Relative Controllability of Hybrid Delay Multi-agent Systems

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This paper considers relative controllability of leader-follower hybrid delay multi-agent systems under fixed communication topology, where two kinds of state delays are existed and each agent subjects to one of them. Some agents with unidirectional signal flows are assigned as leaders and the others are followers. With neighbor-based protocols adopted, the multi-agent systems are represented as a higher dimensional two-delay system without pairwise matrices permutation. Fundamental solution matrix of the two-delay system is constructed by improving the methods in literature, further solution of the system is obtained. Based on the solution Gramian criterion on relative controllability of the system is established. Whereafter, a sufficient condition on relative controllability of the system is presented. An example is attached to support the work.

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Abstract

This paper considers relative controllability of leader-follower hybrid delay multi-agent systems under fixed communication topology, where two kinds of state delays are existed and each agent subjects to one of them. Some agents with unidirectional signal flows are assigned as leaders and the others are followers. With neighbor-based protocols adopted, the multi-agent systems are represented as a higher dimensional two-delay system without pairwise matrices permutation. Fundamental solution matrix of the two-delay system is constructed by improving the methods in literature, further solution of the system is obtained. Based on the solution Gramian criterion on relative controllability of the system is established. Whereafter, a sufficient condition on relative controllability of the system is presented. An example is attached to support the work.

Keywords: Multi-agent systems, Relative controllability, Two delays, Digraph, Solution.

1. Introduction

The cluster behavior of multi-agents is of interest in recent years. This is because of the wide applications of multi-agents, such as spacecraft formation, unmanned air vehicles, autonomous underwater vehicles, etc. The main attentions for multi-agents are paid to consensus [1, 2, 3, 4, 5], formation [6, 7], flocking [8, 9, 10] and controllability [11, 12], etc.

Controllability of the multi-agent systems is a basic problem, which determines whether we can operate and control the cluster behaviors of the group of agents. Since Tanner [11] put forward the controlled agreement problem of multi-agent systems with interaction topology fixation, great improvements have been made in this field and abundant criterions have been established in various conditions and ways, such as fixed topology [8, 9], switching topologies [12, 13], etc.

The above mentioned dynamics of agents are mainly single integrator. However, generic dynamics of agents is another important branch for the controllability of multi-agent systems. Ji et al. [14] consider the controllability of homogeneous multi-agent systems, where each agent shares a common generic linear dynamics. Similar result can be found in [15]. Structural controllability of the multi-

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agent systems with each agent updating its state by a common generic linear dynamics is also tackled in [16]. Zhao et al. [17] generalize the control problem of homogeneous multi-agent systems to heterogeneous ones, where each agent follows different generic linear dynamics. More information about the controllability of generic linear dynamic agents one can find in [18].

Besides, the dynamics of agents always exhibit time delays because of desensitization or burn-in of the sensors. Thus, distributed cooperative control of the delay multi-agent systems is an important topic (see, [3, 4, 19, 20]). For instance, Liu et al. [20] consider the controllability of discrete-time multi-agent systems with delays in communication topology. Ji et al. [21] convert the continuous-time multi-agent systems with communication delay to the delay system and deal with controllability of the delay system by analysing the eigenvalue of matrix. For more conclusion about controllability of the delay multi-agent systems it is referred to [19].

The delay aforementioned is mainly existed in communication topology. Whereas, each agent in the multi-agent systems may subject to different delay because the degree of disturbance is different for each agent (see more in [20, 22, 23, 24]), rendering it more reasonable to model each subsystem in the multi-agent systems by a different delay differential equation. For instance, Si et al. [25] consider the controllability of delay multi-agent systems by using the delay matrix exponential, where the dynamics of each agent obeys a generic delay differential equation.

We call a delay differential system relatively controllable, if there exists a controller such that for any initial function on the delay interval, the state of system can be steered to any designation in a finite time (see [26, 27]). In this study, relative controllability of the hybrid delay multi-agent systems is considered, where two types of delays in all are existed and the dynamics of each agent subjects to one of them. The agents with unidirectional information flows are assigned as leaders and the others are followers. Based on the kinds of delays that the followers subjected, we classify the followers into two groups and index them separately. We assume the dynamics of leaders are output controllable. Neighbor-based protocol is adopted to construct the communication network among agents. Under such protocol the multi-agent systems are written as two-delay system without pairwise permutation of matrices. We combine with the methods of Medved [28] and Mahmudov [31] to solve the two-delayed system. Gramian criterion on relative controllability of the system is established based on the solution. Whereafter, we construct a matrix sequence and prove that the derivative up to any order for the fundamental solution matrix of the two-delay system can be represented by such a matrix sequence established before. Based on this result and Gramian criterion, a sufficient condition on relative controllability of the two-delay system is presented.

2. Preliminaries

Hereinafter, we define a weighted adjacent digraph as $\mathfrak{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$, where $\mathcal{V}$ is the vertex set, $\mathcal{E}$ is the edge set and $\mathcal{A}$ represents the weighted adjacency matrix. If we denote that $\mathcal{V} = \{1, \cdots, n\}$

and $\mathcal{A} = [a_{ij}]$, then there is an direct edge from vertex i to vertex j if and only if $a_{ji} > 0$. For an edge $(i, j) \in \mathcal{E}$, we call j the out neighbor of i. The collection of all nearest out neighbors is denoted by $\mathcal{N}_i = \{j \in \mathcal{V} : \varepsilon_{ij} = (i, j) \in \mathcal{E}\}$. For the given graph $\mathfrak{G}$, the associated Laplacian matrix $\mathcal{L}$ is as follows: $\mathcal{L} = (\mathcal{L}_{ij})_{N\times N}$, where $\mathcal{L}_{ij} = -a_{ij}$ for $i \neq j$ and $\mathcal{L}_{ij} = \sum_{j\neq i}^{N} a_{ij}$ for $i = j$. More properties about the Laplacian matrix it is referred to [29].

3. Description

In this section, we formulize the controlled agreement problem of hybrid delay multi-agent systems with fixed interconnection topology, in which two types of delays are existed in all and each agent suffers from one of them. The interconnection topology among agents is abstracted as a weighted digraph. Each vertex in the digraph represents a subsystem of the multi-agent systems and the edge is associated with the information flow among agents.

The multi-agent systems are assumed to be consisted of $M + L$ members, where L agents which transform information to others unidirectionally are as leaders and the others are followers. Based on the kinds of delays, we separate the rest M agents (followers) into two groups. The agents in one group suffer from delay τ_1 and are labelled from 1 to $N(< M)$. The ones in another group suffer from delay τ_2 and are labelled from $N + 1$ to M. The indices from $M + 1$ to $M + L$ are left to leaders.

In what follows we assume that the followers update their states by the following delay differential equations

$$\dot{x}_i(t) = A_i x_i(t) + B_i x_i(t - \tau_1) + C_i u_i(t), \tag{1a}$$

$$\dot{x}_j(t) = A_j x_j(t) + B_j x_j(t - \tau_2) + C_j u_j(t), \tag{1b}$$

where $x_{i(j)} \in \mathbf{R}^n$ is the state of agent $i(j)$, $A_{i(j)}$, $B_{i(j)} \in \mathbf{R}^{n\times n}$, $C_{i(j)} \in \mathbf{R}^{n\times p}$, $u_{i(j)} \in \mathbf{R}^p$ is the steering input and τ_1, τ_2 are the two kinds of delays, $i = 1, \cdots, N$ $(j = N + 1, \cdots, M)$. We assume that $0 < \tau_1 < \tau_2 \leq 2\tau_1$.

Motivated by Liu et al. [12], we introduce the following neighbor-based protocols to establish the interconnection network among agents

$$\begin{aligned} u_i(t) = & K \sum_{k\in\mathcal{N}_i} w_{ik}(x_i(t) - x_k(t)) \\ & + P \sum_{k=1}^{L} a_{ik}\delta_{ik}(y_k(t) - x_i(t)), i = 1, \cdots, M, \end{aligned} \tag{2}$$

where $y_k(t) = x_{M+k}(t)$ represents the state of leader which receives exogenous control input, K and P are the gain matrices in $\mathbf{R}^{p\times n}$, w_{ik} is the weight of edge (k, i) among followers, a_{ik} is the coupled weight between leader and follower, and δ_{ik} is equal to one if the i-th follower can directly receive signal from the k-th leader, otherwise it is zero.

