Lacunary Statistical Convergence on L - Fuzzy Normed Space

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Abstract

The idea of lacunary statistical convergence sequences, which is a development of statistical convergence, is examined and expanded in this study on L - fuzzy normed spaces, which is a generalization of fuzzy spaces. On L - fuzzy normed spaces, the definitions of lacunary statistical Cauchy and completeness, as well as associated theorems, are provided. The link between lacunary statistical Cauchyness and lacunary statistical boundedness with regard to L - fuzzy norm is also shown.

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Lacunary Statistical Convergence on $\mathscr{L}-$ Fuzzy Normed Space

Reha Yapalı[1], Erdal Korkmaz[2], Muhammed Çınar[3] and Hüsamettin Çoşkun[4]

Abstract: The idea of lacunary statistical convergence sequences, which is a development of statistical convergence, is examined and expanded in this study on $\mathscr{L}-$ fuzzy normed spaces, which is a generalization of fuzzy spaces. On $\mathscr{L}-$ fuzzy normed spaces, the definitions of lacunary statistical Cauchy and completeness, as well as associated theorems, are provided. The link between lacunary statistical Cauchyness and lacunary statistical boundedness with regard to $\mathscr{L}-$ fuzzy norm is also shown.

1 Introduction

Academic investigations have proven for many years that kinds of convergence play an essential role in the field of mathematics analysis and function theory. Statistical convergence and its variants have been investigated and are now being explored in a variety of settings. [1, 3, 5–7, 14–16, 18, 22, 24–30]

$\mathscr{L}-$ fuzzy normed spaces are natural generalizations of normed spaces, fuzzy normed spaces and intuitionistic fuzzy normed spaces [2, 11–13, 17, 19, 31] based on some logical algebraic structures, which also enrich the notion of a $\mathscr{L}-$ fuzzy metric space [9, 10].

There is a vast literature of studies on this structure. In particular, some properties of a variant of the statistical convergence of sequences on $\mathscr{L}-$ fuzzy normed spaces are given [4, 5, 8–10, 20, 21, 23].

In this study, we give some results regarding lacunary statistical convergence of sequences and investigate the relationship between lacunary statistical convergent, lacunary statistical Cauchy and lacunary statistical bounded sequences, which will be newly introduced on $\mathscr{L}-$ fuzzy normed spaces.

In this regard, here we give a characterization of the lacunary statistical convergence of a sequence through the convergence of certain subsequences in the classical sense on $\mathscr{L}-$ fuzzy normed spaces. Then, we introduce and discuss the notion of a statistical bounded sequence on $\mathscr{L}-$ fuzzy normed spaces. And finally we reveal some implications between lacunary statistical convergence, lacunary statistical Cauchyness and lacunary statistical boundedness of a sequence on a $\mathscr{L}-$ fuzzy normed space.

The aim of the present paper is to investigate the lacunary statistical convergence, which was first introduced by Fridy, John Albert, and Cihan Orhan [6], on $L-$fuzzy normed spaces. Then we give a useful characterization for lacunary statistically convergent sequences on $\mathscr{L}-$ fuzzy normed spaces. Also we display an example such that our method of convergence is stronger than the usual convergence on $\mathscr{L}-$ fuzzy normed spaces.

2 Preliminaries

Preliminaries on $\mathscr{L}-$ fuzzy normed spaces are presented in this section.

Definition 2.1. [23] Assume that $K : [0, 1] \times [0, 1] \to [0, 1]$ is a function that satisfies the following

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1. $K(a,b) = K(a,b)$
2. $K(K(a,b),c) = K(a,K(b,c))$
3. $K(a,1) = K(1,a) = x$
4. $a \leq b, c \leq d$ then $K(a,c) \leq K(b,d)$

is known as a $t-$ norm.

Example 2.2. [23] K_1, K_2 and K_3 are the functions that given with,

$K_1(a,b) = \min\{a,b\}$,
$K_2(a,b) = ab$,
$K_3(a,b) = \max\{a+b-1, 0\}$

are the samples, which are well known of $t-$ norms.

Definition 2.3. [23] Assume that $\mathscr{L} = (L, \preceq)$ be a complete lattice and a set A be called the universe. On A, an $L-$ fuzzy set is defined with a function

$$X : A \to L.$$

On a set A, the family of all $L-$sets is denoted by L^A.

Two $L-$ sets on A intersect and union is shown by,

$$(C \cap D)(x) = C(x) \wedge D(x),$$

$$(C \cup D)(x) = C(x) \vee D(x)$$

for all $x \in A$. On the other hand, union and intersection of a family $\{B_i : i \in I\}$ of $L-$ fuzzy sets is given by

$$\Big(\bigcup_{i\in I} B_i\Big)(x) = \bigvee_{i\in I} B_i(x)$$

$$\Big(\bigcap_{i\in I} B_i\Big)(x) = \bigwedge_{i\in I} B_i(x)$$

respectively.

