Existence and uniqueness of solutions for stochastic differential equations with locally one-sided Lipschitz condition

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Existence and uniqueness of solutions for stochastic differential equations with locally one-sided Lipschitz condition

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Abstract

This paper investigates stochastic differential equations (SDEs) with locally one-sided Lipschitz coefficients. Apart from the local one-sided Lipschitz condition, a more general condition is introduced to replace the monotone condition. Then, in terms of the Euler's polygonal line method, the existence and uniqueness of solutions for SDEs is established. In the meanwhile, the pth moment boundedness of solutions is also provided.

Keywords: Stochastic differential equations; Locally one-sided Lipschitz; Existence and uniqueness; Moment boundedness

1. Introduction

Stochastic differential equations (SDEs) have been widely applied in many fields, such as biology, economics and physics for modeling (see, e.g., [2, 4, 7, 9, 14, 18]). More and more people have showed their interests in SDEs. So far, many results of solutions for SDEs have been obtained, such as the existence and uniqueness (see, e.g.,[3, 6, 8, 11, 12, 16]), Markov property (see, e.g.,[15]) and even the long-term behavior (see, e.g.,[20]). In addition, to describe a wide variety of natural and man-made systems precisely, various types of SDEs are developed (see,e.g.,[11, 17]). And the theory of thses SDEs has always been a focus.

On the other hand, one of the popular topics of SDEs is the existence and uniqueness of solutions. Generally, the classical existence and uniqueness theorem for SDEs requires the coefficients to satisfy the global Lipschitz condition(see,e.g., [5, 19]). Under the local Lipschitz condition and the linear growth condition, Arnold [1] has showed the existence of the unique solutions for SDEs. However, there are many interesting SDEs that their coefficients are only superlinear. For such SDEs, Mao [11] has derived that there exists a unique regular solution under locally Lipschitz condition and the monotone condition. Based on these existence and uniqueness results for the classical SDEs, many

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authors have studied the existence and uniqueness problems for other types of SDEs. For instance, Zvonkin [21] have investigated the strong solutions of SDEs with singular coefficients. Mao and Yuan [13] have introduced the existence and uniqueness of solutions for SDEs with Markovian swithching.

Furthermore, although many SDEs have been showed that they each have a unique solution, it is important to determine precisely under which conditions one obtains a unique solution for SDEs. Compared with more restrictive conditions, general conditions can provide the existence and uniqueness of solutions for a larger class of SDEs. By using the Euler method, Krylov [8] have established the existence and uniqueness theorem under the monotone condition and a more general condition which is known as local one-sided Lipschitz condition. Then, Gyöngy and Sabanis [3] have developed this result to stochastic differential delay equations. Recently, Ji and Yuan [6] have established the existence and uniqueness result for neutral stochastic differential delay equations. In this paper, inspired by Li et al. [10] and Krylov [8], we aim to study the existence and uniqueness of solutions for SDEs under weaker conditions compared with what we have mentioned above. Also, we can obtain the pth moment boundedness. And the main contribution of this paper is that we have included the case of $0 < p < 2$ in our conditions.

The rest organization of this paper is as follows: In section 2, some notations and preliminaries are introduced. In section 3, the existence and uniqueness of solutions is provided by deriving a localization lemma, and the pth moment is further estimated. In section 4, an example is given to illustrate our results.

2. Notations and Preliminaries

In this paper, let $(\Omega,\ \mathcal{F},\ \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all P-null sets). Let $\mathbb{N}$ be the set of natural numbers and $m, d \in \mathbb{N}$. Let $\{B(t)\}_{t\geq 0}$ be a standard m-dimensional Brownian motion defined on the probability space. Let $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, $\mathbb{R}^d$ be d-dimensional Euclidean space, and $\mathbb{R}^{d\times m}$ be the space of real $d \times m$-matrices. If $x \in \mathbb{R}^d$, then $|x|$ is the Euclidean norm. For any matrix A, define its trace norm by $|A| = \sqrt{\text{trace}(AA^T)}$. If A is a vector or matrix, A^T denotes its transpose. Moreover, for any $a \in \mathbb{R}$ and $b \in \mathbb{R}$, define $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For a set G, let $I_G(x) = 1$ if $x \in G$ and otherwise 0. Let $\inf \emptyset = \infty$(as usual $\emptyset$ denotes the empty set). For any $x \in \mathbb{R}$, let $\lfloor x \rfloor$ be the integer part of x.

