The Generic Nonlocal Fractal Calculus

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Abstract

The generic nonlocal fractal calculus scheme have been formulated in this work. A unified derivative operator which employs an interpolated characteristic between the generic nonlocal derivative in Riemann–Liouville and Caputo senses has also been derived. For being generic, an arbitrary kernel function has been adopted. The condition on fractional order has been derived so that it is not related to the γ-dimension of the fractal set. The fractal Laplace transforms of our operators have been derived. A simple illustrative example and practical ones have been presented. Unlike the previous power law kernel-based nonlocal fractal calculus operators, ours are generic, consistent with the local fractal derivative and employ higher degree of freedom. The inverse relationships between our derivative and integral operators can be achieved. The results obtained from the examples are significantly different from such previous operator-based counterparts and significantly depended on the kernel function. The unified operator displays an interpolated characteristic as expected.

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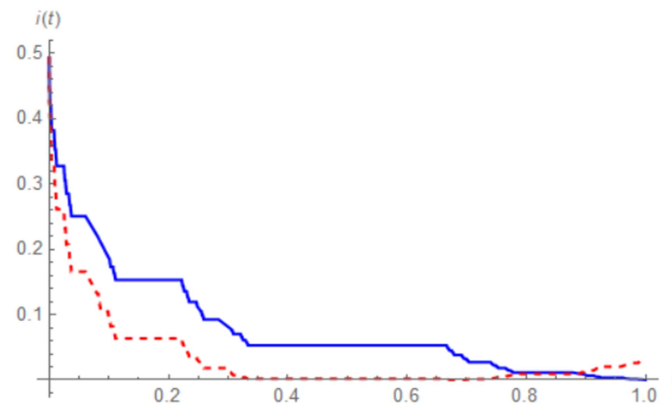

Fig. 1: $i(t)$ vs. t: The proposed generic nonlocal fractional calculus (blue line) and Golmankhaneh-Baleanu nonlocal fractal calculus (red-dashed line)

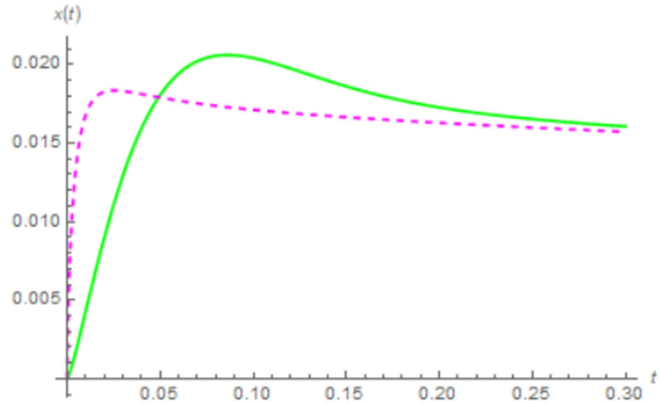

Fig. 2: $x(t)$ vs. t: (2) (based on $k(t;\beta,m)$ as given by (3) (green line)) and (30)-(32) (based on $k(t;\beta,m)$ as given by (5) (magenta-dashed line))

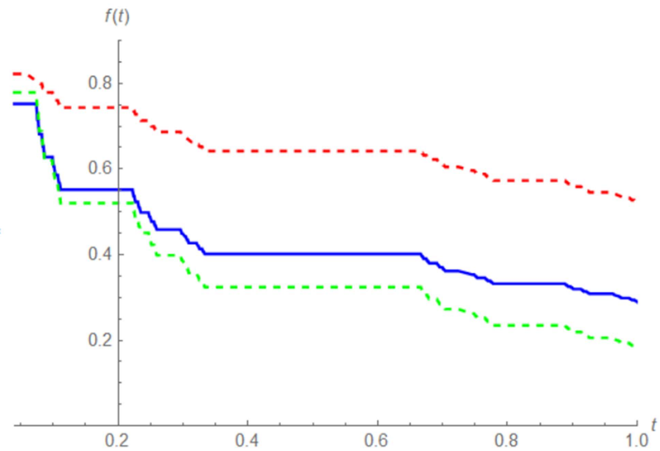

Fig. 3: $f(t)$ vs. t: The unified generic nonlocal fractal derivative (blue line), The generic nonlocal Caputo fractal derivative (red dashed-line), the generic nonlocal Riemann–Liouville fractal derivative (green-dashed line)

The Generic Nonlocal Fractal Calculus

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Abstract

The generic nonlocal fractal calculus scheme have been formulated in this work. A unified derivative operator which employs an interpolated characteristic between the generic nonlocal derivative in Riemann–Liouville and Caputo senses has also been derived. For being generic, an arbitrary kernel function has been adopted. The condition on fractional order has been derived so that it is not related to the γ-dimension of the fractal set. The fractal Laplace transforms of our operators have been derived. A simple illustrative example and practical ones have been presented. Unlike the previous power law kernel-based nonlocal fractal calculus operators, ours are generic, consistent with the local fractal derivative and employ higher degree of freedom. The inverse relationships between our derivative and integral operators can be achieved. The results obtained from the examples are significantly different from such previous operator-based counterparts and significantly depended on the kernel function. The unified operator displays an interpolated characteristic as expected.

Keywords –fractal Laplace transform, fractal set, kernel, local fractal calculus, nonlocal fractal calculus.

I. INTRODUCTION

In the fractal concept related physical phenomena e.g., the anomalous diffusion in fractal structure [1], [2], the charge transportation in porous media [3], [4] and the electromagnetism in fractal time/space [5]-[7], the conventional derivative become invalid. This is because the associated functions are nondifferentiable and thus the fractal set dedicated derivative such as the local fractal calculus [8], [9] and local fractional calculus [10], [11] become necessary. The local fractional derivative has been successfully applied to the studies of the abovementioned electromagnetism in fractal time/space [12], the vibration in fractal media [13] and the analyses of electrical circuits defined on fractal set [14]-[19]. On the other hand, the local fractal derivative has also been widely applied to many fractal concept related issues e.g., the fractal Fokker-Planck equation [20], the fractal stochastic processes [21], the diffusion in fractal structure [22] and of course the analyses of fractal set defined electrical circuits [23]-[25].

However, both local fractal derivative and local fractional derivative are local operators thus they are inapplicable to any system with memory effect e.g., seismograph [26], ground water in the confined aquifer [27] and the electrochemical double layer capacitor (EDLC) [28], [29] etc. This is because they employ non-conservative features involving the irreversible dissipative effects such as ohmic friction, thermal memory and nonlinearities due to the effects of the electric and magnetic fields [30]. Since the nonlocality like that of the fractional calculus is necessary for modelling such memory effect, the nonlocal fractal derivatives have been introduced by Golmankhaneh and Baleanu [31], [32] based on the classical Riemann-Liouville and Caputo fractional derivatives which employ a power law-based kernel. These nonlocal fractal derivatives have been successfully applied to various applications e.g., the mathematical modelling of fractional Brownian motion with fractal support [21] and the analysis of fractional electrical circuits defined on fractal set [23], [33] etc. However, they are inconsistent with the local fractal derivative as will be shown later unlike the fractional derivative that is consistent with the conventional operator. Their order also employs a γ-dimension related condition which in turn limits the degree of freedom.

Therefore, the novel nonlocal fractal derivatives defined in both Riemann–Liouville and Caputo senses without such inconsistency and limited condition thus employ more degree of freedom, have been proposed in this work. Motivated by the generality of those fractional derivatives with arbitrary kernel [34]-[37], the arbitrary kernel function defined on fractal set has been assumed for our operators. As a result, an even more degree of freedom has been obtained. In addition, the associated nonlocal fractal integral has been formulated and the fractal Laplace transforms of these newly developed nonlocal fractal calculus operators have also been derived. A simple illustrative example and the analyses of a fractional electrical and mechanical systems defined on fractal set by means of the nonlocal fractal integrodifferential equation based on these novel nonlocal fractal calculus operators have been presented. The obtained results have been found to be different from their previous nonlocal fractal calculus operator-based counterparts. Finally, motivated by the interpolated characteristic between the Riemann-Liouville and Caputo fractional derivatives of the Hilfer fractional derivative [38], [39], a unified generic nonlocal fractal derivative which employs an interpolated characteristic between the proposed nonlocal fractal derivatives in Riemann–Liouville and Caputo senses has also been presented. In summary, a novel nonlocal fractal calculus scheme has been proposed in this work. The associated operators are generic, consistent with the local ones and also employ higher degree of freedom than their predecessors.

