Investigation of Exact Solutions of M-fractional Ivancevic Option Pricing Model Based on Three Different Methods

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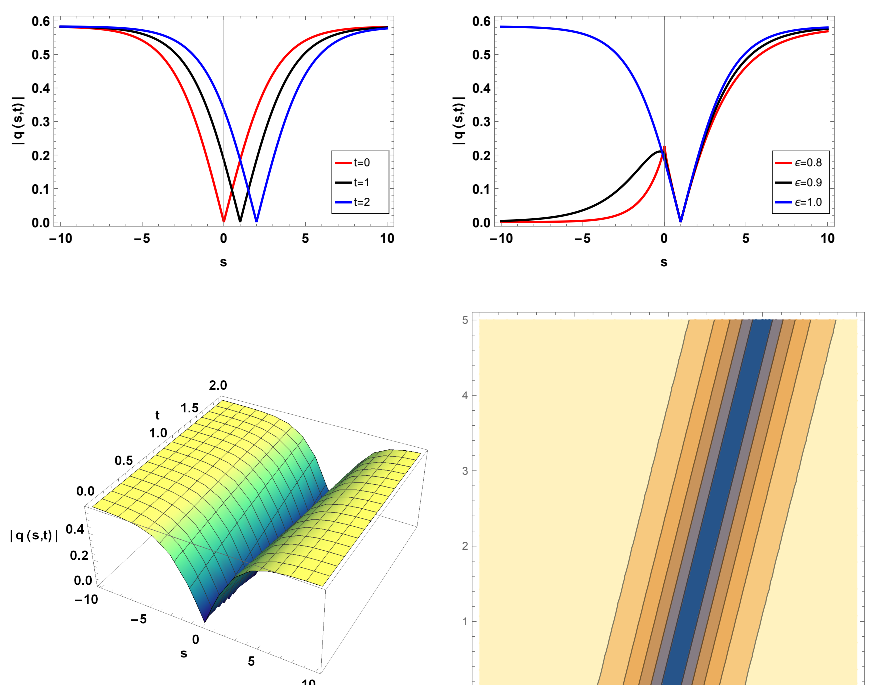
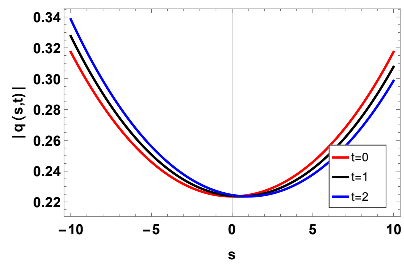
Abstract

This paper is about the investigation of exact solutions of important economic model; Ivancevic option pricing model (IOPM) with M-fractional derivative. To achieve this aim, three different methods; expa function method, extended Sinh-Gordon equation expansion method (EShGEEM) and extended (G'/G)-expansion method are used. Obtained solutions consisting of trigonometric, hyperbolic trigonometric, rational and exponential. The obtained solutions are new than the existing solutions in the literature. The got solutions are also verified by using Mathematica tool. Graphically justification are also done by plotting 2-D,3-D and contour graphs. The importance of this paper is that M-fractional derivative is first time use for this model. On the bases of achieved results it is suggested that these methods are simple, reliable and fruitful than the other methods.

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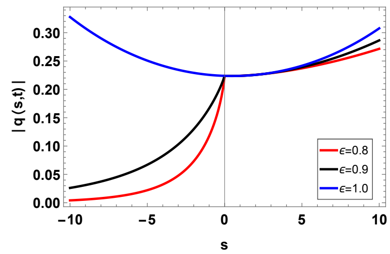
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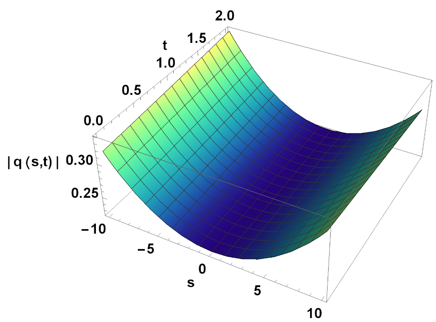
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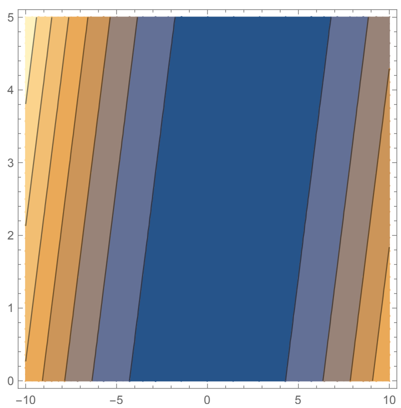
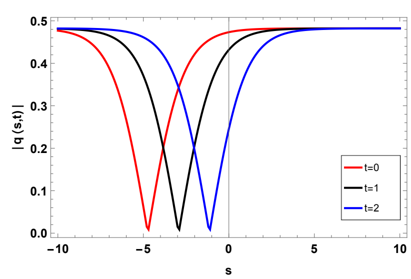
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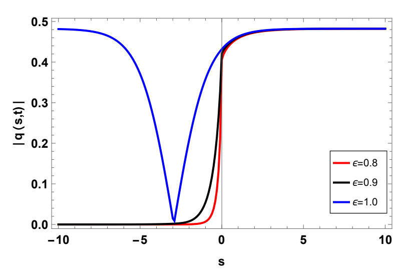
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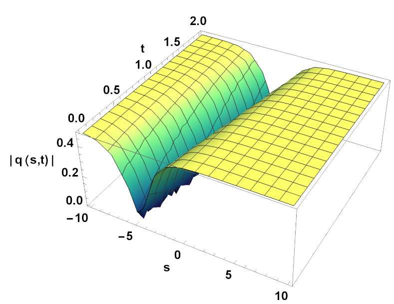
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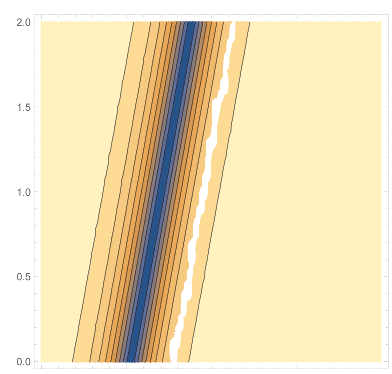