1_L and 0_L are the biggest and smallest elements of the full Lattice L, respectively. On a given lattice $(L, \preceq)$, we also illustrate the symbols $\succeq$, $\prec$, and $\succ$ in the obvious meanings.

Definition 2.4. [23] Let $\mathscr{L} = (L, \preceq)$ be a complete lattice. Therefore, $t-$ norm is a function $\mathscr{K} : L \times L \to L$ that satisfies the following for all $a, b, c, d \in L$:

1. $\mathscr{K}(a,b) = \mathscr{K}(b,a)$
2. $\mathscr{K}(\mathscr{K}(a,b),c) = \mathscr{K}(a,\mathscr{K}(b,c))$
3. $\mathscr{K}(a,1_L) = \mathscr{K}(1_L,a) = a$
4. $a \preceq b$ and $c \preceq d$, then $\mathscr{K}(a,c) \preceq \mathscr{K}(b,d)$.

Definition 2.5. [23] For sequences (a_n) and (b_n) on L such that $(a_n) \to a \in L$ and $(b_n) \to b \in L$, if the property that $\mathscr{K}(a_n,b_n) \to \mathscr{K}(a,b)$ satisfies on L, then a $k-$norm $\mathscr{K}$ on a complete lattice $\mathscr{L} = (L, \preceq)$ is called continuous.

Definition 2.6. [23] The function $\mathscr{N} : L \to L$ is defined as a negator on $\mathscr{L} = (L, \preceq)$ if,

$N_1)$ $\mathscr{N}(0_L) = 1_L$

$N_2)$ $\mathscr{N}(1_L) = 0_L$

$N_3)$ $a \preceq b$ implies $\mathscr{N}(b) \preceq \mathscr{N}(a)$ for all $a, b \in L$.

If in addition,

$N_4)$ $\mathscr{N}(\mathscr{N}(a)) = a$ for all $a \in L$.

Therefore, $\mathscr{N}$ is known as an involutive.

On the lattice $([0,1], \leq)$, the mapping $\mathscr{N}_s : [0,1] \to [0,1]$ defined as $\mathscr{N}_s(x) = 1 - x$ is very common sample of an involutive negator. In the concept of standard fuzzy sets, this type of negator is used. In addition, with the order

$$(\mu_1, \nu_1) \preceq (\mu_2, \nu_2) \iff \mu_1 \leq \mu_2 \quad and \quad \nu_1 \geq \nu_2$$

given the lattice $([0,1]^2, \preceq)$ with for all $i = 1, 2$, $(\mu_i, \nu_i) \in [0,1]^2$. Therefore, the function $\mathscr{N}_1 : [0,1]^2 \to [0,1]^2$,

$$\mathscr{N}_1(\mu, \nu) = (\nu, \mu)$$

in the sense of Atanassov [2], is known as a involutive negator. This type of negator are using in the notion of intuitionistic fuzzy sets.

Definition 2.7. [23] Let $\mathscr{L} = (L, \preceq)$ be a complete lattice and V be a real vector space. $\mathscr{K}$ be a continuous $t-$norm on $\mathscr{L}$ and ν be an $L-$set on $V \times (0, \infty)$ satisfying the following

(a) $\mu(a, t) \succ 0_L$ for all $a \in V, t > 0$

(b) $\mu(a, t) = 1_L$ for all $t > 0$ if and only if $a = \theta$

(c) $\mu(\alpha a, t) = \mu(a, \frac{t}{|\alpha|})$ for all $a \in V, t > 0 \quad and \quad \alpha \in \mathbb{R} - \{0\}$

(d) $\mathscr{K}(\mu(a,t), \mu(b,s)) \preceq \nu(a + b, t + s)$, for all $a, b \in V \quad and \quad t, s > 0$

(e) $\lim_{t\to\infty} \mu(a,t) = 1_L$ and $\lim_{t\to 0} \mu(a,t) = 0_L$ for all $a \in V - \{\theta\}$

(f) The functions $f_a : (0, \infty) \to L$ which is $f(t) = \mu(a, t)$ are continuous.

The triple $(V, \mu, \mathscr{K})$ is referred to as an $\mathscr{L}-$ fuzzy normed space or $\mathscr{L}-$ normed space in this context.

Definition 2.8. [23] A sequence (a_n) is said to be Cauchy sequence in a $\mathscr{L}-$ fuzzy normed space $(V, \mu, \mathscr{K})$ if, there exists $n_0 \in \mathbb{N}$ such that, for all $m, n > n_0$

$$\mu(a_n - a_m, t) \succ \mathscr{N}(\epsilon)$$

where $\mathscr{N}$ is a negator on $\mathscr{L}$, for each $\epsilon \in L - \{0_L\}$ and $t > 0$.

