An Eight Order Block Multistep Method for solution of Second Order Initial Value Problems

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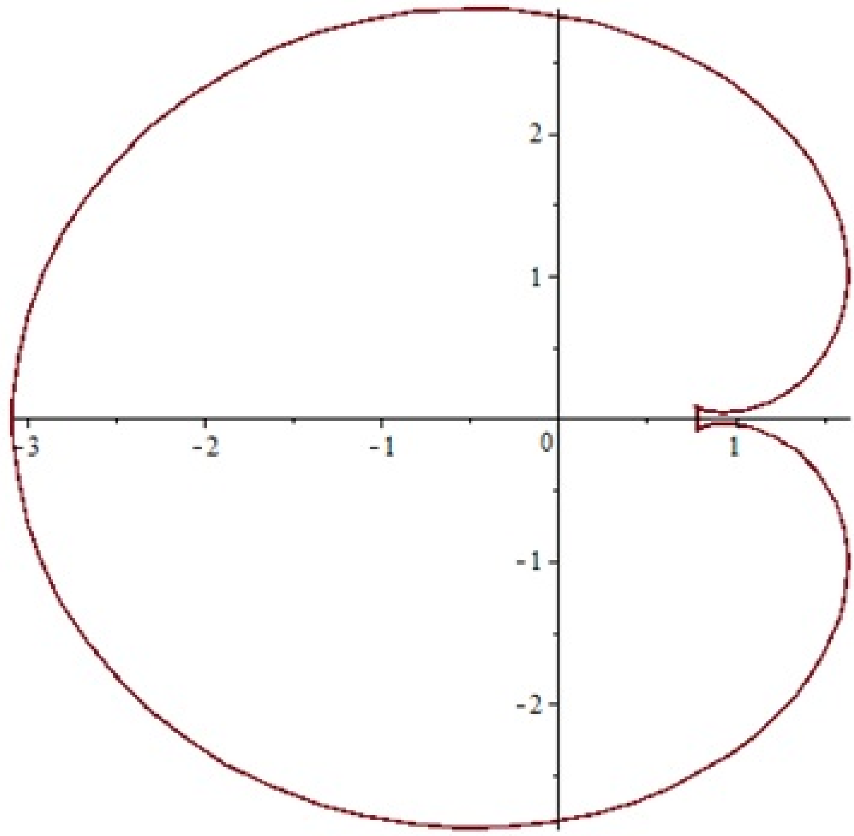
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An Eight Order Block Multistep Method for solution of Second Order Initial Value Problems

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Abstract

An eight order block multistep method (BMM) is put forward for the solution of second order problems of ordinary differential equations with oscillatory solutions. It uses both polynomial and trigonometric functions as bases in the derivation of the method, to produce three discrete formula which is applied to the second order equation by assembling them into a block method known as Third Derivative Trigonometric Fitted Block Method TDTFBM to generate the approximate solutions. The stability, consistency and convergence properties of the TDTFBM are well discussed. To show the performance, it was demonstrated on some classical problems for its accuracy and efficiency advantages over some known methods in the literature.

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1 Introduction

Consider the Second order ordinary differential equation of initial value problems (IVPs) given by

$$q'' = w(t,q), \quad q(t_0) = q_0 \tag{1}$$

$$q'' = w(t,q,q'), \quad q(t_0) = q_0 \ , \ q'(t_0) = q'_0 \tag{2}$$

Equation (1)and (2) can be found in several areas of science and engineering, such as biology, control theory, chemical kinetics, tracking, circuit theory, and celestial mechanics whose theoretical solutions are usually highly oscillatory. In the literature, diverse methods have been proposed for the solution of equation (1) and (2) See(Abdulganiy et al. [2],Coleman and Duxbury [5],D'Ambrosio et al. [6], Monovasilis et al. [16],panopoulos and Simos [25], Ramos et.al[23], [24],[21], Singh and Ramos [11] and Twizell and Khaliq[33]). Numerical methods for the solutions of equations (1) and (2) are applied by converting these equations to a system of first-order differential equations via a step-by-step approach that takes advantage of the peculiar properties of the known solution, see (Brugnano and Trigiante[14], Hairer et al. [12],Lambert [15], Onumanyi et al. [18]). Ramos et.al[22] proposed an optimized two-step hybrid block method used directly to solve second-order problems with oscillating solutions. The scheme was shown to be order two and zero stable. Although the method competes favorably with existing methods of higher-order, it has low order. It was noted by Vigo-Aguiar and Ramos [34] that for oscillatory problems, non-trigonometrically fitted schemes are less efficient than trigonometrically fitted ones. Despite the advantages of these schemes, some are still been affected by fluctuation in frequency, others require the Jacobian to have eigenvalues that are purely imaginary (Neta, [17]), while some have lower order of accuracy, some are expensive to implement (Ndukum et.al,[19]). Following S. N. Jator et.al [13] and (Abdulganiy et.al [2],[1], [4]) we proposed an eight order block multistep method that does not require starting values for solving equation (1)and (2), by first converting them to system of first order equations of the form

$$q' = w(t,q), q(t_0) = q_0 \tag{3}$$

2 Derivation of the method

To integrate equation (1) and (2), assume that the true solution $q(t)$ can be approximated by a fitted function $Q(t,u)$, then the k step TDTFM can be obtained in the form

$$q_{n+k} = q_{n+k-1} + h\sum_{r=0}^{k}\beta_r(u)w_{n+r} + h^3\sum_{r=0}^{k}\gamma_r(u)\zeta_{n+r}(u) \tag{4}$$

where the parameter u is given by $u = \omega h$, ω is the frequency, while β_r, and γ_r are parameters to be uniquely determined, with both relying on the step size h and the frequency ω.

To obtain (4), we begin by searching for an approximation to the exact solution $q(t)$ by assuming a fitted solution $Q(t,u)$ of the form

$$Q(t,u) = \sum_{r=0}^{2k} b_r \varphi_r(t) + b_{2k+1} \sin \omega t + b_{2k+2} \cos \omega t \tag{5}$$

where $t \in (t_n, t_{n+k})$, $q_{n+r} = Q(t_n + rh)$ is the numerical approximation to the analytical solution $q(t_{n+r}), q'_{n+r} = w(t_{n+r}, y_{n+r})$ is an approximation to $q'(t_{n+r})$,, and $\zeta_{n+r} = \frac{d^2w}{dt^2}(t_{n+r}, q(t_{n+r}))$, and $k = 3$.

