On a non-local non-homogeneous fractional Timoshenko system with frictional and viscoelastic damping terms

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Abstract

We are devoted to the study of a non-local nonhomogeneous time fractional timoshenko system with frictional and viscoelastic damping terms. We are concerned with the well-posedness of the given problem. The approach relies on some functional analysis tools, operator theory, a prori estimates, and density arguments.

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ON A NON-LOCAL NON-HOMOGENEOUS FRACTIONAL TIMOSHENKO SYSTEM WITH FRICTIONAL AND VISCOELASTIC DAMPING TERMS

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ABSTRACT. We are devoted to the study of a non-local nonhomogeneous time fractional timoshenko system with frictional and viscoelastic damping terms. We are concerned with the well-posedness of the given problem. The approach relies on some functional analysis tools, operator theory, a prori estimates and density arguments.

1. INTRODUCTION

Vibrations of beams are not always safe and welcomed because of their great and irreparable damages effects. In this situation, researchers try to introduce some damping mechanisms (viscous damping, thermoelastic damping, modal damping, frictional damping, Kelvin-Voigt damping) in such a way that these damaging and destructive vibrations are perfectly reduced. In other words, an intensive investigation has been carried out to impose minimal conditions to provide and guarantee stability of Timoshenko systems using several types of dissipative mechanisms. Several authors studied and investigated problems involving the previous mentioned type of dampings (local or global) where different kind of stability have been showed. In this regard, we refer the reader to the references [52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62] and the references therein.

As a classical and a simple model [1], Timoshenko studied the following coupled hyperbolic system

$$\begin{cases} \rho_1\theta_{tt} - \kappa(\theta_x - \phi)_x = 0, & (x,t) \in \ (0,L) \times (0,\infty) \\ \rho_2\phi_{tt} = \kappa^*\phi_{xx} + \kappa(\theta_x - \phi) & (x,t) \in \ (0,L) \times (0,\infty), \\ (\theta_x - \phi) \mid_{x=0}^{x=L} = 0, \quad \phi_x \mid_{x=0}^{x=L} = 0, & \end{cases} \tag{1.1}$$

describing the transverse vibration of a beam. where L is the length of the beam in its equilibrium configuration. The function θ models the transverse displacement of the beam and ϕ models the rotation angle of its filament. The coefficients ρ_1, ρ_2, κ and κ^* are respectively the density, the polar moment of inertia of a cross section, the shear modulus and the Young's modulus of elasticity. Timoshenko system (1.1) was generalized and studied by many authors. As mentioned at the beginning of the introduction, different types of dampings were added to the Timoshenko system for the purpose of its

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stabilization. For example, in [4], researchers investigated the exponential stability for a Timoshenko system having two weak dampings

$$\begin{cases} \rho_1\theta_{tt} = \kappa(\theta_x - \phi)_x - \theta_t, \quad \text{in } (0,L)\times(0,\infty), \\ \rho_2\phi_{tt} = \kappa^*\phi_{xx} - \kappa(\theta_x - \phi)_x - \phi_t, \quad \text{in } (0,L)\times(0,\infty), \\ \theta(0,t) = \theta(L,t) = \phi(0,t) = \phi(L,t) = 0, \quad t > 0. \end{cases} \tag{1.2}$$

In [2], authors proved some exponential decay results for a Timoshenko system with a memory damping term

$$\begin{cases} \rho_1\theta_{tt} - \kappa_1(\theta_x + \phi)_x = 0, \quad \text{in } (0,L)\times(0,\infty) \\ \rho_2\phi_{tt} - \kappa_2\phi_{xx} + \kappa_1(\theta_x + \phi) + h * \phi_{xx}(x,t) = 0, \ \text{in } (0,L)\times(0,\infty) \\ \theta(0,t) = \theta(L,t) = \phi(0,t) = \phi(L,t) = 0, \\ \theta(x,0) = \theta_0, \ \theta_t(x,0) = \theta_1, \ \phi(x,0) = \phi_0, \ \phi_t(x,0) = \phi_1. \end{cases} \tag{1.3}$$

Authors considered and studied in [5] the effect of frictional and viscoelastic dampings, and proved some exponential and polynomial decay results for the system

$$\begin{cases} \theta_{tt} - (\theta_x + \phi)_x = 0, \\ \phi_{tt} - \phi_{xx} + \theta_x + \phi + \int_0^t g(t-s)(a(x)\phi_x(x,s))_x ds + b(x)h(\phi_t) = 0, \\ \theta(0,t) = \theta(1,t) = \phi(0,t) = \phi(1,t) = 0, \quad t > 0. \end{cases} \tag{1.4}$$

We also mention that in [43], the authors investigated the exponential stabilization of a Timoshenko system by a thermal effect damping.

$$\begin{cases} \rho_1\theta_{tt} - \kappa_1(\theta_x + \phi)_x = 0, \quad \text{in } (0,L)\times(0,\infty) \\ \rho_2\phi_{tt} - \kappa_2\phi_{xx} + \kappa_1(\theta_x + \phi) + \gamma\omega_x, \ \text{in } (0,L)\times(0,\infty) \\ \rho_2\omega_{tt} - \kappa_3\omega_{xx} + \beta\int_0^t g(t-s)\omega_{xx}(x,s)ds + \gamma\phi_{ttx}, \ \text{in } (0,L)\times(0,\infty). \end{cases} \tag{1.5}$$

In [3], the author considered a Timoshenko linear thermoelastic system with linear frictional damping and a distributed delay. He proved the well-posedness, and proved that the system is exponentially stable regardless of the speeds of wave propagation. There are many other papers in the literature dealing with the stabilization of different version of Timoshenko systems. For more results concerning the stabilization and controllability of Timoshenko systems, we refer the reader to [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 31].

Recently, a generalization of the Timoshenko system (1.1) into fractional setting is studied in [44] by using a fractional version of resolvents. The author established the well posedness of a fractional Timoshenko system, and proved that lower order fractional terms can stabilize the system in a Mittag-Leffler fashion. More precisely, the author considered the initial boundary value problem

$$\begin{cases} \rho_1\partial_t^\alpha(\partial_t^\alpha\theta) - \kappa_1(\theta_x + \phi)_x = 0, \quad \text{in } (0,1)\times(0,\infty) \\ \rho_2\partial_t^\alpha(\partial_t^\alpha\phi + a\phi) - \kappa_2\phi_{xx} + \kappa_1(\theta_x + \phi), \ \text{in } (0,1)\times(0,\infty) \\ \theta(0,t) = \theta(1,t) = 0, \ \phi(0,t) = \phi(1,t) = 0, \quad t > 0 \\ \theta(x,0) = \theta_0(x)\ , \ \phi(x,0) = \psi(x). \end{cases}$$

For some other fractional and integer order Timoshenko systems, the reader can refer to [63, 64, 65, 66, 67, 68]. Motivated by the above results on Timoshenko systems, we consider a non local initial boundary value problem for a non-homogeneous fractional Timoshenko system with a frictional damping in the first equation and a viscoelastic memory damping term in the second equation. The system is complemented with initial conditions and non local purely boundary integral conditions. At the beginning of the year 1963, Cannon [20] was the first researcher to investigate a non local problem with a non local constraint (energy specification) of the form $\int_0^l \chi(x)U(x,t)dt = \tau(t)$, where $\chi(x)$, and $\tau(t)$ are given functions. More precisely, he used the potential method to investigate the well posedness of the heat equation subject to the specification of energy. This type of conditions arise mainly when the data cannot be measured directly on the boundary, but only their averages (weighted averages) are known. Due to their importance, physical significance (mean, total flux, total energy,..) and numerous applications in different fields of science and engineering, such as underground water flow, vibration problems, heat conduction, medical science, nuclear reactor dynamics, thermoelasticity, and plasma physics and control theory, several authors extensively studied this type of problems. We can cite for example [16, 21, 22, 23, 24, 25, 26, 28, 29, 30, 31]. Note that theoretical study of non local problems is connected with great difficulties, since the presence of integral terms in the boundary conditions can greatly complicate the application of classical methods of functional analysis method, especially when it comes to the fractional case . A functional analysis method based on some a priori bounds and on the density of the range of the unbounded operator corresponding to the abstract formulation of the given problem is used to prove the well posedness of the posed problem. This is shown through the introduction of some multiplier operators, some classical and fractional inequalities, and the establishment of some properties, involving fractional derivatives.

