epsilon\int_\Omega|\nabla d||\nabla^2 d|^2\eta^k+C\int_\Omega|u|^2|\nabla d|^3\eta^k+C\int_\Omega|\nabla d|^3\eta^{k-3}|\nabla\eta|^2dx\\
\le&\epsilon\int_\Omega|\nabla d||\nabla^2 d|^2\eta^k+\epsilon\int_\Omega|\nabla(|w|^{\frac{3}{2}}\eta^{\frac{k}{2}})|^2dx+C\int_\Omega|\nabla d|^3\eta^{k-3}|\nabla\eta|^2dx+C.
\end{aligned}\tag{3.10}
$$

By (3.5), (3,6) and (3,10) we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega|\nabla d|^3\eta^kdx+\int_\Omega|\nabla d||\nabla^2 d|^2\eta^kdx\le\\
&\epsilon\int_\Omega|\nabla(|w|^{\frac{3}{2}}\eta^{\frac{k}{2}})|^2dx+C\int_\Omega|\nabla d|^3\eta^{k-3}|\nabla\eta|^2dx+C\int_\Omega|\nabla d|^3\eta^{k-1}|\eta_t|dx\\
&+C\int_\Omega|\nabla d|^3\eta^kdx+C.
\end{aligned}\tag{3.11}
$$

Multiplying by test function $\eta^k|w|w$ and integrating over Ω, we obtain

$$
\begin{aligned}
&\frac{1}{3}\frac{d}{dt}\int_\Omega|w|^3\eta^kdx-\frac{1}{3}\int_\Omega|w|^3k\eta^{k-1}\eta_tdx+\int_\Omega\nabla w\nabla(\eta^k|w|wdx\\
=&-\int_\Omega U\nabla w\eta^k|w|wdx-\int_\Omega\frac{\partial p}{\partial x_3}\eta^k|w|wdx-\int_\Omega\frac{\partial d}{\partial x_3}\Delta d\eta^k|w|wdx\\
&-\int_\Omega\frac{\partial}{\partial x_3}\Big(\frac{|\nabla d|^2}{2}\Big)\eta^k|w|wdx.
\end{aligned}\tag{3.12}
$$

By Hölder's inequality and Sobolev's inequality, we have

$$
\begin{aligned}
\int_\Omega\nabla w\nabla(\eta^k|w|w)dx=&\int_\Omega\nabla w[\nabla|w|w\eta^k+|w|\nabla w\eta^k+k\eta^{k-1}\nabla\eta|w|w]dx\\
\ge&\frac{1}{2}\int_\Omega|\nabla w|^2|w|\eta^kdx-C\int_\Omega|w|^3|\nabla\eta|^2\eta^{k-2}dx.
\end{aligned}\tag{3.13}
$$

Since $\nabla\cdot u=0$, we have

$$
\begin{aligned}
-\int_\Omega U\nabla w\eta^k|w|wdx=&\int_\Omega w[\nabla\cdot U\eta^k|w|w+U(\eta^k\nabla|w|w+\eta^k|w|\nabla w+k\eta^{k-1}\nabla\eta|w|w)]dx\\
\le&C\int_\Omega U|w|^3k\eta^{k-1}\nabla\eta+2U|w|^2\nabla|w|\eta^kdx.
\end{aligned}\tag{3.14}
$$

The last term of the above equality (3.12)

$$
-\int_\Omega U\nabla w\eta^k|w|w)dx\le C\int_\Omega U|w|^3k\eta^{k-1}\nabla\eta
$$

$$
\begin{aligned}
&\le C\|\nabla\eta\|_{L^\infty}\Big(\int_\Omega |w|^9\eta^{3(k-1)}dx\Big)^{\frac{1}{3}}\Big(\int_{\Omega_\eta}|U|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}}\\
&\le C\|u\|_{L^2}|\Omega_\eta|^{\frac{1}{6}}\int_\Omega |\nabla(|w|^{\frac{3}{2}}\eta^{\frac{k-1}{2}})|^2dx. \qquad (3.15)
\end{aligned}
$$