We therefore construct the k-step trigonometric method with $\varphi_r(t) = t^r, r = 0, \ldots, 2k$ by imposing the following conditions

$$Q(t_{n+r}, u) = q_{n+r}, \qquad r = k-1 \tag{6}$$

$$\frac{\partial(Q(t,u))}{\partial t}|_{t=t_{n+r}} = w_{n+r}, \qquad r = 0(1)k \tag{7}$$

$$\frac{\partial^3(Q(t,u))}{\partial t^3}|_{t=t_{n+r}} = \zeta_{n+r}, \qquad r = 0(1)k \tag{8}$$

Equations (6),(7) and (8) steer us to a system of $2k+3$ equations that are solved to get the coefficients $b_r, \quad r = 0, 1, \ldots 2k+2$ which are then replaced in (5). Following a few algebraic simplification the continuous form of the three-step trigonometrically fitted integrator is acquired as given in equation (9)

$$Q(t,u) = q_{n+2} + h \sum_{r=0}^{3} \beta_r(t,u) w_{n+r} + h^3 \sum_{r=0}^{3} \Upsilon_r(t,u) \zeta_{n+r}. \tag{9}$$

Where $\beta_r(t,u)$, $\Upsilon_r(t,u)$ are continuous coeficients. Evaluating equation (9) at $t = t_{n+r}$, $r = 0, 1$,and 3, yeild equation 10.

$$\left.\begin{array}{l}
q_n = q_{n+2} + h(\beta_0(\sin u, \cos u) w_n + \beta_1(\sin u, \cos u) w_{n+1} + \beta_2(\sin u, \cos u) w_{n+2} + \\
\beta_3(\sin u, \cos u) w_{n+3}) + h^3(\Upsilon_0(\sin u, \cos u)\zeta n + \Upsilon_1(\sin u, \cos u)\zeta_{n+1} + \\
\Upsilon_2(\sin u, \cos u)\zeta_{n+2} + \Upsilon_3(\sin u, \cos u)\zeta_{n+3}). \\
\\
q_{n+1} = q_{n+2} + h(\beta_{0,1}(\sin u, \cos u) w_n + \beta_{1,1}(\sin u, \cos u) w_{n+1} + \beta_{2,1}(\sin u, \cos u) w_{n+2} + \\
\beta_{3,1}(\sin u, \cos u) w_{n+3}) + h^3(\Upsilon_{0,1}(\sin u, \cos u)\zeta n + \Upsilon_{1,1}(\sin u, \cos u)\zeta_{n+1} + \\
\Upsilon_{2,1}(\sin u, \cos u)\zeta_{n+2} + \Upsilon_{3,1}(\sin u, \cos u)\zeta_{n+3}). \\
\\
q_{n+3} = q_{n+2} + h(\beta_{0,2}(\sin u, \cos u) w_n + \beta_{1,2}(\sin u, \cos u) w_{n+1} + \beta_{2,2}(\sin u, \cos u) w_{n+2} + \\
\beta_{3,2}(\sin u, \cos u) w_{n+3}) + h^3(\Upsilon_{0,2}(\sin u, \cos u)\zeta n + \Upsilon_{1,2}(\sin u, \cos u)\zeta_{n+1} + \\
\Upsilon_{2,2}(\sin u, \cos u)\zeta_{n+2} + \Upsilon_{3,2}(\sin u, \cos u)\zeta_{n+3}).
\end{array}\right\} \tag{10}$$

Equation (10)is the Third Derivative Trigonometric-Fitted Block method (TDTFBM) to procure the numerical solution $q_(n+1)$ $q_(n+2)$ and $q_(n+3)$ at the same time without any overlap in the sub-interval of integration. The coefficients β_s, Υ_s $\beta_{s,l}, \Upsilon_{s,l}$, $s = 0, 1, 2, 3, l = 1, 2$ are given in Appendix A

3 Analysis of TDTFBM

This section discussed the local truncation error, order, zero-stability, consistency, and convergence of the TDTFBM .

3.1 Local truncation error

Theorem

The eight order mustistep method given by equation (4) has a Local Truncation Error(LTE)of $C_{2k+3}h^{2k+3}(\omega^2 q^{(2k+1)}(t_n) + q^{(2k+3)}) + O(h^{2k+4})$

Proof

Following Ngwane and Jator([20])we give the proof of the theorem.

Given the Taylor series expansion of $q(t_n + rh)$ **,** $q'(r_n + rh)$**,** $q'''(r_n + rh)$ **and assuming that** $q(t_n + rh) = q_{n+r}$**,** $q'(t_n + rh) = w_{n+r}$**, and** $q'''(t_n + rh) = \gamma_{n+r}$**. Thus by substituting into the method in equation (10) and simplifying we obtain**

$$LTE = q(x_{n+k}) - q_{n+k}$$

$$= C_{2k+3}h^{2k+3}(\omega^2 y^{(2k+1)}(x_n) + y^{(2k+3)}) + O(h^{2k+4})$$

the Local Truncation Error (LTE) of the TDTFBM are respectively obtained as

$$LTE = \begin{pmatrix} \frac{-3223h^9}{25401600}(q^{(9)}(t_n) + \omega^2 q^{(7)}(t_n)) + O(h^{10}) \\ \frac{-3287h^9}{25401600}(q^{(9)}(t_n) + \omega^2 q^{(7)}(t_n)) + O(h^{10}) \\ \frac{-h^9}{396900}(q^{(9)}(t_n) + \omega^2 q^{(7)}(t_n)) + O(h^{10}) \end{pmatrix}$$

with error constants

$$C_9 = \left(\frac{-3223}{25401600}, \frac{3287}{25401600}, \frac{-h^9}{396900} \right)^T$$

and order $p = (8, 8, 8)^T$ **where** T **is a transpose. By the definition of consistency given in Lambert [15] that, if the order of a numerical method is greater than 1 then it is consistent. Therefore the TDTFBM of order 8 greater than 1, is consistent.**