To the best of our knowledge, the treated fractional system problem (2.1)-(2.4) has never been studied and explored in the literature. This work can be considered as a contribution in the development of the traditional functional analysis method, the so called energy inequality method used to prove the well posedness of mixed problems with integral boundary conditions. For some classical cases, the reader can refer for example to example [16, 17, 18, 19, 27], and for some fractional cases, the reader should refer to [32, 33, 34, 36, 37, 38, 39, 40]. We should also mention here that there are some important papers dealing with numerical aspects for Timoshenko systems, and having many applications, for which the reader can refer to [46 47, 48, 49]. There are some papers dealing with Timoshenko system with fractional operator in the memory [41, 42].

2. Formulation of the problem and function spaces

Given the interval $I=(0,L)$, we consider the non-homogeneous fractional viscoelastic beam model with frictional damping of Timoshenko type

$$\begin{cases} \mathcal{L}_1(\theta,\phi)=\rho_1\partial_t^{\alpha+1}\theta-\kappa_1(\theta_x+\phi)_x+\theta_t=F(x,t) \\ \mathcal{L}_2(\theta,\phi)=\rho_2\partial_t^{\alpha+1}\phi-\kappa_2\phi_{xx}+\kappa_1(\theta_x+\phi)+\int\limits_0^t m(t-s)\phi_{xx}(x,s)ds=G(x,t), \end{cases} \tag{2.1}$$

in the unknowns $(\theta,\phi):(x,t)\in I\times[0,T]\rightarrow\mathbb{R}$, the strictly positive constants ρ_1,ρ_2,κ_1 and κ_2 satisfy the relation

$$\frac{\rho_1}{\kappa_1}=\frac{\rho_2}{\kappa_2},$$

and f , g , φ , ψ, F, and G are given functions, and $m:\mathbb{R}^+\rightarrow\mathbb{R}^+$ is a twice differentiable function such that

$$\kappa_2-\int\limits_0^T m(t)dt=l>0,\quad m'(t)<0,\ \forall t\geq 0. \tag{2.2}$$

The system (2.1) is complemented with the initial conditions

$$\begin{cases} \Gamma_1\theta=\theta(x,0)=\varphi(x)\ ,\ \Gamma_2\theta=\theta_t(x,0)=\psi(x)\ , \\ \Gamma_1\phi=\phi(x,0)=f(x)\ ,\ \Gamma_2\phi=\phi_t(x,0)=g(x), \end{cases} \tag{2.3}$$

and the non local boundary integral conditions

$$\int\limits_0^L \theta dx=0\ ,\ \int\limits_0^L x\theta dx=0, \int\limits_0^L \phi dx=0\ ,\ \int\limits_0^L x\phi dx=0. \tag{2.4}$$

This system of coupled hyperbolic equations represents a Timoshenko model for a thick beam of length L,where θ is the transverse displacement of the beam and ϕ is the rotation angle of the filament of the beam. The coefficients ρ_1,ρ_2,κ_1 and κ_2 are respectively the density, the polar moment of inertia of a cross section, the shear modulus and the Young's modulus of elasticity. The integral conditions represent the averages (weighted averages) of the total transverse displacement of the beam and the rotation angle of the filament of the beam.

Our aim is to study the well posedness of the solution of problem (2.1), (2.4). That is on the basis of some a priori bounds and on the density of the range of the operator generated by the problem under consideration, we prove the existence, uniqueness and continuous dependence of the solution on the given data of problem (2.1), (2.4). We now introduce some function spaces needed throughout the sequel. Let $L^2(Q^T)$ be the Hilbert space of square integrable functions on $Q^T=(0,1)\times(0,T)$, $T<\infty$, with scalar product and norm respectively

$$(Z,S)_{L^2(Q^T)}=\int_{Q^T} ZSdxdt,\qquad \|Z\|^2_{L^2(Q^T)}=\int_{Q^T} Z^2dxdt. \tag{2.5}$$

We also use the space $L^2((0,1))$ on the interval $(0,1)$, whose definition is analogous to the space on Q. Let $B_2^1(0,L)$ be the space obtained by completion of the space $C^0(0,L)$ of real continuous functions with compact support in the interval $(0,L)$ with respect to the inner product

$$(\gamma,\gamma^*)_{B_2^1(0,L)} = \int_0^L \mathcal{I}_x\gamma.\Im_x\gamma^* dx,$$

where $\mathcal{I}_x\gamma = \int_0^x \gamma(\zeta)d\zeta$ for every fixed $x \in (0,L)$. The associated norm is $\|\gamma\|^2_{B_2^1(0,L)} = \sqrt{(\gamma,\gamma)_{B_2^1(0,L)}} = \int_0^L (\mathcal{I}_x\gamma)^2 dx$. We denote by $C(\overline{J};L^2(0,L))$ with $J=(0,T)$ the set of all continuous functions $\gamma(.,t): J \to L^2(0,L)$ with norm

$$\|\gamma\|^2_{C(J;L^2(0,L))} = \sup_{0\le t\le T} \|\gamma(.,t)\|^2_{L^2(0,L)} < \infty, \tag{2.6}$$

and $C(\overline{J};B_2^1(0,L))$ the set of functions $\gamma(.,t): \overline{J} \to B_2^1(0,L)$ with norm

$$\|\gamma\|^2_{C(\overline{J};B_2^1(0,L))} = \sup_{0\le t\le T} \|\mathcal{I}_x\gamma(.,t)\|^2_{L^2(0,L)} = \sup_{0\le t\le T} \|\gamma(.,t)\|^2_{B_2^1(0,L)} < \infty. \tag{2.7}$$

To obtain a priori estimate for the solution, we write down our problem (2.1), (2.4) in its operator form: $\mathcal{GZ} = H$ with $\mathcal{Z} = (\theta,\phi)$, $\mathcal{GZ} = (\mathcal{S}_1(\theta,\phi), \mathcal{S}_2(\theta,\phi))$ and $H = (H_1, H_2)$ where

$$\begin{cases} \mathcal{S}_1(\theta,\phi) = \{\mathcal{L}_1(\theta,\phi), \Gamma_1\theta, \Gamma_2\theta\} \\ \mathcal{S}_2(\theta,\phi) = \{\mathcal{L}_2(\theta,\phi), \Gamma_1\phi, \Gamma_2\phi\} \\ H_1 = \{F,\varphi,\psi\},\ \ H_2 = \{G,f,g\}. \end{cases}, \tag{2.8}$$

The operator $\mathcal{G}$ is an unbounded operator of domain of definition $D(\mathcal{G})$ consisting of elements $(\theta,\phi) \in \left(L^2(\overline{J};L^2(0,L))\right)^2$ such that $\theta_x, \phi_x, \theta_t, \phi_t, \theta_{tt}, \phi_{tt}, \theta_{xx}, \phi_{xx}$ belonging to $L^2(\overline{J};L^2(0,L))$ verifying initial and boundary conditions (2.3) and (2.4). The operator $\mathcal{G}$ is acting from the Banach space $\mathcal{B}$ into the Hilbert space $\mathcal{E}$, where $\mathcal{B}$ is the Banach space obtained by completing $D(\mathcal{G})$ with respect to the norm

$$\|\mathcal{Z}\|^2_{\mathcal{B}} = \|\theta(.,t)\|^2_{C(\overline{J};L^2(0,L))} + \|\phi(.,t)\|^2_{C(\overline{J};L^2(0,L))}. \tag{2.9}$$

And $\mathcal{E} = \left[L^2(Q^T)\times(L^2(0,L))^2\right]\times\left[L^2(Q^T)\times(L^2(0,L))^2\right]$ is the Hilbert space consisting of vector-valued functions $H = (\{F,\varphi,\psi\},\{G,f,g\})$ for which the norm

$$\begin{aligned}\|H\|^2_{\mathcal{E}} &= \|F\|^2_{L^2(Q^T)} + \|\varphi\|^2_{L^2(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|G\|^2_{L^2(Q^T)} \\ &\quad + \|f\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)}.\end{aligned} \tag{2.10}$$

is finite.

3. Preliminaries (Definitions and lemmas)

In this section, we provide some definitions and lemmas needed for establishing different proves in the sequel.

Definition 1. [50] The time fractional derivative of order β, with $\beta \in (1,2)$ for a function V is defined by

$$ {}^{C}\partial_t^{\beta} V(x,t) = \frac{1}{\Gamma(2-\beta)} \int_0^t \frac{V_{\tau\tau}(x,\tau)}{(t-\tau)^{\beta-1}} d\tau, \tag{3.1} $$

and for $\beta \in (0,1)$ it is defined by

$$ {}^{C}\partial_t^{\beta} V(x,t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{V_{\tau}(x,\tau)}{(t-\tau)^{\beta}} d\tau $$

where $\Gamma(1-\beta)$ is the Gamma function.