Definition 2.9. A sequence $a = (a_n)$ is said to be bounded with respect to fuzzy norm in a $\mathscr{L}-$ fuzzy normed space $(V, \mu, \mathscr{K})$, provided that, for each $r \in L - \{0_L, 1_L\}$ and $t > 0$,

$$\mu(a_n, t) \succ \mathscr{N}(r)$$

for all $n \in \mathbb{N}$.

We will first look at the concept of statistical convergence in $\mathscr{L}-$ fuzzy normed spaces. But first, let's give the concept of statistical convergence defined on real numbers [5].

If $K \subseteq \mathbb{N}$, the set of natural numbers, then $\delta\{A\}$ is the asymptotic density of A, is

$$\delta\{A\} := \lim_k \frac{1}{k} \big|\{n \leq k : n \in A\}\big|$$

the limit exists the cardinality of the set A is given by $|A|$.

If the set $K(\epsilon) = \{n \leq k : |a_n - l| > \epsilon\}$ has the asymptotic density zero, i.e.

$$\lim_{k} \frac{1}{k}\{n \leq k : |a_n - l| > \epsilon\} = 0,$$

then the sequence $a = (a_n)$ is known as a statistically convergent to the number ℓ. In this case, we will write $st - \lim a = \ell$.

Despite the notion that every convergent sequence converges to the same limit statistically, the contrary is not always true.

Definition 2.10. A sequence $a = (a_n)$ is statistically convergent to $l \in V$ with respect to ρ fuzzy norm in a $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{K})$ if provided that, for each $\epsilon \in L - \{0_L\}$ and $t > 0$,

$$\delta\{n \in \mathbb{N} : \rho(a_n - l, t) \not\succ \mathscr{N}(\epsilon)\} = 0$$

or equivalently

$$\lim_{m} \frac{1}{m}\{j \leq m : \rho(a_n - l, t) \not\succ \mathscr{N}(\epsilon)\} = 0.$$

In this case, we will write $st_{\mathscr{L}} - \lim a = l$.

Definition 2.11. A sequence $a = (a_k)$ is said to be statistically Cauchy with respect to fuzzy norm ρ in a $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{K})$, if provided that

$$\delta\{k \in \mathbb{N} : \rho(a_k - a_m, t) \not\succ \mathscr{N}(\epsilon)\} = 0$$

for each $\epsilon \in L - \{0_L\}$, $m \in \mathbb{N}$ and $t > 0$.

Definition 2.12. A sequence $a = (a_k)$ is said to be statistically bounded with respect to fuzzy norm ρ in a $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{K})$ if provided that there exists $r \in L - \{0_L, 1_L\}$ and $t > 0$ such that

$$\delta\{k \in \mathbb{N} : \rho(a_k, t) \not\succ \mathscr{N}(r)\} = 0$$

for each positive integer k.

3 Lacunary Statistical Convergence on $\mathscr{L}-$Fuzzy Normed Space

The notion of lacunary statistical convergence has been presented and investigated in many fields [6], [7]. We define and investigate lacunary statistical convergence on the $\mathscr{L}-$ fuzzy normed space in this section.

Definition 3.1. By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by q_r.

For any set $K \subseteq \mathbb{N}$, the number

$$\delta_\theta(K) = \lim_{r\to\infty} \frac{1}{h_r}|\{k \in I_r : k \in N\}|$$

is called the θ density of the set K, provided the limit exists.

A sequence $a = (a_k)$ is said to be lacunary statistically convergent or S_θ convergent to a number ℓ provided that for each $\epsilon > 0$,

$$\lim_{r\to\infty} \frac{1}{h_r}|\{k \in \mathbb{N} : |a_k - \ell| \geq \epsilon\}| = 0.$$

In other words, the set $K(\epsilon) = \{k \in \mathbb{N} : |a_k - \ell| \geq \epsilon\}$ has $\theta-$ density zero. In this case the number ℓ is called lacunary statistical limit of the sequence $x = (x_k)$ and we write $S_\theta - \lim_{r\to\infty} a_k = \ell$ or $a_k \to \ell(S_\theta)$.

Now, let us give the definition of lacunary statistical convergence on $\mathscr{L}-$ fuzzy norm space.

Definition 3.2. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then a sequence $a = (a_k)$ is lacunary statistically convergent to $\ell \in V$ with respect to μ fuzzy norm, provided that, for each $\epsilon \in L - \{0_L\}$ and $t > 0$,

$$\delta_\theta\{k \in \mathbb{N} : \rho(a_k - l, t) \nsucc \mathscr{N}(\epsilon)\} = 0.$$

In this scenario, $S_\theta^{\mathscr{L}} - \lim a = \ell$.