II. PRELIMINARIES

Definition 1 [8]: Let $F \subset \Re$ be a fractal set and $f(t)$ be defined on F such that $t \in F$. If $f(t)$ is F-integrable on $[b_1, b_2]$ where $b_1 \in F$ and $b_2 \in F$ then the local fractal integral of $f(t)$ from b_1 to b_2 can be given by

$$\begin{aligned}\int_{b_1}^{b_2} f(t)\, d_F^{\alpha} t &= \sup_{P[b_1,b_2]} \sum_{i=1}^{n} m[f, F, [t_i, t_{i-1}]]\left(S_F^{\alpha}(t_i) - S_F^{\alpha}(t_{i-1})\right) \\ &= \inf_{P[b_1,b_2]} \sum_{i=1}^{n} M[f, F, [t_i, t_{i-1}]]\left(S_F^{\alpha}(t_i) - S_F^{\alpha}(t_{i-1})\right)\end{aligned}$$

where

$$M[f,F,[b_1,b_2]] = \begin{cases} \sup\limits_{t\in F\cap[b_1,b_2]} f(t) & \kappa_b \cap [b_1,b_2] \neq \{\} \\ 0 & \kappa_b \cap [b_1,b_2] = \{\} \end{cases}$$

$$m[f,F,[b_1,b_2]] = \begin{cases} \inf\limits_{t\in F\cap[b_1,b_2]} f(t) & \kappa_b \cap [b_1,b_2] \neq \{\} \\ 0 & \kappa_b \cap [b_1,b_2] = \{\} \end{cases}$$

Note also that $0 \le \alpha < 1$ and $S_F^\alpha(t)$ denote the γ-dimension of F [32] and integral staircase function respectively.

Definition 2 [8]: The local fractal derivative of $f(t)$ can be given by

$$D_{F,t}^{\alpha}[f(t)] = \begin{cases} F\text{-}\lim\limits_{t'\to t} \frac{f(t')-f(t)}{S_F^\alpha(t')-S_F^\alpha(t)} & t \in F \\ 0 & t \notin F \end{cases}$$

Theorem 1[8]: If $f(t) = \int_0^t g(\tau)\, d_F^\alpha \tau$ then we have

$$D_{F,t}^{\alpha}[f(t)] = g(t)\chi_F(t)$$

where as

$$\chi_F(t) = \begin{cases} 1 & t \in F \\ 0 & t \notin F \end{cases}$$

Theorem 2[8]: If $f(t)$ is F-continuous on $F \cap [a,b]$ and F-differentiable such that

$$g(t)\chi_F(t) = D_{F,t}^{\alpha}[f(t)]$$

then we have

$$\int_a^b g(t)\, d_F^\alpha t = f(b) - f(a)$$

See [8] for the proofs of these theorems.

Definition 3 [31], [32], [40]: The fractal Laplace transform of $f(t)$ can be given by

$$F_s^\alpha(s) = \mathcal{L}_F^\alpha[f(t)] = \int_0^\infty f(t)\exp[-S_F^\alpha(s)S_F^\alpha(t)]d_F^\alpha t$$

Corollary 1[40]:

$$\mathcal{L}_F^\alpha\left[D_{F,t}^{\alpha}[f(t)]\right] = S_F^\alpha(s)F_s^\alpha(s) - f(0)$$

See [40] for the proof.

Corollary 2[24], [25]:

$$\mathcal{L}_F^\alpha\left[\int_0^t f(\tau)\, d_F^\alpha\tau\right] = \frac{F_s^\alpha(s)}{S_F^\alpha(s)}$$

See [25] for the proof.

Corollary 3 [31],[32]: If $g(t)$ is defined on F and

$$h(t) = f(t) *_F g(t) = \int_0^t f(S_F^\alpha(t) - S_F^\alpha(\tau))\, g(\tau) d_F^\alpha\tau$$

then we have

$$H_s^\alpha(s) = \mathcal{L}_F^\alpha[h(t)] = F_s^\alpha(s)G_s^\alpha(s)$$

See [31] and [32] for the proof.

III. The Generic Nonlocal Fractal Calculus

A. *Generic nonlocal fractal derivative*

Definition 4.1: Let $m \in \mathbb{Z}^+$, $m-1 \le \beta \le m$ and $k(t;\beta,m)$ be arbitrary function defined on F where $\lim\limits_{\beta\to m-1} k(t;\beta,m) = 1$ and $\lim\limits_{\beta\to m} \mathcal{L}_F^\alpha[k(t;\beta,m)] = \lim\limits_{\beta\to m} K_s^\alpha(s;\beta,m) = 1$. The β^{th} order generic nonlocal fractal derivative in the Riemann–Liouville sense of $f(t)$ can be given by

$$ {}_{\alpha}^{RL}D_G^\beta[f(t)] = D_{F,t}^{m\alpha}\left[\int_0^t f(\tau)\, k(S_F^\alpha(t) - S_F^\alpha(\tau);\beta,m) d_F^\alpha\tau\right] \quad (1)$$

Definition 4.2: The β^{th} order generic nonlocal fractal derivative in the Caputo sense of $f(t)$ can be given by

$$ {}_{\alpha}^{C}D_G^\beta[f(t)] = \int_0^t D_{F,\tau}^{m\alpha}[f(\tau)]\, k(S_F^\alpha(t) - S_F^\alpha(\tau);\beta,m) d_F^\alpha\tau] \quad (2)$$

At this point, it should be mentioned here that the right generic nonlocal fractal derivative can also be defined by these definitions yet with $-D_{F,t}^{m\alpha}[\,]$ and $\int_t^T [\,] d_F^\alpha \tau$ where $T \in F$ instead. In addition, our nonlocal fractal derivatives employ a higher degree of freedom than the previous ones. Firstly, this is because such previous nonlocal fractal derivatives strictly assume a power law-based kernel which can be given here by

$$k_{GB}(t;\beta,m) = \frac{S_F^\alpha(t)^{-\beta+(m-1)\alpha}}{\Gamma_F(m-\beta)}$$

where $\Gamma_F(\,)$ denotes the Gamma function defined on F [32].

Secondly, the previous nonlocal fractal derivatives assume $(m-1)\alpha \le \beta \le m\alpha$ thus $\beta > m-1$ and $\beta < m$ definitely as $\alpha < 1$. This is unlike our fractal derivatives which $\beta = m-1$ and $\beta = m$ are possible due to our allowed range of β. In addition, it has been shown that the assumed range β of the previous operators causes a serious flaw in the Hamiltonian formalism-based analysis (Banchuin, 2021).

According to definitions 4.1 and 4.2, $k(t;\beta,m)$ can be freely chosen as long as both $\lim\limits_{\beta\to m-1} k(t;\beta,m) = 1$ and $\lim\limits_{\beta\to m} K_s^\alpha(s;\beta,m) = 1$ are satisfied. As an example, a fractional power law based-kernel as given by (3) which its fractal Laplace transform can be given by (4), can be chosen for modelling any system with a fractional power law-based memory effect e.g., the practical electrical circuit component such as the EDLC [28], [29] etc. This is because such fractional power law based-kernel simultaneously satisfies $\lim\limits_{\beta\to m-1} k(t;\beta,m) = 1$ and $\lim\limits_{\beta\to m} K_s^\alpha(s;\beta,m) = 1$ as can be immediately seen from (3) and (4) that

$$k(t;\beta,m)=\frac{|\cos[2\pi(m-1)]|}{|\cos[2\pi\beta]|}\frac{S_F^\alpha(t)^{-\beta+m-1}}{\Gamma_F(m-\beta)} \quad (3)$$

$$K_s^\alpha(s;\beta,m)=\frac{|\cos[2\pi(m-1)]|}{|\cos[2\pi\beta]|}S_F^\alpha(s)^{\beta-m} \quad (4)$$

In addition, motivated by a Caputo-Fabrizio fractional derivative [41] which is suitable to diffusion process [42] and most transport phenomena in practice e.g., the flow of ground water in confined aquifer [27] etc., due to their exponential memory effects [43], we introduce another example of possible kernels as given by (5) where $M(0)=M(1)=1$. Its fractal Laplace transform can be given by (6). Obviously, this Caputo-Fabrizio fractional derivative-based kernel also satisfies both $\lim_{\beta\to m-1} k(t;\beta,m)=1$ and $\lim_{\beta\to m} K_s^\alpha(s;\beta,m) = 1$.

$$k(t;\beta,m)=\frac{M(\beta-m+1)}{m-\beta}\exp\left[-\frac{(\beta-m+1)S_F^\alpha(t)}{m-\beta}\right] \quad (5)$$

$$K_s^\alpha(s;\beta,m)=\frac{M(\beta-m+1)}{m-\beta}\frac{1}{S_F^\alpha(s)+\frac{1-m+\beta}{m-\beta}} \quad (6)$$

Now, it is worthy to introduce the following lemmas.