Definition 2. [50]. The fractional Riemann-Liouville integral of order $0 < \beta < 1$ [52] which is given by

$$ D_t^{-\beta} \upsilon(x,t) = \frac{1}{\Gamma(\beta)} \int_0^t \frac{\upsilon(x,\tau)}{(t-\tau)^{1-\beta}} d\tau. \tag{2.6} $$

Lemma 2.1 [35]. Let $E(s)$ be nonnegative and absolutely continuous on $[0,T]$, and suppose that for almost all $s \in [0,T]$, R satisfies the inequality

$$ \frac{dE}{ds} \leq A(s)E(s) + B(s), \tag{3.2} $$

where the functions $A(s)$ and $B(s)$ are summable and nonnegative on $[0,T]$. Then

$$ E(s) \leq \exp\left\{ \int_0^s A(t)dt \right\} \left(E(0) + \int_0^s B(t)dt \right). \tag{3.3} $$

Lemma 2.3. [34] Let a nonnegative absolutely continuous function $\mathcal{Q}(t)$ satisfy the inequality

$$ {}^{C}\partial_t^{\beta} \mathcal{Q}(t) \leq b_1 \mathcal{Q}(t) + b_2(t), \quad 0 < \beta < 1, \tag{3.4} $$

for almost all $t \in [0,T]$, where b_1 is a positive constant and $b_2(t)$ is an integrable nonnegative function on $[0,T]$. Then

$$ \mathcal{Q}(t) \leq \mathcal{Q}(0) E_{\beta}(b_1 t^{\beta}) + \Gamma(\beta) E_{\beta,\beta}(b_1 t^{\beta}) D_t^{-\beta} b_2(t), \tag{3.5} $$

where

$$ E_{\beta}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + 1)} \text{ and } E_{\beta,\mu}(x) = \sum_{n=0}^{\infty} \frac{x^n}{\Gamma(\beta n + \mu)}, $$

are the MIttag-Leffler functions

4. A priori estimate and its Consequences

In this section, we establish an energy inequality from which we deduce the uniqueness and continuous dependence of solution of problem (2.1)-(2.4) on the given data.

Theorem 3.1. For any function $\mathcal{Z} = (\theta, \phi) \in D(\mathcal{G})$ the following a priori estimates holds

$$\begin{aligned} & \|\theta(.,t)\|^2_{C(\bar{J};L^2(0,L))} + \|\phi(.,t)\|^2_{C(\bar{J};L^2(0,L))} \\ \leq\ & \mathcal{F}^* \Big(\|\varphi\|^2_{L^2(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} \\ & + \|F\|^2_{L^2(0,t;L^2(0,L))} + \|G\|^2_{L^2(0,t;L^2(0,L))} \Big), \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} & D^{\alpha-1} \left(\|\theta_t\|_{B^1_2(0,L)} + \|\phi_t\|_{B^1_2(0,L)} \right) \\ \leq\ & \mathcal{F}^* \Big(\|\varphi\|^2_{L^2(0,L)} + \|\psi\|^2_{L^2(0,L)} + \|g\|^2_{L^2(0,L)} + \|f\|^2_{L^2(0,L)} \\ & + \|F\|^2_{L^2(0,t;L^2(0,L))} + \|G\|^2_{L^2(0,t;L^2(0,L))} \Big). \end{aligned} \tag{4.2}$$

where $\mathcal{F}^*$ is a positive constant independent of $\mathcal{Z}$ =(θ, ϕ) given by

$$\mathcal{F}^* = \mathcal{M}\omega \max\left\{1, \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right\},$$

with

$$\begin{aligned} \mathcal{M} &= \Gamma(\alpha) E_{\alpha,\alpha}(\omega t^\alpha) \left(\max\left\{1, \frac{T^\alpha}{\alpha\Gamma(\alpha)}\right\} \right) \\ \omega &= \mathcal{W}^*(\mathcal{W}^* e^{\mathcal{W}^* T} + 1), \end{aligned}$$

and $\mathcal{W}^*$ is given by (4.24).

Proof. Define the integro-differential operators $\mathcal{M}_1\theta = -\mathcal{I}^2_x \theta_t$ and $\mathcal{M}_2\phi = -\mathcal{I}^2_x \phi_t$, where

$$\mathcal{I}^2_x\theta(x,t) = \int_0^x \int_0^\xi \theta(\eta,t) d\eta d\xi,\ \mathcal{I}^2_x\phi(x,t) = \int_0^x \int_0^\xi \phi(\eta,t) d\eta d\xi,$$

and consider the identity

$$\begin{aligned} & \left(\rho_1 \partial_t^{\alpha+1}\theta, \mathcal{M}_1\theta\right)_{L^2(0,L)} - \kappa_1((\theta_x+\phi)_x, \mathcal{M}_1\theta)_{L^2(0,L)} + (\theta_t, \mathcal{M}_1\theta)_{L^2(0,L)} \\ & + (\rho_2 \partial_t^{\alpha+1}\phi, \mathcal{M}_2\phi)_{L^2(0,L)} - \kappa_2((\phi_{xx}, \mathcal{M}_2\phi)_{L^2(0,L)} + \kappa_1((\theta_x+\phi), \mathcal{M}_2\phi)_{L^2(0,L)} \\ + & \left(\int_0^t m(t-s)\phi_{xx}(x,s)ds, \mathcal{M}_2\phi \right)_{L^2(0,L)} \\ = & \ (F(x,t), \mathcal{M}_1\theta)_{L^2(0,L)} + (G(x,t), \mathcal{M}_2\phi)_{L^2(0,L)}. \end{aligned} \tag{4.3}$$

The standard integration by parts of each term in (4.3) and conditions (2.3), (2.4) give

$$\begin{aligned}\left(\rho_1\partial_t^{\alpha+1}\theta,\mathcal{M}_1\theta\right)_{L^2(0,L)} &= \frac{\rho_1}{2}(\partial_t^{\alpha}(\mathcal{I}_x\theta_t),\mathcal{I}_x\theta_t)_{L^2(0,L)} \\ &\geq \frac{\rho_1}{2}\partial_t^{\alpha}\left\|\mathcal{I}_x\theta_t\right\|_{L^2(0,L)}, \end{aligned} \tag{4.4}$$

$$\begin{aligned}\left(\rho_2\partial_t^{\alpha+1}\phi,\mathcal{M}_2\phi\right)_{L^2(0,L)} &= \frac{\rho_2}{2}(\partial_t^{\alpha}(\mathcal{I}_x\phi_t),\mathcal{I}_x\phi_t)_{L^2(0,L)} \\ &\geq \frac{\rho_2}{2}\partial_t^{\alpha}\left\|\mathcal{I}_x\phi_t\right\|_{L^2(0,L)}, \end{aligned} \tag{4.5}$$

$$-\left(\theta_t,\mathcal{M}_1\theta\right)_{L^2(0,L)} = \|\mathcal{I}_x\theta_t\|^2_{L^2(0,L)}, \tag{4.6}$$

$$\begin{aligned}\kappa_1(\theta_{xx},\mathcal{I}_x^2\theta_t)_{L^2(0,L)} &= \kappa_1\mathcal{I}_x^2\theta_t.\theta_x]_0^L-\kappa_1\int_0^L\mathcal{I}_x\theta_t.\theta_x dx=\kappa_1\int_0^L\theta_t\theta dx \\ &= \frac{\kappa_1}{2}\frac{\partial}{\partial t}\|\theta\|^2_{L^2(0,L)}, \end{aligned} \tag{4.7}$$

and in the same manner, we have

$$\kappa_2(\phi_{xx},\mathcal{I}_x^2\phi_t)_{L^2(0,L)} = \frac{\kappa_2}{2}\frac{\partial}{\partial t}\|\phi\|^2_{L^2(0,L)}, \tag{4.8}$$

$$\begin{aligned}\kappa_1(\phi_x,\mathcal{I}_x^2\theta_t)_{L^2((0,L))} &= \kappa_1\mathcal{I}_x^2\theta_t.\phi]_0^L dt-\kappa_1\int_0^L\mathcal{I}_x\theta_t.\phi dx \\ &= -\kappa_1\int_0^L\mathcal{I}_x\theta_t.\phi dx, \end{aligned} \tag{4.9}$$

$$-\kappa_1(\theta_x,\mathcal{I}_x^2\phi_t)_{L^2(Q^\tau)} = \kappa_1\int_0^L\mathcal{I}_x\phi_t.\theta dx, \tag{4.10}$$

$$\begin{aligned} & -\kappa_1(\phi,\mathcal{I}_x^2\phi_t)_{L^2(0,L)} \\ =\ & -\kappa_1\mathcal{I}_x^2\phi_t.\Im_x\phi]_0^L+\kappa_1\int_0^\tau\int_0^L\mathcal{I}_x\phi_t.\mathcal{I}_x\phi dxdt \\ =\ & \frac{1}{2}\frac{\partial}{\partial t}\|\mathcal{I}_x\phi\|^2_{L^2(0,L)}, \end{aligned} \tag{4.11}$$

$$
\begin{aligned}
& -\left(\int_0^t m(t-s).\phi_{xx}(x,s)ds, \mathcal{I}_x^2\phi_t\right)_{L^2(0,L)} \\
= & -\int_0^L \left(\int_0^t m(t-s).\phi_{xx}(x,s)ds\right)\mathcal{I}_x^2\phi_t dx \\
= & -\int_0^t m(t-s).\phi_x(x,s)ds).\mathcal{I}_x^2\phi_t]_0^L dx + \int_0^L \left(\int_0^t m(t-s).\phi_x(x,s)ds\right)\mathcal{I}_x\phi_t dx \\
= & \left(\int_0^t m(t-s).\phi(x,s)ds)\right)\mathcal{I}_x\phi_t]_0^L dx - \int_0^L \left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi_t dx \\
= & -\int_0^L \left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi_t dx. \qquad (4.12)
\end{aligned}
$$