3.2 Stability of TDTFBM

The three step TDTFBM can be repositioned and rewritten as a matrix finite difference equation of the form

$$V^{(1)}Q_{\tau+1} = V^{(0)}Q_{\tau} + h(C^{(1)}W_{\tau+1} + C^{(0)}W_{\tau}) + h^3(D^{(1)}G_{\tau+1} + D^{(0)}G_{\tau}) \tag{11}$$

where $Q_{\tau+1} = (q_{n+1}, q_{n+2}, q_{n+3})^T$ **,** $Q_{\tau} = (q_{n-2}, q_{n-1}, q_n)^T$ **,** $W_{\tau+1} = (w_{n+1}, w_{n+2}, w_{n+3})^T$ **,** $W_{\tau} = (w_{n-2}, w_{n-1}, w_n)^T$ **,** $G_{\tau+1} = (\zeta_{n+1}, \zeta_{n+2}, \zeta_{n+3})^T$ **,** $G_{\tau} = (\zeta_{n-2}, \zeta_{n-1}, \zeta_n)^T$ **,**

for $\tau = 0, 1, ...andn = 0, 3, 6, ...N - 3$**. And the matrices** $V^{(1)}, V^{(0)}, C^{(1)}, C^{(0)}, D^{(1)}$ **and** $D^{(0)}$ **are** 3 **by** 3 **matrices whose entries are**

$$V^{(1)} = \begin{pmatrix} 0 & -1 & 0 \\ 1 & -1 & 0 \\ 0 & -1 & 1 \end{pmatrix}$$

$$V^{(0)} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$C^{(1)} = \begin{pmatrix} \beta_{1,1} & \beta_{2,1} & \beta_{3,1} \\ \beta_{1,2} & \beta_{2,2} & \beta_{3,2} \\ \beta_1 & \beta_2 & \beta_3 \end{pmatrix}$$

$$C^{(0)} = \begin{pmatrix} 0 & 0 & \beta_{0,1} \\ 0 & 0 & \beta_{0,2} \\ 0 & 0 & \beta_0 \end{pmatrix}$$

$$D^{(1)} = \begin{pmatrix} \Upsilon_{1,1} & \Upsilon_{2,1} & \Upsilon_{3,1} \\ \Upsilon_{1,2} & \Upsilon_{2,2} & \Upsilon_{3,2} \\ \Upsilon_1 & \Upsilon_2 & \Upsilon_3 \end{pmatrix}$$

$$D^{(0)} = \begin{pmatrix} 0 & 0 & \Upsilon_{0,1} \\ 0 & 0 & \Upsilon_{0,2} \\ 0 & 0 & \Upsilon_0 \end{pmatrix}$$

.

3.3 Zero Stability

According to [7] and [15], The $TDTFBM$ is said to be zero stable if the modulus of the roots of the first characteristic polynomial are less than or equal to one and those with modulus one are simple. That is, $\rho(J) = det[JV^{(1)} - V^{(0)} = 0$ and $|J| \leq 1$. Thus, for the $TDTFBM$ we have $J^2(J+1) = 0$ which implies that $|J| = (0,0,1)^T$. Therefore, the $TDTFBM$ is zero stable.

3.4 Linear Stability

The linear stability of the just developed block method can be established by writing it in the form (11) and administering it to the test equations

$$q' = \Lambda q, \qquad q''' = \Lambda^3 q, \qquad \Lambda < 0$$

to yeild

$$Q_{\tau+1} = \sigma(z) Q_\tau, z = \Lambda h \tag{12}$$

where $\sigma(z)$ is given by equation (13)

$$\sigma(z) = (V^{(1)} - zC^{(1)} - z^3 D^{(1)})^{-1}.(V^{(0)} + zC^{(0)} - z^3 D^{(0)}) \tag{13}$$

The matrix $\sigma(z)$ for $TDTFBM$ has eigenvalues $\{\theta_1, \theta_2, \theta_3\} = \{0, 0, \theta_3\}$ with the leading eigenvalue θ_3 called the stability function, obtained as $\theta_3(z, u) = \frac{\nu_3(z,u)}{\mu_3(z,u)}$.

With appropriate values for u in a wide interval the $TDTFBM$ can handle problems with approximated frequencies, see(Ndukum et al.[19]). We noticed that for the $TDTFBM$, $u\varepsilon[\pi, 2\pi]$ is sufficient.

The Stability region of the TDTFBM is plotted for $u = 2\pi$ using the boundary locus method and shown in figure 1

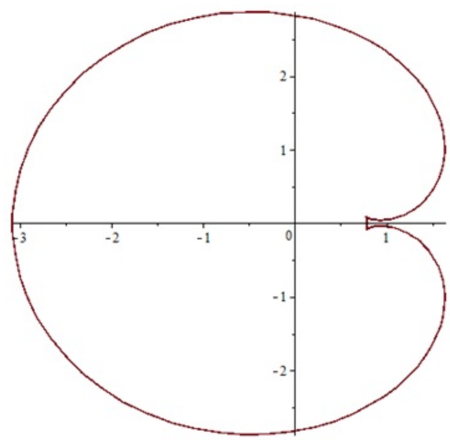

Figure 1: Stability Region

4 Numerical Examples

Example 4.1 ***First we take the well known model problem which has been solved by Simos and Tsitouras [27] using** $Numerov9_{6lin}$ **and** $Numerov9_{7lin}$ **in [28].***

$$q'' = -100q, \qquad q(0) = 1, q'(0) = 0$$

at the interval** $t = [0, 10\pi]$ **whose exact solution is q= Cos10t.

The results for correct digits of accuracy $-logMax|q(t_i - q_i|$ for the TDTFBM compared with those in ([27] and [28]) and displayed in Table1.

Table 1: Computed values for -log of the maximum global error (-log(maxerr)) for correct digits of accuracy taking $\omega = 10$, for Example4.1

$TDTFBM$		$Numerov9_{6lin}$ [27]		$Numerov9_{7lin}$[28]	
NFE	$-log(maxerr)$	NFE	$-log(maxerr)$	NFE	$-log(maxerr)$
362	33.3	36000	19.0	36000	18.1
722	29.9	54000	20.8	54000	19.8
1082	27.0	72000	22.0	72000	21.1
1442	26.6	90000	23.0	90000	22.1
1802	24.3	108000	23.8	108000	22.8
2162	23.1	126000	24.4	126000	23.5
2562	22.1	144000	25.0	144000	24.1
2562	22.1	162000	25.5	162000	24.6

The results show that the TDTFBM of order eight is more accurate and efficient with lower NFEs. More so, even with large step size the method though with fewer number of accurate digits at the last three row on the Table 1 still compete well with the Ninth order methods that used very small step size.