Let $T \in [0, \infty)$. For $p \in (0, \infty)$, let $L^p = L^p(\Omega; \mathbb{R}^d)$ be the family of $\mathbb{R}^d$-valued random variables Z with $\mathbb{E}\big[|Z(\omega)|^p\big] < +\infty$. Let

$$f : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^d,\ g : \mathbb{R}^d \times \mathbb{R}^+ \to \mathbb{R}^{d\times m}$$

be Borel-measurable and continuous mappings. And $\mathcal{L} = \mathcal{L}([0, T]; \mathbb{R})$ denotes the set of $\mathbb{R}$-valued nonnegative integrable functions on $[0, T]$. Furthermore, we consider a d-dimensional SDE described by

$$\mathrm{d}X(t) = f(X(t),t)\mathrm{d}t + g(X(t),t)\mathrm{d}B(t), \quad t \in [0,T], \tag{2.1}$$

with the initial value $X(0) = X_0 \in L^p$.

Moreover, in order for the main results we impose the following assumptions.

Assumption 1. *For any $R,\ T \in [0,\infty)$,*

$$\int_0^T \sup_{|x|\le R} \left\{|f(x,t)| \vee |g(x,t)|^2\right\}\mathrm{d}t < \infty \ a.s.$$

Assumption 2. *For any $R,\ T \in [0,\infty)$, there exists a $K_R \in \mathcal{L}$ such that*

$$2(x_1 - x_2)^T\big(f(x_1,t) - f(x_2,t)\big) + \big|g(x_1,t) - g(x_2,t)\big|^2 \le K_R(t)|x_1 - x_2|^2,$$

for all $t \in [0,T]$, $x_1, x_2 \in \mathbb{R}^d$ and $|x_1| \vee |x_2| \le R$.

Assumption 3. *For any $T \in [0,\infty)$ and $p \in (0,\infty)$, there exists a $K \in \mathcal{L}$ such that*

$$\big(1 + |x|^2\big)\big(2x^T f(x,t) + |g(x,t)|^2\big) - (2-p)\big|x^T g(x,t)\big|^2 \le K(t)\big(1 + |x|^2\big)^2,$$

for all $t \in [0,T]$ and $x \in \mathbb{R}^d$.

For the sake of simplicity, throughout the paper, unless otherwise stated, C denotes a generic constant, whose value may be changed in different appearance.

3. Existence and Uniqueness of Solution

In this section, we shall show that there exists a unique regular solution to (2.1). And according to [6, 8, 16], we prepare a localization lemma below.

Lemma 3.1. *Let Assumptions 1-3 hold with $p > 0$ and $T \in [0,\infty)$. For $n \in \mathbb{N}$, $\{X^n(t)\}_{t\in[0,T]}$ is a continuous, $\mathbb{R}^d$-valued, and $\mathcal{F}_t$-adapted process on $(\Omega,\ \mathcal{F},\ \mathbb{P})$ such that for $X^n(0) = X(0)$,*

$$\mathrm{d}X^n(t) = f(X^n(t) + P^n(t),t)\mathrm{d}t + g(X^n(t) + P^n(t),t)\mathrm{d}B(t), \quad t \in [0,T],$$

where $P^n(t)$ is a progressively measurable process. Moreover, for $n \in \mathbb{N}$ and $R \in [0,\infty)$, suppose that there exists a function $r : [0,\infty) \to [0,\infty)$ such that $\lim_{R\to\infty} r(R) = \infty$, and let $\tau_n(R)$ be $\mathcal{F}_t$-stopping times such that