By integrating apart and Hölder's inequality, we get

$$
\begin{aligned}
-\int_\Omega \frac{\partial p}{\partial x_3}\eta^k|w|wdx &\le C\int_\Omega |p||w||w_{x_3}|\eta^k dx + C\int_\Omega |p||w|^2k\eta^{k-1}|\nabla\eta|dx\\
&\le \epsilon\int_\Omega |w||\nabla w|^2\eta^k dx + \frac{C}{\epsilon}\int_\Omega |p|^2|w|\eta^k dx + C\int_\Omega |w|^3|\nabla\eta|^2\eta^{k-2}dx,
\end{aligned} \qquad (3.16)
$$

$$
\begin{aligned}
-\int_\Omega \frac{\partial d}{\partial x_3}\Delta d\eta^k|w|wdx - \int_\Omega \frac{\partial}{\partial x_3}\Big(\frac{|\nabla d|^2}{2}\Big)\eta^k|w|wdx &\le C\int_\Omega |\nabla d||\nabla^2 d||w|^2\eta^k dx\\
&\le \epsilon\int_\Omega |\nabla d||\nabla^2 d|^2\eta^k dx + C\int_\Omega |\nabla d||w|^4\eta^k dx. \qquad (3.17)
\end{aligned}
$$

For the term $\int_\Omega |\nabla d||w|^4\eta^k dx$, using Sobolev's , Hölder's inequality and the assumption ($u \in L^\infty(0,T;L^{2+\delta})$), we have

$$
\begin{aligned}
\int_\Omega |\nabla d||w|^4\eta^k dx &\le \Big(\int_\Omega |w|^9\eta^{3k}dx\Big)^{\frac{1}{3}}\Big(\int_{\Omega_\eta}|\nabla d|^{\frac{3}{2}}|w|^{\frac{3}{2}}dx\Big)^{\frac{2}{3}}\\
&\le \Big(\int_\Omega |w|^9\eta^{3k}dx\Big)^{\frac{1}{3}}\Big(\int_{\Omega_\eta}|\nabla d|^6dx\Big)^{\frac{1}{6}}\Big(\int_{\Omega_\eta}|w|^2dx\Big)^{\frac{1}{2}}\\
&\le \Big(\int_\Omega |w|^9\eta^{3k}dx\Big)^{\frac{1}{3}}\|\nabla d\|_{L^6(\Omega_\eta)}\Big(\int_{\Omega_\eta}|w|^{2+\delta}dx\Big)^{\frac{1}{2+\delta}}|\Omega_\eta|^{\frac{6}{2(2+\delta)}}\\
&\le \epsilon\int_\Omega |\nabla(|w|^{\frac{3}{2}}\eta^{\frac{k}{2}})|^2dx. \qquad (3.18)
\end{aligned}
$$

here we take $|\Omega_\eta|$ small enough, thus the above inequality is valid. Combining (3.12)-(3.18), we have

$$
\begin{aligned}
&\frac{d}{dt}\int_\Omega |w|^3\eta^k dx + \int_\Omega |\nabla w|^2\eta^k|w|dx\\
\le& C\int_\Omega |w|^3\eta^{k-1}|\eta_t|dx + C\int_\Omega |p|^2|w|\eta^k dx + C\int_\Omega |w|^3|\nabla\eta|^2\eta^{k-2}dx\\
&+\epsilon\int_\Omega |\nabla d||\nabla^2 d|\eta^k dx + \epsilon\int_\Omega |\nabla(|w|^{\frac{3}{2}}\eta^{\frac{k}{2}})|^2dx. \qquad (3.19)
\end{aligned}
$$

Plugging (3.11) and (3.19), we have

$$
\frac{d}{dt}\int_\Omega (|w|^3 + |\nabla d|^3)\eta^k dx + \int_\Omega (|\nabla w|^2|w| + |\nabla^2 d|^2|\nabla d|)\eta^k dx
$$

$$
\begin{aligned}
\leq & C\int_\Omega (|w|^3+|\nabla d|^3)\eta^{k-1}|\eta_t|dx + C\int_\Omega |p|^2|w|\eta^k dx + C\int_\Omega (|w|^3+|\nabla d|^3)|\nabla\eta|^2\eta^{k-2}dx \\
& + C\int_\Omega |\nabla d|^3\eta^k dx + C. \qquad (3.20)
\end{aligned}
$$

Integrating to both side of (3.20), by the fact $u, \nabla d \in L^{\frac{10}{3},\frac{10}{3}}$, we get

$$
\begin{aligned}
J := & \sup_t \frac{d}{dt}\int_\Omega (|w|^3+|\nabla d|^3)\eta^k dx + \int\int (|\nabla w|^2|w| + |\nabla^2 d|^2|\nabla d|)\eta^k dxdt \\
& \leq C + C\int\int |p|^2|w|\eta^k dxdt. \qquad (3.21)
\end{aligned}
$$