Under (2), systems (1a)–(1b) are transformed into the following two-delay system

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}_1 x(t-\tau_1) + \tilde{B}_2 x(t-\tau_2) + \tilde{C}u(t), \tag{3}$$

where $u(t) = (y_1^T(t), \cdots, y_L^T(t))^T$, $x(t) = (x_1^T(t), \cdots, x_M^T(t))^T$, and $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$, $\tilde{C}$ are defined as follows: $\tilde{A} = \tilde{A}_1 - \tilde{A}_2 + \tilde{A}_3$, $\tilde{A}_1 = diag(A_1, \cdots, A_M)$, $\tilde{A}_2 = diag(D_1, \cdots, D_M)$ with $D_m = \sum\limits_{k=1}^{L} a_{mk}\delta_{mk}C_m P$, $m = 1, \cdots, M$, and

$$\tilde{A}_3 = \begin{bmatrix} l_{11}C_1K & \cdots & l_{1M}C_1K \\ \vdots & \ddots & \vdots \\ l_{M1}C_MK & \cdots & l_{MM}C_MK \end{bmatrix}$$

with $l_{ij} = -w_{ij}$ for $i \neq j$ and $l_{ij} = \sum\limits_{j\neq i}^{M} w_{ij}$ for $i = j$;

$$\tilde{B}_1 = \left[\begin{array}{c|c} \hat{B}_1 & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} \end{array}\right], \tilde{B}_2 = \left[\begin{array}{c|c} \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \hat{B}_2 \end{array}\right],$$

where $\hat{B}_1 = diag(B_1 \cdots, B_N)$, $\hat{B}_2 = diag(B_{N+1} \cdots, B_M)$, and $\mathbf{0}$ is corresponding dimensional zero matrix;

$$\tilde{C} = \begin{bmatrix} a_{11}\delta_{11}C_1P & \cdots & a_{1L}\delta_{1L}C_1P \\ \vdots & \ddots & \vdots \\ a_{M1}\delta_{M1}C_MP & \cdots & a_{ML}\delta_{ML}C_MP \end{bmatrix}.$$

The matrices $\tilde{A}$, $\tilde{B}_1$, and $\tilde{B}_2$ are pairwise nonpermutable. If system (3) is controllable, then we call the multi-agent systems (1a)-(1b) under (2) are controllable. For the controllability of system (3), the analysis of matrix eigenvalue [12] or the graphical theory approach [30] might not be the prefer choice because it is still left to us to establish the controllability criterion for system (3). Alternatively, we will apply the method of mathematical analysis to tackle this problem. To this end, firstly we consider the solution of (3).

Remark 3.1. *For delay differential equations, Khusainov & Shuklin [27] gave the delay matrix exponential and solved a single-delay differential equations. Mahmudov [31] presented a matrix function of Mittag-Leffler type and constructed solution of the linear nonhomogeneous fractional delay differential equations. Medved & Pospíšil [28] extended the results of Khusainov & Shuklin and established a multi-delay exponential function to solve multi-delay differential equations with pairwise matrices permutation. System* (3) *is a linear two-delay dynamics without $\tilde{A}$, $\tilde{B}_1$, $\tilde{B}_2$ pairwise permutation. We will improve the methods of Mahmudov and Khusainov to solve the solution of* (3).

3.1. Solution

Next, we focus on the solution of (3) for arbitrary initial function. To this end, with referring to [31] we firstly introduce the following matrix sequence

$$\begin{cases} Q_0(s) = Q_k(-\tau_1) = \Theta, \\ Q_1(0) = I, \\ Q_{k+1}(s) = \tilde{A}Q_k(s) + \tilde{B}_1 Q_k(s-\tau_1), \ k \in \mathbb{N}, \end{cases}$$

where $\tilde{A}$, $\tilde{B}_1$ are defined in (3), Θ and I are zero and unit matrices of appropriate dimensions, respectively, and $s = 0, \tau_1, 2\tau_1, \cdots$. Further introduce a matrix polynomial function as follows

$$X(t) = \begin{cases} \Theta, \ -\tau_1 \leq t < 0, \\ \sum\limits_{i=0}^{\infty} \sum\limits_{j=0}^{p-1} Q_{i+1}(j\tau_1) \frac{(t-j\tau_1)^i}{\Gamma(i+1)}, \ (p-1)\tau_1 \leq t < p\tau_1, \end{cases} \tag{4}$$

where $p \in \mathbb{N}^+$ and $\Gamma(\cdot)$ is the gamma function. It follows from Mahmudov [31] that (4) is a fundamental solution matrix of

$$\dot{X}(t) = \tilde{A}X(t) + \tilde{B}_1 X(t-\tau_1). \tag{5}$$

Apply (4) to construct a function $\Phi_{\tau_2}(\cdot)$ as follows

$$\Phi_{\tau_2}(t) = \begin{cases} \Theta, \ t < -\tau_2, \\ X(t+\tau_2), \ -\tau_2 \leq t < 0, \\ X(t+\tau_2) + \int_0^t X(t-s_1)\tilde{B}_2 X(s_1) ds_1 + \cdots \\ + \int_{(k-1)\tau_2}^t \int_{(k-1)\tau_2}^{s_1} \cdots \int_{(k-1)\tau_2}^{s_{k-1}} X(t-s_1)\tilde{B}_2 X(s_1-s_2)\tilde{B}_2 \cdots \\ \times X(s_{k-1}-s_k)\tilde{B}_2 X(s_k-(k-1)\tau_2) ds_k \cdots ds_2 ds_1, \ (k-1)\tau_2 \leq t < k\tau_2, \ k \in \mathbb{N}^+. \end{cases} \tag{6}$$

Remark 3.2. *To obtain a fundamental solution matrix of* (3)*, we make an improvement of Eq. (2.3) in Medved et al. [28]. Namely, we replace the function $X(\cdot)$ in Eq. (2.3) of Medved et al. [28] by* (4) *and insert $\tilde{B}_2$ into the integrand of the repeated integral discretely (in Eq. (2.3) of Medved et al. [28]) to yield* (6)*. These improvements help us to construct the fundamental solution matrix of* (3) *without pairwise matrices permutation.*

Define a function as follows

$$Y(t) = \Phi_{\tau_2}(t-\tau_2), \ t \in \mathbf{R}, \tag{7}$$

where $\Phi_{\tau_2}(\cdot)$ is defined by (6). The following lemma shows that $Y(\cdot)$ is exactly the fundamental solution matrix of (3) with zero forcing, proof of which is referring to [28].

Lemma 3.3. *Let $Y(\cdot)$ be the function in* (7)*. Then we have*

$$\dot{Y}(t) = \tilde{A}Y(t) + \tilde{B}_1 Y(t-\tau_1) + \tilde{B}_2 Y(t-\tau_2) \tag{8}$$

with

$$Y(0) = I,\ Y(t) = \Theta,\ t \in [-\tau_2, 0). \tag{9}$$

Proof. Denote $\gamma(t) = t - \tau_1 - \tau_2$. For $t \in [-\tau_2, 0)$, it is trivial for (9). For $t = 0$, it has $X(0) = I$. Thus we obtain $Y(0) = I$. For $0 \le t < \tau_2$, $Y(t) = X(t)$. Thus, we obtain $Y(t - \tau_2) = \Theta$. We consider this interval in two cases. For $0 \le t < \tau_1$, $Y(t - \tau_1) = \Theta$, thus (8) holds. For $\tau_1 \le t < \tau_2$, $Y(t - \tau_1) = X(t - \tau_1)$. Thus it holds that

$$\begin{aligned}
\dot{Y}(t) &= \dot{X}(t) \\
&= \tilde{A}X(t) + \tilde{B}_1 X(t - \tau_1) \\
&= \tilde{A}Y(t) + \tilde{B}_1 Y(t - \tau_1) + \tilde{B}_2 Y(t - \tau_2).
\end{aligned}$$

For $k\tau_2 \le t < (k+1)\tau_2$, taking the derivative of $Y(\cdot)$ and following from (5)–(6), we have

$$\dot{Y}(t) = \tilde{A}Y(t) + \tilde{B}_1 \tilde{Y}_1(t) + \tilde{B}_2 \tilde{Y}_2(t)$$

with

$$\begin{aligned}
\tilde{Y}_1(t) =& X(t - \tau_1) + \int_0^{t-\tau_2} X(\gamma(t) - s_1)\tilde{B}_2 X(s_1) ds_1 \\
&+ \int_{\tau_2}^{t-\tau_2} \int_{\tau_2}^{s_1} X(\gamma(t) - s_1)\tilde{B}_2 X(s_1 - s_2)\tilde{B}_2 X(s_2 - \tau_2) ds_2 ds_1 + \cdots \\
&+ \int_{(k-1)\tau_2}^{t-\tau_2} \int_{(k-1)\tau_2}^{s_1} \cdots \int_{(k-1)\tau_2}^{s_{k-1}} X(\gamma(t) - s_1)\tilde{B}_2 X(s_1 - s_2)\tilde{B}_2 \cdots \\
&\times X(s_{k-1} - s_k)\tilde{B}_2 X(s_k - (k-1)\tau_2) ds_k \cdots ds_2 ds_1
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
\tilde{Y}_2(t) =& X(t - \tau_2) + \int_{\tau_2}^{t-\tau_2} X(t - \tau_2 - s_2)\tilde{B}_2 X(s_2 - \tau_2) ds_2 + \cdots \\
&+ \int_{(k-1)\tau_2}^{t-\tau_2} \int_{(k-1)\tau_2}^{s_2} \cdots \int_{(k-1)\tau_2}^{s_{k-1}} X(t - \tau_2 - s_2)\tilde{B}_2 X(s_2 - s_3)\tilde{B}_2 \cdots \\
&\times X(s_{k-1} - s_k)\tilde{B}_2 X(s_k - (k-1)\tau_2) ds_k \cdots ds_3 ds_2.
\end{aligned} \tag{11}$$

Next, we will compute $\tilde{Y}_1(t)$ and $\tilde{Y}_2(t)$, respectively, by separating the interval into two parts.