Substituting equalities (4.4)-(4.12) into ((4.3)),we obtain

$$
\begin{aligned}
& \frac{\rho_1}{2}\partial_t^\alpha \left\|\mathcal{I}_x\theta_t\right\|_{L^2(0,L)} + \frac{\rho_2}{2}\partial_t^\alpha \left\|\mathcal{I}_x\phi_t\right\|_{L^2(0,L)} + \frac{\kappa_1}{2}\frac{\partial}{\partial t}\|\theta\|^2_{L^2(0,L)} \\
& +\frac{\kappa_2}{2}\frac{\partial}{\partial t}\|\phi\|^2_{L^2(0,L)} + \|\mathcal{I}_x\theta_t\|^2_{L^2(0,L)} + \frac{1}{2}\frac{\partial}{\partial t}\|\mathcal{I}_x\phi\|^2_{L^2(0,L)} \\
= & \ \kappa_1\int_0^L \phi\mathcal{I}_x\theta_t dx - \kappa_1\int_0^L \theta\mathcal{I}_x\phi_t dx - \int_0^L \left(\int_0^t m(t-s).\phi(x,s)ds\right)\phi_t dx \\
& -\int_0^L F\mathcal{I}_x^2\theta_t dx - \int_0^L G\mathcal{I}_x^2\phi_t dx. \qquad (4.13)
\end{aligned}
$$

Replacing t by τ, integrating with respect to τ from zero to t and using the given conditions, we obtain

$$\begin{aligned}
& \frac{\rho_1}{2} D^{\alpha-1} \left\| \mathcal{I}_x \theta_t \right\|_{L^2(0,L)} + \frac{\rho_2}{2} D^{\alpha-1} \left\| \mathcal{I}_x \phi_t \right\|_{L^2(0,L)} + \frac{\kappa_1}{2} \|\theta(.,t)\|^2_{L^2(0,L)} \\
& + \frac{\kappa_2}{2} \|\phi(.,t)\|^2_{L^2(0,L)} + \|\mathcal{I}_x \theta_\tau\|^2_{L^2(0,t;L^2(0,L))} + \frac{1}{2} \|\mathcal{I}_x \phi(.,t)\|^2_{L^2(0,L)} \\
= \; & \kappa_1 \left(\phi, \mathcal{I}_x \theta_\tau\right)_{L^2(0,t;L^2(0,L))} - \kappa_1 \left(\theta, \mathcal{I}_x \phi_\tau\right)_{L^2(0,t;L^2(0,L))} \\
& - \left(F, \mathcal{I}_x^2 \theta_\tau\right)_{L^2(0,t;L^2(0,L))} - \left(G, \mathcal{I}_x^2 \phi_\tau\right)_{L^2(0,t;L^2(0,L))} \\
& + \frac{\rho_1 t^{1-\alpha}}{2(\Gamma(1-\alpha)(1-\alpha))} \|\mathcal{I}_x \psi\|^2_{L^2(0,L)} + \frac{\kappa_1}{2} \|\varphi\|^2_{L^2(0,L)} \\
& + \frac{\rho_2 t^{1-\alpha}}{2(\Gamma(1-\alpha)(1-\alpha))} \|\mathcal{I}_x g\|^2_{L^2(0,L)} + \frac{\kappa_2}{2} \|f\|^2_{L^2(0,L)} \\
& + \frac{1}{2} \|\mathcal{I}_x f\|^2_{L^2(0,L)} - \int_0^t \int_0^L \left(\int_0^\tau m(\tau - s).\phi(x,s) ds \right) \phi_\tau dx d\tau. \qquad (4.14)
\end{aligned}$$

The last term on the right-hand side needs to be evaluated as follows

$$\begin{aligned}
& - \int_0^t \int_0^L \left(\int_0^\tau m(\tau - s).\phi(x,s) ds \right) \phi_\tau dx d\tau \\
= \; & - \int_0^L \left(\int_0^\tau m(\tau - s).\phi(x,s) ds \right) \phi dx]_0^t + \int_0^\tau \int_0^L m(0) \phi^2 dx d\tau \\
& + \int_0^t \int_0^L \left(\int_0^\tau m'(\tau - s).\phi(x,s) ds \right) \phi(x,\tau) dx d\tau \\
= \; & - \int_0^L \left(\int_0^t m(t - s).\phi(x,s) ds \right) \phi(x,t) dx + m(0) \|\phi\|^2_{L^2(0,t;L^2(0,L))} \\
& + \int_0^t \int_0^L \left(\int_0^\tau m'(\tau - s).\phi(x,s) ds \right) \phi dx d\tau. \qquad (4.15)
\end{aligned}$$

By replacing (4.15) into (4.14), and estimating different terms on the right-hand side of (by using Cauchy ϵ inequality, a Poincare type inequality) (4.14) as follows

$$\begin{aligned}
& \kappa_1 \left(\phi, \mathcal{I}_x \theta_t\right)_{L^2(0,t;L^2(0,L))} \\
\leq \; & \frac{\kappa_1 \epsilon_1}{2} \|\phi\|^2_{L^2(0,t;L^2(0,L))} + \frac{\kappa_1}{2\epsilon_1} \|\mathcal{I}_x \theta_\tau\|^2_{L^2(0,t;L^2(0,L))}, \qquad (4.16)
\end{aligned}$$

$$\begin{aligned}
& -\kappa_1 \left(\theta, \mathcal{I}_x \phi_\tau\right)_{L^2(0,t;L^2(0,L))} \\
\leq \; & \frac{\kappa_1 \epsilon_2}{2} \|\theta\|^2_{L^2(0,t;L^2(0,L))} + \frac{\kappa_1}{2\epsilon_2} \|\mathcal{I}_x \phi_\tau\|^2_{L^2(0,t;L^2(0,L))}, \qquad (4.17)
\end{aligned}$$

$$
\begin{aligned}
& -\int_0^L \left(\int_0^t m(t-s).\phi(x,s)ds \right) \phi(x,t)dx \\
\leq \quad & \frac{\epsilon_3}{2} \int_0^L \phi^2(x,t)dx + \frac{1}{2\epsilon_3} \int_0^L \left(\int_0^t m(t-s)\phi(x,s)ds \right)^2 dx \\
\leq \quad & \frac{\epsilon_3}{2} \int_0^L \phi^2(x,t)dx + \frac{1}{2\epsilon_3} \int_0^L \left(\int_0^t m^2(t-s)ds \right) \left(\int_0^t \phi^2(x,s)ds \right) dx \\
\leq \quad & \frac{\epsilon_3}{2} \int_0^L \phi^2(x,t)dx + \frac{T}{2\epsilon_3} \sup_{0\leq t\leq T} m^2(t) \int_0^L \int_0^t \phi^2 dx d\tau, \qquad (4.18)
\end{aligned}
$$

