Nonexistence of ground state sign-changing solutions for autonomous Schr\”{o}dinger-Poisson system with critical growth

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Abstract

The paper is concerned with the following Schr\”{o}dinger-Poisson system $$ \left\{\begin{array}{ll} -\Delta u+u+\phi u=|u|^{p-1}u+|u|^4u, &x\in \mathbb{R}^3, \\[0.25cm] -\Delta\phi=u^2, &x\in \mathbb{R}^3, \end{array} \right. $$ where $3

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NONEXISTENCE OF GROUND STATE SIGN-CHANGING SOLUTIONS FOR AUTONOMOUS SCHRÖDINGER-POISSON SYSTEM WITH CRITICAL GROWTH

YING WANG* AND RONG YUAN

Abstract. The paper is concerned with the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u + |u|^4 u, & x \in \mathbb{R}^3, \\ -\Delta \phi = u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $3 < p < 5$. With the help of an odd Nehari manifold and "energy doubling" property, we prove the nonexistence of ground state sign-changing solutions on $H^1(\mathbb{R}^3)$. In this sense, our result explains why the existing literature can only consider the existence of the ground state sign-changing solutions in the radial Sobolev space $H_r^1(\mathbb{R}^3)$.

1. Introduction

In the present paper we are interested in the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + u + \phi u = |u|^{p-1}u + |u|^4 u, & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $3 < p < 5$. Due to its physical relevance as described in [4], system (1.1) or its more general form has been extensively investigated via the variational methods in the past decades, such as [1,3,5,12,18] and the references listed therein.

Recently, many authors began to focus on sign-changing solutions for Schrödinger-Poisson system. For this topic, there are many interesting works for the case that the nonlinearity is of subcritical growth, such as [2,9,10,13,15] and the references mentioned therein. Due to the lack of compactness of $H^1(\mathbb{R}^3) \hookrightarrow L^6(\mathbb{R}^3)$, the investigations on Schrödinger-Poisson system with critical growth are more complicated and challenging from the mathematical point view. For this case, as far as we know, only [6,8,11,14,16, 17,19] dealt with the existence of sign-changing solutions. Meanwhile, it must be pointed out that, in [14,17], to verify the (PS) condition conveniently, the radial space $H_r^1(\mathbb{R}^3)$ is a good choice for autonomous case as the energy space due to the compactness of the embedding $H_r^1(\mathbb{R}^3) \hookrightarrow L^q(\mathbb{R}^3)$, $q \in (2,6)$. Therefore, it is a very interesting problem whether there exist ground state sign-changing solutions for autonomous system (1.1) in the usual Sobolev space $H^1(\mathbb{R}^3)$.

The aim of this paper is to supplement the existing results in [14, 17] and give a negative answer to the above problem. In fact, our main result is stated as follows.

Theorem 1.1. *Assume that $p \in (3,5)$, then system (1.1) does not possess ground state sign-changing solutions in $H^1(\mathbb{R}^3)$.*

As is known to all, weak solutions of system (1.1) can be obtained as critical points of the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx + \frac{1}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{1}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx - \frac{1}{6}\int_{\mathbb{R}^3} |u|^6 dx,$$

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where ϕ_u is the unique solution satisfying $-\Delta\phi = u^2$ obtained by the Lax-Milgram theorem. Recall that u is a weak solution of system (1.1) if $u \in H^1(\mathbb{R}^3)$ satisfies $\langle I'(u), \varphi\rangle = 0$ for any $\varphi \in H^1(\mathbb{R}^3)$. Moreover, if u is a solution of system (1.1) with $u^{\pm} \not\equiv 0$, then u is called a sign-changing solution (nodal solution), where $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = \min\{u(x), 0\}$. A solution is called a ground state solution if its energy is minimal among all nontrivial solutions.