For $k\tau_2 \le t < k\tau_2 + \tau_1$, from $j\tau_2 \le s_1 \le t - \tau_2$, $j = 0, 1, \cdots, k-1$, we have

$$-\tau_1 \le t - \tau_1 - \tau_2 - s_1 \le t - \tau_1 - \tau_2 - j\tau_2.$$

From

$$(k-1)\tau_2 - \tau_1 \le t - \tau_1 - \tau_2 < (k-1)\tau_2,$$

we have

$$(k-j-1)\tau_2 - \tau_1 \le t - \tau_1 - \tau_2 - j\tau_2 < (k-j-1)\tau_2.$$

Thus, we further arrive at

$$-\tau_1 \le t - \tau_1 - \tau_2 - s_1 \le t - \tau_1 - \tau_2 - j\tau_2 < (k-j-1)\tau_2,$$

where $j = 0, 1, \cdots, k-1$. Thus if $j = k-1$, it holds that

$$X(t - \tau_1 - \tau_2 - s_1) = \Theta.$$

For $-\tau_1 \le t - \tau_1 - \tau_2 - s_1 < 0$, we have

$$X(t - \tau_1 - \tau_2 - s_1) = \Theta.$$

For $0 \le t - \tau_1 - \tau_2 - s_1 \le t - \tau_1 - \tau_2 - j\tau_2$, $j = 0, 1, \cdots, k-2$, we obtain

$$X(t - \tau_1 - \tau_2 - s_1) \ne \Theta.$$

Thus, we arrive at

$$\begin{aligned}
\tilde{Y}_1(t) =& X(t-\tau_1) + \int_0^{\gamma(t)} X(\gamma(t)-s_1)\tilde{B}_2 X(s_1)ds_1 \\
&+ \int_{\tau_2}^{\gamma(t)} \int_{\tau_2}^{s_1} X(\gamma(t)-s_1)\tilde{B}_2 X(s_1-s_2)\tilde{B}_2 X(s_2-\tau_2)ds_2 ds_1 + \cdots \\
&+ \int_{(k-2)\tau_2}^{\gamma(t)} \int_{(k-2)\tau_2}^{s_1} \cdots \int_{(k-2)\tau_2}^{s_{k-2}} X(\gamma(t)-s_1)\tilde{B}_2 X(s_1-s_2)\tilde{B}_2 \cdots \\
&\times X(s_{k-2}-s_{k-1})\tilde{B}_2 X(s_{k-1}-(k-2)\tau_2)ds_{k-1}\cdots ds_2 ds_1 \\
=& Y(t-\tau_1).
\end{aligned}$$

For (11), making changes of variables, $s_i = s_{i+1} - \tau_2$, $i = 1, 2, \cdots, k-1$, it yields that

$$\begin{aligned}
\tilde{Y}_2(t) =& X(t-\tau_2) + \int_0^{t-2\tau_2} X(t-2\tau_2-s_1)\tilde{B}_2 X(s_1)ds_1 + \cdots \\
&+ \int_{(k-2)\tau_2}^{t-2\tau_2} \int_{(k-2)\tau_2}^{s_1} \cdots \int_{(k-2)\tau_2}^{s_{k-2}} X(t-2\tau_2-s_1)\tilde{B}_2 X(s_1-s_2)\tilde{B}_2 \cdots \\
&\times X(s_{k-2}-s_{k-1})\tilde{B}_2 X(s_{k-1}-(k-2)\tau_2)ds_{k-1}\cdots ds_2 ds_1 \\
=& Y(t-\tau_2).
\end{aligned} \tag{12}$$

Thus (8) holds for $k\tau_2 \le t < k\tau_2 + \tau_1$.

Next, we consider (10) in another subinterval. For $k\tau_2 + \tau_1 \le t < (k+1)\tau_2$, from $j\tau_2 \le s_1 \le t - \tau_2$, we also obtain

$$-\tau_1 \le t - \tau_1 - \tau_2 - s_1 \le t - (j+1)\tau_2 - \tau_1 < (k-j)\tau_2 - \tau_1,$$

where $j = 0, 1, \cdots, k-1$. Thus, for $-\tau_1 \le t - \tau_1 - \tau_2 - s_1 < 0$, it holds that

$$X(t - \tau_1 - \tau_2 - s_1) = \Theta.$$

Further (10) becomes

$$\begin{aligned}\tilde{Y}_1(t) =& X(t-\tau_1) + \int_0^{\gamma(t)} X(\gamma(t) - s_1)\tilde{B}_2 X(s_1) ds_1 \\ &+ \int_{\tau_2}^{\gamma(t)} \int_{\tau_2}^{s_1} X(\gamma(t) - s_1)\tilde{B}_2 X(s_1 - s_2)\tilde{B}_2 X(s_2 - \tau_2) ds_2 ds_1 + \cdots \\ &+ \int_{(k-1)\tau_2}^{\gamma(t)} \int_{(k-1)\tau_2}^{s_1} \cdots \int_{(k-1)\tau_2}^{s_{k-1}} X(\gamma(t) - s_1)\tilde{B}_2 X(s_1 - s_2)\tilde{B}_2 \cdots \\ &\times X(s_{k-1} - s_k)\tilde{B}_2 X(s_k - (k-1)\tau_2) ds_k \cdots ds_2 ds_1 \\ =& Y(t - \tau_1). \end{aligned} \tag{13}$$

Thus, from (12) and (13), we know that (8) holds for $k\tau_2 + \tau_1 \le t < (k+1)\tau_2$. The proof is completed. $\square$

Lemma 3.4. *Solution of the following homogeneous Cauchy problem*

$$\dot{x}(t) = \tilde{A}x(t) + \tilde{B}_1 x(t-\tau_1) + \tilde{B}_2 x(t - \tau_2), \tag{14a}$$

$$x(t) = \varphi(t),\ t \in [-\tau_2, 0] \tag{14b}$$

can be represented as

$$\begin{aligned}\hat{x}(t) =& Y(t + \tau_1 + \delta(t))\varphi(-\tau_1 - \delta(t)) \\ &+ \int_{-\tau_1}^{0} Y(t - s + \delta(t)) \Big(\varphi'(s - \delta(t)) - \tilde{A}\varphi(s - \delta(t))\Big) ds, \end{aligned} \tag{15}$$

where $\delta(t) = 0$ *for* $t \in [-\tau_1, \infty)$ *and* $\delta(t) = \tau_1$ *for* $t \in (-\infty, -\tau_1)$.