- *(i)* $|X^n(t)| + |P^n(t)| \le R$ *if* $t \in [0,\tau_n(R)]$ *a.s.*
- *(ii)* $\lim_{n\to\infty} \mathbb{E}\big[\int_0^{T\wedge\tau_n(R)} |P^n(t)|\mathrm{d}t\big] = 0$ *for all* $T \in [0,\infty)$.
- *(iii)* *For any* $T \in [0,\infty)$,

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}}\, \mathbb{P}\Big\{\tau_n(R) \le T,\ \sup_{t\in[0,\tau_n(R)]} |X^n(t)| < r(R)\Big\} = 0.$$

Then for any $T \in [0,\infty)$, we have

$$\sup_{t\in[0,T]} |X^n(t) - X^m(t)| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n,m \to \infty. \tag{3.1}$$

Proof. We borrow the techniques from [16] mainly and divide the proof into 2 steps.

Step 1. For $R \in [0,\infty)$ and $t \in [0,T]$, from Assumption 1 we assume that

$$\sup_{|x|\le R} \left\{|f(x,t)| \vee |g(x,t)|^2\right\} \le K_R(t),$$

(Otherwise, we regard $K_R(t)$ as the maximum of $K_R(t)$ and the integrand in Assumption 1). Fix $R \in [0,\infty)$ and define the $\mathcal{F}_t$-stopping time

$$\tau(R,u) = \inf\left\{t \ge 0 | \alpha_R(t) > u\right\},\ u \in (0,\infty),$$

where $\alpha_R(t) = \int_0^t K_R(s)\mathrm{d}s < \infty$. Clearly, $\tau(R,u) \uparrow \infty$ as $u \to \infty$. In particular, there exists $u(R) \in (0,\infty)$ such that

$$\mathbb{P}\left\{\tau(R,u(R)) \le R\right\} \le \frac{1}{R}.$$

Now, we let $\tau(R) = \tau(R,u(R))$, then $\tau(R) \to \infty$ in probability as $R \to \infty$ and $\alpha_R(t\wedge\tau(R)) \le u(R)$. Moreover, referring to [6] and [16], it is easy to prove that all three conditions (i)-(iii) still hold if we replace $\tau_n(R)$ by $\tau_n(R) \wedge \tau(R)$. So we can further assume that $\tau_n(R) \le \tau(R)$, then we have $\alpha_R(t \wedge \tau_n(R)) \le u(R)$. For a fixed $R \in [0,\infty)$, we define

$$\lambda_n^R(t) = \int_0^t |P^n(s)| K_R(s)\mathrm{d}s, \quad t \in [0, T\wedge\tau_n(R)],\ n \in \mathbb{N},$$

and $\tau_{(n,m)}(R) = \tau_n(R) \wedge \tau_m(R)$ for $m,n \in \mathbb{N}$. Then we can obtain

$$\lim_{n\to\infty} \mathbb{E}\left[\lambda_n^R(T \wedge \tau_n(R))\right] = 0. \tag{3.2}$$

And under Assumption 2, we have

$$\sup_{t\in[0,T\wedge\tau_{(n,m)}(R)]} |X^n(t) - X^m(t)| \xrightarrow{\mathbb{P}} 0, \quad \text{as } n,m \to \infty. \tag{3.3}$$

We omit the proof of (3.2) and (3.3) there as the reader can refer to [6] and [16] for more details.