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Abstract

This paper is about the investigation of exact solutions of important economic model; Ivancevic option pricing model (IOPM) with M-fractional derivative. To achieve this aim, three different methods; exp_a function method, extended Sinh-Gordon equation expansion method (EShGEEM) and extended (G'/G)-expansion method are used. Obtained solutions consisting of trigonometric, hyperbolic trigonometric, rational and exponential. The obtained solutions are new than the existing solutions in the literature. The got solutions are also verified by using Mathematica tool. Graphically justification are also done by plotting 2-D,3-D and contour graphs. The importance of this paper is that M-fractional derivative is first time use for this model. On the bases of achieved results it is suggested that these methods are simple, reliable and fruitful than the other methods.

Keywords: Ivancevic option pricing model; M-fractional derivative; exp_a function method; EShGEEM; Extended (G'/G)-expansion method; Exact solutions.

1 Introduction

Many mathematical models have been developed in these areas in the form of nonlinear partial differential equations (NLPDEs). Numerous schemes are made to gain exact solutions of NLPDEs like; generalized exponential rational function scheme (GERFS) [1–3],

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Liu's extended trial function method (LETFM) [4], generalized unified method (GUM) [5], sine-Gordon expansion technique (SGET) [6], enhanced modified simple equation scheme [7],unified technique [8], extended tanh function scheme [9], Lie symmetry technique [10], symbolic computational method, Hirota's simple method and long wave technique [11], Jacobi elliptic function expansion scheme [12], Elzaki transform decomposition technique [13], $(m + \frac{1}{G})$-expansion and adomian decomposition schemes [14], new generalized expansion method [15], simplest equation and kudryashov's new function techniques [16], modified simple equation scheme [17], modified kudryashov simple equation technique [18], first integral technique [19], Bäcklund transformation scheme [20], extended jacobi elliptic function expansion technique [21], extended (G/G)-expansion and improved (G'/G)-expansion schemes [30] etc.
There are three other methods; exp_a function, extended Sinh-Gordon equation expansion (EShGEE) and extended (G'/G)-expansion methods. These methods have various applications. Likely, some new kind of analytical results of perturbed Gerdjikov-Ivanov model (pGIM) has been achieved by using exp_a function and extended tanh function expansion methods in [31]. By applying exp_a function and hyperbolic function methods, various types of wave solutions of two non-linear Schrödinger equations are gained in [32]. New trigonometry and hyperbolic function type soliton solutions of (2+1)-dimensional hyperbolic and cubic-quintic non-linear Schrödinger equations are achieved by applying extended sinh-Gordon equation expansion scheme in [33]. Bright, dark and bright-dark soliton solutions of generalized non-linear Schrödinger equation has been determined by implementing extended sinh-Gordon equation expansion method in [34]. Some exact solitons of (2+1)-dimensional improved Eckhaus equation have been calculated by using extended (G'/G)-expansion technique in [35]. Various kinds of wave solutions of time-fractional parabolic equations have been obtained by applying the extended (G'/G)-expansion scheme in [36].
Fractional calculus have gained much importance due to its various applications in different fields. Therefore, different definition of derivatives have been used like; conformable fractional derivative [37,38], beta derivative [39], caputo-Fabrizio fractional derivative [40], truncated M-fractional derivative [41,42] etc.
Our considering model is one of the important and interesting economical model named as; Ivancevic option pricing model (IOPM). In the literature, few techniques have been used on this model to get different exact solutions. For example, new solutions have been achieved of this model by applying the fractional reduced differential transform technique in [43]. Dark, bright, dark-bright, complex, travelling, periodic, trigonometric ,hyperbolic function solutions have been gained with the help of rational sine-Gordon expansion method and modified exponential method in [44]. Rogue wave and dark wave solitons of Ivancevic option pricing equation have been obtained with the use of trial function technique in [45].
The basic focus of the work is to investigate exact wave solutions of truncated M-fractional Ivancevic option pricing model based on exp_a function method, extended Sinh-Gordon equation expansion and extended (G'/G)-Expansion methods.
Paper have different sections; Section 2: truncated M-fractional derivative and its charac-

teristics, Section 3: model description, Section 4: description of methodologies, Section 5: mathematical treatment of model and exact solutions of model. Section 6: Illustrations with graphics. Section 7: conclusion.

2 Truncated M-derivative:

2.1 Definition:

Suppose $u(t) : [0, \infty) \to \mathbb{R}$, then truncated M-derivative of u of order ϵ is given [22]:

$$D^{\epsilon,\varrho}_{M,t} u(t) = \lim_{\tau \to 0} \frac{u\left(t\ E_\varrho(\tau t^{1-\epsilon})\right) - u(t)}{\tau}, \quad 0 < \epsilon < 1, \ \ \varrho > 0, \tag{1}$$

where $E_\varrho(.)$ shows truncated Mittag-Leffler function of one parameter that is defined as [23]:

$$E_\varrho(z) = \sum_{j=0}^{i} \frac{z^j}{\Gamma(\varrho j + 1)}, \qquad \varrho > 0 \ \ and \ \ \ z \in \mathbf{C}. \tag{2}$$