In what follows, we shall estimate the term $\int\int |p|^2|w|\eta^k dxdt$. We take divergence to both side of the following equation

$$
u_t + (U\cdot\nabla)u - \Delta u + \nabla p = -\nabla\cdot(\nabla d\odot\nabla d),
$$

then

$$
\Delta p = -(U^i u^j)_{x_i x_j} + \nabla\cdot g, \quad \text{in } \Omega\times(0,\infty), \qquad (3.22)
$$

where $g = -\nabla\cdot(\nabla d\odot\nabla d)$. Let $\zeta = \frac{k}{3}$ and $\Gamma(x) = \frac{-1}{4\pi|x|}$ be the fundamental solution of Laplace equation. For $p\eta^\zeta$ has a compact support in Ω, one obtains

$$
\begin{aligned}
p\eta^\zeta &= \lim_{\epsilon\to 0}\int_{\Omega\setminus B_\epsilon(x)} \Gamma(x-y)\Delta_y(p\eta^\zeta(y,t))dy \\
&= \lim_{\epsilon\to 0}\int_{\Omega\setminus B_\epsilon(x)} \Gamma(x-y)(\Delta_y p\eta^\zeta)dy + 2\lim_{\epsilon\to 0}\int_{\Omega\setminus B_\epsilon(x)} \Gamma(x-y)(\nabla p\cdot\nabla\eta^\zeta)dy \\
&\quad + \lim_{\epsilon\to 0}\int_{\Omega\setminus B_\epsilon(x)} \Gamma(x-y)(p\Delta_y\eta^\zeta)dy. \qquad (3.23)
\end{aligned}
$$

By (3.22) and $\nabla\cdot u = 0$, we have

$$
\begin{aligned}
& \int \Gamma(x-y)(\Delta_y p\eta^\zeta)dy \\
= & -\int \Gamma(x-y)(U^i u^j)_{y_i y_j}\eta^\zeta dy + \int \Gamma(x-y)\eta^\zeta\nabla_y\cdot g dy \\
= & -\sum_{i=1}^{3}\sum_{j=1}^{2}\int [\Gamma_{y_i y_j}(x-y)\eta^\zeta + \Gamma_{y_i}(x-y)(\eta^\zeta)_{y_j} + \Gamma_{y_j}(x-y)(\eta^\zeta)_{y_i} \\
& + \Gamma(x-y)(\eta^\zeta)_{y_i y_j}]U^i u^j dy - \int \Gamma(x-y)\eta^\zeta(U^3u^3)_{y_3 y_3}dy + \int \Gamma(x-y)\eta^\zeta\nabla_y\cdot g dy, \\
& \qquad (3.24)
\end{aligned}
$$

and

$$\int \Gamma(x-y)\nabla_y p\nabla_y(\eta^\zeta)dy = -\int \nabla_y\Gamma(x-y)p\nabla_y(\eta^\zeta)dy - \int \Gamma(x-y)p\Delta_y(\eta^\zeta)dy. \tag{3.25}$$

Combining (3.23)-(3.25), we have

$$\begin{aligned}
p\eta^\zeta = &-\sum_{i=1}^{3}\sum_{j=1}^{2}\Big[\int \Gamma_{y_iy_j}(x-y)\eta^\zeta U^i u^j dy + \int \Gamma_{y_i}(x-y)\eta^\zeta_{y_j} U^i u^j dy \\
&+ \int \Gamma_{y_j}(x-y)\eta^\zeta_{y_i} U^i u^j dy + \int \Gamma(x-y)(\eta^\zeta)_{y_iy_j} U^i u^j dy\Big] \\
&- 2\int \nabla_y\Gamma(x-y)p\nabla_y(\eta^\zeta)dy - \int \Gamma(x-y)p\Delta_y(\eta^\zeta)dy \\
&+ \int \Gamma(x-y)\nabla_y\cdot g\eta^\zeta dy - \int \Gamma(x-y)\eta^\zeta(U^3u^3)_{y_iy_j}dy \\
:= &\sum_{i=1}^{3}\sum_{j=1}^{2}(P_{1,ij}+P_{2,ij}+P_{3,ij}+P_{4,ij}) + P_5 + P_6 + P_7 + P_8.
\end{aligned}$$