Step 2. In order for (3.1), we need to show that

$$\lim_{R\to\infty} \overline{\lim_{n\to\infty}} \mathbb{P}\{\tau_n(R) \le T\} = 0,$$

for any given $T \in [0,\infty)$. For $t \in [0,T]$, let κ be a negative constant and define

$$\psi(t) = \exp\left(\kappa\beta(t) - |X(0)|\right),$$

where $\beta(t) = \int_0^t K(s)\mathrm{d}s$. For $t \in [0, T\wedge\tau_n(R)]$, applying the Itô formula, we have

$$\left(1 + |X^n(t)|^2\right)^{\frac{p}{2}} \psi(t)$$

$$
\begin{aligned}
=&\big(1+|X(0)|^2\big)^{\frac{p}{2}}\psi(0)+\int_0^t \kappa K(s)\big(1+|X^n(s)|^2\big)^{\frac{p}{2}}\psi(s)\mathrm{d}s+\frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}}\\
&\times\bigg\{\big(1+|X^n(t)|^2\big)\Big(2\big(X^n(t)\big)^T f(X^n(s)+P^n(s),s)+|g(X^n(s)+P^n(s),s)|^2\Big)\\
&\qquad-(2-p)\big|\big(X^n(t)\big)^T g(X^n(s)+P^n(s),s)\big|^2\bigg\}\mathrm{d}s+J_n^R(t),
\end{aligned}
$$

where

$$
J_n^R(t)=p\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}}\big(X^n(t)\big)^T g(X^n(s)+P^n(s),s)\mathrm{d}B(s).
$$

Then, we further write that

$$
\begin{aligned}
&\big(1+|X^n(t)|^2\big)^{\frac{p}{2}}\psi(t)\\
=&\big(1+|X(0)|^2\big)^{\frac{p}{2}}\psi(0)+\int_0^t \kappa K(s)\big(1+|X^n(s)|^2\big)^{\frac{p}{2}}\psi(s)\mathrm{d}s\\
&+\frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}}\bigg\{\Big(1+|X^n(s)+P^n(s)|^2-2\big(X^n(s)+P^n(s)\big)^T P^n(s)+|P^n(s)|^2\Big)\\
&\times\Big(2\big(X^n(s)+P^n(s)\big)^T f(X^n(s)+P^n(s),s)+|g(X^n(s)+P^n(s),s)|^2\\
&-2(P^n(s))^T f(X^n(s)+P^n(s),s)\Big)-(2-p)\bigg[\big|\big(X^n(s)+P^n(s)\big)^T g(X^n(s)+P^n(s),s)\big|^2\\
&-2(P^n(s))^T\big(X^n(s)+P^n(s)\big)|g(X^n(s)+P^n(s),s)|^2\\
&+\big|(P^n(s))^T g(X^n(s)+P^n(s),s)\big|^2\bigg]\bigg\}\mathrm{d}s+J_n^R(t)\\
=&\big(1+|X(0)|^2\big)^{\frac{p}{2}}\psi(0)+\int_0^t \kappa K(s)\big(1+|X^n(s)|^2\big)^{\frac{p}{2}}\psi(s)\mathrm{d}s+\sum_{i=1}^5 J_i(t)+J_n^R(t),
\end{aligned}
\tag{3.4}
$$

where

$$
\begin{aligned}
J_1(t)=&\frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}}\bigg\{\big(1+|X^n(s)+P^n(s)|^2\big)\\
&\times\Big[2\big(X^n(s)+P^n(s)\big)^T f(X^n(s)+P^n(s),s)+|g(X^n(s)+P^n(s),s)|^2\Big]\\
&-(2-p)\big|\big(X^n(s)+P^n(s)\big)^T g(X^n(s)+P^n(s),s)\big|^2\bigg\}\mathrm{d}s,\\
J_2(t)=&\frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}}\Big(-2\big(X^n(s)+P^n(s)\big)^T P^n(s)+|P^n(s)|^2\Big)\\
&\times\Big[2\big(X^n(s)+P^n(s)\big)^T f(X^n(s)+P^n(s),s)+|g(X^n(s)+P^n(s),s)|^2\Big]\mathrm{d}s,\\
J_3(t)=&\frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}}\big(1+|X^n(s)+P^n(s)|^2\big)\Big(-2(P^n(s))^T f(X^n(s)+P^n(s),s)\Big)\mathrm{d}s,\\
J_4(t)=&\frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(s)^2\big)^{\frac{p-4}{2}}\Big(-2\big(X^n(s)+P^n(s)\big)^T P^n(s)+|P^n(s)|^2\Big)\\
&\times\Big(-2(P^n(s))^T f(X^n(s)+P^n(s),s)\Big)\mathrm{d}s,
\end{aligned}
$$