Lemma 1: $\quad {}^{RL}_{\alpha}D_G^{m-1}[f(t)]=D_{F,t}^{(m-1)\alpha}[f(t)] \quad (7)$

Lemma 2: $\quad {}^{C}_{\alpha}D_G^{m-1}[f(t)]=D_{F,t}^{(m-1)\alpha}[f(t)] \quad (8)$

Proof: 1)

$${}^{RL}_{\alpha}D_G^{m-1}[f(t)]=D_{F,t}^{m\alpha}[\int_0^t f(\tau)\,k(S_F^\alpha(t)-S_F^\alpha(\tau);m-1,m)d_F^\alpha\tau]$$

Since $\lim_{\beta\to m-1} k(t;\beta,m)=1$, we have

$${}^{RL}_{\alpha}D_G^{m-1}[f(t)]=D_{F,t}^{(m-1)\alpha}[D_{F,t}^{\alpha}[\int_0^t f(\tau)\,d_F^\alpha\tau]]$$

i.e.,

$${}^{RL}_{\alpha}D_G^{m-1}[f(t)]=D_{F,t}^{(m-1)\alpha}[f(t)]$$

which completes the proof lemma 1, due to theorem 1 because $t\in F$.

2)

$$\begin{aligned}{}^{C}_{\alpha}D_G^{m-1}[f(t)]&=\int_0^t D_{F,\tau}^{m\alpha}[f(\tau)]\,k(S_F^\alpha(t)-S_F^\alpha(\tau);m-1,m)d_F^\alpha\tau\\&=\int_0^t D_{F,\tau}^{(m-1)\alpha}D_{F,\tau}^{\alpha}[[f(\tau)]]d_F^\alpha\tau\\&=D_{F,\tau}^{(m-1)\alpha}[\int_0^t D_{F,\tau}^{\alpha}[f(\tau)]]d_F^\alpha\tau\end{aligned}$$

Thus, by keeping in mind that $t\in F$, we have

$${}^{C}_{\alpha}D_G^{m-1}[f(t)]=D_{F,t}^{(m-1)\alpha}[f(t)]$$

which completes the proof lemma 2, due to theorem 2.

Before we proceed further, it should be mentioned here that the previous nonlocal fractal derivatives are unfortunately unable to satisfy these lemmas even though $m-1\le\beta\le m$ has been allowed because

$$\lim_{\beta\to m-1} k_{GB}(t;\beta,m)=S_F^\alpha(t)^{(m-1)(\alpha-1)}\neq 1$$

B. Generic nonlocal fractal integral

Definition 5: Let $l(t;\beta,m)$ be defined on F such that $\int_0^t k(\tau;\beta,m)\,l(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau=\frac{S_F^\alpha(t)^{m-1}}{\Gamma_F(m)}$. The β^{th} order generic nonlocal fractal integral of $f(t)$ can be given by

$${}_{\alpha}J_G^{\beta}[f(t)]=\int_0^t f(\tau)\,l(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau \quad (9)$$

In order to formulate $l(t;\beta,m)$, corollary 3 must be invoked. As a result, $l(t;\beta,m)$ can be obtained from the inverse fractal Laplace transformation of $L_s^\alpha(s;\beta,m)=\mathcal{L}_F^\alpha[l(t;\beta,m)]$ whereas

$$L_s^\alpha(s;\beta,m)=\frac{1}{S_F^\alpha(s)^m K_s^\alpha(s;\beta,m)} \quad (10)$$

Therefore, we have

$$l(t;\beta,m)=\frac{|\cos[2\pi\beta]|}{|\cos[2\pi(m-1)]|}\frac{S_F^\alpha(t)^{\beta-1}}{\Gamma_F(\beta)} \quad (11)$$

$$l(t;\beta,m)=\frac{(1-m+\beta)S_F^\alpha(t)^{m-1}+(m-\beta)(m-1)S_F^\alpha(t)^{m-2}}{M(\beta-m+1)\Gamma_F(m)} \quad (12)$$

for $k(t;\beta,m)$ as given by (3) and (5) respectively.

C. Fractal Laplace transforms of generic nonlocal fractal calculus operators

Lemma 3:

$$\begin{aligned}&\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^{\beta}[f(t)]\right] \quad (13)\\&=K_s^\alpha(s;\beta,m)(S_F^\alpha(s)^m F_s^\alpha(s)\\&\quad-\frac{F_-^\alpha\lim_{t\to0}\sum_{k=1}^m S_F^\alpha(s)^{m-k}D_{F,t}^{\alpha(k-1)}\left[\int_0^t f(\tau)k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau\right]}{K_s^\alpha(s;\beta,m)})\end{aligned}$$

Lemma 4:

$$\begin{aligned}&\mathcal{L}_F^\alpha\left[{}^{C}_{\alpha}D_G^{\beta}[f(t)]\right] \quad (14)\\&=K_s^\alpha(s;\beta,m)(S_F^\alpha(s)^m F_s^\alpha(s)-\\&\quad F_-^\alpha\lim_{t\to0}\sum_{k=1}^m S_F^\alpha(s)^{m-k}D_{F,t}^{\alpha(k-1)}[f(t)])\end{aligned}$$

Lemma 5: $\quad \mathcal{L}_F^\alpha\left[{}_{\alpha}J_G^{\beta}[f(t)]\right]=\frac{F_s^\alpha(s)}{S_F^\alpha(s)^m K_s^\alpha(s;\beta,m)} \quad (15)$

Proof: 3)

$$\begin{aligned}&\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^{\beta}[f(t)]\right]\\&=\mathcal{L}_F^\alpha[D_{F,t}^{m\alpha}\left[\int_0^t f(\tau)\,k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau\right]]\end{aligned}$$

According to the conjugacy of the fractal and conventional calculus (Parvate & Gangal, 2011), the corollary 1 can be extended as

$$\mathcal{L}_F^\alpha\left[D_{F,t}^{m\alpha}[f(t)]\right]=S_F^\alpha(s)^m F_s^\alpha(s)-F_-\lim_{t\to0}\sum_{k=1}^m S_F^\alpha(s)^{m-k}D_{F,t}^{\alpha(k-1)}[f(t)] \quad (16)$$

As a result, we have

$$\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^\beta[f(t)]\right] = S_F^\alpha(s)^m \mathcal{L}_F^\alpha\left[\int_0^t f(\tau)\, k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau\right]$$
$$-F_-^\alpha \lim_{t\to 0}\sum_{k=1}^m S_F^\alpha(s)^{m-k}\, D_{F,t}^{\alpha(k-1)}\left[\int_0^t f(\tau)\, k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau\right]$$

Then, after applying corollary 3, we obtain

$$\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^\beta[f(t)]\right] = S_F^\alpha(s)^m F_s^\alpha(s) K_s^\alpha(s;\beta,m)$$
$$-F_-^\alpha \lim_{t\to 0}\sum_{k=1}^m S_F^\alpha(s)^{m-k} D_{F,t}^{\alpha(k-1)}\left[\int_0^t f(\tau)\, k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau\right]$$

therefore, we have

$$\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^\beta[f(t)]\right]$$
$$= K_s^\alpha(s;\beta,m)(S_F^\alpha(s)^m F_s^\alpha(s)$$
$$-\frac{F_-^\alpha \lim_{t\to 0}\sum_{k=1}^m S_F^\alpha(s)^{m-k} D_{F,t}^{\alpha(k-1)}\left[\int_0^t f(\tau) k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau\right]}{K_s^\alpha(s;\beta,m)})$$

right after some algebraic manipulation thus completes the proof lemma 3.

4)

$$\mathcal{L}_F^\alpha\left[{}^{C}_{\alpha}D_G^\beta[f(t)]\right]$$
$$= \mathcal{L}_F^\alpha[\int_0^t D_{F,\tau}^{m\alpha}[f(\tau)]\, k(S_F^\alpha(t)-S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau]$$

By recalling corollary 3, we have

$$\mathcal{L}_F^\alpha\left[{}^{C}_{\alpha}D_G^\beta[f(t)]\right] = K_s^\alpha(s;\beta,m)\mathcal{L}_F^\alpha\left[D_{F,\tau}^{m\alpha}[f(\tau)]\right]$$

i.e.,

$$\mathcal{L}_F^\alpha\left[{}^{C}_{\alpha}D_G^\beta[f(t)]\right]$$
$$= K_s^\alpha(s;\beta,m)(S_F^\alpha(s)^m F_s^\alpha(s) -$$
$$\sum_{k=1}^m S_F^\alpha(s)^{m-k} F_-^\alpha \lim_{t\to 0} D_{F,t}^{\alpha(k-1)}[f(t)])$$

due to (16) thus completes the proof lemma 4.

5)

$$\mathcal{L}_F^\alpha\left[{}_{\alpha}J_G^\beta[f(t)]\right] = \mathcal{L}_F^\alpha[\int_0^t f(\tau)\, l\big(S_F^\alpha(t)-S_{\kappa_b}^\alpha(\tau);\beta,m\big)d_F^\alpha\tau]$$

By recalling corollary 3, we have

$$\mathcal{L}_F^\alpha\left[{}_{\alpha}J_G^\beta[f(t)]\right] = F_s^\alpha(s) L_s^\alpha(s;\beta,m)$$

i.e.,

$$\mathcal{L}_F^\alpha\left[{}_{\alpha}J_G^\beta[f(t)]\right] = \frac{F_s^\alpha(s)}{S_F^\alpha(s)^m K_s^\alpha(s;\beta,m)}$$

according to (10) thus completes the proof lemma 5.