By Hölder's inequality, when $i = 1, 2$, we have

$$\int\int |u^i|^4|w|\eta^\zeta dxdt \le C\Big(\int\int |u^i|^5 dxdt\Big)^{\frac{4}{5}}\Big(\int\int |w\eta^\zeta|^5 dxdt\Big)^{\frac{1}{5}}.$$

Thus,

$$\int\int (U^1u^1 + U^2u^2)^2|w|\eta^\zeta dxdt \le C(\|u^1\|_{L^{5,5}} + \|u^2\|_{L^{5,5}(Q_\eta)})^4\|w\eta^\zeta\|_{L^{5,5}}. \tag{3.26}$$

Applying Calderon-Zygmund estimate for singular integral type operator, we have

$$\|P_{1;ij}\|_{L^{\frac{5}{2},\frac{5}{2}}} \le C\|U^iu^j\eta^\zeta\|_{\frac{5}{2},\frac{5}{2}} \le C\|U^i\eta^\zeta\|_{L^{5,5}(Q_\eta)}\|u^j\|_{L^{5,5}(Q_\eta)},$$

For $j \neq 3$, we have

$$\int\int |P_{1,ij}|^2|w|\eta^\zeta dxdt \le C\|P_{1,ij}\|^2_{\frac{5}{2},\frac{5}{2}}\|w\eta^\zeta\|_{L^{5,5}} \le C\|U^i\eta^\zeta\|^2_{L^{5,5}}\|u^j\|^2_{L^{5,5}(Q_\eta)}\|w\eta^\zeta\|_{L^{5,5}},$$

Then, we have

$$\begin{aligned}
\sum_{i=1}^{3}\sum_{j=1}^{2}\int\int |P_{1,ij}|^2|w|\eta^\zeta dxdt \le &C(\|u^1\|_{L^{5,5}(Q_\eta)} + \|u^2\|_{L^{5,5}(Q_\eta)})^4\|w\eta^\zeta\|_{L^{5,5}} \\
&+ C(\|u^1\|_{L^{5,5}(Q_\eta)} + \|u^2\|_{L^{5,5}(Q_\eta)})^2\|w\eta^\zeta\|^3_{L^{5,5}}.
\end{aligned} \tag{3.27}$$

Now, we shall consider the terms $P_{2,ij}$ and $P_{3,ij}$. By Sobolev's , Hölder's inequality, $|\nabla\Gamma(x)| \leq \frac{C}{|x|^2}$ and Calderon-Zygmund theory, we have for $j \neq 3$

$$\begin{aligned}\int\int |P_{2,ij}|^{\frac{20}{7}}dxdt &\leq \int\Big(\int |\nabla P_{2,ij}|^{\frac{60}{21}}dx\Big)^{\frac{41}{21}}dt \leq C\int\Big(\int_{\Omega_\eta}|U^iu^j|^{\frac{60}{21}}dx\Big)^{\frac{41}{21}}dt\\ &\leq C\int\Big(\int_{\Omega_\eta}|U^i|^{\frac{10}{3}}dx\Big)^{\frac{6}{7}}dt\int\Big(\int_{\Omega_\eta}|u^j|^{\frac{60}{23}}dx\Big)^{\frac{23}{21}}dt\\ &\leq \sup_t\|u^j\|^{\frac{20}{7}}_{L^{\frac{60}{23}}(\Omega_\eta)}\int\Big(\int_{\Omega_\eta}|U^i|^{\frac{10}{3}}dx\Big)^{\frac{6}{7}}dt \leq C. \qquad (3.28)\end{aligned}$$

By (3.28) and Hölder's inequality, we have

$$\int\int |P_{2,ij}|^2|w|\eta^\zeta dxdt \leq \Big(\int\int |P_{2,ij}|^{\frac{20}{7}}dxdt\Big)^{\frac{7}{10}}\Big(\int\int |w\eta^\zeta|^{\frac{10}{3}}dxdt\Big)^{\frac{3}{10}} \leq C. \qquad (3.29)$$