$$
\begin{aligned}
J_5(t) = \frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}} \\
\times\Big\{ -(2-p)\Big(-2\big(X^n(s)+P^n(s)\big)^T P^n(s)|g(X^n(s)+P^n(s),s)|^2 \\
+\big|(P^n(s))^T g(X^n(s)+P^n(s),s)\big|^2\Big)\Big\}\mathrm{d}s.
\end{aligned}
$$

By Assumption 3, we have

$$
\begin{aligned}
J_1(t) &\le \frac{p}{2}\int_0^t \psi(s)\big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}} K(s)\big(1+|X^n(s)+P^n(s)|^2\big)^2 \mathrm{d}s \\
&\le C\int_0^t \psi(s)\big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}} K(s)\Big(\big(1+|X^n(s)|^2\big)^2+|P^n(s)|^4\Big)\mathrm{d}s \\
&\le C\int_0^t \psi(s)K(s)\big(1+|X^n(t)|^2\big)^{\frac{p}{2}}\mathrm{d}s + C\int_0^t \psi(s)K(s)\big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}}|P^n(s)|^4\mathrm{d}s.
\end{aligned}
$$

For $0<p\le 4$, we have $\big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}} \le 1$. For $t\in[0,T\wedge\tau_n(R)]$, using the condition (i), we have $|X^n(t)|+|P^n(t)|\le R$ $a.s.$. Then we can derive that

$$
J_1(t) \le C\int_0^t \psi(s)K(s)\big(1+|X^n(t)|^2\big)^{\frac{p}{2}}\mathrm{d}s + C_R\int_0^t \psi(s)K(s)|P^n(s)|\mathrm{d}s, \tag{3.5}
$$

where C_R denotes a generic positive constant related to R in this paper. While $p>4$, using Young's inequality, we have

$$
\begin{aligned}
J_1(t) &\le C\Big(\int_0^t \psi(s)K(s)\big(1+|X^n(t)|^2\big)^{\frac{p}{2}}\mathrm{d}s + \int_0^t \psi(s)K(s)|P^n(s)|^p\mathrm{d}s\Big) \\
&\le C\int_0^t \psi(s)K(s)\big(1+|X^n(t)|^2\big)^{\frac{p}{2}}\mathrm{d}s + C_R\int_0^t \psi(s)K(s)|P^n(s)|\mathrm{d}s.
\end{aligned} \tag{3.6}
$$

For $p>0$, combining (3.5) and (3.6), we have

$$
J_1(t) \le C\int_0^t \psi(s)K(s)\big(1+|X^n(t)|^2\big)^{\frac{p}{2}}\mathrm{d}s + C_R\int_0^t \psi(s)K(s)|P^n(s)|\mathrm{d}s, \tag{3.7}
$$

Next, we compute $J_2(t)$, that is

$$
\begin{aligned}
J_2(t) &\le C\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s)\Big(|X^n(s)+P^n(s)||P^n(s)|+\big|P^n(s)\big|^2\Big) \\
&\quad\times\Big(|X^n(s)+P^n(s)|+1\Big)\mathrm{d}s \\
&\le C\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s)\Big(|X^n(s)+P^n(s)|+\big|P^n(s)\big|\Big)|P^n(s)| \\
&\quad\times\Big(|X^n(s)|+|P^n(s)|+1\Big)\mathrm{d}s \\
&\le C\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s)\Big(1+|X^n(s)|+|P^n(s)|\Big)^2|P^n(s)|\mathrm{d}s \\
&\le C\int_0^t \psi(s)\big(1+|X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s)\Big(1+|X^n(s)|^2+|P^n(s)|^2\Big)|P^n(s)|\mathrm{d}s.
\end{aligned} \tag{3.8}
$$