To find ground state sign-changing solutions of system (1.1), the usual strategy is to deal with the following minimizing problem

$$c_{\mathcal{M}} := \inf_{\mathcal{M}} I(u), \tag{1.2}$$

where $\mathcal{M}$ is the corresponding sign-changing Nehari manifold

$$\mathcal{M} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'(u), u^+\rangle = 0 = \langle I'(u), u^-\rangle, u^{\pm} \neq 0\}.$$

In our context, to reach the conclusion, we not only need to consider the Nehari manifold $\mathcal{N}$ but also seek for the help of an odd Nehari manifold $\mathcal{N}_{odd}$ defined as $\mathcal{N}_{odd} := \mathcal{N} \cap H^1_{odd}(\mathbb{R}^3)$. Here, $\mathcal{N}$ is the usual Nehari manifold given by $\mathcal{N} := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : \langle I'(u), u\rangle = 0\}$ and $H^1_{odd}(\mathbb{R}^3)$ is the Sobolev space of odd functions respect to the third component (introduced in [7]) denoted by $H^1_{odd}(\mathbb{R}^3) := \{u \in H^1(\mathbb{R}^3)\backslash\{0\} : u(x', -x_3) = -u(x', x_3), \forall x = (x', x_3) \in \mathbb{R}^3\}$. At this point, define the corresponding levels on $\mathcal{N}$ and $\mathcal{N}_{odd}$ by

$$c_{\mathcal{N}} := \inf_{\mathcal{N}} I(u) \quad \text{and} \quad c_{odd} := \inf_{\mathcal{N}_{odd}} I(u), \tag{1.3}$$

respectively. Then, by contradiction and discussing the relationship among these three levels $c_{\mathcal{M}}$, $c_{\mathcal{N}}$ and c_{odd}, we are able to accomplish the proof of Theorem 1.1.

Throughout this paper, we denote by C a positive constant whose value may change from line to line, by $\|\cdot\|$ the norm of $H^1(\mathbb{R}^3)$, and by $x = (x', x_3)$ the point in $\mathbb{R}^3$. For any $R > 0$ and any $x \in \mathbb{R}^3$, $B_R(x)$ is the ball of radius R centered at x. The rest of the present paper is organized as follows. In Section 2 we state two useful lemmas related to the energy functional and the Nehari manifold. In Section 3, we give the proof of Theorem 1.1.

2. Preliminaries

Firstly, we present a technical lemma which is essentially related to the energy functional.

Lemma 2.1. *Define $\psi_\beta : \mathbb{R}_0^+ \to \mathbb{R}$ by $\psi_\beta(t) = \alpha t^2 + \beta t^4 - \gamma t^{p+1} - \delta t^6$, $\forall t \geq 0$, where $\alpha, \beta, \gamma, \delta$ be positive constants and $p > 3$. Then*

(i) ψ_β has a unique critical point t^* corresponding to its maximum. Moreover, $\psi_\beta(t)$ is strictly increasing in $(0, t^*)$ and strictly decreasing in $(t^*, +\infty)$;

(ii) if $\beta_1 > \beta_2 > 0$ and $\psi'_{\beta_1}(t_1^*) = \psi'_{\beta_2}(t_2^*) = 0$ with $t_1^*, t_2^* > 0$, we have $t_1^* > t_2^*$.

Proof. A direct calculation gives the derivatives of ψ_β up to five order:

$$\begin{aligned}
\psi'_\beta(t) &= 2\alpha t + 4\beta t^3 - (p+1)\gamma t^p - 6\delta t^5,\\
\psi''_\beta(t) &= 2\alpha + 12\beta t^2 - (p+1)p\gamma t^{p-1} - 30\delta t^4,\\
\psi^{(3)}_\beta(t) &= 24\beta t - (p+1)p(p-1)\gamma t^{p-2} - 120\delta t^3,\\
\psi^{(4)}_\beta(t) &= 24\beta - (p+1)p(p-1)(p-2)\gamma t^{p-3} - 360\delta t^2,\\
\psi^{(5)}_\beta(t) &= -(p+1)p(p-1)(p-2)(p-3)\gamma t^{p-4} - 720\delta t.
\end{aligned}$$