For $P_{3,ij}$, in the same way, we have

$$\int\int |P_{3,ij}|^2|w|\eta^\zeta dxdt \leq C. \qquad (3.30)$$

For $P_{4,ij}$, using Sobolev's inequality, we have

$$\begin{aligned}\int\int |P_{4,ij}|^2|w|\eta^\zeta dxdt &\leq \int\Big(\int |P_{4,ij}|^{\frac{15}{2}}dx\Big)^{\frac{4}{15}}dt\Big(\int\int |w\eta^\zeta|^{\frac{15}{11}}dxdt\Big)^{\frac{11}{15}}\\ &\leq C\int\Big(\int |\nabla P_{4,ij}|^{\frac{5}{4}}dx\Big)^{\frac{8}{5}}dt\Big(\int\int |w\eta^\zeta|^{\frac{15}{11}}dxdt\Big)^{\frac{11}{15}}\\ &\leq C\int\Big(\int |U^i|^{\frac{10}{3}}dx\Big)^{\frac{3}{5}}dt\int\Big(\int |u^j|^2dx\Big)dt \leq C. \qquad (3.31)\end{aligned}$$

For P_5, by Lemma 3.1 and Lemma 3.3, we have $p \in L^{\frac{15}{11},\frac{5}{2}}$, thus, $\nabla P_5 \in L^{\frac{15}{11},\frac{5}{2}}$. By Sobolev's inequality, we get $P_5 \in L^{5,5}$ and the following valid

$$\begin{aligned}\int\int |P_5|^2|w|\eta^\zeta dxdt &\leq \int\Big(\int |P_5|^{\frac{5}{2}}dx\Big)^{\frac{4}{5}}dt\Big(\int\int |w\eta^\zeta|^5dxdt\Big)^{\frac{1}{5}}\\ &\leq C\|P_5\|^2_{L^{\frac{5}{2},\frac{5}{2}}(Q_\eta)}\|w\eta^\zeta\|_{L^{5,5}}. \qquad (3.32)\end{aligned}$$

For P_6, by similar way, we have $\Delta P_6 \in L^{\frac{15}{11},\frac{5}{2}}$ and $P_6 \in L^{15,\frac{5}{2}}$. Then we have

$$\|P_6\|_{L^{\frac{5}{2},\frac{5}{2}}} \leq C\|P_6\|_{L^{15,\frac{5}{2}}(Q_\eta)},$$

and

$$\int\int |P_6|^2|w|\eta^\zeta dxdt \leq C\|P_6\|_{L^{15,\frac{5}{2}}(Q_\eta)}\|w\eta^\zeta\|_{L^{5,5}}. \qquad (3.33)$$

The estimate of P_7 is similar to $P_{1,ij}$, $P_{2,ij}$, $P_{3,ij}$ and $P_{4,ij}$, thus

$$\int\int |P_7|^2|w|\eta^\zeta dxdt \le C. \tag{3.34}$$

For the last term P_8, the integral P_8 containing $(U^3u^3)_{y_3y_3}$ is the most troublesome. Since $\nabla\cdot u = 0$, then we have

$$\begin{aligned}(U^3u^3)_{y_3y_3} =&(U^1(u^3)_{y_1})_{y_3} + (u^1(U^3)_{y_1})_{y_3} + (U^2(u^3)_{y_2})_{y_3} + (u^2(U^3)_{y_2})_{y_3}\\ &- (U^1u^3)_{y_1y_3} - (u^1U^3)_{y_1y_3} - (U^2u^3)_{y_2y_3} - (u^2U^3)_{y_2y_3}.\end{aligned}$$

Thus

$$\begin{aligned}P_8 =& \int(\Gamma(x-y)\eta^\zeta)_{y_1y_3}U^1(y,t)u^3(y,t)dy + \int(\Gamma(x-y)\eta^\zeta)_{y_1y_3}u^1(y,t)U^3(y,t)dy\\ &+ \int(\Gamma(x-y)\eta^\zeta)_{y_2y_3}U^2(y,t)u^3(y,t)dy + \int(\Gamma(x-y)\eta^\zeta)_{y_2y_3}u^2(y,t)U^3(y,t)dy\\ &+ \int(\Gamma(x-y)\eta^\zeta)_{y_3}U^1(y,t)(u^3(y,t))_{y_1}dy + \int(\Gamma(x-y)\eta^\zeta)_{y_3}u^1(y,t)(U^3(y,t))_{y_1}dy\\ &+ \int(\Gamma(x-y)\eta^\zeta)_{y_3}U^2(y,t)(u^3(y,t))_{y_2}dy + \int(\Gamma(x-y)\eta^\zeta)_{y_3}u^2(y,t)(U^3(y,t))_{y_2}dy\\ :=&R_1 + R^1 + R_2 + R^2 + R_3 + R^3 + R_4 + R^4.\end{aligned}$$