Obviously, we also need to consider (3.8) in two cases respectively: $0 < p \le 4$ and $p > 4$. By the condition (i), for $p > 0$, it is easy to show that

$$J_2(t) \le \big(1 + C_R\big) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s. \tag{3.9}$$

For $J_3(t)$, we can write that

$$J_3(t) \le C \int_0^t \psi(s) \big(1 + |X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s) \big(1 + |X^n(s) + P^n(s)|^2\big) |P^n(s)| \mathrm{d}s.$$

In the same way as discussed above, we have

$$J_3(t) \le \big(1 + C_R\big) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s. \tag{3.10}$$

Repeating the similar procedures, we also have

$$J_4(t) \le C \int_0^t \psi(s) \big(1 + |X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s) \Big(|X^n(s) + P^n(s)||P^n(s)| + |P^n(s)|^2\Big) |P^n(s)| \mathrm{d}s,$$

and

$$\begin{aligned} J_5(t) \le{}& C \int_0^t \psi(s) \big(1 + |X^n(s)|^2\big)^{\frac{p-4}{2}} \Big(|X^n(s) + P^n(s)||P^n(s)||g(X^n(s) + P^n(s), s)|^2 \\ &\qquad + |(P^n(s))|^2 |g(X^n(s) + P^n(s), s)|^2\Big) \mathrm{d}s \\ \le{}& C \int_0^t \psi(s) \big(1 + |X^n(s)|^2\big)^{\frac{p-4}{2}} K_R(s) \Big(|X^n(s) + P^n(s)||P^n(s)| + |(P^n(s))|^2\Big) \mathrm{d}s. \end{aligned}$$

Therefore, for $t \in [0, T \wedge \tau_n(R)]$ and $p > 0$, by virtue of the condition (i), we derive that

$$J_4(t) \le \big(1 + C_R\big) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s, \tag{3.11}$$

and

$$J_5(t) \le \big(1 + C_R\big) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s. \tag{3.12}$$

Substituting (3.7), (3.9), (3.10), (3.11) and (3.12) into (3.4), we have

$$\begin{aligned} \big(1 + |X^n(t)|^2\big)^{\frac{p}{2}} \psi(t) \le{}& \big(1 + |X(0)|^2\big)^{\frac{p}{2}} \psi(0) + \int_0^t \kappa K(s) \big(1 + |X^n(s)|^2\big)^{\frac{p}{2}} \psi(s) \mathrm{d}s \\ &+ C \int_0^t \psi(s) K(s) \big(1 + |X^n(t)|^2\big)^{\frac{p}{2}} \mathrm{d}s + C_R \int_0^t \psi(s) K(s) |P^n(s)| \mathrm{d}s \\ &+ \big(1 + C_R\big) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s + J_n^R(t). \end{aligned} \tag{3.13}$$

Choosing $\kappa = -C$ and then replacing $K_R(s)$ by $K(s) \vee K_R(s)$, we have

$$\big(1 + |X^n(t)|^2\big)^{\frac{p}{2}} \psi(t) \le \big(1 + |X(0)|^2\big)^{\frac{p}{2}} \psi(0) + \big(1 + C_R\big) \int_0^t \psi(s) K_R(s) |P^n(s)| \mathrm{d}s + J_n^R(t). \tag{3.14}$$