Since $p > 3$, it is obvious that $\psi^{(5)}_\beta(t) < 0$ for any $t \in (0, +\infty)$, which means that $\psi^{(4)}_\beta(t)$ decreasing on the interval $[0, +\infty)$. Note that $\psi^{(4)}_\beta(0) = 24\beta > 0$, then there exists a unique $t_4 > 0$ such that $\psi^{(4)}_\beta(t_4) = 0$ and $\psi^{(4)}_\beta(t)(t_4 - t) > 0$ for $t \neq t_4$. As for $\psi^{(3)}_\beta(t)$, since $\psi^{(3)}_\beta(0) = 0$ and $\psi^{(4)}_\beta(t) > 0$, $\forall t \in (0, t_4)$, $\psi^{(3)}_\beta(t)$ increases and takes positive values for $t \in (0, t_4]$; $\psi_\beta^{(3)}(t)$ decreases for $t > t_4$ and tends to $-\infty$ thanks to $\psi_\beta^{(4)}(t) < 0$, $\forall t > t_4$. Then, there exists a unique $t_3 > t_4$ such that $\psi_\beta^{(3)}(t_3) = 0$ and $\psi_\beta^{(3)}(t)(t_3 - t) > 0$ for $t \neq t_3$.

Repeating the argument for ψ_β'' and ψ_β' as we did for $\psi_\beta^{(4)}$ and $\psi_\beta^{(3)}$, we can conclude the existence of $t^* > 0$ such that $\psi_\beta'(t^*) = 0$ and $\psi_\beta'(t)(t^* - t) > 0$ for $t \neq t^*$. Therefore, t^* is the unique critical point of ψ_β corresponding to its maximum. Moreover, $\psi_\beta(t)$ is strictly increasing in $(0, t^*)$ and strictly decreasing in $(t^*, +\infty)$.

Next, we turn to *(ii)*. Note that $\psi_\beta'(t) = 2\alpha t + 4\beta t^3 - (p+1)\gamma t^p - 6\delta t^5$. Clearly, the assumption $\beta_2 < \beta_1$ implies that $\psi_{\beta_2}'(t_1^*) < \psi_{\beta_1}'(t_1^*) = \psi_{\beta_2}'(t_2^*) = 0$. As a direct result of *(i)*, it brings that $t_1^* > t_2^*$. □

Next, we show some properties related to the Nehari manifold $\mathcal{N}$.

Lemma 2.2. *The following statements are true:*

(i) *there exists $\rho > 0$ such that $\|u\| \geq \rho$ for any $u \in \mathcal{N}$;*

(ii) *for each $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, there exists a unique $t_u > 0$ such that $t_u u \in \mathcal{N}$ and $I(t_u u) = \max_{t\geq 0} I(tu)$. Moreover, the map $H^1(\mathbb{R}^3)\backslash\{0\} \to (0, +\infty) : u \mapsto t_u$ is continuous;*

(iii) *if $u \in H^1(\mathbb{R}^3)\backslash\{0\}$ satisfies $I'(u)u < 0$, then there exists a unique $t_u \in (0, 1)$ such that $t_u u \in \mathcal{N}$;*

(iv) *$c_{\mathcal{N}} > 0$.*

Proof. *(i)* For any fixed $u \in \mathcal{N}$, using Sobolev inequality, we have

$$\|u\|^2 < \int_{\mathbb{R}^3} (|\nabla u|^2 + |u|^2)dx + \int_{\mathbb{R}^3} \phi_u u^2 dx = \int_{\mathbb{R}^3} |u|^{p+1}dx + \int_{\mathbb{R}^3} |u|^6 dx \leq C(\|u\|^{p+1} + \|u\|^6).$$

The above inequality indicates that there exists $\rho > 0$ independent of $u \in \mathcal{N}$ such that $\|u\| \geq \rho$.