2.2 Characteristics

Let $\epsilon \in (0,1]1, \varrho > 0, r, s \in \Re$, and g, f $\epsilon-$differentiable at a point $t > 0$, then by [22]:

$$(i)\ D^{\epsilon,\varrho}_{M,t}(rg(t) + sf(t)) = rD^{\epsilon,\varrho}_{M,t} g(t) + sD^{\epsilon,\varrho}_{M,t} f(t). \tag{3}$$

$$(ii)\ D^{\epsilon,\varrho}_{M,t}(g(t).f(t)) = g(t)D^{\epsilon,\varrho}_{M,t} f(t) + f(t)D^{\epsilon,\varrho}_{M,t} g(t). \tag{4}$$

$$(iii)\ D^{\epsilon,\varrho}_{M,t}(\frac{g(t)}{f(t)}) = \frac{f(t)D^{\epsilon,\varrho}_{M,t} g(t) - g(t)D^{\epsilon,\varrho}_{M,t} f(t)}{(f(t))^2}. \tag{5}$$

$$(iv)\ D^{\epsilon,\varrho}_{M,t}(A) = 0, \ \ where\ A\ is\ a\ constant. \tag{6}$$

$$(v)\ D^{\epsilon,\varrho}_{M,t} g(t) = \frac{t^{1-\epsilon}}{\Gamma(\varrho + 1)}\ \frac{dg(t)}{dt}. \tag{7}$$

3 Model Description:

Let's assume M-fractional Ivancevic option pricing model(IOPM) [45] given as follows:

$$\iota\ D^{\epsilon,\varrho}_{M,t} q + \frac{\delta}{2}\ D^{2\epsilon,\varrho}_{M,2s} q + \Omega\ q|q|^2 = 0, \qquad \iota = \sqrt{-1}. \tag{8}$$

This model was first time developed by Ivancevic [24] to fulfill both behavioral and efficient markets. Where $q = q(s,t)$ shows the option price wave profile. While t is the time variable and s is asset price of model. Parameter δ represents the volatility which shows either stochastic process itself or only a constant. Where $\Omega = \Omega(r, \omega)$ is called Landau coefficient which describes adaptive market potential. In nonadaptive simplest case Ω and r become equal which shows the interest rate while in adaptive case $\Omega(r, \omega)$ may be connected to market temperature and it depends on the set of tractable parameters $\{W_i\}$. In third term $|q|^2$ shows the probability density function which denotes the potential field.

3.1 Summary of exp_a function scheme:

Here, we will give complete concept of this scheme.
Assuming the non-linear PDE;

$$G(q, q^2 q_t, q_x, q_{tt}, q_{xx}, q_{xt}, ...) = 0. \tag{9}$$

Eq.(9) transformed in non-linear partial differential equation:

$$\Lambda(Q, Q^{'}, Q^{''}, ...,) = 0. \tag{10}$$

By using following transformations:

$$q(x,\ y,\ t) = Q(\zeta), \zeta = ax + by + rt. \tag{11}$$

Considering root of Eq. (10) is shown in [25–28]:

$$Q(\zeta) = \frac{\alpha_0 + \alpha_1 d^{\zeta} + ... + \alpha_m d^{m\zeta}}{\beta_0 + \beta_1 d^{\zeta} + ... + \beta_m d^{m\zeta}}, \quad d \neq 0, 1. \tag{12}$$

where α_i and $\beta_i (0 \leq i \leq m)$ are undetermined. Positive integral value of m is calculate by utilizing homogeneous balance technique into Eq.(10). Putting Eq. (12) into Eq. (10), gives

$$\wp(d^{\zeta}) = \ell_0 + \ell_1 d^{\zeta} + ... + \ell_t d^{t\zeta} = 0. \tag{13}$$

Taking ℓ_i $(0 \leq i \leq t)$ in Eq. (13) equal to 0, a system of algebraic equations is achieved as fellows.

$$\ell_i = 0, \qquad\qquad where \ \ i = 0, ..., t. \tag{14}$$

By using the got solutions, we achieve the exact results of Eq.(9).

3.2 Presentation of EShGEEM:

In this part, there are some fundamental steps of this method:
Step 1:
Assuming the NLPDE shown as follow:

$$G(q, D^{\alpha,\gamma}_{M,t} q, q^2 q_x, q_y, q_{yy}, q_{xx}, q_{xy}, ...) = 0, \tag{15}$$

where $q = q(x, y, t)$ represents wave function.
Supposing the wave transform given as follow:

$$q(x, y, t) = Q(\zeta), \quad \zeta = x - \nu y + \frac{\Gamma(\gamma + 1)}{\alpha}(\kappa t^{\alpha}). \tag{16}$$

Inserting Eq. (16) into Eq. (15), resulting ordinary differential equation shown as follows:

$$\Lambda(Q(\zeta), Q^2(\zeta)Q^{'}(\zeta), Q^{''}(\zeta), ...) = 0. \tag{17}$$

Step 2:
Let's assume the root of Eq. (16) in the form:

$$Q(p) = \alpha_0 + \sum_{i=1}^{m}(\beta_i \sinh(p) + \alpha_i \cosh(p))^i, \tag{18}$$

where $\alpha_0,\ \alpha_i,\ \beta_i\ (i = 1, 2, 3, ...,\ m)$ are undetermined. A new function p of ζ that fulfill the following equation:

$$\frac{dp}{d\zeta} = \sinh(w). \tag{19}$$

Positive integral value of m may be obtained with the use of homogenous balance method. Eq. (19) is achieved from sinh-Gordon equation given as follows:

$$q_{xt} = \kappa \sinh(v). \tag{20}$$

Resultantly given in [29], one may get the roots of Eq. (20) shown as:

$$\sinh p(\zeta) = \pm \operatorname{csch}(\zeta) \qquad or \qquad \cosh p(\zeta) = \pm \coth(\zeta), \tag{21}$$

and

$$\sinh p(\zeta) = \pm \iota\ sech(\zeta) \qquad or \qquad \cosh p(\zeta) = \pm \tanh(\zeta), \tag{22}$$

where $\iota = \sqrt{-1}$.
Step 3:
Substituting Eq. (18) along Eq. (20) into Eq. (17), result in the form of algebraic expressions in $p^{'k}(\zeta) \sinh^l p(\zeta) \cosh^m p(\zeta)$ $(k = 0, 1;\ \ l = 0, 1;\ \ m = 0, 1, 2, ...)$. Putting each coefficient of $p^{'k}(\zeta) \sinh^l p(\zeta) \cosh^m p(\zeta)$ equal to 0, to obtain a set of algebraic equations having $\nu, \kappa, \alpha_0, \alpha_i$ and $\beta_i (i = 1, 2, 3, ..., m)$.
Step 4:
By manipulating the gained set of algebraic equations by using Mathematica software, one may get the results of unknowns, $\nu, \kappa, \alpha_0, \alpha_i$ and β_i.
Step 5:

With the help of obtained solutions and Eqs. (21) and (22), we may achieve the roots of Eq. (17) shown as:

$$Q(\zeta) = \alpha_0 + \sum_{i=1}^{m} (\pm\beta_i \operatorname{csch}(\zeta) \pm \alpha_i \coth(\zeta))^i. \tag{23}$$

and

$$Q(\zeta) = \alpha_0 + \sum_{i=1}^{m} (\pm\iota\beta_i sech(\zeta) \pm \alpha_i \tanh(\zeta))^i. \tag{24}$$

3.3 Presentation of Extended $(G'/G)-$expansion Method:

In this part, there are some fundamental steps of this method given in [30].
Step 1: Supposing the NLPDE shown as follows:

$$G(q, D_{M,t}^{\alpha,\gamma} q, q^2 q_x, q_y, q_{yy}, q_{xx}, q_{xy}, ...) = 0, \tag{25}$$

where $q = q(x, y, t)$ show the wave function.
Assuming the wave transform shown as follows:
Step 2:

$$q(x, y, t) = Q(\zeta), \quad \zeta = x - \nu y + \frac{\Gamma(\gamma+1)}{\alpha}(\kappa t^{\alpha}). \tag{26}$$

Putting Eq. (26) into Eq. (25), results in the form of ODE shown as:

$$\Lambda(Q(\zeta), Q^2(\zeta)Q'(\zeta), Q''(\zeta), ...) = 0. \tag{27}$$

Step 3:
Considering toots of Eq. (27) in the form given as:

$$Q(\zeta) = \sum_{i=-m}^{m} \alpha_i \left(\frac{G'(\zeta)}{G(\zeta)}\right)^i. \tag{28}$$

In Eq. (28), α_0 and $\alpha_i, (i = \pm1, \pm2, \pm3, ..., \pm m)$ are unknowns and $\alpha_i \neq 0$. Using homogenous balance method into Eq. (27), one can calculate positive integer m. Function $G = G(\zeta)$ fulfill the Riccati differential equation shown as follows:

$$dGG'' - aG^2 - bGG' - c(G')^2 = 0, \tag{29}$$

where a, b, c and d are constants.
Step 4:
Suppose Eq. (29) have results shown as:

Case 1: if $b \neq 0$ and $b^2 + 4ad - 4ac > 0$, then

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right) = \frac{b}{2(d-c)} + \frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)} \left(\frac{C_1 \sinh\left(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d}\right) + C_2 \cosh\left(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d}\right)}{C_1 \cosh\left(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d}\right) + C_2 \sinh\left(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d}\right)}\right). \tag{30}$$

Case 2: if $b \neq 0$ and $b^2 + 4ad - 4ac < 0$, then

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right) = \frac{b}{2(d-c)} + \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)} \left(\frac{C_2 \cos\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right) - C_1 \sin\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1 \cos\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right) + C_2 \sin\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)}\right). \tag{31}$$

Case 3: if $b \neq 0$ and $b^2 + 4ad - 4ac = 0$, then

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right) = \frac{b}{2(d-c)} + \frac{dD}{(d-c)(C-D\zeta)}. \tag{32}$$

Case 4: if $b = 0$ and $ad - ac > 0$, then

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right) = \frac{\sqrt{ad-ac}}{(d-c)} \left(\frac{C_1 \sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right) + C_2 \cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}{C_1 \cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right) + C_2 \sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}\right). \tag{33}$$

Case 5: if $b = 0$ and $ad - ac < 0$, then

$$\left(\frac{G'(\zeta)}{G(\zeta)}\right) = \frac{\sqrt{ac-ad}}{d-c} \left(\frac{C_2 \cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right) - C_1 \sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}{C_1 \cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right) + C_2 \sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}\right). \tag{34}$$

where a, b, c, d, C_1 and C_2 are constants.

Step 5:

Substituting Eq. (28) along Eq. (29) into Eq. (27) and collecting coefficients of each power of $\left(\frac{G'(\zeta)}{G(\zeta)}\right)$. Putting each coefficient equal to zero, we achieve a set of algebraic equations involving ν, κ, $\alpha_i, (i = 0, \pm1, \pm2, ..., \pm m)$ and other parameters.

Step 6:

Solving the obtained set of equations by using Mathematica software.

Step 7:

Putting the gained solutions into Eq. (28) and we get exact solutions of Eq. (25).

4 Mathematical Treatment of the Model:

Let's suppose the travelling wave transform given as follows;

$$q(s,t) = Q(\zeta)\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}+\rho\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})), \quad \zeta = \lambda\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}+\tau\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}. \tag{35}$$

where $Q(\zeta)$ shows the amplitude of wave function while ρ and τ represent the time velocity. Parameters μ and λ are obtaining from asset price of the product.
Inserting Eq.(35) into Eq.(8), result in the form of real and imaginary parts given as follows:
Real part:

$$2\Omega Q^3 + \delta\lambda^2 Q'' - \left(\delta\mu^2 + 2\rho\right) Q = 0. \tag{36}$$

Imaginary part:

$$(\delta\mu\lambda + \tau)Q^{'} = 0. \tag{37}$$

From Eq.(37), we get the velocity of wave function given as follows:

$$\tau = -\delta\mu\lambda. \tag{38}$$

Applying the homogenous balance method into Eq.(36), we get $m = 1$
Now we will gain the exact solutions of Eq.(36) by using above mentioned three methods.