$$
\begin{aligned}
& \int_0^t \int_0^L \left(\int_0^\tau m'(\tau-s).\phi(x,s)ds \right) \phi dx d\tau \\
\leq \quad & \frac{\epsilon_4}{2} \int_0^t \int_0^L \phi^2 dx d\tau + \frac{1}{2\epsilon_4} \int_0^t \int_0^L \left(\int_0^\tau m'^2(\tau-s)ds \right) \left(\int_0^\tau \phi^2(x,s)ds \right) dx d\tau \\
\leq \quad & \frac{\epsilon_4}{2} \int_0^t \int_0^L \phi^2 dx d\tau + \frac{T}{2\epsilon_4} \sup_{0\leq t\leq T} m'^2(t) \int_0^t \int_0^L \left(\int_0^\tau \phi^2(x,s)ds \right) dx d\tau \\
= \quad & \frac{\epsilon_4}{2} \int_0^t \int_0^L \phi^2 dx d\tau + \frac{T}{2\epsilon_4} \sup_{0\leq t\leq T} m'^2(t) \int_0^L \left[\left(\tau \int_0^\tau \phi^2(x,s)ds \right)_0^t - \int_0^t \tau\phi^2 d\tau \right] dx \\
= \quad & \frac{\epsilon_4}{2} \int_0^t \int_0^L \phi^2 dx d\tau + \frac{T}{2\epsilon_4} \sup_{0\leq t\leq T} m'^2(t) \int_0^L \left[\int_0^t (t-\tau)\phi^2(x,\tau)d\tau \right] dx \\
\leq \quad & \frac{\epsilon_4}{2} \int_0^t \int_0^L \phi^2 dx d\tau + \frac{T^2}{2\epsilon_4} \sup_{0\leq t\leq T} m'^2(t) \int_0^L \int_0^t \phi^2 d\tau dx \\
= \quad & \left(\frac{\epsilon_4}{2} + \frac{T^2}{2\epsilon_4} \sup_{0\leq t\leq T} m'^2(t) \right) \int_0^t \int_0^L \phi^2 dx d\tau, \qquad (4.19)
\end{aligned}
$$

$$
-\left(F, \mathcal{I}_x^2 \theta_\tau\right)_{L^2(0,t;L^2(0,L))} \leq \frac{\epsilon_5}{2} \|F\|^2_{L^2(0,t;L^2(0,L))} + \frac{L^2}{2\epsilon_5} \|\mathcal{I}_x \theta_\tau\|^2_{L^2(0,t;L^2(0,L))}, \quad (4.20)
$$

$$
-\left(G, \mathcal{I}_x^2 \phi_\tau\right)_{L^2(0,t;L^2(0,L))} \leq \frac{\epsilon_6}{2} \|G\|^2_{L^2(0,t;L^2(0,L))} + \frac{L^2}{4\epsilon_6} \|\mathcal{I}_x \phi_\tau\|^2_{L^2(0,t;L^2(0,L))}, \quad (4.21)
$$

Combination of (4.16)-(4.21) and (4.14), leads to

$$\begin{aligned}
&\frac{\rho_1}{2}D^{\alpha-1}\left\|\mathcal{I}_x\theta_t\right\|_{L^2(0,L)}+\frac{\rho_2}{2}D^{\alpha-1}\left\|\mathcal{I}_x\phi_t\right\|_{L^2(0,L)}+\frac{\kappa_1}{2}\|\theta(.,t)\|^2_{L^2(0,L)}\\
&+\frac{\kappa_2}{2}\|\phi(.,t)\|^2_{L^2(0,L)}+\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\frac{1}{2}\|\mathcal{I}_x\phi(.,t)\|^2_{L^2(0,L)}\\
\leq\;&\frac{\rho_1T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))}\|\psi\|^2_{L^2(0,L)}+\frac{\kappa_1}{2}\|\varphi\|^2_{L^2(0,L)}\\
&+\frac{\rho_2T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))}\|g\|^2_{L^2(0,L)}+\left(\frac{\kappa_2}{2}+\frac{L^2}{4}\right)\|f\|^2_{L^2(0,L)}\\
&+\left(\frac{\kappa_1\epsilon_1}{2}+\frac{T}{2\epsilon_3}\sup_{0\leq t\leq T}m^2(t)+\frac{\epsilon_4}{2}+\frac{T^2}{2\epsilon_4}\sup_{0\leq t\leq T}m'^2(t)+m(0)\right)\|\phi\|^2_{L^2(0,t;L^2(0,L))}\\
&+\left(\frac{\kappa_1}{2\epsilon_1}+\frac{L^2}{2\epsilon_5}\right)\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\left(\frac{\kappa_1}{2\epsilon_2}+\frac{L^2}{4\epsilon_6}\right)\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}\\
&+\frac{\kappa_1\epsilon_2}{2}\|\theta\|^2_{L^2(0,t;L^2(0,L))}+\frac{\epsilon_3}{2}\|\phi(.,t)\|^2_{L^2(0,L)}+\frac{\epsilon_5}{2}\|F\|^2_{L^2(0,t;L^2(0,L))}\\
&+\frac{\epsilon_6}{2}\|G\|^2_{L^2(0,t;L^2(0,L))}. \qquad (4.22)
\end{aligned}$$

The choice $\varepsilon_1=\kappa_1$, $\varepsilon_5=L^2/2$, $\varepsilon_3=\kappa_2/2$, $\varepsilon_2=\varepsilon_4=\varepsilon_6=1$, and cancellation of the last term on the left-hand side of (4.22) reduce it to the following estimate

$$\begin{aligned}
&D^{\alpha-1}\left\|\mathcal{I}_x\theta_t\right\|_{L^2(0,L)}+D^{\alpha-1}\left\|\mathcal{I}_x\phi_t\right\|_{L^2(0,L)}+\|\theta(.,t)\|^2_{L^2(0,L)}+\|\phi(.,t)\|^2_{L^2(0,L)}\\
\leq\;&\mathcal{W}^*\Big(\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\theta\|^2_{L^2(0,t;L^2(0,L))}+\|\phi\|^2_{L^2(0,t;L^2(0,L))}\\
&+\|\varphi\|^2_{L^2(0,L)}+\|\psi\|_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)}+\|F\|^2_{L^2(0,t;L^2(0,L))} \qquad (4.23)\\
&+\|G\|^2_{L^2(0,t;L^2(0,L))}\Big),
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{W}^*\;=\;&\max\left(\frac{\kappa_1^2}{2}+\frac{T}{\kappa_2}\sup_{0\leq t\leq T}m^2(t)+\frac{1}{2}+\frac{T^2}{2}\sup_{0\leq t\leq T}m'^2(t)+m(0),\ \frac{3}{2},\ \frac{\kappa_1}{2}+\frac{L^2}{4},\frac{\kappa_2}{4},\right.\\
&\left.\frac{\rho_1T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))},\frac{\rho_2T^{1-\alpha}L^2}{4(\Gamma(1-\alpha)(1-\alpha))}\right)/\min\left(\frac{\rho_1}{2},\frac{\rho_2}{2},\frac{\kappa_1}{2},\frac{\kappa_2}{2},\frac{1}{2}\right). \qquad (4.24)
\end{aligned}$$

By omitting the first and second term on the left-hand side of (4.23), and applying the Gronwall-Bellman lemma ([45]), where

$$\begin{cases} E(t)=\|\theta\|^2_{L^2(0,t;L^2(0,L))}+\|\phi\|^2_{L^2(0,t;L^2(0,L))}\\ \frac{dE}{dt}=\|\theta(.,t)\|^2_{L^2(0,L)}+\|\phi(.,t)\|^2_{L^2(0,L)},\\ \mathcal{Q}(0)=0.\end{cases} \qquad (4.25)$$

we obtain

$$\begin{aligned}
y(t)\;\leq\;&\mathcal{W}^*e^{\mathcal{W}^*t}\Big(\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}\\
&+\|\varphi\|^2_{L^2(0,L)}+\|\psi\|_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)} \qquad (4.26)\\
&+\|F\|^2_{L^2(0,t;L^2(0,L))}+\|G\|^2_{L^2(0,t;L^2(0,L))}\Big).
\end{aligned}$$

Then by omitting the last two terms on the left-hand side of (4.23), and using (4.26), we have

$$\begin{aligned} & D^{\alpha-1}\left(\|\mathcal{I}_x\theta_t\|_{L^2(0,L)}+\|\mathcal{I}_x\phi_t\|_{L^2(0,L)}\right) \\ \leq\; & \mathcal{W}^*(\mathcal{W}^*e^{\mathcal{W}^*T}+1)\left(\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))}\right. \\ & +\|\varphi\|^2_{L^2(0,L)}+\|\psi\|_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)} \\ & \left.+\|F\|^2_{L^2(0,t;L^2(0,L))}+\|G\|^2_{L^2(0,t;L^2(0,L))}\right). \end{aligned} \tag{4.27}$$

Now, Lemma 3.2, can be applied to remove the first two terms on the right-hand side of (4.27), by taking

$$\begin{cases} \mathcal{Q}(t)=\|\mathcal{I}_x\theta_\tau\|^2_{L^2(0,t;L^2(0,L))}+\|\mathcal{I}_x\phi_\tau\|^2_{L^2(0,t;L^2(0,L))} \\ {}^C\partial_t^\beta\mathcal{Q}(t)=D^{\alpha-1}\left(\|\mathcal{I}_x\theta_t\|_{L^2(0,L)}+\|\mathcal{I}_x\phi_t\|_{L^2(0,L)}\right) \\ \mathcal{Q}(0)=0. \end{cases} \tag{4.28}$$