(ii) For any given $u \in H^1(\mathbb{R}^3)\backslash\{0\}$, consider the fibering map $g_u : \mathbb{R}^+ \to \mathbb{R}$ defined by

$$g_u(t) := I(tu) = \frac{t^2}{2}\|u\|^2 + \frac{t^4}{4}\int_{\mathbb{R}^3} \phi_u u^2 dx - \frac{t^{p+1}}{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx - \frac{t^6}{6}\int_{\mathbb{R}^3} |u|^6 dx, \ t \geq 0.$$

Therefore, apply Lemma 2.1 *(i)*, g_u has a unique maximum point $t_u > 0$ such that $g_u'(t_u) = 0$ and $g_u(t_u) = \max_{t\geq 0} g_u(t)$. Namely, $t_u u \in \mathcal{N}$ and $I(t_u u) = \max_{t\geq 0} I(tu)$.

To show the continuity of t_u, suppose $u_n \to u$ in $H^1(\mathbb{R}^3)$ as $n \to +\infty$. Note that, for each u_n, there exists a unique $t_{u_n} > 0$ such that $t_{u_n} u_n \in \mathcal{N}$. Since $u_n \to u \neq 0$ and $\langle I'(t_{u_n} u_n), t_{u_n} u_n\rangle = 0$, we infer that $\{t_{u_n}\}$ is bounded. Up to a subsequence if necessary, still denoted by $\{t_{u_n}\}$, there exists $t_0 \geq 0$ such that $t_{u_n} \to t_0$ as $n \to +\infty$. In fact, using *(i)*, we deduce that $\|t_{u_n} u_n\| \geq \rho$, which implies that $t_0 > 0$. Then, from $\langle I'(t_0 u), t_0 u\rangle = 0 = \langle I'(t_u u), t_u u\rangle$ and the uniqueness of t_u, we readily derive that $t_u = t_0$.

(iii) In view of $t_u u \in \mathcal{N}$, we have

$$t_u^2\|u\|^2 + t_u^4 \int_{\mathbb{R}^3} \phi_u u^2 dx - t_u^{p+1}\int_{\mathbb{R}^3} |u|^{p+1}dx - t_u^6 \int_{\mathbb{R}^3} |u|^6 dx = 0,$$

which equals to

$$\frac{1}{t_u^2}\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx - t_u^{p-3}\int_{\mathbb{R}^3} |u|^{p+1}dx - t_u^2 \int_{\mathbb{R}^3} |u|^6 dx = 0. \tag{2.1}$$

Since $I'(u)u < 0$, there exists

$$\|u\|^2 + \int_{\mathbb{R}^3} \phi_u u^2 dx - \int_{\mathbb{R}^3} |u|^{p+1}dx - \int_{\mathbb{R}^3} |u|^6 dx < 0,$$

from which and (2.1), it immediately yields that

$$\left(\frac{1}{t_u^2} - 1\right)\|u\|^2 - \left(t_u^{p-3} - 1\right)\int_{\mathbb{R}^3} |u|^{p+1}dx - \left(t_u^2 - 1\right)\int_{\mathbb{R}^3} |u|^6 dx > 0.$$

If $t_u \geq 1$, the above inequality is meaningless. Thus, it must be $0 < t_u < 1$.

(iv) To show $c_0 > 0$, it is sufficient to notice that

$$I(u) = I(u) - \frac{1}{4}I'(u)u = \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3}|u|^{p+1}dx + \frac{1}{12}\int_{\mathbb{R}^3}|u|^6dx \geq \frac{1}{4}\|u\|^2,$$

and use *(i)*. □

3. Proof of Theorem 1.1

In this section, we are dedicated to finishing the proof of Theorem 1.1. To accomplish this purpose, we use the "energy doubling" property and estimate the level c_{odd} defined in (1.3). To make the proof clear, we divide it into two steps.