Proof. From Lemma 3.3, it is easy to obtain that (15) satisfies (14a). It remains to verify that (14b) is satisfied. For $t \in [-\tau_2, -\tau_1)$, $\delta(t) = \tau_1$. From $-\tau_1 \le s \le 0$, we have

$$-\tau_2 + \tau_1 \le t + \tau_1 \le t - s + \delta(t) \le t + 2\tau_1 < \tau_1.$$

Thus for $t + \tau_1 < s \le 0$, we have

$$Y(t - s + \delta(t)) = \Theta.$$

For $-\tau_1 \le s \le t + \tau_1$, it holds that

$$Y(t - s + \delta(t)) = e^{\tilde{A}(t - s + \tau_1)}.$$

From $-\tau_2 + 2\tau_1 \le t + \tau_1 + \delta(t) < \tau_1$ and $\tau_1 < \tau_2 < 2\tau_1$, we arrive at

$$Y(t + \tau_1 + \delta(t)) = e^{\tilde{A}(t + 2\tau_1)}.$$

Thus, (15) is simplified as

$$\begin{aligned}\hat{x}(t) =&\int_{-\tau_1}^{t+\tau_1} e^{\tilde{A}(t-s+\tau_1)}(\varphi'(s-\tau_1) - \tilde{A}\varphi(s-\tau_1))ds \\ &+ e^{\tilde{A}(t+2\tau_1)}\varphi(-2\tau_1) \\ =&\varphi(t).\end{aligned}$$

For $t \in [-\tau_1, 0]$, $\delta(t) = 0$. From an analogous process, it holds that for $t \leq s < 0$, $Y(t-s) = \Theta$. For $-\tau_1 \leq s \leq t$, another result is yielded

$$Y(t-s) = e^{\tilde{A}(t-s)}.$$

From $0 \leq t + \tau_1 \leq \tau_1$, we have

$$Y(t+\tau_1) = e^{\tilde{A}(t+\tau_1)}.$$

Thus, the following relation holds

$$\hat{x}(t) = e^{\tilde{A}(t+\tau_1)}\varphi(-\tau_1) + \varphi(t) - e^{\tilde{A}(t+\tau_1)}\varphi(-\tau_1) = \varphi(t).$$

The proof is ended. □

Lemma 3.5. *Solution of system* (3) *associated with the zero initial value can be represented as*

$$\tilde{x}(t) = \int_0^t Y(t-s)\tilde{C}u(s)ds,$$

where $Y(\cdot)$ *is the function defined by* (7)*.*

Proof. Assume solution of (3) with the zero initial value can be represented as

$$\tilde{x}(t) = \int_0^t Y(t-s)g(s)ds.$$

Based on Lemma 3.3, the derivative of $\tilde{x}(t)$ is further arrived at

$$\begin{aligned}\dot{\tilde{x}}(t) =&g(t) + \int_0^t \tilde{A}Y(t-s)g(s)ds \\ &+ \int_0^t \left(\tilde{B}_1 Y(t-s-\tau_1) + \tilde{B}_2 Y(t-s-\tau_2)\right) g(s)ds.\end{aligned}$$

For $0 \leq s \leq t$, it holds $-\tau_1 \leq t-s-\tau_1 \leq t-\tau_1$. So, for $-\tau_1 \leq t-s-\tau_1 < 0$, we arrive at

$$Y(t-s-\tau_1) = \Theta.$$

Further for $0 \leq t-s-\tau_1 \leq t-\tau_1$, we obtain that

$$Y(t-s-\tau_1) \neq \Theta.$$

Similarly, for $t-\tau_2 \le s < t$, it holds

$$Y(t-s-\tau_2)=\Theta.$$

For $0 \le s \le t-\tau_2$, we obtain

$$Y(t-s-\tau_2)\neq\Theta.$$

Thus, we further arrive at

$$\begin{aligned}\dot{\tilde{x}}(t)=&\tilde{B}_1\int_0^{t-\tau_1}Y(t-s-\tau_1)g(s)ds+\tilde{B}_2\int_0^{t-\tau_2}Y(t-s-\tau_2)g(s)ds\\&+g(t)+\tilde{A}\int_0^t Y(t-s)g(s)ds\\=&g(t)+\tilde{A}\tilde{x}(t)+\tilde{B}_1\tilde{x}(t-\tau_1)+\tilde{B}_2\tilde{x}(t-\tau_2).\end{aligned}$$

Comparing this formula with (3), we obtain that $g(t)=\tilde{C}u(s)$. This proof is ended. □

Lemma 3.6. *Solution of system* (3) *with original function*

$$x(t)=\varphi(t),\ t\in[-\tau_2,0]$$

enjoys the following form

$$\begin{aligned}x(t)=&\int_{-\tau_1}^0 Y(t-s+\delta(t))\left(\varphi'(s-\delta(t))-\tilde{A}\varphi(s-\delta(t))\right)ds\\&+Y(t+\tau_1+\delta(t))\varphi(-\tau_1-\delta(t))\\&+\int_0^t Y(t-s)\tilde{C}u(s)ds,\end{aligned}$$

where $\delta(t)=0$ *for* $t\in[-\tau_1,\infty)$ *and* $\delta(t)=\tau_1$ *for* $t\in(-\infty,-\tau_1)$.

Proof. Following from Lemmas 3.3–3.5 we can deduce the result. □

4. Controllability

Relative controllability of system (3) is considered in this section.

Definition 4.1. *System* (3) *is called relatively controllable if, for any initial function* $\varphi(t)$, $t\in[-\tau_2,0]$, *and any final state* x_f, *there exist a terminal time* $t_f>0$ *and a measurable function* $u^*(t)$ *such that system* (3) *has a solution* $x^*(t)$ *on* $[-\tau_2,t_f]$ *which satisfies* $x^*(t_f)=x_f$ *and* $x^*(t)\equiv\varphi(t)$, $t\in[-\tau_2,0]$.

Next, we establish Gramian criterion for relative controllability of system (3). For some $t_f>0$, construct the following matrix

$$G(0,t_f)=\int_0^{t_f}Y(t_f-s)\tilde{C}\tilde{C}^TY^T(t_f-s)ds. \tag{16}$$

Denote that

$$\eta = Y(t_f+\tau_1)\varphi(-\tau_1) + \int_{-\tau_1}^{0} Y(t_f-s)\left(\varphi'(s) - \tilde{A}\varphi(s)\right) ds. \tag{17}$$

For relative controllability of (3), the following assertion holds.

Theorem 4.2. *System* (3) *is relatively controllable on* $[0, t_f]$ *if and only if* (16) *is nonsingular.*

Proof. Sufficiency. Suppose (16) is nonsingular. Then for any differentiable function $\varphi(t)$, $t \in [-\tau_2, 0]$, construct the following control input

$$u^*(s) = \tilde{C}^T Y^T(t_f - s) G^{-1}(0, t_f)(x_f - \eta). \tag{18}$$

Under (18), system (3) always has a solution of the form in Lemma 3.6 and this solution automatically satisfies the initial condition. Thus we have

$$\begin{aligned} x^*(t_f) =& \eta + \int_0^{t_f} Y(t_f-s)\tilde{C}\tilde{C}^T Y^T(t_f-s) ds G^{-1}(0,t_f)(x_f-\eta) \\ =& \eta + G(0,t_f)G^{-1}(0,t_f)(x_f-\eta) \\ =& x_f. \end{aligned}$$

Thus, (3) is relatively controllable.

Necessity. Suppose that system (3) is relatively controllable, but (16) is singular. Then a nonzero constant vector $\tilde{x}$ exists which renders the quadratic form vanishing, i.e.

$$\begin{aligned} \tilde{x}^T G(0,t_f)\tilde{x} =& \int_0^{t_f} \tilde{x}^T Y(t_f-s)\tilde{C}\tilde{C}^T Y^T(t_f-s)\tilde{x} ds \\ =& \int_0^{t_f} \|\tilde{x}^T Y(t_f-s)\tilde{C}\|^2 ds \\ =& 0. \end{aligned}$$

Thus, we further arrive at $\tilde{x}^T Y(t_f - s)\tilde{C} = \theta, s \in [0, t_f]$, where θ is a zero vector of appropriate dimension. System (3) being relatively controllable, two measurable control functions exist which steer the trajectories of (3) to $\tilde{x}$ and θ, respectively, namely

$$\begin{aligned} x^*(t_f) = \eta + \int_0^{t_f} Y(t_f-s)\tilde{C}u_1^*(s) ds = \tilde{x}, \\ x^*(t_f) = \eta + \int_0^{t_f} Y(t_f-s)\tilde{C}u_2^*(s) ds = \theta. \end{aligned}$$

Thus, we have

$$\tilde{x} = \int_0^{t_f} Y(t_f-s)\tilde{C}\left(u_1^*(s) - u_2^*(s)\right) ds.$$

Further we obtain

$$\|\tilde{x}\|^2 = \int_0^{t_f} \tilde{x}^T Y(t_f-s)\tilde{C}\left(u_1^*(s) - u_2^*(s)\right) ds = 0,$$

which yields that $\tilde{x} = \theta$. Thus (16) is nonsingular. This ends the proof. □