Furthermore, since $\psi(t) \leq 1$ and $J_n^R(t)$ is a continuous local $\mathcal{F}_t$-martingale with $J_n^R(0) = 0$, according to [11], for any $R,\ T \in [0, \infty)$, taking expectations on both sides of (3.14), it is easy to see that

$$\mathbb{E}\Big[\Big(1 + |X^n(\varsigma)|^2\Big)^{\frac{p}{2}}\psi(\varsigma)\Big] \leq \psi(0)\mathbb{E}\big[(1 + |X(0)|^2)^{\frac{p}{2}}\big] + \big(1 + C_R\big)\mathbb{E}\big[\lambda_n^R(T \wedge \tau_n(R)\big].$$

where ς represents any $\mathcal{F}_t$-stopping time satisfying $\varsigma \leq T \wedge \tau_n(R)$. Then, based on [8, pp.584, Lemma 1], for any $l \in (0, \infty)$, we have

$$l\mathbb{P}\Big\{\sup_{t \in [0, T \wedge \tau_n(R)]} |X^n(t)|^p \psi(t) \geq l\Big\} \leq \big(1 + C_R\big)\Big(1 + \mathbb{E}\big[\lambda_n^R(T \wedge \tau_n(R))\big]\Big).$$

We then have

$$\mathbb{P}\Big\{\sup_{t \in [0, T \wedge \tau_n(R)]} |X^n(t)|^p \psi(t) \geq l\Big\} \leq \frac{\big(1 + C_R\big)\Big(1 + \mathbb{E}\big[\lambda_n^R(T \wedge \tau_n(R))\big]\Big)}{l}.$$

Thanks to (3.2), it is easy to derive that

$$\lim_{l \to \infty} \sup_{R \in [0, \infty)} \varlimsup_{n \to \infty} \mathbb{P}\Big\{\sup_{t \in [0, T \wedge \tau_n(R)]} |X^n(t)|^p \psi(t) \geq l\Big\} = 0. \tag{3.15}$$

Recalling that $r(R) \to \infty$ *as* $R \to \infty$ and choosing $l = r^p(R)\psi(t)$ in (3.15) , we have

$$\lim_{R \to \infty} \varlimsup_{n \to \infty} \mathbb{P}\Big\{\sup_{t \in [0, T \wedge \tau_n(R)]} |X^n(t)| \geq r(R)\Big\} = 0,$$

which implies

$$\lim_{R \to \infty} \varlimsup_{n \to \infty} \mathbb{P}\Big\{\sup_{t \in [0, \tau_n(R)]} |X^n(t)| \geq r(R), \tau_n(R) \leq T\Big\} = 0.$$

Under condition (iii), we obtain

$$\lim_{R \to \infty} \varlimsup_{n \to \infty} \mathbb{P}\big\{\tau_n(R) \leq T\big\} = 0.$$

Hence for any $\varepsilon > 0$, thanks to (3.3), we have

$$\begin{aligned}
&\mathbb{P}\Big\{\sup_{t \in [0, T]} |X^n(t) - X^m(t)| > \varepsilon\Big\} \\
&= \mathbb{P}\Big\{\sup_{t \in [0, T]} |X^n(t) - X^m(t)| > \varepsilon, \tau_{(n,m)}(R) \leq T\Big\} \\
&\quad + \mathbb{P}\Big\{\sup_{t \in [0, T \wedge \tau_{(n,m)}(R)]} |X^n(t) - X^m(t)| > \varepsilon, \tau_{(n,m)}(R) > T\Big\} \\
&\leq \mathbb{P}\Big\{\tau_n(R) \leq T\Big\} + \mathbb{P}\Big\{\tau_m(R) \leq T\Big\} + \mathbb{P}\Big\{\sup_{t \in [0, T \wedge \tau_{(n,m)}(R)]} |X^n(t) - X^m(t)| > \varepsilon\Big\},
\end{aligned}$$

which leads to (3.1). $\square$

We now give the theorem about the existence and uniqueness of the exact solution to (2.1).