Proof of Theorem 1.1.

Step 1. If c is achieved, then $c_{\mathcal{M}} > 2c_{\mathcal{N}}$.

Assuming that there exists some $\hat{u} \in \mathcal{M}$ such that $I(\hat{u}) = c_{\mathcal{M}}$, from the definition of $\mathcal{M}$, we have

$$\begin{aligned}
0 = I'(\hat{u})\hat{u}^{\pm} &= \int_{\mathbb{R}^3}(|\nabla\hat{u}^{\pm}|^2 + |\hat{u}^{\pm}|^2 + \phi_{\hat{u}^{\pm}}(\hat{u}^{\pm})^2 + \phi_{\hat{u}^{\mp}}(\hat{u}^{\pm})^2 - |\hat{u}^{\pm}|^{p+1} - |\hat{u}^{\pm}|^6)dx \\
&> \int_{\mathbb{R}^3}(|\nabla\hat{u}^{\pm}|^2 + |\hat{u}^{\pm}|^2 + \phi_{\hat{u}^{\pm}}(\hat{u}^{\pm})^2 - |\hat{u}^{\pm}|^{p+1} - |\hat{u}^{\pm}|^6)dx \\
&= I'(\hat{u}^{\pm})\hat{u}^{\pm}.
\end{aligned}$$

Then, using Lemma 2.2, we get the existence of $t_{\hat{u}^{\pm}} \in (0,1)$ such that $t_{\hat{u}^{\pm}}\hat{u}^{\pm} \in \mathcal{N}$. Therefore, we deduce that

$$\begin{aligned}
I(\hat{u}) =& I(\hat{u}) - \frac{1}{4}\langle I'(\hat{u}), \hat{u}\rangle \\
=& \frac{1}{4}\int_{\mathbb{R}^3}(|\nabla\hat{u}|^2 + |\hat{u}|^2)dx + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3}|\hat{u}|^{p+1}dx + \frac{1}{12}\int_{\mathbb{R}^3}|\hat{u}|^6dx \\
>& \frac{1}{4}\int_{\mathbb{R}^3}(|t_{\hat{u}^+}\nabla\hat{u}^+|^2 + |t_{\hat{u}^+}\hat{u}^+|^2)dx + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3}|t_{\hat{u}^+}\hat{u}^+|^{p+1}dx + \frac{1}{12}\int_{\mathbb{R}^3}|t_{\hat{u}^+}\hat{u}^+|^6dx \\
&+ \frac{1}{4}\int_{\mathbb{R}^3}|t_{\hat{u}^-}\nabla\hat{u}^-|^2 + |t_{\hat{u}^-}\hat{u}^-|^2dx + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3}|t_{\hat{u}^-}\hat{u}^-|^{p+1}dx + \frac{1}{12}\int_{\mathbb{R}^3}|t_{\hat{u}^-}\hat{u}^-|^6dx \\
=& I(t_{\hat{u}^+}\hat{u}^+) + I(t_{\hat{u}^-}\hat{u}^-) \geq 2c_{\mathcal{N}},
\end{aligned}$$

which indicates that $c_{\mathcal{M}} > 2c_{\mathcal{N}}$.

Step 2. $c_{odd} \leq 2c_{\mathcal{N}}$.