The terms R^1, R^2, $R^3$and R^4 be treated with in same ways to R_1, R_2, $R_3$and R_4, respectively. The estimates of R_1 and R_2 are similar to $P_{1,ij}$, $P_{2,ij}$, $P_{3,ij}$ and $P_{4,ij}$. Thus, we have

$$\begin{aligned}\int\int(|R_1|^2 + |R_2|^2)|w|\eta^\zeta dxdt \le& C + C(\|u^1\|_{L^{5,5}(Q_\rho)} + \|u^2\|_{L^{5,5}(Q_\rho)})^4\|w\eta^\zeta\|_{L^{5,5}}\\ &+ C(\|u^1\|_{L^{5,5}(Q_\rho)} + \|u^2\|_{L^{5,5}(Q_\rho)})^2\|w\eta^\zeta\|^3_{L^{5,5}}.\end{aligned} \tag{3.35}$$

Next, we shall estimate R_3 and R_4. By Hölder's inequality and the fact

$$|(\Gamma(x-y)\eta^\zeta)_{y_3}| \le \frac{C}{|x-y|^2}, \tag{3.36}$$

we get

$$\|U^1\nabla u^3\eta\|_{L^{\frac{3}{2}}(\Omega)} \le \|u^1\eta\|_{L^6(\Omega)}\|\nabla u\|_{L^2(\Omega_\eta)}, \tag{3.37}$$

hence, according to (3.36) (3.37) and the embedding theorem for the Riese potential, for a.a t, we have

$$\|R_3\|_{L^3(\Omega)} \le \Big(\int\Big(\int\frac{C}{|x-y|^2}|U^1\nabla u^3|\eta dy\Big)^3 dx\Big)^{\frac{1}{3}}$$

$$
\begin{aligned}
&\leq C\Big(\int |\int \frac{1}{|x-y|^6}dy|| \int |U^1\nabla u^3\eta|^{\frac{3}{2}}dy|^2dx\Big)^{\frac{1}{3}}\\
&\leq C\|u^1\|_{L^6}(t)\|\nabla u\|_{L^2}(t),
\end{aligned}
$$

and

$$
\begin{aligned}
\int\int |R_3|^2|w|\eta^\zeta dxdt &\leq \int\Big(\int |R_3|^3dx\Big)^{\frac{2}{3}}\Big(\int |w\eta^\zeta|^3\Big)^{\frac{1}{3}}dt\\
&\leq C\int \|u^1\eta\|^2_{L^6(\Omega)}\|\nabla u\|^2_{L^2(\Omega)}\|w\eta^\zeta\|_{L^3(\Omega)}dt\\
&\leq C\sup_t \|u^1\|^2_{L^{6,\infty}_{loc}}\Big(\sup_t |w\eta^\zeta|^3dx\Big)^{\frac{1}{3}}\Big(\int\int |\nabla u|^2dxdt\Big)\\
&\leq CJ^{\frac{1}{3}}. \qquad (3.38)
\end{aligned}
$$

Similarly, we have

$$
\int\int |R_4|^2|w|\eta^\zeta dxdt \leq CJ^{\frac{1}{3}}.
$$

If the Lebesgue measure of Q_η is small, then there exists a sufficient small $\epsilon > 0$ such that

$$
\|u^1\|_{L^{5,5}(Q_\eta)} + \|u^2\|_{L^{5,5}(Q_\eta)} + \|P^5\|_{L^{\frac{5}{2},\frac{5}{2}}(Q_\eta)} + \|P^6\|_{L^{15,\frac{5}{2}}(Q_\eta)} < \epsilon.
$$

Therefore, we get

$$
\int\int |p|^2|w|\eta^k dxdt \leq C + \frac{1}{2}J.
$$

Finally, combining (3.20) and (3.39), we have

$$
J \leq C + \frac{1}{2}J.
$$

and J is bounded. Therefor we conclude that $u \in L^{5,5}_{loc}$. This complete the proof of Theorem 3.4.

Finally, we prove the Theorem 1.2. Two limiting functions u and ∇d of u_N and ∇d_N belong to $L^{5,5}_{loc}$, thus the suitable weak solution of the system (1.1) meets local Serrin's condition. By Liu-Min-Wang-Zhang [9], the suitable weak solution to the system (1.1) has local regularity. This complete Theorem 1.2.

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