Example 4.2 ***Next we consider the highly oscillatory inhomogeneous problem with interval** $[0, 10\pi]$**,***

$$q'' = -100q + 99Sint, \qquad q(0) = 1, q'(0) = 11$$

with its exact solution q=Cos10t+Sin10t+Sint.

The solution consists of rapid and slow functions due to its inhomogeneous term which vary slowly (Sallam and Anwar [26]). This problem has

been solved using the 18th phase-lag order, eighth algebraic order, 7 stages methods (NEW8ph18) and (MEW8zero) of Simos and Tsitouras [29] as well as (STF8ph16) Simos et al [30] in the interval $[0, 10\pi]$.

The numerical results for correct digits of the TDTFBM at $t = 10\pi$ when compared with those in ([29] and [30]) show that the TDTFBM is more accurate as contained in Table 2.

Table 2: Computed values of correct digits (CD) using TDTFBM for Example4.2

Steps	$TDTFBM$ CD	Steps	$NEW8ph18$[29] CD	$MEW8zero$[29] CD	$STF8ph16$[30] CD
67	8.6	200	5.6	4.0	3.8
134	11.5	400	9.3	7.2	7.5
201	14.4	600	10.4	9.0	8.6
267	15.9	800	11.4	10.2	9.6
334	14.9	1000	11.8	11.2	10.4
400	13.6	1200	12.5	11.9	11.0

Example 4.3 ***The third example is the harmonic oscillator with frequency*** $\omega = 1$ ***and small perturbation*** δ ***in the interval*** $[0, 1000]$ ***solved by Franco[8] and Guo & Yan[9] using ARKN and MLGC method respectively***

$$q'' + \delta q' + \omega^2 q = 0, \qquad q(0) = 0, \qquad q'(0) = \frac{-\delta}{2}$$

with exact solution $q(t) = e^{\frac{\delta}{2}} Cos(\sqrt{(}\omega^2 - \frac{\delta^2}{4})t)$

where$\omega = 1$,$\delta = 10^{-6}$,**and** $\delta = 10^{-10}$.

Table 3: Computed values of $error = |y(t) - y|$, using TDTFBM for Example4.3

	TDTFBM		*MLGC*[9]		*ARKN* [8]	
h	$\delta = 10^{-6}$	$\delta = 10^{-10}$	$\delta = 10^{-6}$	$\delta = 10^{-10}$	$\delta = 10^{-6}$	$\delta = 10^{-10}$
$\frac{1}{2}$	2.84×10^{-11}	2.84×10^{-15}	3.62×10^{-14}	1.32×10^{-14}	9.05×10^{-8}	9.00×10^{-12}
$\frac{1}{4}$	3.21×10^{-14}	3.21×10^{-18}	5.29×10^{-15}	2.67×10^{-14}	5.43×10^{-9}	7.06×10^{-13}
$\frac{1}{8}$	2.36×10^{-16}	2.34×10^{-20}	3.27×10^{-14}	2.77×10^{-15}	2.03×10^{-10}	2.87×10^{-13}
$\frac{1}{16}$	5.66×10^{-19}	5.66×10^{-23}	1.22×10^{-14}	1.39×10^{-15}	7.25×10^{-12}	3.56×10^{-13}
$\frac{1}{32}$	1.16×10^{-21}	1.16×10^{-25}	7.74×10^{-16}	2.13×10^{-14}	3.45×10^{-13}	5.19×10^{-13}

The results show as expected that the TDTFBM of order eight is superior to the methods Franco[8] and Guo and Yan[9]which are of order five method.

Below we show the results of the TDTFBM for the harmonic oscillators when the frequency $\omega = 10$ in Table 4

Table 4: Computed values of $error = |y(t) - y|$, using TDTFBM for Example4.3

	TDTFBM	
h	$\delta = 10^{-6}$	$\delta = 10^{-10}$
$\frac{1}{2}$	1.57×10^{-5}	1.569×10^{-9}
$\frac{1}{4}$	8.40×10^{-6}	8.40×10^{-10}
$\frac{1}{8}$	3.41×10^{-8}	3.41×10^{-12}
$\frac{1}{16}$	6.82×10^{-12}	6.82×10^{-16}
$\frac{1}{32}$	7.64×10^{-15}	7.64×10^{-19}

Example 4.4 ***The fourth example is the periodically forced nonlinear IVP in the interval*** $[0, 1000]$ ***where the first derivative does not appear explicitly.***

$$q'' + q^3 + q = (Cos(t) + \epsilon Sin(10t))^3 - 99\epsilon Sin(10t), \qquad q0 = 1, \qquad q'(0) = 10\epsilon$$

with exact solution $q(t) = Cos(t) + \epsilon Sin(10t)$***,*** $\epsilon = 10^{-10}$ ***and*** $\omega = 1$ ***is chosen as the principal frequency. Table 5 shows the performance of TDTFBM in comparison with the BFFM by Ramos et al.[31], the TFARKN by Fang et al.[10], the EFRK and EFRKN by Franco [8]. The TDTFBM show dominance over the other methods in [10] and [8].***

Table 5: Computed values of $Maxerror = Max|q(t_i) - q_i|$, $\omega = 1$ using TDTFBM for Example4.4

h	$TDTFBM$	$BFFM$	$TFARKN$	$EFRK$	$EFRKN$
$\frac{1}{8}$	6.44×10^{-11}	1.21×10^{-11}	4.77×10^{-6}	0.75×10^{-7}	0.75×10^{-7}
$\frac{1}{16}$	7.97×10^{-16}	6.09×10^{-13}	3.72×10^{-8}	0.75×10^{-8}	1.26×10^{-7}
$\frac{1}{32}$	3.03×10^{-18}	3.66×10^{-14}	1.17×10^{-13}	6.31×10^{-9}	1.00×10^{-9}

Example 4.5 ***Lastly, is the well known two-body problem which was solved by Senu et al. (2010) in the interval*** $0 \leq t \leq 10$ ***with*** $\omega = 10$***. The TDTFBM is compared with the DIRKNNew of Senuet al.(2010) and BHTM of Abdulganiy etal.[3].***

$$q_1''(t) + \frac{q_1}{r^3} = 0, \qquad q_1(0) = 1, \qquad q_1'(0) = 0$$

$$q_2''(t) + \frac{q_2}{r^3} = 0, \qquad q_2(0) = 1, \qquad q_2'(0) = 0$$

Where $r = \sqrt{q_1^2 + q_2^2}$ ***whose exact solution is given by*** $q_1(t) = Cos(t) \qquad q_2(t) = Sin(t)$***.***

The results in comparison with those of BHTM and fourth-order DIRKNNew of Senuet al.(2010) are displayed in Table 6.