5 Exact Solutions Through exp_a function Method:

For $m = 1$, Eq.(12) changes into:

$$Q(\zeta) = \frac{\alpha_0 + \alpha_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}. \tag{39}$$

Inserting Eq.(39) into Eq.(36) and solving the system of equations, we obtain different solution sets given as follows:
Set 1:

$$\left\{\alpha_0 = -\frac{i\beta_0\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}, \alpha_1 = \frac{i\beta_1\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\right\}. \tag{40}$$

From Eqs.(40), (39) and (35), we get

$$\begin{aligned} q(s,t) = &-\frac{i\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}\left(\frac{\beta_0 - \beta_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}\right) \\ &\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \end{aligned} \tag{41}$$

Set 2:

$$\left\{\alpha_0 = \frac{i\beta_0\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}, \alpha_1 = -\frac{i\beta_1\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\right\}. \quad (42)$$

From Eqs.(42), (39) and (35), we get

$$\begin{aligned} q(s,t) = &\frac{i\sqrt{\delta}\lambda\log(d)}{2\sqrt{\Omega}}\left(\frac{\beta_0 - \beta_1 d^{\zeta}}{\beta_0 + \beta_1 d^{\zeta}}\right) \\ &\times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2\log^2(d) + 2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \end{aligned} \quad (43)$$

where $\zeta = \lambda\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \delta\mu\lambda\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}$.

5.1 Exact Solutions Through EShGEEM:

For $m = 1$, Eq. (23), Eq. (24) and Eq. (16) become:

$$Q(\zeta) = \alpha_0 \pm \beta_1\operatorname{csch}(\zeta) \pm \alpha_1\coth(\zeta). \quad (44)$$

$$Q(\zeta) = \alpha_0 \pm \iota\beta_1 sech(\zeta) \pm \alpha_1\tanh(\zeta). \quad (45)$$

$$Q(\zeta) = \alpha_0 + \beta_1\sinh(p) + \alpha_1\cosh(p). \quad (46)$$

where α_0, α_1 *and* β_1 are unknowns. Subtituting Eq. (46) into Eq. (36), we obtain the algebraic equations having $\alpha_0, \alpha_1, \beta_1$ and other parameters. Now with the help of software, we get different solution sets given as:

Set 1:

$$\left\{\alpha_0 = 0, \alpha_1 = -\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \beta_1 = 0, \rho = -\frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\right\}. \quad (47)$$

From Eqs.(47), (44) and (35), we get

$$q_1(s,t) = \mp\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\coth(\zeta) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (48)$$

From Eqs.(47), (45) and (35), we get

$$q_2(s,t) = \mp\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\tanh(\zeta) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (49)$$

Set 2:

$$\left\{\alpha_0 = 0, \alpha_1 = \frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \beta_1 = 0, \rho = -\frac{1}{2}\delta\left(2\lambda^2 + \mu^2\right)\right\}. \quad (50)$$

From Eqs.(50), (44) and (35), we get

$$q_1(s,t)=\pm\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\coth(\zeta)\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{2}\delta\left(2\lambda^2+\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{51}$$

From Eqs.(50), (45) and (35), we get

$$q_2(s,t)=\pm\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\tanh(\zeta)\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{2}\delta\left(2\lambda^2+\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{52}$$

Set 3:

$$\left\{\alpha_0=0,\alpha_1=-\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}},\beta_1=-\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}},\rho=-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\right\}. \tag{53}$$

From Eqs.(53), (44) and (35), we get

$$q_1(s,t)=\mp\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\coth(\zeta)+\operatorname{csch}(\zeta))\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{54}$$

From Eqs.(53), (45) and (35), we get

$$q_2(s,t)=\mp\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\iota\operatorname{sech}(\zeta)+\tanh(\zeta))\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{55}$$

Set 4:

$$\left\{\alpha_0=0,\alpha_1=\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}},\beta_1=-\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}},\rho=-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\right\}. \tag{56}$$

From Eqs.(56), (44) and (35), we get

$$q_1(s,t)=\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\coth(\zeta)\mp\operatorname{csch}(\zeta))\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{57}$$

From Eqs.(56), (45) and (35), we get

$$q_2(s,t)=\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\tanh(\zeta)\mp\iota\ \operatorname{sech}(\zeta))\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{58}$$

Set 5:

$$\left\{\alpha_0=0,\alpha_1=-\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}},\beta_1=\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}},\rho=-\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\right\}. \tag{59}$$

From Eqs.(59), (44) and (35), we get

$$q_1(s,t) = -\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\coth(\zeta) \mp \operatorname{csch}(\zeta)) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{60}$$

From Eqs.(59), (45) and (35), we get

$$q_2(s,t) = -\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\pm\tanh(\zeta) \mp \iota\operatorname{sech}(\zeta)) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{61}$$

Set 6:

$$\left\{\alpha_0 = 0, \alpha_1 = \frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}, \beta_1 = \frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}, \rho = -\frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\right\}. \tag{62}$$

From Eqs.(62), (44) and (35), we get

$$q_1(s,t) = \pm\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\coth(\zeta) + \operatorname{csch}(\zeta)) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{63}$$

From Eqs.(62), (45) and (35), we get

$$q_2(s,t) = \pm\frac{i\sqrt{\delta}\lambda}{2\sqrt{\Omega}}(\iota\operatorname{sech}(\zeta) + \tanh(\zeta)) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{1}{4}\delta\left(\lambda^2+2\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{64}$$

Set 7:

$$\left\{\alpha_0 = 0, \alpha_1 = 0, \beta_1 = -\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \rho = \frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\right\}. \tag{65}$$