At this point, we are ready to introduce the following lemmas

Lemma 6: $\quad {}^{RL}_{\alpha}D_G^m[f(t)] = D_{F,t}^{m\alpha}[f(t)] \qquad (17)$

Lemma 7: $\quad {}^{C}_{\alpha}D_G^m[f(t)] = D_{F,t}^{m\alpha}[f(t)] \qquad (18)$

Proof: 6)

$${}^{RL}_{\alpha}D_G^m[f(t)] = D_{F,t}^{m\alpha}[\int_0^t f(\tau)\, k(S_F^\alpha(t)-S_F^\alpha(\tau);m,m)d_F^\alpha\tau]$$

Therefore, by applying lemma 3, we have

$$\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^\beta[f(t)]\right]$$
$$= K_s^\alpha(s;m,m)(S_F^\alpha(s)^m F_s^\alpha(s)$$
$$-\frac{F_-^\alpha \lim_{t\to 0}\sum_{k=1}^m S_F^\alpha(s)^{m-k} D_{F,t}^{\alpha(k-1)}\left[\int_0^t f(\tau) k(S_F^\alpha(t)-S_F^\alpha(\tau);m,m)d_F^\alpha\tau\right]}{K_s^\alpha(s;m,m)})$$

Since $\lim_{\beta\to m} K_s^\alpha(s;\beta,m) = 1$, $k(t;m,m) = \delta_F(t)$ and thus

$$\mathcal{L}_F^\alpha[{}^{RL}_{\alpha}D_G^m[f(t)]]$$
$$= S_F^\alpha(s)^m F_s^\alpha(s) - \sum_{k=1}^m S_F^\alpha(s)^{m-k} F_- \lim_{t\to 0} D_{F,t}^{\alpha(k-1)}[f(t)]$$

i.e.,

$$\mathcal{L}_F^\alpha[{}^{RL}_{\alpha}D_G^m[f(t)]] = \mathcal{L}_F^\alpha\left[D_{F,\tau}^{m\alpha}[f(\tau)]\right]$$

according to (16). As a result, it has been found that

$${}^{RL}_{\alpha}D_G^m[f(t)] = D_{F,t}^{m\alpha}[f(t)]$$

which completes the proof lemma 6.

7)

$${}^{C}_{\alpha}D_G^m[f(t)] = \int_0^t D_{F,\tau}^{m\alpha}[f(\tau)]\, k(S_F^\alpha(t)-S_F^\alpha(\tau);m,m)d_F^\alpha\tau$$

Thus, by applying lemma 4 and keeping in mind that $\lim_{\beta\to m} K_s^\alpha(s;\beta,m) = 1$, we have

$$\mathcal{L}_F^\alpha[{}^{C}_{\alpha}D_G^m[f(t)]]$$
$$= S_F^\alpha(s)^m F_s^\alpha(s) - \sum_{k=1}^m S_F^\alpha(s)^{m-k} F_-^\alpha \lim_{t\to 0} D_{F,t}^{\alpha(k-1)}[f(t)]$$

i.e.,

$${}^{C}_{\alpha}D_G^m[f(t)] = D_{F,t}^{m\alpha}[f(t)]$$

which completes the proof lemma 7, after applying (16) and taking the inverse fractal Laplace transformation to both sides of the equation.

Unlike our generic nonlocal operators, the Golmankhaneh-Baleanu nonlocal fractal derivatives are unable to satisfy lemmas 6 and 7 albeit $m-1 \le \beta \le m$ has been allowed because

$$\lim_{\beta\to m} \mathcal{L}_F^\alpha[k_{GB}(t;\beta,m)] = \frac{\Gamma_F((m-1)\alpha)}{\Gamma_F(m-1) S_F^\alpha(s)^{(m-1)\alpha}} \neq 1$$

By the satisfactions of lemmas 1, 2, 6 and 7, it can be stated that our generic nonlocal fractal derivatives are consistent with the local one unlike the previous nonlocal fractal derivatives as they fail to do so. This consistency is another advantage of our generic operators beside their higher degree of freedom.

At this point, it is worthy to introduce the following relationship between ${}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]$ and ${}^{C}_{\alpha}D^{\beta}_{G}[f(t)]$.

Lemma 8:

$$
\begin{aligned}
&{}^{C}_{\alpha}D^{\beta}_{G}[f(t)] \\
&= {}^{RL}_{\alpha}D^{\beta}_{G}[f(t)] - F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}\frac{D^{\alpha(k-1)}_{F,t}[f(t)]\int_0^t S^{\alpha}_{F}(\tau)^{k-m-1}k\left(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m\right)d^{\alpha}_{F}\tau}{\Gamma_F(k-m)}
\end{aligned}
\tag{19}
$$

Proof: Firstly, we rewrite lemma 4 as

$$
\begin{aligned}
\mathcal{L}^{\alpha}_{F}\left[{}^{C}_{\alpha}D^{\beta}_{G}[f(t)]\right] &= K^{\alpha}_{s}(s;\beta,m)S^{\alpha}_{F}(s)^{m}F^{\alpha}_{s}(s) \\
&-F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}K^{\alpha}_{s}(s;\beta,m)S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}[f(t)]
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&\mathcal{L}^{\alpha}_{F}\left[{}^{C}_{\alpha}D^{\beta}_{G}[f(t)]\right] = K^{\alpha}_{s}(s;\beta,m)S^{\alpha}_{F}(s)^{m}F^{\alpha}_{s}(s) \\
&-F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[\int_0^t f(\tau)\,k(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m)d^{\alpha}_{F}\tau\right] \\
&+F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[\int_0^t f(\tau)\,k(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m)d^{\alpha}_{F}\tau\right] \\
&-F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}K^{\alpha}_{s}(s;\beta,m)S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}[f(t)]
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&\mathcal{L}^{\alpha}_{F}\left[{}^{C}_{\alpha}D^{\beta}_{G}[f(t)]\right] \\
&= \mathcal{L}^{\alpha}_{F}\left[{}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]\right] - F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}K^{\alpha}_{s}(s;\beta,m)S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}[f(t)] \\
&\quad +F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[\int_0^t f(\tau)\,k(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m)d^{\alpha}_{F}\tau\right]
\end{aligned}
$$

according to lemma 3.

After taking the inverse fractal Laplace transformation to both sides of the equation, we obtain

$$
\begin{aligned}
&{}^{C}_{\alpha}D^{\beta}_{G}[f(t)] \\
&= {}^{RL}_{\alpha}D^{\beta}_{G}[f(t)] - F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}\frac{D^{\alpha(k-1)}_{F,t}[f(t)]\int_0^t S^{\alpha}_{F}(\tau)^{k-m-1}k\left(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m\right)d^{\alpha}_{F}\tau}{\Gamma_F(k-m)} \\
&\quad +F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}\frac{D^{\alpha(k-1)}_{F,t}\left[\int_0^t f(\tau)k\left(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m\right)d^{\alpha}_{F}\tau\right]S^{\alpha}_{F}(t)^{k-m-1}}{\Gamma_F(k-m)}
\end{aligned}
$$

Since $S^{\alpha}_{F}(0)=0$, we have

$$
\begin{aligned}
&{}^{C}_{\alpha}D^{\beta}_{G}[f(t)] \\
&= {}^{RL}_{\alpha}D^{\beta}_{G}[f(t)] - F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}\frac{D^{\alpha(k-1)}_{F,t}[f(t)]\int_0^t S^{\alpha}_{F}(\tau)^{k-m-1}k\left(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m\right)d^{\alpha}_{F}\tau}{\Gamma_F(k-m)}
\end{aligned}
$$

thus completes the proof.

Finally, the following lemmas which show that ${}_{\alpha}J^{\beta}_{G}[\,]$ can serve as the inverse of ${}^{RL}_{\alpha}D^{\beta}_{G}[\,]$ and ${}^{C}_{\alpha}D^{\beta}_{G}[\,]$, are worthy of mentioned before we proceed to the subsesequent section.

Lemma 9: If $F_{-}\lim_{t\to 0}k(t;\beta,m)=0$ then

$$
{}^{RL}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=f(t) \tag{20}
$$

Lemma 10: If $F_{-}\lim_{t\to 0}k(t;\beta,m)=0$ then

$$
{}_{\alpha}J^{\beta}_{G}\left[{}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]\right]=f(t) \tag{21}
$$

Lemma 11: If $F^{\alpha}_{-}\lim_{t\to 0}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=0$ then

$$
{}^{C}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=f(t) \tag{22}
$$

Lemma 12: If $F^{\alpha}_{-}\lim_{t\to 0}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=0$ then

$$
{}_{\alpha}J^{\beta}_{G}\left[{}^{C}_{\alpha}D^{\beta}_{G}[f(t)]\right]=f(t) \tag{23}
$$

Proof: 9)

By recalling lemmas 3 and 5, we can formulate

$$
\begin{aligned}
&\mathcal{L}^{\alpha}_{F}\left[{}^{RL}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]\right]=F^{\alpha}_{s}(s) \\
&-F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[\int_0^t {}_{\alpha}J^{\beta}_{G}[f(t)]\,k(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m)d^{\alpha}_{F}\tau\right]
\end{aligned}
$$

Thus, after taking the inverse fractal Laplace transformation, we have

$$
\begin{aligned}
&{}^{RL}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=f(t)- \\
&F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}\frac{S^{\alpha}_{F}(t)^{k-m-1}}{\Gamma_F(k-m)}D^{\alpha(k-1)}_{F,t}\left[\int_0^t {}_{\alpha}J^{\beta}_{G}[f(t)]\,k(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m)d^{\alpha}_{F}\tau\right]
\end{aligned}
$$

i.e.,

$$
{}^{RL}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=f(t)
$$

if $F^{\alpha}_{-}\lim_{t\to 0}k(t;\beta,m)=0$ therefore completes the proof lemma 9.