Table 6: Computed values of $(MaxError = |y(t_i) - y_i|)$, using TDTFBM for Example4.5

TDTFBM		BHTM		DIRKNNew	
NFE	MaxError	NFE	$MaxError$	NFE	$MaxError$
122	1.30×10^{-41}	1600	5.13×10^{-27}	5000	7.49×10^{-4}
242	9.47×10^{-37}	3200	1.30×10^{-29}	10000	5.62×10^{-7}
482	4.41×10^{-32}	6400	4.00×10^{-30}	5000	1.00×10^{-9}
962	3.70×10^{-28}	12800	7.43×10^{-28}	60000	1.78×10^{-7}

5 Conclusion

We have derived and implemented the block multistep method known as $TDTFBM$ **for the solving oscillatory second order equations of IVPs. The method is of order**

eight and was found to be efficient with lower number of function evaluations and accurate with large step size. The method proved to be superior when compared to some other methods in the literature.

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References

[1] R.I. Abdulganiy, O.A. Akinfenwa & S.A. Okunuga,(2018). Construction of $\mathcal{L}$-stable second derivative trigonometrically fitted block backward differentiation formula for the solution of oscillatory initial value problems, African Journal of Science, Technology, Innovation and Development, Vol. 10, No. 4, 411419

[2] R. I. Abdulganiy O. A. Akinfenwa H. Ramos and S. A. Okunuga,(2021). A second-derivative functionally fitted method of maximal order for oscillatory initial value problems Computational and Applied Mathematics 40:188

[3] Abdulganiy, R. I., Akinfenwa, O. A., Okunuga, S. A. and Oladimeji, G. O. A Robust Block Hybrid Trigonometric Method for the Numerical Integration of Oscillatory Second Order Nonlinear Initial Value Problems. AMSE JOURNALS-AMSE IIETA publication-2017-Series: Advances A, 2017, 54,497-518.

[4] R. I. Abdulganiy, O. A. Akinfenwa, O. A. Yusuff , O. E. Enobaborc , Okunuga, S. A., Block Third Derivative Trigonometrically-Fitted Methods for Stiff and Periodic Problems. J. Nig. Soc. Phys. Sci. 2 (2020) 12 - 25

[5] J. P. Coleman and S. C. Duxbury. Mixed collocation methods for y00 = f(x, y), J. Comput. Appl. Math. 126, (2000), 47-75.

[6] R . DAmbrosio, M. Ferro, and B. Paternoster, Two-step hybrid collocation methods for y00 = f(x, y), Appl. Math. Lett. 22, (2009), 1076-1080.

[7] S. O. Fatunla, Block methods for second order IVPs, Intern. J. Comput. Math. 41,(1991), pp 55 - 63.

[8] J.M. Franco, (2002). Runge-Kutta-Nystrom methods adapted to the numericalintegration of perturbed oscillators. Comput.Phys.Comm.147,770-787.

[9] Ben-yu Guo & Jian-ping Yan (2009). LegendreGauss collocation method for initial value problems of second order ordinary differential equations. , 59(6), 13861408.

[10] Y. Fang, Y. Song, & Wu, X., A robust trigonometrically fitted embedded pair for perturbed oscillators, Journal of Computational and Applied Mathematics, vol. 225, No. 2, pp. 347355, 2009.

[11] Gurjinder Singh and Higinio Ramos,(2015) An Optimized Two-Step Hybrid Block Method Formulated in Variable Step-Size Mode for Integrating y ” = f (x, y, y’) Numerically, Numer. Math. Theor. Meth. Appl. Vol. 12, No. 2, 640-660

[12] E. Hairer, S. P. Norsett and G. Wanner, Solving Ordinary Differential Equations I, Nonstiff problems, Springer-Verlag Berlin Heidelberg, 1993.

[13] S. N. Jator , A. O. Akinfenwa , S. A. Okunuga and A. B. Sofoluwe (2013) High-order continuous third derivative formulas with block extensions for y”=f(x,y,y’) International Journal of Computer Mathematics, Volume 90 Issue 9, pp 18991914

[14] L. Brugnano and D. Trigiante, Solving Differential Problems by Multitep Initial and Boundary Value Methods, Gordon and Breach Science Publishers, Amsterdam, (1998), 280-299.

[15] J. D. Lambert, Numerical methods for ordinary differential systems, John Wiley, New York, (1991).

[16] T Monovasilis, Z Kalogiratou, H Ramos, TE Simos (2017). Modified two-step hybrid methods for the numerical integration of oscillatory problems Mathematical Methods in the Applied Sciences 40 (14), 5286-5294

[17] Neta, B. 1986. Families of Backward Differentiation Methods Based on Trigonometric Polynomials. International Journal of Computer Mathematics 20: 6775.

[18] P. Onumanyi, U. W. Sirisena and S.N.Jator, Continuous finite difference approximations for solving differential equations,Inter. J. Compt. Maths. 72, (1999), 15-27.

[19] Ndukum, P. L., T. A. Biala, S. N. Jator, and R. B. Adeniyi. (2015)A Fourth Order Trigonometrically Fitted Method with the Block Unification Implementation Approach for Oscillatory Initial Value Problems. International Journal of Pure and Apllied Mathematics 103 (2): 201213.

[20] F.F. Ngwane and Jator S.N.(2015) Solving Telegraph and Oscillatory Differential Equations by a Block hybrid Trigonometrically Fitted Algorithm. Journal of Differential Equation, Vol.2015 pp. 1-15.