Next, we present a sufficient condition about relative controllability of system (3), which is based on the rank of controllability matrix and avoids to compute (16) mathematically. To this end, we construct the following bivariate matrix sequence

$$\begin{aligned}
&\tilde{Q}_1(0,0) = I,\\
&\tilde{Q}_0(s,t) = \tilde{Q}_k(-\tau_1,t) = \tilde{Q}_k(s,-\tau_2) = \Theta,\\
&\tilde{Q}_{k+1}(s,t) = \tilde{Q}_k(s,t)\tilde{A} + \tilde{Q}_k(s-\tau_1,t)\tilde{B}_1 + \tilde{Q}_k(s,t-\tau_2)\tilde{B}_2
\end{aligned}\tag{19}$$

with $\tilde{A}$, $\tilde{B}_1$, and $\tilde{B}_2$ defined in (3), where $k \in \mathbb{N}$, $s = 0, \tau_1, 2\tau_1, \cdots$, and $t = 0, \tau_2, 2\tau_2, \cdots$. We have

$$\begin{aligned}
\tilde{Q}_{k+1}(0,0) =& \tilde{A}^k,\\
\tilde{Q}_{k+1}(\tau_1,0) =& \sum_{i=0}^{k-1} \tilde{A}^i \tilde{B}_1 \tilde{A}^{k-i-1},\\
\tilde{Q}_{k+1}(0,\tau_2) =& \sum_{i=0}^{k-1} \tilde{A}^i \tilde{B}_2 \tilde{A}^{k-i-1},
\end{aligned}$$

where $k \in \mathbb{N}^+$. Besides, the following assertions hold

$$\tilde{Q}_k(k\tau_1,0) = \Theta, \tilde{Q}_k(0,k\tau_2) = \Theta,\tag{20}$$

where $k \in \mathbb{N}$. In fact, from (19) we have

$$\begin{aligned}
\tilde{Q}_k(k\tau_1,0) =& \tilde{Q}_{k-1}(k\tau_1,0)\tilde{A} + \tilde{Q}_{k-1}((k-1)\tau_1,0)\tilde{B}_1\\
=& \tilde{Q}_{k-2}(k\tau_1,0)\tilde{A}^2 + \tilde{Q}_{k-2}((k-1)\tau_1,0)\tilde{B}_1\tilde{A}\\
&+ \tilde{Q}_{k-2}((k-1)\tau_1,0)\tilde{A}\tilde{B}_1 + \tilde{Q}_{k-2}((k-2)\tau_1,0)\tilde{B}_1^2\\
=& \cdots\\
=& \tilde{Q}_1(k\tau_1,0)\tilde{A}^{k-1} + \cdots + \tilde{Q}_1(\tau_1,0)\tilde{B}_1^{k-1}\\
=& \Theta.
\end{aligned}$$

By an analogous process we obtain the other equation in (20). More generally, we have

$$\tilde{Q}_k(i\tau_1,0) = \tilde{Q}_k(0,j\tau_2) = \Theta$$

as long as $i \geq k$, $j \geq k$. Based on this, it holds that

$$
\begin{aligned}
\tilde{Q}_k(i\tau_1,(k-i)\tau_2) =& \tilde{Q}_{k-1}(i\tau_1,(k-i)\tau_2)\tilde{A} \\
& + \tilde{Q}_{k-1}((i-1)\tau_1,(k-i)\tau_2)\tilde{B}_1 \\
& + \tilde{Q}_{k-1}(i\tau_1,(k-i-1)\tau_2)\tilde{B}_2 \\
=& \cdots \\
=& \tilde{Q}_{k-i}(i\tau_1,(k-i)\tau_2)\tilde{A}^i \\
& + \cdots + \tilde{Q}_{k-i}(0,(k-i)\tau_2)\tilde{B}_1^i + \cdots \\
=& \cdots \\
=& \tilde{Q}_1(i\tau_1,(k-i)\tau_2)\tilde{A}^{k-1} + \cdots \\
& + \tilde{Q}_1(0,\tau_2)\tilde{B}_1^i\tilde{B}_2^{k-i-1} + \cdots \tilde{Q}_1(\tau_1,0)\tilde{B}_2^{k-i}\tilde{B}_1^{i-1} \\
=& \Theta.
\end{aligned} \tag{21}
$$

Lemma 4.3. *For the matrix sequence* (19)*, the following assertions hold*

$$\tilde{Q}_{k+j}(\tau_1,0) = \tilde{Q}_{k+1}(0,0)\tilde{Q}_j(\tau_1,0) + \tilde{Q}_{k+1}(\tau_1,0)\tilde{Q}_j(0,0), \tag{22a}$$

$$\tilde{Q}_{k+j}(0,\tau_2) = \tilde{Q}_{k+1}(0,0)\tilde{Q}_j(0,\tau_2) + \tilde{Q}_{k+1}(0,\tau_2)\tilde{Q}_j(0,0), \tag{22b}$$

where $j,k \in \mathbb{N}^+$.

Proof. For $k=1$, $j=1$, we have $\tilde{Q}_{k+1}(0,0)=\tilde{A}$, $\tilde{Q}_j(\tau_1,0)=\Theta$, and $\tilde{Q}_{k+j}(\tau_1,0)=\tilde{B}_1$. For $k=1$ and any integer j, it holds that

$$
\begin{aligned}
\tilde{Q}_{k+1}(0,0)\tilde{Q}_j(\tau_1,0) =& \sum_{r=0}^{j-2} \tilde{A}^{r+1}\tilde{B}_1\tilde{A}^{j-r-2} \\
=& \sum_{r=0}^{j-1} \tilde{A}^{r}\tilde{B}_1\tilde{A}^{j-r-1} - \tilde{B}_1\tilde{A}^{j-1} \\
=& \tilde{Q}_{k+j}(\tau_1,0) - \tilde{Q}_{k+1}(\tau_1,0)\tilde{Q}_j(0,0).
\end{aligned}
$$

Suppose (22a) holds for any k, $j \in \mathbb{N}^+$, then we have

$$
\begin{aligned}
\tilde{Q}_{k+2}(0,0)\tilde{Q}_j(\tau_1,0) =& \tilde{A}\tilde{Q}_{k+1}(0,0)\tilde{Q}_j(\tau_1,0) \\
=& \tilde{A}\tilde{Q}_{k+j}(\tau_1,0) - \tilde{A}\tilde{Q}_{k+1}(\tau_1,0)\tilde{Q}_j(0,0) \\
=& \sum_{r=0}^{k+j-1} \tilde{A}^{r}\tilde{B}_1\tilde{A}^{k+j-r-2} - \sum_{r=0}^{k} \tilde{A}^{r}\tilde{B}_1\tilde{A}^{k+j-r-2} \\
=& \tilde{Q}_{k+j+1}(\tau_1,0) - \tilde{Q}_{k+2}(\tau_1,0)\tilde{Q}_j(0,0).
\end{aligned}
$$

Thus (22a) holds. Similarly, we have (22b) hold. □

Remark 4.4. *Based on the definition and simple calculation of $\tilde{Q}_{k+1}(i\tau_1, j\tau_2)$, we know that $\tilde{Q}_{k+1}(i\tau_1, j\tau_2)$ can be regarded as a combination with k positions in a stack written into k matrices, where i matrices $\tilde{B}_1$ are wrote into i positions, j matrices $\tilde{B}_2$ are into j positions and $(k-i-j)$ matrices $\tilde{A}$ are into the rest $(k-i-j)$ ones, respectively. The total number of writing methods can be simply calculated by combination of mathematics. Based on the principles of classification counting and step-by-step counting we can obtain a series of equations, such as*

$$\begin{aligned} Q_{k+i+1}(0, j\tau_2) =& Q_{k+1}(0,0)Q_{i+1}(0, j\tau_2) + Q_{k+1}(0, \tau_2)Q_{i+1}(0, (j-1)\tau_2) \\ &+ \cdots + Q_{k+1}(0, j\tau_2)Q_{i+1}(0,0) \end{aligned}$$

and

$$\begin{aligned} Q_{k+i+1}(\tau_1, \tau_2) =& Q_{k+1}(\tau_1, 0)Q_{i+1}(0, \tau_2) + Q_{k+1}(0, \tau_2)Q_{i+1}(\tau_1, 0) \\ &+ Q_{k+1}(\tau_1, \tau_2)Q_{i+1}(0,0) + Q_{k+1}(0,0)Q_{i+1}(\tau_1, \tau_2), \end{aligned}$$

even more.