From (4.27), it follows that

$$\begin{aligned} \mathcal{Q}(t) \;\leq\; & \mathcal{M}\left\{D^{-1-\alpha}\left(\|F\|^2_{L^2(0,L)}+\|G\|^2_{L^2(0,L)}\right)\right. \\ & \left.+\|\varphi\|^2_{L^2(0,L)}+\|\psi\|_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)}\right\}, \end{aligned} \tag{4.29}$$

where

$$\mathcal{M}=\Gamma(\alpha)E_{\alpha,\alpha}(\omega t^\alpha)\left(\max\left\{1,\frac{T^\alpha}{\alpha\Gamma(\alpha)}\right\}\right),$$

with

$$\omega=\mathcal{W}^*(\mathcal{W}^*e^{\mathcal{W}^*T}+1).$$

Now since

$$D^{-1-\alpha}\left(\|F\|^2_{L^2(0,L)}+\|G\|^2_{L^2(0,L)}\right)\leq\frac{T^\alpha}{\Gamma(\alpha+1)}\int_0^t\left(\|F\|^2_{L^2(0,L)}+\|G\|^2_{L^2(0,L)}\right)d\tau, \tag{4.30}$$

then, we infer from (4.29), (4.30), and (4.23) the following inequality

$$\begin{aligned} & D^{\alpha-1}\|\mathcal{I}_x\theta_t\|_{L^2(0,L)}+D^{\alpha-1}\|\mathcal{I}_x\phi_t\|_{L^2(0,L)}+\|\theta(.,t)\|^2_{L^2(0,L)}+\|\phi(.,t)\|^2_{L^2(0,L)} \\ \leq\; & \mathcal{F}^*\left(\|\psi\|^2_{L^2(0,L)}+\|\varphi\|^2_{L^2(0,L)}+\|f\|^2_{L^2(0,L)}+\|g\|^2_{L^2(0,L)}\right. \\ & \left.+\|F\|^2_{L^2(0,T;L^2(0,L))}+\|G\|^2_{L^2(0,T;L^2(0,L))}\right). \end{aligned} \tag{4.31}$$

The first estimate (4.1) follows, if we disregard the first and second term on the left-hand side of (4.31), and pass to the supremum with respect to t over $(0,T)$. The second estimate (4.2) follows from (4.31) if we drop the last two terms on the left-hand side of the inequality (4.31).

Since the range of the operator $\mathcal{G}$ is subset of $\mathcal{E}$, that is $R(\mathcal{G})\subset\mathcal{E}$, so we extend $\mathcal{G}$ so that inequality (4.31) holds for the extension, and $R\left(\overline{\mathcal{G}}\right)=\mathcal{E}$. We can easily show that the following

Proposition 3.2. The unbounded operator $\mathcal{G}:\mathcal{B}\to\mathcal{E}$ admits a closure $\overline{\mathcal{G}}$ with domain of definition $D(\overline{\mathcal{G}})$.

Definition 3.3. The solution of the equation $\overline{\mathcal{G}}\,\mathcal{Z} = H = (\{F,\varphi,\psi\},\{G,f,g\})$ is called a strong solution of problem (2.1), (2.3), (2.4).

The a priori estimate (4.1), can be extended to

$$\|\mathcal{Z}\|^2_{\mathcal{B}} \le \mathcal{F}^* \|\overline{\mathcal{G}}\,\mathcal{Z}\|^2_{\mathcal{E}}, \quad \forall \mathcal{Z} \in D(\overline{\mathcal{G}}). \tag{4.32}$$

The estimate (4.32) shows that the operator $\overline{\mathcal{G}}$ is one to one and that $\overline{\mathcal{G}}^{-1}$ is continuous from $R(\overline{\mathcal{G}})$ onto $\mathcal{B}$. Consequently if a strong solution of problem (2.1), (2.3), (2.4) exists, it is unique and depends continuously on the input data φ,ψ,f,g and the external forces F and G. Also as a consequence of (4.32), the set $R(\overline{\mathcal{G}}) \subset \mathcal{E}$ is closed and $R(\overline{\mathcal{G}}) = \overline{R(\mathcal{G})}$.

5. Existence of solution

Theorem 4.1. Problem (2.1), (2.3), (2.4) admits a unique strong solution $\mathcal{Z} = \left(\overline{\mathcal{G}}\right)^{-1}(\{F,\varphi,\psi\},\{G,f,g\}) = \overline{\mathcal{G}^{-1}}(\{F,\varphi,\psi\},\{G,f,g\})$,which depend continuously on the given data, for all $F,G \in L^2(0,T;L^2(0,L))$,and $\varphi,\psi,f,g \in L^2(0,L)$.

Proof. It follows from above that in order to prove the existence of the strongly generalized solution of problem (2.1), (2.3), (2.4), it suffices to prove that $\overline{R(\mathcal{G})} = \mathcal{E}$. To this end, we first prove the density in the following special case.

Theorem 4.2. If for some function $W(x,t) = (\Lambda_1(x,t),\Lambda_2(x,t)) \in (L^2(0,T;L^2(0,L)))^2$ and for elements $\mathcal{Z} \in D_0(\mathcal{G}) = \{\mathcal{Z} : \mathcal{Z} \in D(\mathcal{G}) \text{ and } \Gamma_i\theta = \Gamma_i\phi = 0\ ,\ i = 1,2\}$ we have

$$(\mathcal{S}_1(\theta,\phi),\Lambda_1)_{L^2(0,T;L^2(0,L))} + (\mathcal{S}_2(\theta,\phi),\Lambda_2)_{L^2(0,T;L^2(0,L))} = 0, \tag{5.1}$$

then $W(x,t) = (\Lambda_1(x,t),\Lambda_2(x,t)) = (0,0)$ a.e in Q^T.

Proof. The identity (5.1) is equivalent to

$$\begin{aligned}
&\int_0^T (\rho_1\partial_t^{\alpha+1}\theta,\Lambda_1)_{L^2(0,L)}dt - \kappa_1\int_0^T(\theta_{xx},\Lambda_1)_{L^2(0,L)}dt - \kappa_1\int_0^T(\phi_x,\Lambda_1)_{L^2(0,L)}dt + \int_0^T(\theta_t,\Lambda_1)_{L^2(0,L)}dt \\
&+\int_0^T(\rho_2\partial_t^{\alpha+1}\phi,\Lambda_2)_{L^2(0,L)}dt - \kappa_2\int_0^T(\phi_{xx},\Lambda_2)_{L^2(0,L)}dt + \kappa_1\int_0^T((\theta_x,\Lambda_2)_{L^2(0,L)}dt \\
&+\kappa_1\int_0^T(\phi,\Lambda_2)_{L^2(0,L)}dt + \int_0^T(\int_0^t m(t-s)\phi_{xx}(x,s)ds,\Lambda_2)_{L^2(0,L)}dsdt \qquad (5.2)\\
= &\ 0.
\end{aligned}$$

Assume that the functions $\xi(x,t)$, $\eta(x,t)$and satisfy the boundary and the initial conditions (2.3), and (2.4) and such that ξ,η, ξ_x,η_x, $\mathcal{I}_t\xi$, $\mathcal{I}_t\eta$, $\mathcal{I}_t\mathcal{I}_x^2\xi$, $\mathcal{I}_t\mathcal{I}_x^2\eta$, $\mathcal{I}_t^2\xi$, $\mathcal{I}_t^2\eta$ and $\partial_t^{\beta+1}\xi$, $\partial_t^{\beta+1}\eta \in L^2(0,T;L^2(0,L))$, we then set

$$\theta(x,t) = \mathcal{I}_t^2\xi = \int_0^t\int_0^s \xi(x,z)dzds,\ \ \phi(x,t) = \mathcal{I}_t^2\eta = \int_0^t\int_0^s \eta(x,z)dzds, \tag{5.3}$$

and introduce the functions

$$\Lambda_1(x,t) = \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi,\ \ \Lambda_2(x,t) = \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta. \tag{5.4}$$