Definition 3.2. implies the following Proposition.

Proposition 3.3. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. Then, the following statements are equivalent, for every $\epsilon \in L - \{0_L\}$ and $t > 0$:

(a) $S_\theta^{\mathscr{L}} - \lim a = \ell$.

(b) $\delta_\theta\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc \mathscr{N}(\epsilon)\} = 0$.

(c) $\delta_\theta\{k \in \mathbb{N} : \rho(a_k - \ell, t) \succ \mathscr{N}(\epsilon)\} = 1$.

(d) $S_\theta^{\mathscr{L}} - \lim \rho(a_k - \ell, t) = 1_L$.

Proof. The equivalences between (a), (b) and (c) follow directly from the definitions.
$(a) \iff (d)$: Note that $S_\theta^{\mathscr{L}} - \lim a = \ell$ means that, for all $\varepsilon \in L - \{0_L\}$ and $t > 0$ we have

$$\delta_\theta\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc \mathscr{N}(\epsilon)\} = 0.$$

On the other hand, a local base for the open neighborhoods of $1_L \in L$ with respect to the order topology on the lattice $\mathscr{L} = (L, \leq)$, are the sets

$$(b, 1_L] = \{x \in L : b < x \leq 1_L\}$$

for each $b \in L - \{1_L\}$. $S_\theta^{\mathscr{L}} - \lim \rho(a_k - \ell, t) = 1_L$ if and only if, for any given $b \in L - \{1_L\}$,

$$\delta_\theta(\{k \in \mathbb{N} : \rho(a_k - \ell, t) \notin (b, 1_L]\}) = 0$$

or equivalently

$$\delta_\theta(\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc b\}) = 0.$$

Note that, the two statements

$\delta_\theta(\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc \mathscr{N}(\epsilon)\}) = 0$ for all $\varepsilon \in L - \{0_L\}$

$\delta_\theta(\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc b\}) = 0$ for all $b \in L - \{1_L\}$

are equivalent since for each $\varepsilon \in L - \{0_L\}$ we can choose $b \in L - \{1_L\}$ as $b = \mathscr{N}(\varepsilon)$ and conversely for each $b \in L - \{1_L\}$ we can choose $\varepsilon \in L - \{0_L\}$ as $\varepsilon = \mathscr{N}(b)$, so that $b = \mathscr{N}(\mathscr{N}(b)) = \mathscr{N}(\varepsilon)$. This proves that (a) is equivalent to (d). □

Theorem 3.4. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. If $\lim a = l$, then $S_\theta^{\mathscr{L}} - \lim a = l$.

Proof. Let $\lim a = l$. Then for every $\epsilon \in L - \{0_L\}$ and $t > 0$, there is a number $k_0 \in \mathbb{N}$ such that

$$\rho(a_k - \ell, t) \succ \mathscr{N}(\epsilon)$$

for all $k \geq k_0$. Therefore,

$$\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc \mathscr{N}(\epsilon)\}$$

has at most finitely many terms. We can see right away that any finite subset of the natural numbers has double $\theta-$ density zero. Hence,

$$\delta_\theta\{k \in \mathbb{N} : \rho(a_k - \ell, t) \nsucc \mathscr{N}(\epsilon)\} = 0.$$

□

As can be seen in the following example, the converse of this theorem need not be true in general.

Example 3.5. Let $V = \mathbb{R}$ and $\mathscr{L} = (\mathscr{P}(\mathbb{R}^+), \subseteq)$, the lattice of all subsets of the set of non-negative real numbers. Define the function $\rho : \mathbb{R} \times (0, \infty) \to \mathscr{P}(\mathbb{R}^+)$ with

$$\rho(x, t) = \{r \in \mathbb{R}^+ : \mid x \mid < \frac{t}{r}\}.$$

Then, $(\mathbb{R}, \nu, \mathscr{P}(\mathbb{R}^+))$ is a $\mathscr{L}-$ fuzzy normed space. On this space, consider the sequence $a = (a_k)$ given by the rule

$$a_k = \begin{cases} 1, & for\ k \in (k_r - \ln(h_r), k_r],\ r \in \mathbb{N} \\ 0, & otherwise; \end{cases}$$

Then,

$$\lim_{r\to\infty} \delta_\theta = \lim_{r\to\infty} \frac{\ln(h_r)}{h_r} = 0$$

which means $S_\theta^{\mathscr{L}} - \lim a = \ell \in \mathbb{R}$, while the sequence itself is not convergent.