Lemma 4.5. *The derivative of $Y(\cdot)$ in (7) up to any order can be represented as*

$$Y^{(k)}(t) = \sum_{\substack{i,j=0 \\ i+j\le k}}^{k} \tilde{Q}_{k+1}(i\tau_1, j\tau_2)Y(t - i\tau_1 - j\tau_2), \ k \in \mathbb{N}^+. \tag{23}$$

Proof. From Lemma 3.3, we have (23) hold for $k = 1$. Suppose (23) holds for any $k \in \mathbb{N}^+$. Taking the $(k+1)$-th derivative of (23) and following from (8), we have

$$\begin{aligned} Y^{(k+1)}(t) =& \sum_{\substack{i,j=0 \\ i+j\le k}}^{k} \tilde{Q}_{k+1}(i\tau_1, j\tau_2)\tilde{A}Y(t - i\tau_1 - j\tau_2) \\ &+ \sum_{\substack{i=1 \\ i+j\le k+1}}^{k+1}\sum_{j=0}^{k} \tilde{Q}_{k+1}((i-1)\tau_1, j\tau_2)\tilde{B}_1 Y(t - i\tau_1 - j\tau_2) \\ &+ \sum_{\substack{i=0 \\ i+j\le k+1}}^{k}\sum_{j=1}^{k+1} \tilde{Q}_{k+1}(i\tau_1, (j-1)\tau_2)\tilde{B}_2 Y(t - i\tau_1 - j\tau_2). \end{aligned}$$

Decompose the sum to yield

$$
\begin{aligned}
Y^{(k+1)}(t) = &\sum_{\substack{i,j=1\\ i+j\le k}}^{k} \tilde{Q}_{k+2}(i\tau_1, j\tau_2)Y(t-i\tau_1-j\tau_2) + \sum_{i=0}^{k} \tilde{Q}_{k+1}(i\tau_1,0)\tilde{A}Y(t-i\tau_1) \\
&+ \sum_{j=1}^{k} \tilde{Q}_{k+1}(0,j\tau_2)\tilde{A}Y(t-j\tau_2) + \sum_{i=1}^{k+1} \tilde{Q}_{k+1}((i-1)\tau_1,0)\tilde{B}_1Y(t-i\tau_1) \\
&+ \sum_{j=1}^{k} \tilde{Q}_{k+1}((k-j)\tau_1, j\tau_2)\tilde{B}_1Y(t-(k-j+1)\tau_1-j\tau_2) \\
&+ \sum_{j=1}^{k+1} \tilde{Q}_{k+1}(0,(j-1)\tau_2)\tilde{B}_2Y(t-j\tau_2) \\
&+ \sum_{i=1}^{k} \tilde{Q}_{k+1}(i\tau_1,(k-i)\tau_2)\tilde{B}_2Y(t-i\tau_1-(k-i+1)\tau_2) \\
&:= J_1+J_2+J_3+J_4+J_5+J_6+J_7,
\end{aligned}
\tag{24}
$$

where J_i, $i=1,\cdots,7$, is defined to represent each sum function at the right hand of (24), respectively. From (20), it is obtained

$$
\begin{aligned}
\tilde{Q}_{k+2}((k+1)\tau_1,0) =&\tilde{Q}_{k+1}((k+1)\tau_1,0)\tilde{A} + \tilde{Q}_{k+1}(k\tau_1,0)\tilde{B}_1 \\
=&\tilde{Q}_{k+1}(k\tau_1,0)\tilde{B}_1.
\end{aligned}
$$

Thus, it holds that

$$
\begin{aligned}
J_2+J_4 =&\sum_{i=1}^{k} \left(\tilde{Q}_{k+1}(i\tau_1,0)\tilde{A} + \tilde{Q}_{k+1}((i-1)\tau_1,0)\tilde{B}_1\right) Y(t-i\tau_1) \\
&+ \tilde{Q}_{k+1}(0,0)\tilde{A}Y(t) + \tilde{Q}_{k+1}(k\tau_1,0)\tilde{B}_1Y(t-(k+1)\tau_1) \\
=&\sum_{i=1}^{k} \tilde{Q}_{k+2}(i\tau_1,0)Y(t-i\tau_1) + \tilde{Q}_{k+2}(0,0)Y(t) \\
&+ \tilde{Q}_{k+2}((k+1)\tau_1,0)Y(t-(k+1)\tau_1).
\end{aligned}
$$

By an analogous process we have

$$
\begin{aligned}
J_3+J_6 =&\sum_{j=1}^{k} \left(\tilde{Q}_{k+1}(0,j\tau_2)\tilde{A} + \tilde{Q}_{k+1}(0,(j-1)\tau_2)\tilde{B}_2\right)Y(t-j\tau_2) \\
&+ \tilde{Q}_{k+1}(0,k\tau_2)\tilde{B}_2Y(t-(k+1)\tau_2) \\
=&\sum_{j=1}^{k} \tilde{Q}_{k+2}(0,j\tau_2)Y(t-j\tau_2) \\
&+ \tilde{Q}_{k+2}(0,(k+1)\tau_2)Y(t-(k+1)\tau_2).
\end{aligned}
$$

Further combining the terms in (24) and from (21) we obtain

$$\begin{aligned}
J_5 + J_7 =& \sum_{j=1}^{k} \tilde{Q}_{k+1}((j-1)\tau_1, (k-j+1)\tau_2)\tilde{B}_1 Y\left(t - j\tau_1 - (k-j+1)\tau_2\right) \\
&+ \sum_{j=1}^{k} \tilde{Q}_{k+1}(j\tau_1, (k-j)\tau_2)\tilde{B}_2 Y\left(t - j\tau_1 - (k-j+1)\tau_2\right) \\
=& \sum_{j=1}^{k} \tilde{Q}_{k+2}(j\tau_1, (k-j+1)\tau_2) Y\left(t - j\tau_1 - (k-j+1)\tau_2\right).
\end{aligned}$$

Rewrite (24) to yield

$$\begin{aligned}
Y^{(k+1)}(t) =& \sum_{\substack{i,j=1 \\ i+j\le k}}^{k} \tilde{Q}_{k+2}(i\tau_1, j\tau_2)Y(t - i\tau_1 - j\tau_2) + \sum_{i=1}^{k} \tilde{Q}_{k+2}(i\tau_1, 0)Y(t - i\tau_1) \\
&+ \sum_{i=1}^{k} \Big(\tilde{Q}_{k+2}(0,0)Y(t) + \tilde{Q}_{k+2}((k+1)\tau_1, 0)Y(t - (k+1)\tau_1)\Big) \\
&+ \sum_{j=1}^{k} \Big(\tilde{Q}_{k+2}(0, j\tau_2)Y(t - j\tau_2) + \tilde{Q}_{k+2}(0, (k+1)\tau_2)Y(t - (k+1)\tau_2)\Big) \\
&+ \sum_{j=1}^{k} \tilde{Q}_{k+2}(j\tau_1, (k-j+1)\tau_2)Y\left(t - j\tau_1 - (k-j+1)\tau_2\right) \\
=& \sum_{\substack{i,j=0 \\ i+j\le k+1}}^{k+1} \tilde{Q}_{k+2}(i\tau_1, j\tau_2)Y(t - i\tau_1 - j\tau_2).
\end{aligned}$$

Thus, (23) holds for any $k \in \mathbb{N}^+$. □

Theorem 4.6. *If $rank(\check{Q}) = Mn$, then system (3) is relatively controllable on $[0, t_f]$ for some $t_f > 0$, where $\check{Q}$ is the controllability matrix defined by*

$$\begin{aligned}
\check{Q} = \Big[&\hat{Q}_1(0,0), \cdots, \hat{Q}_{Mn}(0,0), \\
&\hat{Q}_2(\tau_1, 0), \cdots, \hat{Q}_{Mn}(\tau_1, 0), \\
&\hat{Q}_2(0, \tau_2), \cdots, \hat{Q}_{Mn}(0, \tau_2), \cdots, \hat{Q}_{Mn}(i\tau_1, j\tau_2)\Big],\ i+j \le Mn-1,
\end{aligned} \tag{25}$$

with

$$\hat{Q}_{k+1}(r\tau_1, s\tau_2) = \tilde{Q}_{k+1}(r\tau_1, s\tau_2)\tilde{C},\ r+s \le k,\ k \in \mathbb{N}. \tag{26}$$

Proof. Assume that $rank(\check{Q}) = Mn$, whereas (3) is relatively uncontrollable on $[0, t_f]$. Then (16) is singular because of Theorem 4.2. Thus a nonzero vector $\hat{x}$ is existed which renders $\hat{x}^T G(0, t_f)\hat{x} = 0$, namely,

$$\hat{x}^T Y(t)\tilde{C} = \theta,\ t \in [0, t_f]. \tag{27}$$

Taking derivative of (27) up to any k-th order and from (23) we arrive at

$$\begin{aligned}\hat{x}^T Y^{(k)}(t)\tilde{C} = &\sum_{\substack{i,j=0\\ i+j\le k}}^{k} \hat{x}^T \tilde{Q}_{k+1}(i\tau_1, j\tau_2) Y(t - i\tau_1 - j\tau_2)\tilde{C}\\ =&\theta,\ k \in \mathbb{N}^+. \end{aligned}\tag{28}$$

Taking $t = 0$ in (28) and from (9), (27), we have

$$\hat{x}^T \tilde{Q}_{k+1}(0,0)\tilde{C} = \theta,\ k \in \mathbb{N}. \tag{29}$$

Taking $t = \tau_1$ in (28) we obtain that

$$\hat{x}^T Y^{(k)}(\tau_1)\tilde{C} = \sum_{i=0}^{\infty} \hat{x}^T \tilde{Q}_{k+i+1}(0,0)\tilde{C}\frac{\tau_1^i}{\Gamma(i+1)} + \hat{x}^T \tilde{Q}_{k+1}(\tau_1, 0)\tilde{C},\ k \in \mathbb{N}.$$

From (29), we have

$$\hat{x}^T \tilde{Q}_{k+1}(\tau_1, 0)\tilde{C} = \theta,\ k \in \mathbb{N}. \tag{30}$$

Following analogous process we obtain

$$\hat{x}^T \tilde{Q}_{k+1}(2\tau_1, 0)\tilde{C} = \theta, \tag{31a}$$

$$\hat{x}^T \tilde{Q}_{k+1}(0, \tau_2)\tilde{C} = \theta, \tag{31b}$$

where $k \in \mathbb{N}$.