Equation (5.2) then reduces to

$$\begin{aligned}
& \int_0^T (\rho_1\partial_t^{\alpha+1}\mathcal{I}_t^2\xi, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)}dt - \kappa_1\int_0^T (\mathcal{I}_t^2\xi_{xx}, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)}dt \\
& -\kappa_1\int_0^T (\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)}dt + \int_0^T (\mathcal{I}_t\xi, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)}dt \\
& + \int_0^T (\rho_2\partial_t^{\alpha+1}\mathcal{I}_t^2\eta, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}dt - \kappa_2\int_0^T (\mathcal{I}_t^2\eta_{xx}, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}dt \\
& +\kappa_1\int_0^T ((\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}dt + \kappa_1\int_0^T (\mathcal{I}_t^2\eta, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}dt \\
& + \int_0^T (\int_0^t m(t-s)\mathcal{I}_s^2\eta_{xx}(x,s)ds, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}dt \\
= \ & 0.
\end{aligned} \tag{5.5}$$

Invoking boundary integral conditions and carrying out appropriate integrations by parts of each inner product term, we have

$$\begin{aligned}
& (\rho_1\partial_t^{\alpha+1}\mathcal{I}_t^2\xi, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)} \\
= \ & (\rho_1\partial_t^{\alpha}\mathcal{I}_t\xi, \mathcal{I}_t\xi) + (\rho_1\partial_t^{\alpha}\mathcal{I}_x\mathcal{I}_t\xi, \mathcal{I}_x\mathcal{I}_t\xi)_{L^2(0,L)},
\end{aligned} \tag{5.6}$$

$$\begin{aligned}
& -\kappa_1(\mathcal{I}_t^2\xi_{xx}, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)} \\
= \ & \frac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\xi_x\|_{L^2(0,L)}^2 + \frac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\xi\|_{L^2(0,L)}^2,
\end{aligned} \tag{5.7}$$

$$\begin{aligned}
& -\kappa_1(\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)} \\
= \ & -\kappa_1(\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi)_{L^2(0,L)} + \kappa_1(\mathcal{I}_t^2\eta, \mathcal{I}_x\mathcal{I}_t\xi)_{L^2(0,L)},
\end{aligned} \tag{5.8}$$

$$\begin{aligned}
& (\mathcal{I}_t\xi, \mathcal{I}_t\xi + \mathcal{I}_x^2\mathcal{I}_t\xi)_{L^2(0,L)} \\
= \ & \|\Im_t\xi\|_{L^2(0,L)}^2 - \|\mathcal{I}_x\Im_t\xi\|_{L^2(0,L)}^2,
\end{aligned} \tag{5.9}$$

$$\begin{aligned}
& (\rho_2\partial_t^{\alpha+1}\mathcal{I}_t^2\eta, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)} \\
= \ & (\rho_2\partial_t^{\alpha}\mathcal{I}_t\eta, \mathcal{I}_t\eta) + (\rho_2\partial_t^{\alpha}\mathcal{I}_x\mathcal{I}_t\eta, \mathcal{I}_x\mathcal{I}_t\eta)_{L^2(0,L)},
\end{aligned} \tag{5.10}$$

$$\begin{aligned}
& -\kappa_2(\mathcal{I}_t^2\eta_{xx}, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)} \\
= \ & \frac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2\eta_x\|_{L^2(0,L)}^2 + \frac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2\eta\|_{L^2(0,L)}^2
\end{aligned} \tag{5.11}$$

$$\begin{aligned}&\kappa_1(\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}\\ = \;&\kappa_1(\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta)_{L^2(0,L)} - \kappa_1(\mathcal{I}_t^2\xi, \mathcal{I}_x\mathcal{I}_t\eta)_{L^2(0,L)},\end{aligned}\tag{5.12}$$

$$\begin{aligned}&(\int_0^t m(t-s)\mathcal{I}_s^2\eta_{xx}(x,s)ds, \mathcal{I}_t\eta + \mathcal{I}_x^2\mathcal{I}_t\eta)_{L^2(0,L)}\\
= \;& -(\int_0^t m(t-s)\mathcal{I}_s^2\eta_x(x,s)ds, \mathcal{I}_t\eta_x)_{L^2(0,L)} + (\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)}\\
= \;& -\frac{d}{dt}\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m\,(t-s)\,(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx\\
&+\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m'\,(t-s)\,(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx + \int_0^L m\,(0)\left(\mathcal{I}_t^2\eta_x\right)^2dx\\
&+(\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)}.\end{aligned}\tag{5.13}$$

Insertion of equations (5.6)-(5.13) into (5.5), and using Lemma 2.2, yields

$$\begin{aligned}&\frac{\rho_1}{2}{}^C\partial_t^\alpha\left\|\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} + \frac{\rho_1}{2}{}^C\partial_t^\alpha\left\|\mathcal{I}_x\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} + \frac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)}\\
&+\frac{\kappa_1\partial}{2\partial t}\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)} + \|\mathcal{I}_t\xi\|^2_{L^2(0,L)} + \frac{\rho_2}{2}{}^C\partial_t^\alpha\left\|\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)}\\
&+\frac{\rho_2}{2}{}^C\partial_t^\alpha\left\|\mathcal{I}_x\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)} + \frac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} + \frac{\kappa_2\partial}{2\partial t}\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}\\
&+\int_0^L h\,(0)\left(\mathcal{I}_t^2\eta_x\right)^2dx\\
\leq\;& \kappa_1(\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi)_{L^2(0,L)} - \kappa_1(\mathcal{I}_t^2\eta, \mathcal{I}_x\mathcal{I}_t\xi)_{L^2(0,L)} + \|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,L)}\\
&-\kappa_1(\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta)_{L^2(0,L)} + \kappa_1(\mathcal{I}_t^2\xi, \mathcal{I}_x\mathcal{I}_t\eta)_{L^2(0,L)}\\
&+\frac{d}{dt}\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m\,(t-s)\,(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx\\
&-\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m'\,(t-s)\,(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx\\
&-(\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta)_{L^2(0,L)}.\end{aligned}\tag{5.14}$$

By using Cauchy ϵ -inequality, we estimate each term of the right-hand side of previous relations to get

$$\kappa_1(\mathcal{I}_t^2\eta_x, \mathcal{I}_t\xi)_{L^2(0,L)} \leq \frac{\kappa_1}{2}\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} + \frac{\kappa_1}{2}\|\mathcal{I}_t\xi\|^2_{L^2(0,L)}, \tag{5.15}$$

$$-\kappa_1(\mathcal{I}_t^2\eta, \mathcal{I}_x\mathcal{I}_t\xi)_{L^2(0,L)} \leq \frac{\kappa_1}{2}\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)} + \frac{\kappa_1}{2}\|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,L)}, \tag{5.16}$$

$$-\kappa_1(\mathcal{I}_t^2\xi_x, \mathcal{I}_t\eta)_{L^2(0,L)} \leq \frac{\kappa_1}{2}\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)} + \frac{\kappa_1}{2}\|\mathcal{I}_t\eta\|^2_{L^2(0,L)}, \tag{5.17}$$

$$\kappa_1(\mathcal{I}_t^2\xi, \mathcal{I}_x\mathcal{I}_t\eta)_{L^2(0,L)} \leq \frac{\kappa_1}{2}\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)} + \frac{\kappa_1}{2}\|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,L)}, \tag{5.18}$$

$$\begin{aligned} & -\int_0^L \mathcal{I}_t^2\eta_x\left(\int_0^t m'(t-s)\,(\mathcal{I}_s^2\eta_x)(x,s)ds\right)dx \\ \leq\ & \frac{1}{2}\sup_{0\leq t\leq T}|m'|\left(1+\frac{T^2}{2}\right)\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)}, \end{aligned} \tag{5.19}$$

$$\begin{aligned} & -\left(\int_0^t m(t-s)\mathcal{I}_s^2\eta(x,s)ds, \mathcal{I}_t\eta\right)_{L^2(0,L)} \\ & \frac{1}{2}\sup_{0\leq t\leq T}|m|\,\|\mathcal{I}_t\eta\|^2_{L^2(0,L)} + \frac{T^2}{2}\sup_{0\leq t\leq T}|m|\,\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}. \end{aligned} \tag{5.20}$$