From Eqs.(65), (44) and (35), we get

$$q_1(s,t) = \mp\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\operatorname{csch}(\zeta) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} + \frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}.t^{\epsilon})) \tag{66}$$

From Eqs.(65), (45) and (35), we get

$$q_2(s,t) = \pm\frac{\sqrt{\delta}\lambda}{\sqrt{\Omega}}\operatorname{sech}(\zeta) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} + \frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{67}$$

Set 8:

$$\left\{\alpha_0 = 0, \alpha_1 = 0, \beta_1 = \frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}, \rho = \frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\right\}. \tag{68}$$

From Eqs.(68), (44) and (35), we get

$$q_1(s,t) = \pm\frac{i\sqrt{\delta}\lambda}{\sqrt{\Omega}}\operatorname{csch}(\zeta) \times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} + \frac{1}{2}\delta\left(\lambda^2-\mu^2\right)\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \tag{69}$$

From Eqs.(68), (45) and (35), we get

$$q_2(s,t) = \mp \frac{\sqrt{\delta}\lambda}{\sqrt{\Omega}} \operatorname{sech}(\zeta) \times \exp(\iota(\mu \frac{\Gamma(\varrho+1)}{\epsilon} s^{\epsilon} + \frac{1}{2}\delta\left(\lambda^2 - \mu^2\right) \frac{\Gamma(\varrho+1)}{\epsilon} t^{\epsilon})), \tag{70}$$

where $\zeta = \lambda \frac{\Gamma(\varrho+1)}{\epsilon} s^{\epsilon} - \delta\mu\lambda \frac{\Gamma(\varrho+1)}{\epsilon} t^{\epsilon}$.

5.2 Exact Solutions Through Extended (G'/G)-expansion Method:

For $m = 1$, Eq. (28) becomes:

$$Q(\zeta) = \alpha_{-1} \left(\frac{G'(\zeta)}{G(\zeta)}\right)^{-1} + \alpha_0 + \alpha_1 \left(\frac{G'(\zeta)}{G(\zeta)}\right), \tag{71}$$

where α_{-1}, α_0 and α_1 are unknowns.
Inserting Eq. (71) along Eq. (29) into Eq. (36) and solving the system for $\alpha_{-1}, \alpha_0, \alpha_1$ and other parameters, we gain different solution sets given as:
Set 1:

$$\left\{\alpha_{-1} = -\frac{ia\sqrt{\delta}\lambda}{d\sqrt{\Omega}}, \alpha_0 = -\frac{ib\sqrt{\delta}\lambda}{2d\sqrt{\Omega}}, \alpha_1 = 0, \rho = -\frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\right\}. \tag{72}$$

From Eqs.(72), (71), (30) and (35), we get

$$\begin{aligned} q(s,t) = &-\frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2} + a(\frac{b}{2(d-c)} \\ &+ \frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}(\frac{C_1\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d}) + C_2\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})}{C_1\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d}) + C_2\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})}))^{-1}) \\ &\times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \end{aligned} \tag{73}$$

From Eqs.(72), (71), (31) and (35), we get

$$\begin{aligned} q(s,t) = &-\frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2} + a(\frac{b}{2(d-c)} \\ &+ \frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right) - C_1\sin\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right) + C_2\sin\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)}\right))^{-1}) \\ &\times \exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon} - \frac{\delta\left(4a\lambda^2(d-c) + b^2\lambda^2 + 2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \end{aligned} \tag{74}$$

From Eqs.(72), (71), (33) and (35), we get

$$q(s,t)=-\frac{ia\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}\right)\right)^{-1}$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (75)$$

From Eqs.(72), (71), (34) and (35), we get

$$q(s,t)=-\frac{ia\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}\right)\right)^{-1}$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (76)$$

Set 2:

$$\left\{\alpha_{-1}=\frac{ia\sqrt{\delta}\lambda}{d\sqrt{\Omega}},\alpha_0=\frac{ib\sqrt{\delta}\lambda}{2d\sqrt{\Omega}},\alpha_1=0,\rho=-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\right\}. \quad (77)$$

From Eqs.(77), (71), (30) and (35), we get

$$q(s,t)=\frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2}+a(\frac{b}{2(d-c)}$$
$$+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}(\frac{C_1\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})+C_2\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})}{C_1\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})+C_2\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})}))^{-1})$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (78)$$

From Eqs.(77), (71), (31) and (35), we get

$$q(s,t)=\frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{b}{2}+a(\frac{b}{2(d-c)}\right.$$
$$\left.+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}\left(\frac{C_2\cos\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)-C_1\sin\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)}{C_1\cos\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)+C_2\sin\left(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d}\right)}\right))^{-1}\right)$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (79)$$

From Eqs.(77), (71), (33) and (35), we get

$$q(s,t)=\frac{ia\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}\right)\right)^{-1}$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (80)$$

From Eqs.(77), (71), (34) and (35), we get

$$q(s,t)=\frac{ia\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}\right)\right)^{-1}$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (81)$$

Set 3:

$$\left\{\alpha_{-1}=0,\alpha_0=-\frac{ib\sqrt{\delta}\lambda}{2d\sqrt{\Omega}},\alpha_1=-\frac{i\sqrt{\delta}\lambda(c-d)}{d\sqrt{\Omega}},\rho=-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\right\}. \quad (82)$$

From Eqs.(82), (71), (30) and (35), we get

$$q(s,t)=\frac{-i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2}-(\frac{b}{2}$$
$$+\frac{\sqrt{-4ac+4ad+b^2}}{2}(\frac{C_1\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})+C_2\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})}{C_1\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})+C_2\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})})))$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (83)$$

From Eqs.(82), (71), (31) and (35), we get

$$q(s,t)=-\frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2}$$
$$+(c-d)(\frac{b}{2(d-c)}+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}(\frac{C_2\cos(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})-C_1\sin(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})}{C_1\cos(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})+C_2\sin(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})})))$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (84)$$