Again taking $t = \tau_1 + \tau_2$ and following from Remark 4.4 we have

$$\begin{aligned}\hat{x}^T Y^{(k)}(\tau_1 + \tau_2)\tilde{C} = &\sum_{i=0}^{\infty} \hat{x}^T \tilde{Q}_{k+i+1}(0,0)\tilde{C}\frac{(\tau_1+\tau_2)^i}{\Gamma(i+1)}\\ &+ \sum_{i=0}^{\infty} \hat{x}^T \tilde{Q}_{k+i+1}(0,\tau_2)\tilde{C}\frac{\tau_1^i}{\Gamma(i+1)}\\ &+ \sum_{i=0}^{\infty} \hat{x}^T \tilde{Q}_{k+i+1}(\tau_1,0)\tilde{C}\frac{\tau_2^i}{\Gamma(i+1)}\\ &+ \sum_{i=0}^{\infty} \hat{x}^T \tilde{Q}_{k+i+1}(2\tau_1,0)\tilde{C}\frac{(\tau_2-\tau_1)^i}{\Gamma(i+1)}\\ &+ \hat{x}^T \tilde{Q}_{k+1}(\tau_1, \tau_2)\tilde{C}.\end{aligned}$$

From (29), (30) and (31), we have

$$\hat{x}^T \tilde{Q}_{k+1}(\tau_1, \tau_2)\tilde{C} = \theta,\ k \in \mathbb{N}. \tag{32}$$

Suppose that

$$\hat{x}^T \tilde{Q}_{k+1}(i\tau_1, j\tau_2)\tilde{C} = \theta,\ i + j \le r, \tag{33}$$

holds for arbitrary positive integer $r \in (1, k-1]$. For $t = i\tau_1 + j\tau_2$, $i + j = r + 1$, denote $r^* = r - i$ to have

$$
\begin{aligned}
&\hat{x}^T Y^{(k)}(i\tau_1 + (r^*+1)\tau_2)\tilde{C} \\
=&\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(0,0)\tilde{C}\frac{(i\tau_1 + (r^*+1)\tau_2)^j}{\Gamma(j+1)} + \cdots \\
&+\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(0,(s_1^*+1)\tau_2)\tilde{C}\frac{(i\tau_1 + (r^*-s_1^*)\tau_2)^j}{\Gamma(j+1)} \\
&+\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(\tau_1,0)\tilde{C}\frac{((i-1)\tau_1 + (r^*+1)\tau_2)^j}{\Gamma(j+1)} + \cdots \\
&+\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(\tau_1,(s_2^*+1)\tau_2)\tilde{C}\frac{((i-1)\tau_1 + (r^*-s_2^*)\tau_2)^j}{\Gamma(j+1)} + \cdots \\
&+\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(i\tau_1,0)\tilde{C}\frac{((r^*+1)\tau_2)^j}{\Gamma(j+1)} + \cdots + \hat{x}^T \tilde{Q}_{k+1}(i\tau_1,(r^*+1)\tau_2)\tilde{C} + \cdots \\
&+\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(s^\dagger\tau_1,0)\tilde{C}\frac{((i-s^\dagger)\tau_1 + (r^*+1)\tau_2)^j}{\Gamma(j+1)} + \cdots \\
&+\sum_{j=0}^{\infty} \hat{x}^T \tilde{Q}_{k+j+1}(s^\dagger\tau_1,(s_p^*+1)\tau_2)\tilde{C}\frac{((i-s^\dagger)\tau_1 + (r^*-s_p^*)\tau_2)^j}{\Gamma(j+1)},
\end{aligned}
$$

where s_i^*, $i = 1, \cdots, p$, $p \in [r+1-i, r-1)$, is the maximum integer such that $(i-j)\tau_1 + (r-i-s_p^*)\tau_2 \geq 0$ and $(i-j)\tau_1 + (r-i-s_p^*-1)\tau_2 < 0$ with $j = 0, 1, \cdots, s^\dagger$, $s^\dagger \in [i+1, r-1)$. Thus, it follows from (29)–(32) and the assumption (33) that

$$
\hat{x}^T \tilde{Q}_{k+1}(i\tau_1, j\tau_2)\tilde{C} = \theta. \tag{34}
$$

Thus, the assumption (33) holds.

From (27)–(34) we obtain that

$$
\hat{x}^T \tilde{Q}_{k+1}(i\tau_1, j\tau_2)\tilde{C} = \theta \tag{35}
$$

holds for $i + j \leq k$, $k \in \mathbb{N}^+$. Rewrite (35) to yield

$$
\begin{aligned}
\hat{x}^T \Big[&\hat{Q}_1(0,0), \hat{Q}_2(0,0), \cdots, \hat{Q}_{k+1}(0,0), \\
&\hat{Q}_2(\tau_1,0), \hat{Q}_3(\tau_1,0), \cdots, \hat{Q}_{k+1}(\tau_1,0), \\
&\hat{Q}_2(0,\tau_2), \hat{Q}_3(0,\tau_2), \cdots, \hat{Q}_{k+1}(0,\tau_2), \\
&\hat{Q}_3(\tau_1,\tau_2), \hat{Q}_4(\tau_1,\tau_2), \cdots, \hat{Q}_{k+1}(\tau_1,\tau_2), \cdots, \hat{Q}_{k+1}(i\tau_1,j\tau_2)\Big] := \hat{x}^T \bar{Q} = \theta
\end{aligned}
$$

for $i + j \leq k$, $k \in \mathbb{N}^+$, implying that $\bar{Q}$ is always row linearly dependent for arbitrary $k \in \mathbb{N}^+$. Thus, for $k = Mn$ we obtain $rank(\check{Q}) < Mn$, which yields a contradiction. This ends the proof. $\square$

5. Example

In this section, an example will be taken to explain our work. To simplify the problem, we assume the topological graph of multi-agent systems is presented in Fig. 1, where agents $5, 6$ are selected as leaders and $1, 2, 3, 4$ are followers. Dynamics of followers obey the following rules

$$\dot{x}_1(t) = a_1 x_1(t) + b_1 x_1(t-\tau_1) + c_1 u_1(t), \tag{36a}$$

$$\dot{x}_2(t) = a_2 x_2(t) + b_2 x_2(t-\tau_1) + c_2 u_2(t), \tag{36b}$$

$$\dot{x}_3(t) = a_3 x_3(t) + b_3 x_3(t-\tau_2) + c_3 u_3(t), \tag{36c}$$

$$\dot{x}_4(t) = a_4 x_4(t) + b_4 x_4(t-\tau_2) + c_4 u_4(t), \tag{36d}$$

where $x_i \in R$, $i = 1, \cdots, 4$. Taking values as $\tau_1 = 2$, $\tau_2 = 3$, finial state $x_f = [10, -20, -30, 24]^T$ and initial function

$$\varphi(t) = 10 \left[-\sin(2t), 2\cos(5t), 3\sin(\pi t), -3e^{-t} \right]^T.$$

From (4), it is obtained

$$\begin{aligned} X(t) = & \sum_{i=0}^{\infty} Q_{i+1}(0) \frac{t^i}{\Gamma(i+1)} + \sum_{i=0}^{\infty} Q_{i+1}(\tau_1) \frac{(t-\tau_1)^i}{\Gamma(i+1)} \\ & + \sum_{i=0}^{\infty} Q_{i+1}(2\tau_1) \frac{(t-2\tau_1)^i}{\Gamma(i+1)} + \sum_{i=0}^{\infty} Q_{i+1}(3\tau_1) \frac{(t-3\tau_1)^i}{\Gamma(i+1)}. \end{aligned}$$