[21] Higinio Ramos,(2019). Development of a new Runge-Kutta method and its economical implementation, Comp and Math Methods. 1-11

[22] H Ramos, Z Kalogiratou, T Monovasilis, & TE Simos (2016). An optimized two-step hybrid block method for solving general second order initial-value problems. Numerical Algorithms 72 (4), 1089-1102

[23] Higinio Ramos, Samuel N. Jator, & Mark I. Modebei,(2020). Efficient k-Step Linear Block Methods to Solve Second Order Initial Value Problems Directly, E Mathematics, 8, 1752; doi:10.3390/math8101752

[24] Higinio Ramos, Ridwanulahi Abdulganiy , Ruth Olowe, & Samuel Jator, A Family of Functionally-Fitted Third Derivative Block Falkner Methods for Solving Second-Order Initial-Value Problems with Oscillating Solutions E Mathematics 2021, 9, 713.

[25] G.A. Panopoulos, T.E. Simos (2015) An eight-step semi-embedded predictorcorrector method for orbital problems and related IVPs with oscillatory solutions for which the frequency is unknown Journal of Computational and Applied Mathematics 290, 1-15

[26] S. Sallam & N. Anwar (2002). Sixth order C^2- spline collocation method for integrating second order ordinary initial value problems, Int. Comput. Math. vol.79, pp 625-635

[27] T.E.Simos,Ch.Tsitouras, (2020). Explicit, ninth order, two step methods for solving inhomogeneous linear problems $x''(t) = \bigwedge x(t) + f(t)$. AppliedNumericalMathematics 153, 344351

[28] V.N. Kovalnogov, T.E. Simos, Ch. Tsitouras,(2020). Ninth order, explicit, two step methods for second order inhomogeneous linear IVPs, Math. Methods Appl. Sci. https://doi .org /10 .1002 /mma .6246.

[29] T.E. Simos, & Tsitouras, C. (2017). A new family of 7 stages, eighth-order explicit Numerov-type methods. Mathematical Methods in the Applied Sciences, 40(18), 78677878. doi:10.1002/mma.4570

[30] Simos TE, & Ch.Tsitouras,(2017) Famelis IT. Explicit Numerov type methods with constant coefficients: a review. Appl Comput Math. vol.16, pp. 89-113.

[31] Ramos, H.; Abdulganiy, R., Olowe, R., Jator, S. A Family of Functionally-Fitted Third Derivative Block Falkner Methods for Solving Second-Order Initial-Value Problems with Oscillating Solutions. Mathematics 2021, 9, 713. https://doi.org/10.3390/math9070713

[32] Tsitouras Ch.,(2006) Explicit eight order two-step methods with nine stages for integrating oscillatory problems, Int. J. Mod. Phys. 17, 861-876.

[33] E.H. Twizell, & A.Q.M. Khaliq (1984),Multiderivative methods for periodic IVPs.SIAM Journal of Numerical Analysis, 21, 111-121.

[34] J. Vigo-Aguiar & Ramos,H. (2014). A strategy for selecting frequency choice in trigonometricallyfitted methods based on the minimization of truncation error and total energy error . J Math Chem (2014) 52:10501058

APPENDIX A

$$
\left.
\begin{array}{l}
\beta_0(\sin u, \cos u) = \frac{-1}{5}\frac{h(\cos(u)u^3+9u^3+60\sin(u)-60u)}{u(\cos(u)u^2+5u^2+12\cos(u)-12)} \\
\\
\beta_1(\sin u, \cos u) = (-8h(18u^5+222u^3-360u)\sin(2u) - 8h(u^5+27u^3+180u)\sin(3u) + (-1200hu^2+2880h)\cos(2u) - \\
8h(81\sin(u)u^5 - 165\sin(u)u^3 - 15\cos(u)u^2 + 15\cos(3u)u^2 - 150u^2 + 180\sin(u)u - 180\cos(u) + 180\cos(3u) + 360)/ \\
((5(u^2+12)^2 u\sin(3u) + (100u^5+960u^3-2880u)\sin(2u) + (505(u^4+\frac{720}{101} - (\frac{456}{101})u^2))\sin(u)u)) \\
\beta_2(\sin u, \cos u) = \frac{1}{15}h^3(20\sin(u)u^5 + 2\sin(2u)u^5 - 291\sin(u)u^3 - 30sin(2u)*u^3 - 3\sin(3*u)u^3 - \\
15\cos(3u)u^2 - 150\cos(2u)u^2 + 15\cos(u)u^2 + 540\sin(u)u - 216\sin(2u)u - 36\sin(3u)u + 150u^2 - 180\cos(3*u) + \\
360\cos(2u) + 180\cos(u) - 360)/(u(101\sin(u)u^4 + 20sin(2u)u^4 + \sin(3*u)u^4 - 456\sin(u)u^2 + 24u^2\sin(3u) \\
-576\sin(2u) + 720\sin(u) + 144\sin(3*u) + 192*u^2\sin(2u))) \\
\\
\beta[3] = 0 \\
\\
\Upsilon_0(\sin u, \cos u) = \frac{1}{15}h^3(20\sin(u)u^5 + 2\sin(2u)u^5 - 291\sin(u)u^3 - 30\sin(2u)u^3 - 3\sin(3u)u^3 - \\
15\cos(3u)u^2 - 150\cos(2u)u^2 + 15\cos(u)u^2 + 540\sin(u)u - 216\sin(2u)u - 36\sin(3u)u + 150u^2 - 180\cos(3u) + \\
360\cos(2u) + 180\cos(u) - 360)/(u(101\sin(u)u^4 + 20\sin(2u)u^4 + \sin(3u)*u^4 - 456\sin(u)u^2 + 24u^2\sin(3u) - \\
576\sin(2u) + 720\sin(u) + 144\sin(3u) + 192u^2\sin(2u))) \\
\Upsilon_1(\sin u, \cos u) = -2h^3((10u^5 + 342*u^3 + 504*u)*sin(2u) + (u^5+39u^3+324u)\sin(3u) + (750u^2 - \\
1800)\cos(2u) + (75*u^2+900)\cos(3*u) + (u^5+999u^3-1980*u)\sin(u) - 75\cos(u)u^2 - 750u^2 - 900\cos(u) + 1800) \\
/((300u^5+2880*u^3-8640u)\sin(2u) + (1515((1/101)(u^2+12)^2\sin(3u) + (u^4+720/101-(456/101)u^2)\sin(u)))u) \\
\\
\Upsilon_2(\sin u, \cos u) = (1/15)*h^3(20\sin(u)u^5 + 2\sin(2u)u^5 - 291\sin(u)u^3 - 30\sin(2u)u^3 - 3\sin(3u)u^3 - \\
15\cos(3*u)u^2 - 150\cos(2u)u^2 + 15\cos(u)u^2 + 540\sin(u)u - 216\sin(2*u)u - 36\sin(3*u)u + 150u^2 - 180\cos(3u) + \\
360\cos(2u) + 180\cos(u) - 360)/(u*(101\sin(u)u^4 + 20\sin(2*u)u^4 + \sin(3u)u^4 - 456\sin(u)u^2 + 24u^2\sin(3u) - \\
576\sin(2u) + 720\sin(u) + 144\sin(3u) + 192u^2\sin(2*u))) \\
\\
\Upsilon_3(\sin u, \cos u) = 0
\end{array}
\right\}
\qquad (14)
$$