Theorem 3.6. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$fuzzy normed space. If a sequence $a = (a_k)$ is lacunary statistically convergent with respect to the $\mathscr{L}-$ fuzzy norm ρ, then $S_\theta^{\mathscr{L}} -$ limit is unique.

Proof. Suppose that $S_\theta^{\mathscr{L}} - \lim a = \ell_1$ and $S_\theta^{\mathscr{L}} - \lim a = \ell_2$, where $\ell_1 \neq \ell_2$. For any given $\epsilon \in L - \{0_L\}$ and $t > 0$, we can choose a $r \in L - \{0_L\}$ such that

$$\mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\epsilon).$$

Define the following sets

$$K_1 = \{k \in \mathbb{N} : \rho(a_k - \ell_1, t)) \nsucc \mathscr{N}(r)\}$$

and

$$K_2 = \{k \in \mathbb{N} : \rho(a_k - \ell_2, t)) \nsucc \mathscr{N}(r)\}$$

for any $t > 0$. Since for elements of the set $K(\epsilon, t) = K_1(\epsilon, t) \cup K_2(\epsilon, t)$ we have

$$\rho(\ell_1 - \ell_2, t) \succeq \mathscr{K}(\rho(a_k - \ell_1, \frac{t}{2}), \rho(a_k - \ell_2, \frac{t}{2})) \succ \mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\epsilon).$$

it can be concluded that $\ell_1 = \ell_2$. $\square$

Theorem 3.7. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. Then, $S_\theta^{\mathscr{L}} - \lim a = \ell$ if and only if there exists a subset $K \subset \mathbb{N}$ such that $\delta_\theta(K) = 1$ and $\mathscr{L} - \lim_{n\to\infty} a_n = \ell$.

Proof. Suppose that $S_\theta^{\mathscr{L}} - \lim a = \ell$. Let (ϵ_n) be a sequence in $L - \{0_L\}$ such that $\mathscr{N}(\epsilon_n) \to 1_L$ in L increasingly, and for any $t > 0$ and $k \in \mathbb{N}$, let

$$K(k) = \{n \in \mathbb{N} : \rho(a_n - \ell, t) \succ \mathscr{N}(\epsilon_k)\}$$

Then observe that, for any $t > 0$ and $k \in \mathbb{N}$,

$$K(k + 1) \subset K(k).$$

Since $S_\theta^{\mathscr{L}} - \lim a = \ell$, it is obvious that

$$\delta_\theta\{K(k)\} = 1, (k \in \mathbb{N} \text{ and } t > 0).$$

Now let p_1 be an arbitrary number of $K(1)$. Then there exist numbers $p_2 \in K(2)$, $p_2 > p_1$, such that for all $n > p_2$,

$$\frac{1}{h_n} |n \in I_n : \rho(a_n - \ell, t) \succ \mathscr{N}(\epsilon_2)\}| > \frac{1}{2}.$$

Further, there is a number $p_3 \in K(3), p_3 > p_2$ such that for all $n > p_3$,

$$\frac{1}{h_n}|\{n \in I_n : \rho(a_n - \ell, t) \succ \mathscr{N}(\epsilon_3)\}| > \frac{2}{3}$$

and so on. So, we can construct, by induction, an increasing index sequence $(p_k)_{k\in\mathbb{N}}$ of the natural numbers such that $p_k \in K(k)$ and that the following statement holds for all $n > p_k$:

$$\frac{1}{h_n}|\{n \in \mathbb{N} : \rho(a_n - \ell, t) \succ \mathscr{N}(\epsilon_k)\}| > \frac{k-1}{k}.$$

Now we construct increasing index sequence as follows:

$$K := \{n \in \mathbb{N} : 1 < n < p_1\} \cup \Big[\bigcup_{k\in\mathbb{N}} \{n \in K(k) : p_k \leq n < p_{k+1}\}\Big]$$

Hence it follows that $\delta_\theta(K) = 1$. Now let $\varepsilon \succ 0_L$ and choose a positive integer k such that $\varepsilon_k \prec \varepsilon$. Such a number k always exists since $(\varepsilon_n) \to 0_L$. Assume that $n \geq p_k$ and $n \in K$. Then by the definiton of K, there exists a number $m \geq k$ such that $p_m \leq n < p_{m+1}$ and $n \in K(k)$. Hence, we have, for every $\varepsilon \succ 0_L$

$$\rho(a_n - \ell, t) \succ \mathscr{N}(\varepsilon_k) \succ \mathscr{N}(\varepsilon)$$

for all $n \geq p_k$ and $n \in K$ and this means

$$\mathscr{L} - \lim_{n\to\infty} a_n = \ell.$$

Conversely, suppose that there exists an increasing index sequence $K = (k_n)_{n\in\mathbb{N}}$ of natural numbers such that $\delta_\theta(K) = 1$ and $\mathscr{L} - \lim_{n\to\infty} a_n = \ell$. Then, for every $\varepsilon \succ 0_L$ there is a number n_0 such that for each $n \geq n_0$ the inequality $\rho(a_n - \ell, t) \succ \mathscr{N}(\varepsilon)$ holds. Now define

$$M(\varepsilon) := \{n \in \mathbb{N} : \rho(a_n - \ell, t) \not\succ \mathscr{N}(\varepsilon)\}.$$