From Eqs.(82), (71), (33) and (35), we get

$$q(s,t) = -\frac{i(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}\right)\right)$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (85)$$

From Eqs.(82), (71), (34) and (35), we get

$$q(s,t) = -\frac{i(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}\right)\right)$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (86)$$

Set 4:

$$\left\{\alpha_{-1}=0, \alpha_0=\frac{ib\sqrt{\delta}\lambda}{2d\sqrt{\Omega}}, \alpha_1=\frac{i\sqrt{\delta}\lambda(c-d)}{d\sqrt{\Omega}}, \rho=-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\right\}. \quad (87)$$

From Eqs.(87), (71), (30) and (35), we get

$$q(s,t) = \frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2}+(c-d)(\frac{b}{2(d-c)}$$
$$+\frac{\sqrt{-4ac+4ad+b^2}}{2(d-c)}(\frac{C_1\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})+C_2\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})}{C_1\cosh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})+C_2\sinh(\frac{\zeta\sqrt{-4ac+4ad+b^2}}{2d})})))$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (88)$$

From Eqs.(87), (71), (31) and (35), we get

$$q(s,t) = \frac{i\sqrt{\delta}\lambda}{d\sqrt{\Omega}}(\frac{b}{2}+(c-d)(\frac{b}{2(d-c)}$$
$$+\frac{\sqrt{4ac-4ad-b^2}}{2(d-c)}(\frac{C_2\cos(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})-C_1\sin(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})}{C_1\cos(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})+C_2\sin(\frac{\zeta\sqrt{4ac-4ad-b^2}}{2d})})))$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+b^2\lambda^2+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (89)$$

From Eqs.(87), (71), (33) and (35), we get

$$q(s,t)=\frac{i(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ad-ac}}{(d-c)}\left(\frac{C_1\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}{C_1\cosh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)+C_2\sinh\left(\frac{\zeta\sqrt{ad-ac}}{d}\right)}\right)\right)$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (90)$$

From Eqs.(87), (71), (34) and (35), we get

$$q(s,t)=\frac{i(c-d)\sqrt{\delta}\lambda}{d\sqrt{\Omega}}\left(\frac{\sqrt{ac-ad}}{d-c}\left(\frac{C_2\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)-C_1\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}{C_1\cos\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)+C_2\sin\left(\frac{\zeta\sqrt{ac-ad}}{d}\right)}\right)\right)$$
$$\times\exp(\iota(\mu\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\frac{\delta\left(4a\lambda^2(d-c)+2d^2\mu^2\right)}{4d^2}\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon})). \quad (91)$$

where $\zeta=\lambda\frac{\Gamma(\varrho+1)}{\epsilon}s^{\epsilon}-\delta\mu\lambda\frac{\Gamma(\varrho+1)}{\epsilon}t^{\epsilon}$ for all above mentioned solutions.

6 Illustrations with graphics

Here, we show some two-dimensional and three-dimensional figures to help clarify the solutions we presented. Figures 1-3 depict some of the analytical solutions. In Figure 1, we use our method to introduce the graph of (41) at $\delta=0.5,\lambda=0.3,\varrho=0.1,\mu=2,\Omega=0.7,\beta_0=0.1,\beta_1=0.1,d=0.1,\epsilon=1$. In addition, Figure 2 shows the graph of (48) $\delta=0.5,\lambda=0.1,\varrho=0.5,\mu=1,\Omega=0.1,\epsilon=1$. Finally, the graph of (73) at $\delta=0.3,\lambda=0.4,\varrho=0.5,\mu=6,\Omega=0.4,d=0.17,a=0.1,c=0.01,b=0.4,C_1=0.4,C_2=0.5$ presented in Figure 3.

7 Conclusion

We have succeed to obtain the modernistic exact solutions of M-fractional Ivancevic option pricing model by utilizing exp_a function, extended Sinh-Gordon equation expansion and extended (G'/G)-Expansion methods. The gained solutions are also verified and demonstrated through graphs by using MATHEMATICA software. The obtained results are also explained graphically by 2-D,3-D and contour plots. Finally, it is suggested, to deal the other non-linear PDEs, the exp_a function, extended Sinh-Gordon equation expansion and extended (G'/G)-Expansion methods are very helpful, reliable and straight forward. Results achieved in this paper may useful for the progress in the supplementary analyzing of this model. Fractional derivatives (local, comformable, truncated M-fractional, beta fractional, caputo-Fabrizio fractional derivative,...) have attracted extensive attention in the field of mathematical physics. How to apply the proposed methods to study the fractional PDEs will be the focus of our future research.