10)

By also recalling lemmas 3 and 5, we can formulate

$$
\begin{aligned}
&\mathcal{L}^{\alpha}_{F}\left[{}_{\alpha}J^{\beta}_{G}\left[{}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]\right]\right] \\
&=F^{\alpha}_{s}(s)-\frac{F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}S^{\alpha}_{F}(s)^{-k}D^{\alpha(k-1)}_{F,t}\left[\int_0^t f(\tau)k(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau);\beta,m)d^{\alpha}_{F}\tau\right]}{K^{\alpha}_{s}(s;\beta,m)}
\end{aligned}
$$

Thus, if $F^{\alpha}_{-}\lim_{t\to 0}k(t;\beta,m)=0$ then we have

$$
\mathcal{L}^{\alpha}_{F}\left[{}_{\alpha}J^{\beta}_{G}\left[{}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]\right]\right]=F^{\alpha}_{s}(s)
$$

i.e.,

$$
{}_{\alpha}J^{\beta}_{G}\left[{}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]\right]=f(t)
$$

which completes the proof lemma 10.

11)

By recalling lemmas 4 and 5, we can derive

$$
\begin{aligned}
&\mathcal{L}^{\alpha}_{F}\left[{}^{C}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]\right] \\
&=F^{\alpha}_{s}(s)-K^{\alpha}_{s}(s;\beta,m)F^{\alpha}_{-}\lim_{t\to 0}\sum_{k=1}^{m}S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]
\end{aligned}
$$

Thus, if $F^{\alpha}_{-}\lim_{t\to 0}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]=0$ then we have

$$
\mathcal{L}^{\alpha}_{F}\left[{}^{C}_{\alpha}D^{\beta}_{G}\left[{}_{\alpha}J^{\beta}_{G}[f(t)]\right]\right]=F^{\alpha}_{s}(s)
$$

i.e.,

$$ {}_{\alpha}^{C}D_{G}^{\beta}\left[{}_{\alpha}J_{G}^{\beta}[f(t)]\right]=f(t) $$

which completes the proof lemma 11.

12)

By recalling lemmas 4 and 5, we can derive

$$ \mathcal{L}_{F}^{\alpha}\left[{}_{\alpha}J_{G}^{\beta}\left[{}_{\alpha}^{C}D_{G}^{\beta}[f(t)]\right]\right] \\ = F_{s}^{\alpha}(s)-F_{-}^{\alpha}\lim_{t\to 0}\sum_{k=1}^{m}S_{F}^{\alpha}(s)^{-k}D_{F,t}^{\alpha(k-1)}[f(t)] $$

Therefore, after taking the inverse fractal Laplace transformation, we have

$$ {}_{\alpha}J_{G}^{\beta}\left[{}_{\alpha}^{C}D_{G}^{\beta}[f(t)]\right] \\ = f(t)-F_{-}^{\alpha}\lim_{t\to 0}\sum_{k=1}^{m}\frac{S_{F}^{\alpha}(t)^{k-1}D_{F,t}^{\alpha(k-1)}[f(t)]}{\Gamma_{F}(k)} $$

i.e.,

$$ {}_{\alpha}J_{G}^{\beta}\left[{}_{\alpha}^{C}D_{G}^{\beta}[f(t)]\right]=f(t) $$

if $F_{-}^{\alpha}\lim_{t\to 0}D_{F,t}^{\alpha(k-1)}[f(t)]=0$ thus completes the proof lemma 11.

IV. Examples

A. A simple illustrative example

Consider a generic nonlocal fractal derivative based differential equation as

$$ {}_{\alpha}^{C}D_{G}^{\beta}[f(t)]+f(t)=0 \tag{24} $$

where $0\leq\beta\leq 1$.

Since $k(t;\beta,m)$ can be arbitrarily chosen unless $\lim_{\beta\to m-1}k(t;\beta,m)=1$ and $\lim_{\beta\to m}K_{s}^{\alpha}(s;\beta,m)=1$ are violated, we may choose $k(t;\beta,m)$ as given by (5) as both $\lim_{\beta\to m-1}k(t;\beta,m)=1$ and $\lim_{\beta\to m}K_{s}^{\alpha}(s;\beta,m)=1$ are satisfied. Thus, by keeping in mind that $m = 1$ due to the given range of β, we have

$$ \frac{M(\beta)}{1-\beta}\frac{S_{F}^{\alpha}(s)F_{s}^{\alpha}(s)-f(0)}{S_{F}^{\alpha}(s)+\frac{\beta}{1-\beta}}+F_{s}^{\alpha}(s)=0 $$

i.e.,

$$ F_{s}^{\alpha}(s)=\frac{f(0)}{(1+\frac{1-\beta}{M(\beta)})S_{F}^{\alpha}(s)+\frac{\beta}{M(\beta)}} $$

As a result, $f(t)$ can be found as

$$ f(t)=\frac{M(\beta)f(0)}{M(\beta)+1-\beta}\exp\left[-\frac{\beta S_{F}^{\alpha}(t)}{M(\beta)+1-\beta}\right] \tag{25} $$

which is totally different from the previous Golmankhaneh-Baleanu nonlocal fractal derivative-based counterpart (see [31]) that is in terms of a fractal Mittag-Leffler function. This is because $k_{GB}(t;\beta,m)$ is based on a fractional power law unlike the chosen $k(t;\beta,m)$.

B. Fractional electrical circuit

Now, consider a source free series fractional RLC circuit defined on *F*. By applying the KVL, the following generic nonlocal fractal calculus-based differential equation in the Caputo sense can be obtained

$$ L\,{}_{\alpha}^{C}D_{G}^{\beta}[i(t)]+Ri(t)+\frac{1}{C}\,{}_{\alpha}J_{G}^{\beta}[i(t)]=0 \tag{26} $$

Since the practical electrical circuit component employ a fractional power law-based memory effect as abovementioned, we choose $k(t;\beta,m)$ as given by (3) as it is based on a fractional power law and obeys $\lim_{\beta\to m-1}k(t;\beta,m)=1$ and $\lim_{\beta\to m}K_{s}^{\alpha}(s;\beta,m)=1$. Based on the experimental data (see [30]), we let $0\leq\beta\leq 1$. As a result, we have

$$ \frac{LS_{F}^{\alpha}(s)^{\beta-1}}{|\cos[2\pi\beta]|}(S_{F}^{\alpha}(s)I_{s}^{\alpha}(s)-i(0))+RI_{s}^{\alpha}(s)+\frac{|\cos[2\pi\beta]|I_{s}^{\alpha}(s)}{CS_{F}^{\alpha}(s)^{\beta}}=0 $$

i.e.,

$$ I_{s}^{\alpha}(s)=\frac{i(0)S_{F}^{\alpha}(s)^{2\beta-1}}{S_{F}^{\alpha}(s)^{2\beta}+\frac{R|\cos[2\pi\beta]|}{L}S_{F}^{\alpha}(s)^{\beta}+\frac{|\cos[2\pi\beta]|^{2}}{LC}} $$

Therefore, *i*(*t*) can be found as

$$ i(t)=i(0)\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{(-\frac{|\cos[2\pi\beta]|^{2}}{LC})^{n}(-\frac{R|\cos[2\pi\beta]|}{L})^{k}}{\Gamma_{F}(\beta(2n+k)+1)}\binom{n+k}{k}S_{F}^{\alpha}(t)^{\beta(2n+k)} \tag{27} $$

By let *F* be a Cantor ternary set thus α = 0.6309 and assuming that *i*(0) = 1 A, *R* = 1 Ω, *L* = 1 H, *C* = 1 F and β = 0.6309, *i*(*t*) can be approximately simulated based on (27) up to *n* = *k* = 99 via MATHEMATICA as depicted in Fig. 1 where a significantly different dynamic from that of its Golmankhaneh-Baleanu nonlocal fractal calculus-based counterpart obtained from the previous analysis of fractional RLC circuit on fractal set (see [33]) which has also been depicted in this figure, can be observed. Such disagreement is also caused by the different between the assumed $k(t;\beta,m)$ and $k_{GB}(t;\beta,m)$.