$$
\left.\begin{array}{l}
\beta_{0,1}(\sin u, \cos u) = (1/20)h(119\sin(u)u^5 + 2\sin(2*u)u^5 - \sin(3u)u^5 + 900\sin(u)u^3 + 288\sin(2u)u^3 - \\
12\sin(3u)u^3 + 240\cos(2u)u^2 + 1920\cos(u)u^2 - 2880\sin(u)u + 1440\sin(2u)u - 2160u^2 + 2880\cos(2u) - \\
11520\cos(u) + 8640)/(u(101\sin(u)u^4 + 20\sin(2u)u^4 + \sin(3u)u^4 - 456\sin(u)u^2 + 24u^2\sin(3u) \\
-576\sin(2u) + 720\sin(u) + 144\sin(3u) + 192u^2\sin(2u))) \\
\\
\beta_{1,1}(\sin u, \cos u) = (-h(202u^5 + 2208u^3 - 4320u)\sin(2u) - h(9u^5 + 228*u^3 + 1440u)\sin(3*u) - \\
240h(u^2 + 12\cos(2u) - h(1129*u^5 - 3660u^3 + 4320u)\sin(u) - 1920h((u^2 - 6)\cos(u) - (9/8)u^2 + 9/2)) \\
/(20(u^2 + 12)^2 u\sin(3u) + (400u^5 + 3840u^3 - 11520*u)\sin(2u) + (2020(u^4 + 720/101 - (456/101)u^2))\sin(u)u) \\
\\
\beta_{2,1}(\sin u, \cos u) = (-h(202u^5 + 2208u^3 - 4320u)sin(2u) - h(9u^5 + 228u^3 + 1440u)\sin(3*u) - \\
240h(u^2 + 12)\cos(2u) - h(1129u^5 - 3660u^3 + 4320u)\sin(u) - 1920h((u^2 - 6)\cos(u) - (9/8)u^2 + 9/2)) \\
/(20(u^2 + 12)^2 u\sin(3u) + (400u^5 + 3840u^3 - 11520u)\sin(2u) + (2020(u^4 + 720/101 - \frac{456}{101}u^2))\sin(u)u) \\
\\
\beta_{3,1}(\sin u, \cos u) = (1/20)h(119\sin(u)u^5 + 2\sin(2u)u^5 - \sin(3u)u^5 + 900\sin(u)u^3 + 288\sin(2u)u^3 - \\
12\sin(3u)u^3 + 240\cos(2u)u^2 + 1920\cos(u)u^2 - 2880\sin(u)u + 1440\sin(2u)u - 2160u^2 + 2880\cos(2u) - \\
11520\cos(u) + 8640)/(u(101\sin(u)u^4 + 20\sin(u)u^4 + \sin(3u)u^4 - 456\sin(u)u^2 + 24u^2\sin(3u) \\
-576\sin(2u) + 720\sin(u) + 144\sin(3u) + 192u^2\sin(2u))) \\
\\
\Upsilon_{0,1}(\sin u, \cos u) = -(1/120)h^3(110\sin(u)u^5 + 11\sin(2u)u^5 + 222\sin(u)u^3 + 240\sin(2u)u^3 + \\
6\sin(3*u)*u^3 + 120\cos(2u)u^2 + 960\cos(u)u^2 - 1080\sin(u)u + 432\sin(2u)u + 72 sin(3u)u - 1080u^2 + \\
1440\cos(2*u) - 5760\cos(u) + 4320)/(u(101\sin(u)*u^4 + 20\sin(2u)u^4 + \sin(3u)u^4 - 456\sin(u)u^2 + \\
24u^2\sin(3u) - 576\sin(2u) + 720\sin(u) + 144\sin(3u) + 192u^2\sin(2u))) \\
\\
\Upsilon_{1,1}(\sin u, \cos u) = -(99((112u + (16/3)u^3 - u^5)\sin(2*u) + (-(1/9)*u^5 - 2*u^3 - 8u)\sin(3*u) + \\
((40/3)u^2 + 160)\cos(2u) + (u^5 + (226/3)u^3 - 200*u)\sin(u) - 120u^2 + 480 + (320/3)\cos(u)u^2 - \\
640\cos(u)))h^3/(120(u^2 + 12)^2 u\sin(3u) + 120u(101*sin(u)u^4 + 20\sin(2u)u^4 - 456\sin(u)u^2 + \\
192u^2\sin(2*u) + 720\sin(u) - 576\sin(2*u))) \\
\\
\Upsilon_{2,1}(\sin u, \cos u) = -(99((112u + (16/3)u^3 - u^5)\sin(2u) + (-(1/9)u^5 - 2u^3 - 8u)\sin(3u) + \\
((40/3)u^2 + 160)\cos(2u) + (u^5 + (226/3)u^3 - 200*u)\sin(u) - 120u^2 + 480 + (320/3)\cos(u)u^2 - \\
640\cos(u)))h^3/(120*(u^2 + 12)^2*u*sin(3*u) + 120*u*(101\sin(u)u^4 + 20\sin(2*u)u^4 - \\
456\sin(u)u^2 + 192u^2\sin(2u) + 720\sin(u) - 576\sin(2*u))) \\
\\
\Upsilon_{3,1}(\sin u, \cos u) = -(99*((112*u + (16/3)*u^3 - u^5)*sin(2*u) + (-(1/9)u^5 - 2u^3 - 8u)\sin(3u) + \\
((40/3)u^2 + 160)\cos(2u) + (u^5 + (226/3)u^3 - 200u)\sin(u) - 120u^2 + 480 + (320/3)\cos(u)u^2 - \\
640\cos(u)))h^3/(120(u^2 + 12)^2 u\sin(3u) + 120u(101\sin(u)u^4 + 20\sin(2u)u^4 - \\
456\sin(u)u^2 + 192u^2\sin(2u) + 720\sin(u) - 576\sin(2u)))
\end{array}\right\} \qquad (15)
$$