Then there exists an $n_0 \in \mathbb{N}$ such that

$$M(\epsilon) \subseteq \mathbb{N} - (K - \{k_n : n \leq n_0\}).$$

Since $\delta_\theta(K) = 1$, we get $\delta_\theta\{(\mathbb{N}) - (K - \{k_n : n \leq n_0\})\} = 0$, which yields that $\delta_\theta\{M(\varepsilon)\} = 0$. In other words, $S_\theta^{\mathscr{L}} - \lim a = l$. □

4 Lacunary Statistical Cauchy and Completeness

Lacunary statistically Cauchy sequences with respect to $\mathscr{L}-$ fuzzy normed space will be given in this section, and also a new concept of lacunary statistical completeness will be defined.

Definition 4.1. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. Then a sequence $a = (a_k)$ is said to be lacunary statistically Cauchy with respect to $\mathscr{L}-$ fuzzy norm ρ, if for every $\epsilon \in L - \{0_L\}$ and $t > 0$, there exist $N = N(\epsilon)$ such that for all $m, k \geq N$ provided that

$$\delta_\theta\{k \in \mathbb{N} : \rho(a_k - a_m, t) \not\succ \mathscr{N}(\epsilon)\} = 0.$$

Theorem 4.2. Every lacunary statistically convergent sequence is lacunary statistically Cauchy.

Proof. Let $a = (a_k)$ be a lacunary statistical convergent to ℓ with respect to $\mathscr{L}-$ fuzzy norm ρ, in other saying $S_\theta^{\mathscr{L}} - \lim a = l$. For a given $\varepsilon > 0$, choose $r > 0$ such that,

$$\mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\epsilon).$$

For $t > 0$ we can write,

$$A = \{k \in \mathbb{N} : \rho(a_k - \ell, \frac{t}{2}) \succ \mathscr{N}(r)\}.$$

Take $m \in A$. Obviously, $\rho(a_m - \ell, \frac{t}{2}) \succ \mathscr{N}(r)$. Also since,

$$\rho(\ell - a_m, \frac{t}{2}) = \rho(a_m - \ell, \frac{\frac{t}{2}}{|-1|}) = \rho(a_m - \ell, \frac{t}{2}) \succ \mathscr{N}(\varepsilon)$$

we have,

$$\begin{aligned} \rho(a_k - a_m, t) &= \rho\big((a_k - \ell) + (\ell - a_m), \frac{t}{2} + \frac{t}{2}\big) \\ &\succ \mathscr{K}\big(\rho(a_k - \ell, \frac{t}{2}), (\rho(\ell - a_m, \frac{t}{2})\big) \\ &\succ \mathscr{K}\big(\mathscr{N}(r), \mathscr{N}(r)\big) \\ &\succ \mathscr{N}(\varepsilon). \end{aligned}$$

If we define a set $B = \{k \in \mathbb{N} : \rho(a_k - a_m, t) \succ \mathscr{N}(\varepsilon)\}$, then $A \subseteq B$. Since $\delta_\theta(A) = 1$, $\delta_\theta(B) = 1$. Thus, the theta density of complement of B equals to zero,i.e. $\delta_\theta(B^c) = 0$, which means $a = (a_k)$ is lacunary statistical Cauchy. □

Definition 4.3. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. $(V, \rho, \mathscr{K})$ is said to be complete if every Cauchy sequence is convergent with respect to $\mathscr{L}-$ fuzzy norm ρ.

Definition 4.4. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. $(V, \rho, \mathscr{K})$ is said to be lacunary statistical complete if every lacunary statistical Cauchy sequence is lacunary statistical convergent with respect to $\mathscr{L}-$ fuzzy norm ρ.

Theorem 4.5. Every $\mathscr{L}-$ fuzzy normed space is lacunary statistically complete but not complete in general.