By combining equality (5.14) and inequalities (5.15)-(5.20), we obtain

$$\begin{aligned} & {}^C\partial_t^\alpha\,\|\mathcal{I}_t\xi)\|^2_{L^2(0,L)} + {}^C\partial_t^\alpha\,\|\mathcal{I}_t\eta)\|^2_{L^2(0,L)} + {}^C\partial_t^\alpha\,\|\mathcal{I}_x\mathcal{I}_t\xi)\|^2_{L^2(0,L)} \\ & + {}^C\partial_t^\alpha\,\|\mathcal{I}_x\mathcal{I}_t\eta)\|^2_{L^2(0,L)} + \frac{\partial}{\partial t}\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)} + \frac{\partial}{\partial t}\|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} \\ & + \frac{\partial}{\partial t}\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)} + \frac{\partial}{\partial t}\|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)} + \int_0^L \left(\mathcal{I}_t^2\eta_x\right)^2dx + \|\mathcal{I}_t\xi\|^2_{L^2(0,L)} \\ \leq\ & \mathcal{W}\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,L)} + \|\mathcal{I}_t\eta\|^2_{L^2(0,L)} + \|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,L)} + \|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,L)} \\ & + \|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}\Big), \end{aligned} \tag{5.21}$$

where

$$\mathcal{W} = \frac{\max\left\{\frac{\kappa_1}{2} + (1+\frac{T^2}{2})\frac{1}{2}\sup|m'|\,, \frac{\kappa_1}{2} + \frac{T^2}{2}\frac{1}{2}\sup|m|\right\}}{\min\left\{1, \frac{\rho_1}{2}, \frac{\rho_2}{2}, \frac{\kappa_1}{2}, \frac{\kappa_2}{2}, h(0)\right\}}. \tag{5.22}$$

By discarding the last two terms from the left hand side of (5.21), replacing t by τ in (5.22) and then integrating with respect to τ over the interval $(0,t)$, we obtain

$$\begin{aligned} & D_t^{\alpha-1}\left\|\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} + D_t^{\alpha-1}\left\|\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)} + D_t^{\alpha-1}\left\|\mathcal{I}_x\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} \\ & +D_t^{\alpha-1}\left\|\mathcal{I}_x\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} \\ & +\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)} \\ \leq\ & \mathcal{W}\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} \\ & +\|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}\Big) \end{aligned} \tag{5.23}$$

If we omit the first four terms on the left hand side of (5.23), and use Gronwall-Bellman lemma, by taking

$$\begin{cases} \mathcal{R}(t) = \|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,t;L^2(0,L))} \\ \quad +\|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}, \\ \frac{\partial\mathcal{R}(t)}{\partial t} = \|\mathcal{I}_t^2\xi_x\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\eta_x\|^2_{L^2(0,L)} \\ \quad +\|\mathcal{I}_t^2\xi\|^2_{L^2(0,L)} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,L)}, \\ \mathcal{R}(t) = 0, \end{cases} \tag{5.24}$$

we obtain

$$\begin{aligned} \mathcal{R}(t) \leq\ & Te^{T\mathcal{W}}\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_x\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} \\ & +\|\mathcal{I}_x\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))}\Big). \end{aligned} \tag{5.25}$$

Next, if we disregard the last four terms on the left-hand side and take into account the inequality (5.25), we end with

$$\begin{aligned} & D_t^{\alpha-1}\left\|\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} + D_t^{\alpha-1}\left\|\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)} + D_t^{\alpha-1}\left\|\mathcal{I}_x\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} \\ & +D_t^{\alpha-1}\left\|\mathcal{I}_x\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)} \\ \leq\ & \mathcal{W}\left(Te^{T\mathcal{W}}+1\right)\Big(\|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} \\ & + \|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}\Big). \end{aligned} \tag{5.26}$$

Now, we are able to apply lemma 2.2, by letting

$$\begin{cases} Q(t) = \|\mathcal{I}_t\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t\eta\|^2_{L^2(0,t;L^2(0,L))} \\ \quad \|\mathcal{I}_t^2\xi\|^2_{L^2(0,t;L^2(0,L))} + \|\mathcal{I}_t^2\eta\|^2_{L^2(0,t;L^2(0,L))}, \\ {}^C\partial_t^\alpha Q(t) = D_t^{\alpha-1}\left\|\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} + D_t^{\alpha-1}\left\|\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)} \\ \quad +D_t^{\alpha-1}\left\|\mathcal{I}_x\mathcal{I}_t\xi)\right\|^2_{L^2(0,L)} + D_t^{\alpha-1}\left\|\mathcal{I}_x\mathcal{I}_t\eta)\right\|^2_{L^2(0,L)}, \\ Q(0) = 0, \end{cases} \tag{5.27}$$

we infer from (5.26) that

$$Q(t) \leq \Gamma(\alpha)E_{\alpha,\alpha}(\mathcal{W}\left(Te^{T\mathcal{W}}+1\right)t^\alpha)D_t^{-\alpha}(0) = 0. \tag{5.28}$$

We conclude from (5.28), and (5.27) that $\xi = 0, \eta = 0$. Consequently, $W(x,t) = (\Lambda_1(x,t), \Lambda_2(x,t)) = (0,0)$ a.e in Q^T.

We now consider the general case for density

Since $\mathcal{E}$ is a Hilbert space, then $\overline{R(\mathcal{G})} = \mathcal{E} \Leftrightarrow R(\mathcal{G})^\perp = \{0\} \Leftrightarrow (\mathcal{G}\mathcal{Z}, \mathcal{K})_\mathcal{E} = 0$, for all $\mathcal{Z} \in \mathcal{B}$, and $\mathcal{K} \in \mathcal{E}$,then $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2) = \{(J_1, J_3, J_4), (J_2, J_5, J_6)\} =$

$(0,0)$, that is $J_1 = J_2 = J_3 = J_4 = J_5 = J_6 = 0$. So suppose that for some element $\mathcal{K} = (\mathcal{K}_1, \mathcal{K}_2) = \{(J_1, J_3, J_4), (J_2, J_5, J_6)\} \in R(\mathcal{G})^{\perp}$

$$\begin{aligned}
& (\mathcal{G}\mathcal{Z}, \mathcal{K})_{\mathcal{E}} \\
= & (\{\mathcal{S}_1(\theta,\phi), \mathcal{S}_2(\theta,\phi\}, \{\mathcal{K}_1, \mathcal{K}_2\})_{\mathcal{E}} \\
= & (\{\mathcal{S}_1(\theta,\phi), \Gamma_1\theta, \Gamma_2\theta\}, \{\mathcal{S}_2(\theta,\phi), \Gamma_1\phi, \Gamma_2\phi\}\}, \{\{J_1, J_2, J_3\}, \{J_4, J_5, J_6\}\})_{\mathcal{E}} \\
= & (\mathcal{S}_1(\theta,\phi), J_1)_{L^2(Q^T)} + (\Gamma_1\theta, J_2)_{L^2(0,L)} + (\Gamma_2\theta, J_3)_{L^2(0,L)} \\
& +(\mathcal{S}_2(\theta,\phi), J_4)_{L^2(Q^T)} + (\Gamma_1\phi, J_5)_{L^2(0,L)} + (\Gamma_2\phi, J_6)_{L^2(0,L)} \\
= & \ 0, \qquad (5.29)
\end{aligned}$$

where $\mathcal{Z}$ runs over the space $\mathcal{B}$, we have to prove that $\mathcal{K} = 0$.

Let $\mathcal{Z} \in D_0(\mathcal{G})$, then equation (5.29) becomes

$$(\mathcal{S}_1(\theta,\phi), J_1)_{L^2(Q^T)} + (\mathcal{S}_2(\theta,\phi), J_4)_{L^2(Q^T)} = 0. \qquad (5.30)$$

Hence, by virtue of Theorem 4.2, it follows from (5.30) that $J_1 = J_4 = 0$. Consequently, equation (5.29) takes the form

$$(\Gamma_1\theta, J_2)_{L^2(0,L)} + (\Gamma_2\theta, J_3)_{L^2(0,L)} + (\Gamma_1\phi, J_5)_{L^2(0,L)} + (\Gamma_2\phi, J_6)_{L^2(0,L)} = 0. \quad (5.31)$$

Since the four terms in (5.31) vanish independently and since the ranges $R(\Gamma_1), R(\Gamma_2)$ of the trace operators Γ_1, Γ_2 are everywhere dense in the space $L^2(0,L)$, then it follows from (5.31) that $J_2 = J_3 = J_5 = J_6 = 0$. Consequently $\mathcal{K} = 0$, that is $R(\mathcal{G})^{\perp} = \{0\}$. Thus $\overline{R(\mathcal{G})} = \mathcal{E}$.

Conclusion: In this paper, we investigated a non-local non-homogeneous fractional Timoshenko system with frictional and viscoelastic damping terms. This fractional order system is supplemented with some initial conditions and classical and non local boundary conditions of integral type. The well posedness of the given non local initial boundary value problem is established. The used approach relies on some functional analysis tools, operator theory, a prori estimates and density arguments. To the best of our knowledge, the treated fractional Timoshenko system problem has never been studied and explored in the literature. This work can be considered as a contribution in the development of the traditional functional analysis method, the so called a priori estimate method or the energy inequalities method used to prove the existence, uniqueness and stability of initial boundary value problems with non local boundary conditions such as integral conditions.

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