Theorem 3.2. *Let Assumptions 1-3 hold with $p > 0$. Then, for any $T \in [0, \infty)$, there exists a unique process $\{X(t)\}_{t\in[0,T]}$ that satisfies equation (2.1) with the property*

$$\sup_{t\in[0,T]} \mathbb{E}[|X(t)|^p] < C. \tag{3.16}$$

Proof. Based on Euler's method, we construct a sequence $\{X^n(\cdot)\}, n \in \mathbb{N}$. For $n \in \mathbb{N}$, we define $\{X^n(t)\}_{t\geq 0}$ as follows

$$\begin{cases} X^n(0) = X(0), \\ X^n(t) = X^n(\frac{k}{n}) + \int_{\frac{k}{n}}^t f(X^n(\frac{k}{n}), s)\mathrm{d}s + \int_{\frac{k}{n}}^t g(X^n(\frac{k}{n}), s)\mathrm{d}B(s), \quad t \in [\frac{k}{n}, \frac{k+1}{n}), k \in \{0\} \cup \mathbb{N} \end{cases}$$

We further define $\iota(n,t) = \lfloor nt \rfloor / n$. Then for $t \geq 0$, we have

$$X^n(t) = X^n(\iota(n,t)) + \int_{\iota(n,t)}^t f(X^n(\iota(n,s)), s)\mathrm{d}s + \int_{\iota(n,t)}^t g(X^n(\iota(n,s)), s)\mathrm{d}B(s),$$

which can be written as

$$X^n(t) = X^n(0) + \int_0^t f(X^n(\iota(n,s)), s)\mathrm{d}s + \int_0^t g(X^n(\iota(n,s)), s)\mathrm{d}B(s). \tag{3.17}$$

This is equivalent to

$$X^n(t) = X^n(0) + \int_0^t f(X^n(s) + P^n(s), s)\mathrm{d}s + \int_0^t g(X^n(s) + P^n(s), s)\mathrm{d}B(s),$$

where $P^n(t) = X^n(\iota(n,t)) - X^n(t) = -\int_{\iota(n,t)}^t f(X^n(\iota(n,s)), s)\mathrm{d}s - \int_{\iota(n,t)}^t g(X^n(\iota(n,s)), s)\mathrm{d}B(s)$. In order for the existence and uniqueness, we need to show that there exists an $\mathcal{F}_t$-adapted continuous process $\{X(t)\}_{t\in[0,T]}$ and

$$X(t) = X(0) + \int_0^t f(X(s), s)\mathrm{d}s + \int_0^t g(X(s), s)\mathrm{d}B(s) \qquad \mathbb{P} - a.s.$$

after taking limits on both sides of (3.17). And in terms of Lemma 3.1, the proofs of these are same as [8] and [16], so we omit it there. It remains to prove the pth moment boundedness. In fact, an application of the Itô formula, we have

$$\begin{aligned} &\big(1 + |X^n(t)|^2\big)^{\frac{p}{2}} \\ &= \big(1+|X(0)|^2\big)^{\frac{p}{2}} + \frac{p}{2}\int_0^t \big(1+|X^n(t)|^2\big)^{\frac{p-4}{2}} \Big\{\big(1+|X^n(t)|^2\big)\Big(2\big(X^n(t)\big)^T f(X^n(s), s) + |g(X^n(s), s)|^2\Big) \\ &\qquad - (2-p)\big|\big(X^n(t)\big)^T g(X^n(s), s)\big|^2\Big\}\mathrm{d}s + H^n(t), \end{aligned}$$

where

$$H^n(t) = p\int_0^t \big(1 + |X^n(s)|^2\big)^{\frac{p-4}{2}} \big(X^n(t)\big)^T g(X^n(s), s)\mathrm{d}B(s).$$

We recall that $\big(1 + |X^n(t)|^2\big)^{\frac{p-4}{2}} \leq 1$ for $0 < p \leq 4$, and Young's inequality can be used in the case of $p > 4$. Therefore, by Assumption 3, (3.16) follows directly from [10, pp.851, Theorem 2.3]. $\square$

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