Proof. Let $a = (a_k)$ be a lacunary statistical Cauchy, but not lacunary statistical convergent with respect to $\mathscr{L}-$ fuzzy norm ρ. For a given $\epsilon > 0$ and $t > 0$, choose $r > 0$ such that

$$\mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \succ \mathscr{N}(\epsilon).$$

Therefore,

$$\begin{aligned} \rho(a_k - a_n, t) &\succ \mathscr{K}(\rho(a_k - \ell, \frac{t}{2}), \rho(a_k - \ell, \frac{t}{2})) \\ &\succ \mathscr{K}(\mathscr{N}(r), \mathscr{N}(r)) \\ &\succ \mathscr{N}(\epsilon). \end{aligned}$$

If we take a set $A = \{k \in \mathbb{N} : \rho(a_k - a_n, t) \succ \epsilon\}$, then $\delta_\theta(A) = 1$ and thus $\delta_{\theta^c} = 1$. Since a was lacunary statistical Cauchy with respect to $\mathscr{L}-$ fuzzy norm ρ, this is a contradiction. So, a has to be lacunary statistical convergent. Therefore, every $\mathscr{L}-$ fuzzy normed space is lacunay statistical complete.

In order to show that an $\mathscr{L}-$ fuzzy normed space is not complete in general, we give the following example:

Example 4.6. Let $X = C[0, 1], L = [0, 1]$ and

$$\rho(f, t) = \frac{t}{\int_0^1 (t + f(x))dx}.$$

Then, (X, ρ, L) is $\mathscr{L}-$ fuzzy normed space. However, in this space if we take (f_n) where,

$$f_n : [0, 1] \to \mathbb{R}, f_n(x) = x^n.$$

It is obvious that even though the sequence (f_n) is Cauchy, not convergent with respect to $\mathscr{L}-$ fuzzy norm ρ.

Theorem 4.7. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space. Then, for any sequence $a = (a_k)$, the following conditions are equivalent:

(a) a is lacunary statistical convergent with respect to $\mathscr{L}-$ fuzzy norm ρ.

(b) a is lacunary statistical Cauchy with respect to $\mathscr{L}-$ fuzzy norm ρ.

(c) $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{K})$ is lacunary statistical complete.

(d) There exists an increasing index sequence $K = (k_n)$ of natural numbers such that $\delta_\theta(K) = 1$ and the subsequence (x_{k_n}) is a lacunary statistical Cauchy with respect to $\mathscr{L}-$ fuzzy norm ρ.

□

5 The Relationship Between Lacunary Statistical Cauchy and Lacunary Statistical Bounded Sequences

In this section, the notion of lacunary statistical bounded sequences will be defined and relationship between lacunary statistical Cauchy and lacunary bounded sequences will be given.

Definition 5.1. Let $(V, \rho, \mathscr{K})$ be a $\mathscr{L}-$ fuzzy normed space and $a = (a_k)$ be a sequence. Then $a = (a_k)$ is said to be lacunary statistically bounded with respect to $\mathscr{L}-$ fuzzy norm ρ, provided that there exists $r \in L - \{0_L, 1_L\}$ and $t > 0$ such that

$$\delta_\theta\{k \in \mathbb{N} : \rho(a_k, t) \nsucc \mathscr{N}(r)\} = 0$$

for each positive integer k.

Theorem 5.2. Every bounded sequence on a $\mathscr{L}-$ fuzzy normed space $(V, \rho, \mathscr{K})$, is lacunary statistically bounded.

Proof. Let $a = (a_k)$ be a bounded sequence on $(V, \rho, \mathscr{K})$. Then there exist $t > 0$ and $r \in L - \{0_L, 1_L\}$ such that $\rho(a_k, t) \succ \mathscr{N}(r)$. In that case we have,

$$\{k \in \mathbb{N} : \rho(a_k, t) \nsucc \mathscr{N}(r)\} = \emptyset$$

which yields

$$\delta_\theta\{k \in \mathbb{N} : \rho(a_k, t) \nsucc \mathscr{N}(r)\} = 0.$$

Thus, (a_k) is lacunary statistically bounded. □

However the converse of this theorem does not hold in general as seen in the example below.

Example 5.3. Let $V = \mathbb{R}$ and $\mathscr{L} = (L, \leq)$ where L is the set of non-negative extended real numbers, that is $L = [0, \infty]$. Then $0_L = 0, 1_L = \infty$. Define a $\mathscr{L}-$ fuzzy norm ρ on V by $\rho(x, t) = \frac{t}{|x|}$ for $x \neq 0$ and $\rho(0, t) = \infty$ for each $t \in (0, \infty)$. Consider the $t-$ norm $\mathscr{K}(a, b) = \min\{a, b\}$ on $\mathscr{L}$. Given the sequence,

$$x_n = \begin{cases} n, & \text{if n is prime number} \\ \frac{1}{\tau(n)-2}, & \text{otherwise.} \end{cases}$$

where $\tau(n)$ denotes the number of positive divisors of n. Note that (x_n) is not bounded since for each $t > 0$ and $r \in L - \{0, \infty\}$, for any prime number n such that $rt \leq n$ we have

$$\rho(x_n, t) = \rho(n, t) = \frac{t}{|\, n \,|} = \frac{t}{n} \nsucc \frac{1}{r} = \mathscr{N}(r).$$

However for $t = 1$ and any non-prime integer n, $r = 2$ satisfies

$$\rho(x_n, 1) = \rho(\frac{1}{\tau(n) - 2}, 1) = \frac{1}{|\frac{1}{\tau(n)-2}|} = |\tau(n) - 2| > \frac{1}{2} = \mathscr{N}(r)$$

since $\tau(n) \neq 2$ for any non-prime n.