C. Fractional mechanical system

In practice, the mechanical system also employs the memory effect as in the case of the seismograph [26]. Therefore, the nonlocal fractal calculus become necessary whenever such system is defined on *F*. Now, consider a mass-spring-damper system defined on *F*. After applying the Newton's 2nd law, the following generic nonlocal fractal calculus-based differential equation in the Riemann–Liouville sense can be obtained

$$ m\,{}_{\alpha}^{RL}D_{G}^{2\beta}[x(t)]+\xi\,{}_{\alpha}^{RL}D_{G}^{\beta}[x(t)]+kx(t)=f(t) \tag{28} $$

where k, m and ξ denote the spring constant, mass and damping coefficient respectively. Note also that $x(t)$ and $f(t)$ stand for the displacement and the forcing function.

Since we assume $0 \leq \beta \leq 1$ and null initial values, it can be seen from lemma 3 that

$$\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^{\beta}[x(t)]\right] = K_S^\alpha(s;\beta,1)S_F^\alpha(s)X_S^\alpha(s)$$

$$\mathcal{L}_F^\alpha\left[{}^{RL}_{\alpha}D_G^{2\beta}[x(t)]\right] = K_S^\alpha(s;2\beta,2)S_F^\alpha(s)^2X(s)$$

Thus, for $f(t) = f$, we have

$$mK_S^\alpha(s;2\beta,2)S_F^\alpha(s)^2X_S^\alpha(s) + \xi K_S^\alpha(s;\beta,1)S_F^\alpha(s)X_S^\alpha(s) + kX_S^\alpha(s) = \frac{f}{S_F^\alpha(s)}$$

i.e.,

$$X_S^\alpha(s) = \frac{f}{S_F^\alpha(s)(MK_S^\alpha(s;2\beta,2)S_F^\alpha(s)^2+\xi K_S^\alpha(s;\beta,1)S_F^\alpha(s)+k)}$$

If we choose a fractional power law-based $k(t;\beta,m)$ which is given by (3), we have

$$K_S^\alpha(s;\beta,1) = \frac{S_F^\alpha(s)^{\beta-1}}{|\cos[2\pi\ \ ]|}$$

$$K_S^\alpha(s;2\beta,2) = \frac{S_F^\alpha(s)^{2(\beta-1)}}{|\cos[2\pi\ \ ]|}$$

and thus

$$X_S^\alpha(s) = \frac{f|\cos[2\pi\beta]|S_F^\alpha(s)^{-1}}{MS_F^\alpha(s)^{2\beta}+\xi S_F^\alpha(s)^{\beta}+k|\cos[2\pi\beta]|}$$

According to [8], $X_S^\alpha(s)$ can be approximated as

$$X_S^\alpha(s) \approx \frac{f|\cos[2\pi\beta]|s^{-\alpha}}{Ms^{2\alpha}\ \ +\xi s^{\alpha\beta}+k|\cos[2\pi\beta]|} \tag{29}$$

On the other hand, if we choose $k(t;\beta,m)$ as given by (5) which is based on an exponential function, we have

$$K_S^\alpha(s;\beta,1) = \frac{M(\beta)}{1-\beta}\frac{1}{S_F^\alpha(s)+\frac{\beta}{1-\beta}}$$

$$K_S^\alpha(s;2\beta,2) = \frac{M(2\beta-1)}{2-2\beta}\frac{1}{S_F^\alpha(s)+\frac{2\beta-1}{2-2\beta}}$$

i.e.,

$$X_S^\alpha(s) = \frac{f}{\frac{\frac{mM(2\beta-1)}{2-2\beta}S_F^\alpha(s)^3}{S_F^\alpha(s)+\frac{2\beta-1}{2-2\beta}}+\frac{\frac{\xi M(\beta)}{1-\beta}S_F^\alpha(s)^2}{S_F^\alpha(s)+\frac{\beta}{1-\beta}}+kS_F^\alpha(s)}$$

which in turn can be approximated as

$$X_S^\alpha(s) \approx \frac{f}{\frac{\frac{mM(2\beta-1)}{2-2\beta}s^{3\alpha}}{s^\alpha+\frac{2\beta-1}{2-2\beta}}+\frac{\frac{\xi M(\beta)}{1-\beta}s^{2\alpha}}{s^\alpha+\frac{\beta}{1-\beta}}+ks^\alpha} \tag{30}$$

Note also that we let

$$M(\beta) = \begin{cases} \frac{2}{2-\beta}; 0 \leq \beta < 1 \\ \frac{1}{2}\left(\frac{2}{2-\beta}\right); \beta = 1 \end{cases} \tag{31}$$

$$M(2\beta-1) = \begin{cases} \frac{2}{3-2\beta}; 0 \leq \beta < 1 \\ \frac{1}{2}\left(\frac{2}{3-2\beta}\right); \beta = 1 \end{cases} \tag{32}$$

which clearly satisfy $M(0) = M(1) = 1$.

By assuming that f = 12 N, k = 800 N/m, m = 3 kg, ξ = 20 Nsec/m, $\alpha = 0.85$ and $\beta = 0.85$, $x(t)$ can be approximately simulated based on the numerical inverse Laplace transforms of (29) and (30)-(32) as depicted in Fig. 2 which shows that $x(t)$ due to the fractional power law-based kernel and its exponential function kernel-based counterpart employ significantly different dynamics.

V. The Unified Generic Nonlocal Fractal Derivative

Motivated by the interpolated characteristic between the Riemann-Liouville and Caputo fractional derivatives of the Hilfer fractional derivative, a unified generic nonlocal fractal derivative which employs an interpolated characteristic between both generic nonlocal fractal derivatives in Riemann–Liouville and Caputo senses will be defined in this section. Firstly, we introduce the following fractal convolution integral operator

$${}_{\alpha}I_G^{\beta}[\,] = \int_0^t [\,]\, k(S_F^\alpha(t) - S_F^\alpha(\tau);\beta,m)d_F^\alpha\tau \tag{33}$$

Therefore, ${}^{RL}_{\alpha}D_G^{\beta}[f(t)]$ and ${}^{C}_{\alpha}D_G^{\beta}[f(t)]$ can be respectively given in terms of ${}_{\alpha}I_G^{\beta}[\ ]$ as

$${}^{RL}_{\alpha}D_G^{\beta}[f(t)] = D_{F,t}^{m\alpha}[{}_{\alpha}I_G^{\beta}[f(t)]]$$

$${}^{C}_{\alpha}D_G^{\beta}[f(t)] = {}_{\alpha}I_G^{\beta}[D_{F,t}^{m\alpha}[f(t)]]$$

Since $\lim_{\beta\to m} K_S^\alpha(s;\beta,m) = 1$, we have

$$\lim_{\beta\to m} k(S_F^\alpha(t);\beta,m) = \delta_F(t)$$

where $\delta_F(t)$ stands for a fractal impulse function. Thus, by the conjugacy between the fractal and conventional calculus, ${}_{\alpha}I_G^{m}[\ ]$ is an identity fractal convolution integral operation i.e., ${}^{RL}_{\alpha}D_G^{\beta}[f(t)]$ and ${}^{C}_{\alpha}D_G^{\beta}[f(t)]$ can be alternatively given by

$${}^{RL}_{\alpha}D_G^{\beta}[f(t)] = {}_{\alpha}I_G^{m}\left[D_{F,t}^{m\alpha}\left[{}_{\alpha}I_G^{\beta}[f(t)]\right]\right]$$

$${}^{C}_{\alpha}D_G^{\beta}[f(t)] = {}_{\alpha}I_G^{\beta}\left[D_{F,t}^{m\alpha}\left[{}_{\alpha}I_G^{m}[f(t)]\right]\right]$$

As a result, the unified generic nonlocal fractal derivative can be finally defined as follows

Definition 6: Let $m-1 \leq \beta_1 \leq m$ and $m-1 \leq \beta_2 \leq m$, the unified generic nonlocal fractal derivative of $f(t)$ can be given by

$$ {}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)] = {}_{\alpha}I^{\beta_1}_{G}\left[D^{m\alpha}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]\right] \tag{34} $$

Remark 1: ${}^{U}_{\alpha}D^{m,\beta}_{G}[f(t)] = {}^{RL}_{\alpha}D^{\beta}_{G}[f(t)]$.

Remark 2: ${}^{U}_{\alpha}D^{\beta,m}_{G}[f(t)] = {}^{C}_{\alpha}D^{\beta}_{G}[f(t)]$.

Lemma 13: ${}^{U}_{\alpha}D^{m,m}_{G}[f(t)] = D^{m\alpha}_{F,t}[f(t)]$.