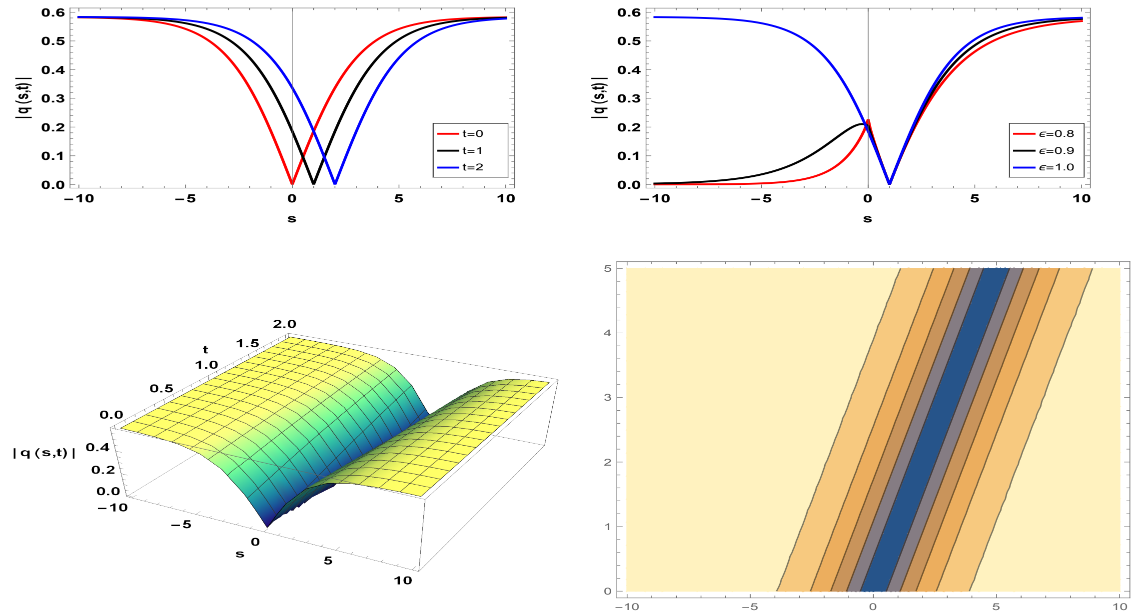

Figure 1: Structure of (41) at $\delta = 0.5, \lambda = 0.3, \varrho = 0.1, \mu = 2, \Omega = 0.7, \beta_0 = 0.1, \beta_1 = 0.1, d = 0.1, \epsilon = 1$

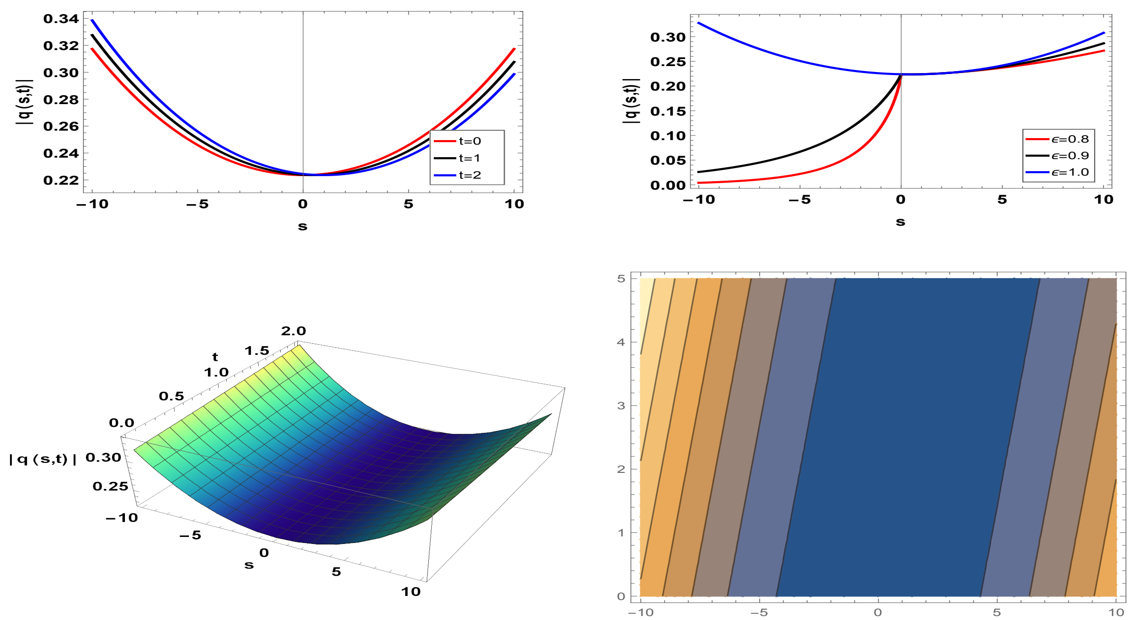

Figure 2: Structure of (48) at $\delta = 0.5, \lambda = 0.1, \varrho = 0.5, \mu = 1, \Omega = 0.1, \epsilon = 1$

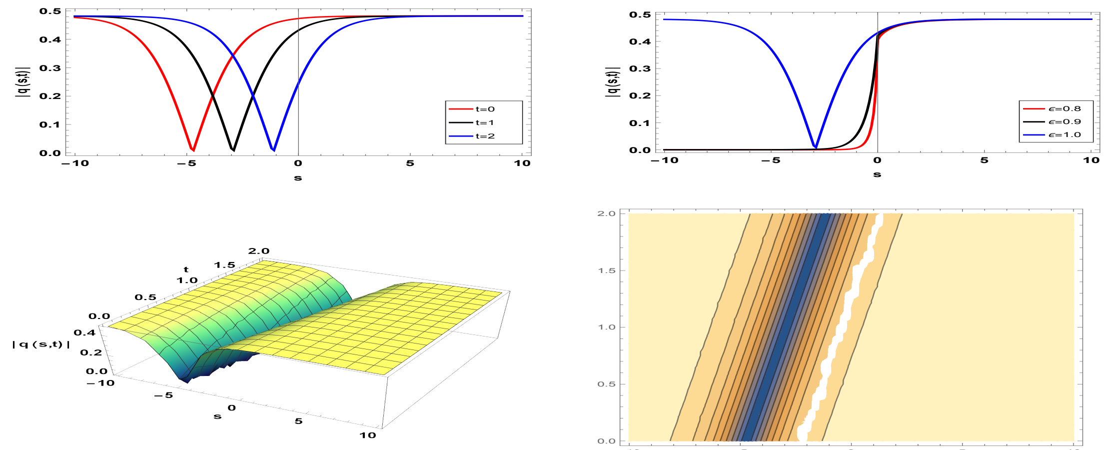

Figure 3: Structure of (73) at $\delta = 0.3, \lambda = 0.4, \varrho = 0.5, \mu = 6, \Omega = 0.4, d = 0.17, a = 0.1, c = 0.01, b = 0.4, C_1 = 0.4, C_2 = 0.5$.

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