Thus, Gramian matrix becomes

$$\begin{aligned} G(0, t_f) = & \int_0^{\tau_2} X(t) C C^T X^T(t) dt + \int_{\tau_2}^{2\tau_2} X(t) C C^T X^T(t) dt \\ & + \int_{\tau_2}^{2\tau_2} X(t) C C^T \tilde{X}_1^T(t) dt + \int_{\tau_2}^{2\tau_2} \tilde{X}_1(t) C C^T X^T(t) dt \\ & + \int_{\tau_2}^{2\tau_2} \tilde{X}_1(t) C C^T \tilde{X}_1^T(t) dt + \int_{2\tau_2}^{t_f} X(t) C C^T X^T(t) dt \\ & + \int_{2\tau_2}^{t_f} X(t) C C^T \tilde{X}_1^T(t) dt + \int_{2\tau_2}^{t_f} X(t) C C^T \tilde{X}_2^T(t) dt \\ & + \int_{2\tau_2}^{t_f} \tilde{X}_1(t) C C^T X^T(t) dt + \int_{2\tau_2}^{t_f} \tilde{X}_1(t) C C^T \tilde{X}_1^T(t) dt \\ & + \int_{2\tau_2}^{t_f} \tilde{X}_1(t) C C^T \tilde{X}_2^T(t) dt + \int_{2\tau_2}^{t_f} \tilde{X}_2(t) C C^T X^T(t) dt \\ & + \int_{2\tau_2}^{t_f} \tilde{X}_2(t) C C^T \tilde{X}_1^T(t) dt + \int_{2\tau_2}^{t_f} \tilde{X}_2(t) C C^T \tilde{X}_2^T(t) dt, \end{aligned}$$

with

$$\begin{aligned} \tilde{X}_1(t) &= \int_0^{t-\tau_2} X(t-s_1-\tau_2) \tilde{B}_2 X(s_1) ds_1, \\ \tilde{X}_2(t) &= \int_{\tau_2}^{t-\tau_2} \int_{\tau_2}^{s_1} X(t-s_1-\tau_2) \tilde{B}_2 X(s_1-s_2) \tilde{B}_2 X(s_2-\tau_2) ds_2 ds_1. \end{aligned}$$

If (16) is nonsingular, then the control input function is

$$\begin{aligned}
u(t) =& C^T X^T(t) G^{-1}(0,t_f)(x_f-\eta), t\in[0,\tau_2),\\
u(t) =& C^T X^T(t) G^{-1}(0,t_f)(x_f-\eta)\\
&+ C^T\int_0^{t-\tau_2} X^T(s_1)\tilde{B}_2^T X^T(t-s_1-\tau_2)ds_1 G^{-1}(0,t_f)(x_f-\eta), t\in[\tau_2, 2\tau_2),
\end{aligned}$$

and

$$\begin{aligned}
u(t) =& C^T X^T(t) G^{-1}(0,t_f)(x_f-\eta)\\
&+ C^T\int_0^{t-\tau_2} X^T(s_1)\tilde{B}_2^T X^T(t-s_1-\tau_2)ds_1 G^{-1}(0,t_f)(x_f-\eta)\\
&+ C^T\int_{\tau_2}^{t-\tau_2}\int_{\tau_2}^{s_1} X^T(s_2-\tau_2)\tilde{B}_2^T X^T(s_1-s_2)\tilde{B}_2^T X^T(t-s_1-\tau_2)ds_2 ds_1\\
&\times G^{-1}(0,t_f)(x_f-\eta), t\in[2\tau_2, t_f],
\end{aligned}$$

where η is defined by (17).

$$\begin{aligned}
\eta =& X(t_f+\tau_1)\varphi(-\tau_1) + \int_{-\tau_1}^0 X(t_f-s)(\varphi'(s)-\tilde{A}\varphi(s))ds\\
&+\int_0^{t_f+\tau_1-\tau_2} X(t_f+\tau_1-\tau_2-s_1)\tilde{B}_2 X(s_1)ds_1\varphi(-\tau_1)\\
&+\int_{\tau_2}^{t_f+\tau_1-\tau_2}\int_{\tau_2}^{s_1} X(t_f+\tau_1-\tau_2-s_1)\tilde{B}_2 X(s_1-s_2)\tilde{B}_2 X(s_2-\tau_2)ds_2 ds_1\varphi(-\tau_1)\\
&+\int_{-\tau_1}^0\int_0^{t_f-s-\tau_2} X(t_f-s-s_1-\tau_2)\tilde{B}_2 X(s_1)(\varphi'(s)-\tilde{A}\varphi(s))ds_1 ds\\
&+\int_{-\tau_1}^0\int_{\tau_2}^{t_f-s-\tau_2}\int_{\tau_2}^{s_1} X(t_f-s-s_1-\tau_2)\tilde{B}_2 X(s_1-s_2)\tilde{B}_2 X(s_1-\tau_2)(\varphi'(s)-\tilde{A}\varphi(s))ds_2 ds_1 ds.
\end{aligned}$$

Other parameters are taken as $a_1 = 0.2$, $a_2 = 0.5$, $a_3 = 0.8$, $a_4 = 0.12$, $b_1 = 0.18$, $b_2 = 0.3$, $b_3 = 0.5$, $b_4 = 0.16$, $c_1 = 0.8$, $c_2 = 0.1$, $c_3 = 0.7$, $c_4 = 0.15$, $w_{12} = 0.12$, $w_{13} = 0.21$, $w_{14} = 0.2$, $w_{21} = 0.18$, $w_{23} = 0.15$, $w_{31} = 0.1$, $w_{32} = 0.12$, $w_{41} = 0.3$, $p_{11} = 0.24$, $p_{22} = 0.2$, $p_{32} = 0.3$, $p_{41} = 0.18$. If we select the terminal time $t_f = 7$, then the determinant of $G(0, t_f)$ in (16) is 56.89 and the rank of $\hat{Q}$ in (25) is 4, thus from Theorems 4.2 and 4.6, respectively, we have that (36) is relatively controllable. Simulations are shown in Fig. 2–3. Fig. 2 is the trajectories of the four agents, all of which are steered to the arbitrary given terminal states. Fig. 3 is the control inputs.

6. Conclusion

Relative controllability of leader-follower multi-agent systems with two kinds of delays in dynamics is considered in this paper. Neighbor-based protocols are used to realize the communication of the group of agents. Solution of two-delayed differential equation without pairwise matrices permutation is presented by improving the fundamental solution matrix in literature and further Gramian

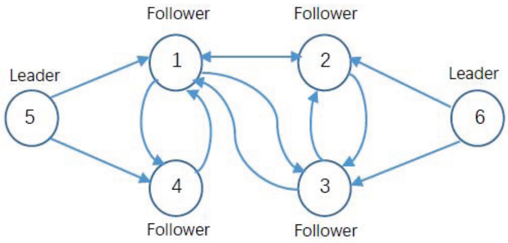

Figure 1: The communication topology of multi-agent systems.

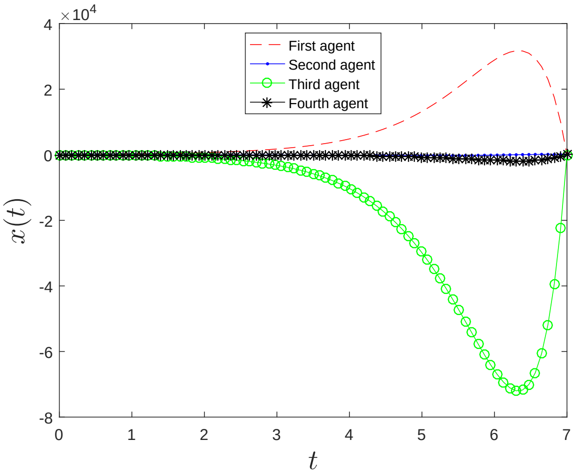

Figure 2: The four states of agents.

criterion is established. Derivative of the fundamental solution matrix up to any order is presented by a matrix sequence and a sufficient condition that rank deficiency or not of controllability matrix determines directly relative controllability of the two-delayed system is presented. Simulation of an example is shown to explain our work. For more information about solutions and controllability of delay differential equations we refer readers to Khusainov et al. [27] and Medved et al. [32].

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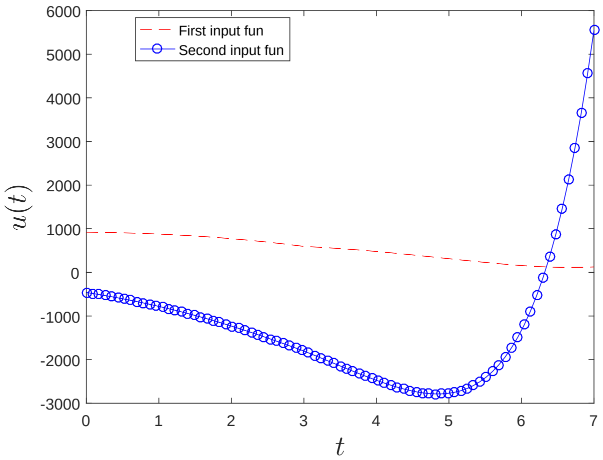

Figure 3: The control inputs from leaders.

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