Lemma 14:

$$ \begin{aligned} &\mathcal{L}^{\alpha}_{F}\left[{}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]\right] \\ &= K^{\alpha}_{S}(s;\beta_1,m)(S^{\alpha}_{F}(s)^{m}K^{\alpha}_{S}(s;\beta_2,m)F^{\alpha}_{S}(s) - \\ &\qquad F^{\alpha}_{-}\lim_{t\to 0}\textstyle\sum_{k=1}^{m} S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]) \end{aligned} \tag{35} $$

Proof: 13)

$$ {}^{U}_{\alpha}D^{m,m}_{G}[f(t)] = {}_{\alpha}I^{m}_{G}\left[D^{m\alpha}_{F,t}\left[{}_{\alpha}I^{m}_{G}[\,f(t)]\right]\right] $$

Since ${}_{\alpha}I^{m}_{G}[\ ]$ is an identity fractal convolution integral operation, we have

$$ {}^{U}_{\alpha}D^{m,m}_{G}[f(t)] = D^{m\alpha}_{F,t}[f(t)] $$

which completes the proof lemma 13.
14)

$$ \mathcal{L}^{\alpha}_{F}\left[{}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]\right] = \mathcal{L}^{\alpha}_{F}\left[{}_{\alpha}I^{\beta_1}_{G}\left[D^{m\alpha}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]\right]\right] $$

By applying (33) and corollary 3, it has been found that

$$ \mathcal{L}^{\alpha}_{F}\left[{}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]\right] = K^{\alpha}_{S}(s;\beta_1,m)\mathcal{L}^{\alpha}_{F}\left[D^{m\alpha}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]\right] $$

due to corollary 3.

By applying (16) to $\mathcal{L}^{\alpha}_{F}\left[D^{m\alpha}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]\right]$, $\mathcal{L}^{\alpha}_{F}\left[{}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]\right]$ become

$$ \begin{aligned} &\mathcal{L}^{\alpha}_{F}\left[{}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]\right] \\ &= K^{\alpha}_{S}(s;\beta_1,m)(S^{\alpha}_{F}(s)^{m}\mathcal{L}^{\alpha}_{F}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right] - F^{\alpha}_{-}\lim_{t\to 0}\textstyle\sum_{k=1}^{m} S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]) \end{aligned} $$

Now, we apply (33) and corollary 3 to $\mathcal{L}^{\alpha}_{F}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]$. As a result, we have

$$ \begin{aligned} &\mathcal{L}^{\alpha}_{F}\left[{}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]\right] \\ &= K^{\alpha}_{S}(s;\beta_1,m)(S^{\alpha}_{F}(s)^{m}K^{\alpha}_{S}(s;\beta_2,m)F^{\alpha}_{S}(s) - F^{\alpha}_{-}\lim_{t\to 0}\textstyle\sum_{k=1}^{m} S^{\alpha}_{F}(s)^{m-k}D^{\alpha(k-1)}_{F,t}\left[{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\right]) \end{aligned} $$

which completes the proof lemma 14.

At this point, the interpolated characteristic of the unified generic nonlocal fractal derivative will be demonstrated. In order to do so, the simple example presented in the previous section will be reconsidered here yet with ${}^{C}_{\alpha}D^{\beta}_{G}[f(t)]$ replaced by ${}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)]$. Thus, we now have

$$ {}^{U}_{\alpha}D^{\beta_1,\beta_2}_{G}[f(t)] + f(t) = 0 \tag{36} $$

where $0 \le \beta_1 \le 1$ and $0 \le \beta_2 \le 1$ i.e., $m = 1$.

After applying the fractal Laplace transformation and lemma 14, (36) become

$$ K^{\alpha}_{S}(s;\beta_1,1)(S^{\alpha}_{F}(s)\,K^{\alpha}_{S}(s;\beta_2,1)F^{\alpha}_{S}(s) - F^{\alpha}_{-}\lim_{t\to 0}{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]\,) + F^{\alpha}_{S}(s) = 0 $$

Since we now choose $k(t;\beta,m)$ as given by (3), we have

$$ K^{\alpha}_{S}(s;\beta_1,1) = \frac{S^{\alpha}_{F}(s)^{\beta_1-1}}{|\cos[2\pi\beta_1]|} $$

$$ K^{\alpha}_{S}(s;\beta_2,1) = \frac{S^{\alpha}_{F}(s)^{\beta_2-1}}{|\cos[2\pi\beta_2]|} $$

i.e.,

$$ F^{\alpha}_{S}(s) = \frac{F_{\beta_2}(0)|\cos[2\pi\beta_2]|S^{\alpha}_{F}(s)^{\beta_1-1}}{S^{\alpha}_{F}(s)^{\beta_1+\beta_2-1}+|\cos[2\pi\beta_1]||\cos[2\pi\beta_2]|} $$

where $F_{\beta_2}(0) = F^{\alpha}_{-}\lim_{t\to 0}{}_{\alpha}I^{\beta_2}_{G}[\,f(t)]$ with $k(t;\beta,m)$ as given by (3) and $m = 1$.

As a result, the following solution can be obtained.

$$ \begin{aligned} f(t) = F_{\beta_2}(0)|\cos[2\pi\beta_2]|S^{\alpha}_{F}(t)^{\beta_2-1} \\ E^{\alpha}_{F,\beta_1+\beta_2-1,\beta_2}\left[-|\cos[2\pi\beta_1]||\cos[2\pi\beta_2]|t^{\beta_1+\beta_2-1}\right] \end{aligned} \tag{37} $$

where $E^{\alpha}_{F,\beta_1+\beta_2-1,\beta_2}[\ ]$ is a generalized two parameter Mittag-Liffler function defined on F [32].

For demonstrating such interpolated characteristic, it is worthy to introduce the benchmarking generic nonlocal Riemann–Liouville and Caputo fractal derivative based solutions which can be respectively given by

$$ \begin{aligned} f(t) = F_{\beta}(0)|\cos[2\pi\beta]|\,S^{\alpha}_{F}(t)^{\beta-1} \\ E^{\alpha}_{F,\beta,\beta}\left[-|\cos[2\pi\beta]|t^{\beta}\right] \end{aligned} \tag{38} $$

$$ f(t) = f(0)E^{\alpha}_{F,\beta}[-|\cos[2\pi\beta]|t^{\beta}] \tag{39} $$

where $F_{\beta}(0) = \dfrac{F^{\alpha}_{-}\lim_{t\to 0}\int_{0}^{t} f(\tau)(S^{\alpha}_{F}(t)-S^{\alpha}_{F}(\tau))^{-\beta}d^{\alpha}_{F}\tau}{|\cos[2\pi\beta]|\Gamma_{F}(1-\beta)}$ and where $E^{\alpha}_{F,\beta}[\ ]$ is a generalized standard Mittag- Liffler function defined on F [31].

Based on (37)-(39), the unified generic nonlocal fractal derivative, the generic nonlocal Riemann–Liouville fractal derivative and the generic nonlocal Caputo fractal derivative based $f(t)$'s can be simulated as depicted in Fig. 3 where $f(0) = F_{\beta}(0) = F_{\beta_2}(0) = 1$ and $\alpha = \beta_1 = \beta_2 = \beta = 0.6309$ have been assumed. Obviously, the unified generic nonlocal fractal derivative based solution displays an interpolated characteristic between the generic nonlocal Riemann–Liouville and Caputo fractal derivative based counterparts.

VI. Conclusion

Motivated by the capability to model memory effect in the fractal time-space of nonlocal fractal calculus and the generality of general fractional calculus such as that of Kochubei [34], [35], a generic nonlocal fractal calculus scheme has been proposed in this work. Both generic nonlocal fractal derivatives in Riemann–Liouville and Caputo senses are consistent with the local one and employ higher degree of freedom than the previous nonlocal fractal calculus. The fractal Laplace transforms of our novel nonlocal fractal calculus operators have also been derived. The consistency with local fractal calculus and inverse relationships between our nonlocal fractal derivatives and integral have been mathematically validated.

In addition, a simple illustrative example and the analyses of a fractional RLC circuit and a fractional mass-spring-damper system defined on a fractal set have been presented. The obtained results have been found to be significantly different from their previous power law kernel-based nonlocal fractal calculus-based counterparts and also depended on the specifically chosen kernel function. Finally, a unified generic nonlocal fractal derivative which employs an interpolated characteristic between the proposed generic nonlocal fractal derivative in Riemann–Liouville and Caputo senses has also been presented. By the generality, consistency with the local fractal calculus and higher degree of freedom, the nonlocal fractal calculus scheme proposed in this work has been found to be more preferable than its predecessors.