$$\left.\begin{array}{l}
\beta_{0,2}(\sin u,\cos u)=(1/20)h(119\sin(u)u^5+2\sin(2u)u^5-\sin(3*u)u^5+900\sin(u)u^3+\\
288sin(2u)u^3-12\sin(3u)u^3+240\cos(2u)u^2+1920\cos(u)u^2-2880\sin(u)u+1440\sin(2u)u-2160u^2+\\
2880\cos(2u)-11520\cos(u)+8640)/(u(101\sin(u)u^4+20\sin(2u)u^4+sin(3u)u^4-456\sin(u)u^2+\\
24u^2\sin(3u)-576\sin(2u)+720\sin(u)+144\sin(3u)+192u^2\sin(2u)))\\
\\
\beta_{1,2}(\sin u,\cos u)=(-h(18u^5+384u^3+288u)\sin(2u)-h(u^5+36u^3+288u)\sin(3u)-\\
h(81\sin(u)u^5+564\sin(u)u^3+336\cos(u)u^2+528\cos(2u)u^2+48\cos(3u)u^2-912u^2\\
-1440\sin(u)u-2880\cos(u)-576\cos(2*u)+576\cos(3u)+2880))/((80u^5+768u^3-2304u)\sin(2u)+\\
(404((1/101)(u^2+12)^2\sin(3u)+(u^4+720/101-\frac{456}{101}u^2\sin(u)))u)\\
\\
\beta_{2,2}(\sin u,\cos u)=(h(374u^5+4896u^3-7200u)\sin(2u)+h(23u^5+636u^3+4320u)\sin(3u)+\\
h(1463\sin(u)u^5-1620\sin(u)u^3-2400\cos(u)u^2+4560\cos(2u)u^2+480\cos(3*u)u^2-2640u^2\\
+1440\sin(u)u+5760\cos(u)-14400\cos(2*u)+5760\cos(3u)+2880))/(20(u^2+12)^2u\sin(3u)+\\
(400u^5+3840u^3-11520u)\sin(2u)+(2020(u^4+720/101-(456/101)u^2))\sin(u)u)\\
\\
\beta_{3,2}(\sin u,\cos u)=(3h(38u^5+192u^3-1440u)\sin(2u)+3hu^3*(u^2+12)\sin(3u)+3h(281\sin(u)u^5-\\
1860\sin(u)u^3+720\cos(u)u^2-720\cos(2u)u^2-80\cos(3u)u^2+80u^2+2880\sin(u)u-2880\cos(u)+2880\cos(2u)\\
-960\cos(3u)+960))/((400u^5+3840u^3-11520u)\sin(2u)+(2020((1/101)(u^2+12)^2\sin(3u)+\\
(u^4+\frac{720}{101}-(\frac{456}{101}u^2\sin(u)))u)\\
\\
\Upsilon 0,2(\sin u,\cos u)=-(1/120)h^3(110\sin(u)u^5+11\sin(2u)u^5+222\sin(u)u^3+240\sin(2u)u^3+\\
6\sin(3u)u^3+120\cos(2u)u^2+960\cos(u)u^2-1080\sin(u)u+432\sin(2u)*u+72\sin(3u)u-1080u^2+\\
1440\cos(2u)-5760\cos(u)+4320)/(u(101\sin(u)u^4+20\sin(2u)u^4+\sin(3u)u^4-456\sin(u)u^2+\\
24u^2\sin(3u)-576\sin(2u)+720\sin(u)+144\sin(3u)+192u^2\sin(2u)))\\
\\
\Upsilon_{1,2}(\sin u,\cos u)=(-h^3(-83u^5+288u^3+9360u)\sin(2*u)-h^3*(-11u^5-222*u^3-1080*u)\sin(3*u)-\\
h^3(259\sin(u)u^5+5130\sin(u)u^3+10680\cos(u)u^2+120\cos(2u)u^2-120\cos(3u)u^2-10680u^2-15480\sin(u)u-\\
61920\cos(u)+18720\cos(2u)-1440\cos(3u)+44640))/(120(u^2+12)^2u\sin(3u)+120u(101\sin(u)u^4+\\
20\sin(2u)u^4-456\sin(u)u^2+192u^2\sin(2u)+720\sin(u)-576\sin(2u)))\\
\\
\Upsilon_{2,2}(\sin u,\cos u)=(-h^3(-259u^5-4944*u^3+3024*u)\sin(2u)-h^3(-27u^5-822u^3-5976u)\sin(3u)-\\
h^3(83\sin(u)u^5-8526\sin(u)u^3+11760\cos(u)u^2-10680\cos(2u)u^2-1200\cos(3u)u^2+120u^2+11880\sin(u)u-\\
48960\cos(u)+44640\cos(2u)-14400\cos(3u)+18720))/(120(u^2+12)^2u\sin(3u)+120u(101\sin(u)u^4+\\
20\sin(2*u)*u^4-456\sin(u)u^2+192u^2\sin(2u)+720\sin(u)-576\sin(2u)))\\
\\
\Upsilon_{3,2}(\sin u,\cos u)=-(1/40)h^3(90\sin(u)u^5+9\sin(2u)u^5-702\sin(u)u^3-6\sin(3u)u^3-\\
40\cos(3u)u^2-360\cos(2u)u^2+360\cos(u)u^2+1080\sin(u)*u-432*sin(2*u)u-72*sin(3u)u+40u^2-\\
480\cos(3u)+1440\cos(2u)-1440\cos(u)+480)/(u(101\sin(u)u^4+20\sin(2u)u^4+\sin(3u)u^4-\\
456\sin(u)u^2+24*u^2\sin(3u)-576\sin(2u)+720\sin(u)+144\sin(3*u)+192*u^2\sin(2u)))
\end{array}\right\} \tag{16}$$