Let $k_0 = 0$ and for $n \geq 1$, $k_n = 10^{n-1}$ and p_n be the number of primes in I_n, where for $n > 1$, $I_n = (10^{n-2}, 10^{n-1}]$ and $I_1 = (0, 1]$. Therefore, we have

$$\delta_\theta\{k \in \mathbb{N} : \rho(x_k, 1) \not\succ \mathscr{N}(2)\} = \lim_{n\to\infty} \frac{p_n}{h_n} = \lim_{n\to\infty} \frac{p_n}{9.10^{n-2}}.$$

Assume that, $\delta_\theta\{k \in \mathbb{N} : \nu(x_k, 1) \not\succ \mathscr{N}(2)\} > 0$, then $\lim_{n\to\infty} \frac{p_n}{9.10^{n-2}} = \varepsilon$ implies that $\lim_{n\to\infty} \frac{p_n}{10^{n-2}} = 9\varepsilon$. Therefore, $\lim_{n\to\infty} \frac{p_n}{10^{n-1}} = \frac{9\varepsilon}{10}$. Hence,

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{p_k}{10^{n-1}} \geq \lim_{n\to\infty} \frac{p_n}{10^{n-1}} = \frac{9\varepsilon}{10} > 0.$$

However, since the limit on the left is the density of primes in natural numbers, this limit should be 0 according to the prime number theorem. In other words, this is a contradiction. So

$$\delta_\theta\{k \in \mathbb{N} : \rho(x_k, 1) \not\succ \mathscr{N}(2)\} = 0$$

suggesting that (x_n) is lacunary statistically bounded.

Theorem 5.4. Every lacunary statistically Cauchy sequence on a $\mathscr{L}-$fuzzy normed space $(V, \rho, \mathscr{K})$ is lacunary statistically bounded.

Proof. Let $a = (a_n)$ be a lacunary statistically Cauchy on $(V, \rho, \mathscr{K})$. Then for every $\epsilon \in L - \{0_L\}$ and $t > 0$, there exist $N = N(\epsilon)$ such that for all $m, k \geq N$ provided that

$$\delta_\theta\{n \in \mathbb{N} : \rho(a_n - a_k, t) \not\succ \mathscr{N}(\epsilon)\} = 0.$$

Then,

$$\delta_\theta\{n \in \mathbb{N} : \rho(a_n - a_k, t) \succ \mathscr{N}(\epsilon)\} = 1.$$

Consider a number $n \in \mathbb{N}$ such that $\rho(a_n - a_k, 1) \succ \mathscr{N}(\epsilon)$. Then for $t = 2$

$$\rho(a_n, 2) = \rho(a_n - a_k + a_k, 2) \succ \mathscr{K}(\rho(a_n - a_k, 1), \rho(a_k, 1)) \succ \mathscr{K}(\mathscr{N}(\epsilon), \rho(a_k, 1)).$$

Say $r := \mathscr{N}(\mathscr{K}(\mathscr{N}(\epsilon), \rho(a_k, 1)))$. Then

$$\rho(a_n, 2) \succ \mathscr{K}(\mathscr{N}(\epsilon), \rho(a_k, 1)) = \mathscr{N}(r),$$

which implies

$$\delta_\theta\{n \in \mathbb{N} : \rho(a_n, 2) \succ \mathscr{N}(r)\} = 1$$

or equivalently

$$\delta_\theta\{n \in \mathbb{N} : \rho(a_n, 2) \not\succ \mathscr{N}(r)\} = 0$$

giving lacunary statistically boundedness of (a_n). □

6 Conclusion

In this study, the concepts of lacunary statistical convergence, lacunary statistical Cauchy, lacunary statistical completeness and lacunary statistical limitation on $L-$ fuzzy normed spaces, which are a generalization of fuzzy normed spaces, are given, and the relationships between them are given. Some new ideas have also been defined, as well as some of the links between them. These findings may be combined with the lattice structure and the normed space structure, allowing a wider range of topological vector spaces to benefit from the convenience afforded by a variant of the notion of norm. In another study, these relationships can be examined by transferring them to double and triple sequences.

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