REFERENCES

1. Ben-Avraham, D. and Havlin, S. (2000), *Diffusion and reactions in fractals and disordered systems*. Cambridge university press, UK.
2. Poirier, D.R. and Geiger, G.H. (2016), “Fick’s Law and Diffusivity of Materials”, In *Transport Phenomena in Materials Processing* (pp. 419-461), Springer, CH
3. Brovelli, A. and Cassiani, G. (2010), “A combination of the Hashin-Shtrikman bounds aimed at modelling electrical conductivity and permittivity of variably saturated porous media”, *Geophysical Journal International*, Vol. 180, No. 1, pp.225-237.
4. Cai, J., Wei, W., Hu, X. and Wood, D.A. (2017), “Electrical conductivity models in saturated porous media: A review”, *Earth-Science Reviews*, Vol. 171, pp.419-433.
5. Tarasov, V.E. (2006), “Electromagnetic fields on fractals”, *Modern Physics Letters A*, Vol. 21, No. 20, pp.1587-1600.
6. Balankin, A.S., Mena, B., Patiño, J. and Morales, D. (2013), “Electromagnetic fields in fractal continua”, *Physics Letters A*, Vol. 377, No. 10-11, pp.783-788.
7. Ostoja-Starzewski, M. (2013), “Electromagnetism on anisotropic fractal media”, *Zeitschrift für angewandte Mathematik und Physik*, Vol. 64, No. 2, pp.381-390.
8. Parvate, A. and Gangal, A.D. (2009), “Calculus on fractal subsets of real line—I: Formulation”, *Fractals*, Vol. 17 No. 01, pp. 53-81.
9. Parvate, A. and Gangal, A.D. (2011), “Calculus on fractal subsets of real line—II: conjugacy with ordinary calculus”, *Fractals*, Vol. 19 No. 03, pp.271-290.
10. Yang, X.J. (2012), *Advanced local fractional calculus and its applications*. World Scientific Publishing, SG.
11. Yang, X.J., Baleanu, D. and Srivastava, H.M. (2015), *Local fractional integral transforms and their applications*. Academic Press, US.
12. Zhao, Y., Baleanu, D., Cattani, C., Cheng, D.F. and Yang, X.J. (2013), “Maxwell’s equations on Cantor sets: a local fractional approach”, *Advances in high energy physics*, Vol. 2013, pp. 1-6.
13. Yang, X.J. and Srivastava, H.M. (2015), “An asymptotic perturbation solution for a linear oscillator of free damped vibrations in fractal medium described by local fractional derivatives” *Communications in Nonlinear Science and Numerical Simulation*, Vol. 29, No. 1-3, pp.499-504.
14. Yang, X.J., Machado, J.T., Cattani, C. and Gao, F. (2017), “On a fractal LC-electric circuit modeled by local fractional calculus”, *Communications in Nonlinear Science and Numerical Simulation*, Vol. 47, pp. 200-206.
15. Zhao, X.H., Zhang, Y., Zhao, D. and Yang, X. (2017), “The RC circuit described by local fractional differential equations”, *Fundamenta Informaticae*, Vol. 151 No. 1-4, pp. 419-429.
16. Wang, K.J. (2020), “On a High-pass filter described by local fractional derivative”, *Fractals*, Vol. 28 No. 03, p. 2050031.
17. Wang, K.J. (2020a), “The transient analysis for zero-input response of fractal RC circuit based on local fractional derivative”, *Alexandria Engineering Journal*, Vol. 59 No. 6, pp. 4669-4675.
18. Wang, K.J. and Li, C.L. (2020), “A$\wp$-order RL high-pass filter modeled by local fractional derivative”, *Alexandria Engineering Journal*, Vol. 59 No. 5, pp. 3255-3259.
19. Wang, K.J., Sun, H.C. and Cui, Q.C. (2020), “The fractional Sallen-Key filter described by local fractional derivative”, *IEEE Access*, Vol. 8, pp. 166377-166383.
20. Satin, S.E., Parvate, A. and Gangal, A.D. (2013), “Fokker–Planck equation on fractal curves”, *Chaos, Solitons & Fractals*, Vol. 52, pp.30-35.
21. Golmankhaneh, A.K. and Sibatov, R.T. (2021), “Fractal stochastic processes on thin cantor-like sets”, *Mathematics*, Vol. 9 No. 6, p.613.
22. Khalili Golmankhaneh, A., Fernandez, A., Khalili Golmankhaneh, A. and Baleanu, D. (2018), “Diffusion on middle-ξ Cantor sets”, *Entropy*, Vol 20, No. 7, p. 504.
23. Ali, K.K., Golmankhaneh, A.K. and Yilmazer, R. (2021), “Battery discharging model on fractal time sets”, *International Journal of Nonlinear Sciences and Numerical Simulation*, Vol. ahead-of-print, No. ahead-of-print.
24. Khalili Golmankhaneh, A., Kamal Ali, K., Yilmazer, R. and Welch, K. (2021), “Electrical circuits involving fractal time”, *Chaos: An Interdisciplinary Journal of Nonlinear Science*, Vol 31, No. 3, p. 033132
25. Banchuin, R. (2022), “Noise analysis of electrical circuits on fractal set”, *COMPEL - The international journal for computation and mathematics in electrical and electronic engineering*, Vol. ahead-of-print, No. ahead-of-print.
26. Caputo, M. (2013), “The memory damped seismograph”, *Bollettino di Geofisica Teorica ed Applicata*, Vol. 54, No. 3, pp. 217-228
27. Atangana, A. and Alkahtani, B.S.T. (2016), “New model of groundwater flowing within a confine aquifer: application of Caputo-Fabrizio derivative”, *Arabian Journal of Geosciences*, Vol. 9, No. 1, pp.1-6.
28. Allagui, A. and Fouda, M.E. (2021), “Inverse problem of reconstructing the capacitance of electric double- layer capacitors”, *Electrochimica Acta*, Vol. 390, p. 138848.
29. Banchuin, R. (2022), “Comparative analysis of suitability of fractional derivatives in modelling the practical capacitor”, *COMPEL - The international journal for computation and mathematics in electrical and electronic engineering*, Vol. 41, No. 1, pp. 304-318.
30. Gómez-Aguilar, J.F. (2018), “Fundamental solutions to electrical circuits of non-integer order via fractional derivatives with and without singular kernels”, *The European Physical Journal Plus*, Vol. 133, No. 5, p.197.
31. Golmankhaneh, A.K. and Baleanu, D. (2016), “Non-local integrals and derivatives on fractal sets with applications”, *Open Physics*, Vol. 14, No. 1, pp. 542-548.
32. Khalili Golmankhaneh, A. and Baleanu, D. (2016), “New derivatives on the fractal subset of real-line”, *Entropy*, Vol. 18, No. 2, p.1.
33. Banchuin, R. (2022), “Nonlocal fractal calculus based analyses of electrical circuits on fractal set”, *COMPEL - The international journal for computation and mathematics in electrical and electronic engineering*, Vol. 41, No. 1, pp. 528-549.
34. Kochubei, A. N. (2011), “General fractional calculus, evolution equations, and renewal processes”, *Integral Equations and Operator Theory,* Vol. 71, No. 4, pp. 583–600.
35. Kochubei, A.N. (2019), “General fractional calculus”, In *Handbook of Fractional Calculus with Applications Vol 1: Basic Theory* (pp. 111–126), De Gruyter, US
36. Katsikadelis, J.T. (2015), “Generalized fractional derivatives and their applications to mechanical systems”, *Archive of Applied Mechanics*, Vol. 85, No. 9, pp.1307-1320.
37. Luchko, Y. and Yamamoto, M. (2020), “The general fractional derivative and related fractional differential equations”, *Mathematics*, Vol. 8, No. 12, p. 2115.

38. Hilfer, R. (2002), “Experimental evidence for fractional time evolution in glass forming materials”, *Chemical physics*, Vol. 284, No. 1-2, pp. 399-408.
39. Hilfer, R., Luchko, Y. and Tomovski, Z. (2009), “Operational method for the solution of fractional differential equations with generalized Riemann-Liouville fractional derivatives”, *Fracttional Calculus and Applied Analysis*, 12(3), pp.299-318.
40. Golmankhaneh, A. K., & Tunç, C. (2019), “Sumudu transform in fractal calculus”, *Applied Mathematics and Computation*, Vol. 350, pp. 386-401.
41. Caputo, M. and Fabrizio, M. (2015), “A new definition of fractional derivative without singular kernel”, *Progress in Fracttional Differentiation and Applications*, Vol. 1, No. 2, pp. 1-13.
42. Alqhtani, M. and Saad, K.M. (2022), “Numerical solutions of space-fractional diffusion equations via the exponential decay kernel”, *AIMS Mathematics*, Vol. 7, No. 4, pp. 6535-6549.
43. Hristov, J. (2017), “Derivatives with non-singular kernels from the Caputo-Fabrizio definition and beyond: Appraising analysis with emphasis on diffusion models”, *Frontiers in Fractional Calculus*, Vol. 1, pp.270-342.

Fig. 1: $i(t)$ vs. t: The proposed generic nonlocal fractional calculus (blue line) and Golmankhaneh-Baleanu nonlocal fractal calculus (red-dashed line)

Fig. 2: $x(t)$ vs. t: (2) (based on $k(t;\beta,m)$ as given by (3) (green line)) and (30)-(32) (based on $k(t;\beta,m)$ as given by (5) (magenta-dashed line))

Fig. 3: $f(t)$ vs. t: The unified generic nonlocal fractal derivative (blue line), The generic nonlocal Caputo fractal derivative (red dashed-line), the generic nonlocal Riemann–Liouville fractal derivative (green-dashed line)