The idea here is to find an element in $\mathcal{N}_{odd}$ such that the value of I is less than or equal to $2c_{\mathcal{N}}$ on this element. We firstly observe that

$$c_{\mathcal{N}} = \inf\{I(u) : u \in \mathcal{N} \cap C_c^{\infty}(\mathbb{R}^3) \text{ and } u \geq 0 \text{ in } \mathbb{R}^3\}. \tag{3.1}$$

Actually, if $u \in \mathcal{N}$, then $|u| \in \mathcal{N}$ and it holds that $c_{\mathcal{N}} = \inf\{I(u) : u \in \mathcal{N} \text{ and } u \geq 0 \text{ in } \mathbb{R}^3\}$. Since $C_c^{\infty}(\mathbb{R}^3)$ is dense in $H^1(\mathbb{R}^3)$, for each $u \in \mathcal{N}$ with $u \geq 0$, there exists a sequence $\{u_n\} \subset C_c^{\infty}(\mathbb{R}^3)$ with $u_n \geq 0$ such that $u_n \to u$ in $H^1(\mathbb{R}^3)$. For each u_n, by *(ii)* of Lemma 2.2, there is a unique $t_{u_n} > 0$ such that $t_{u_n}u_n \in \mathcal{N}$. In view of $u_n \to u$ $H^1(\mathbb{R}^3)$ and the continuity of t_u, we see that $t_{u_n} \to 1$ as $n \to +\infty$. Therefore, we obtain a sequence $\{t_{u_n}u_n\} \subset \mathcal{N} \cap C_c^{\infty}(\mathbb{R}^3)$ satisfying $t_{u_n}u_n > 0$ and $t_{u_n}u_n \to u$ as $n \to \infty$. Based on this point, (3.1) is verified. Consequently, for any $\epsilon > 0$, we can find a positive function $u \in \mathcal{N} \cap C_c^{\infty}(\mathbb{R}^3)$ such that $\langle I'(u), u\rangle = 0$ and $I(u) \leq c_{\mathcal{N}} + \epsilon$. Namely, one has

$$\begin{aligned}
I(u) &= I(u) - \frac{1}{4}I'(u)u \\
&= \frac{1}{4}\|u\|^2 + \left(\frac{1}{4} - \frac{1}{p+1}\right)\int_{\mathbb{R}^3}|u|^{p+1}dx + \frac{1}{12}\int_{\mathbb{R}^3}|u|^6dx \\
&\leq c_{\mathcal{N}} + \epsilon.
\end{aligned} \tag{3.2}$$

For this u, we assume its support is contained in $B_R(0)$ for some $R>0$, that is, $\mathrm{supp}(u)\subset B_R(0)$.

Secondly, for each $r>0$, we define the function $\tilde{u}_r(x):\mathbb{R}^3\to\mathbb{R}$ by

$$\tilde{u}_r(x)=v_r(x)-\bar{v}_r(x),$$

where $v_r(x):=u(x',x_3+R+r)$ and $\bar{v}_r(x):=u(x',-x_3+R+r)$, $\forall x=(x',x_3)\in\mathbb{R}^3$. It is clear that $\mathrm{supp}(v_r)\cap\mathrm{supp}(\bar{v}_r)=\emptyset$ and $\tilde{u}_r\in H^1_{odd}(\mathbb{R}^3)$. From Lemma 2.2 *(ii)*, there exists a unique $t_{\tilde{u}_r}>0$ such that $t_{\tilde{u}_r}\tilde{u}_r\in\mathcal{N}$, that is,

$$t_{\tilde{u}_r}^2\|u\|^2+t_{\tilde{u}_r}^4\left(\int_{\mathbb{R}^3}\phi_u u^2dx+\int_{\mathbb{R}^3}\phi_{v_r}\bar{v}_r^2dx\right)-t_{\tilde{u}_r}^{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx-t_{\tilde{u}_r}^6\int_{\mathbb{R}^3}|u|^6dx=0. \tag{3.3}$$

Combining with Sobolev inequality, we derive that

$$t_{\tilde{u}_r}^2\|u\|^2\le t_{\tilde{u}_r}^{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx+t_{\tilde{u}_r}^6\int_{\mathbb{R}^3}|u|^6dx\le C(t_{\tilde{u}_r}^{p+1}\|u\|^{p+1}+t_{\tilde{u}_r}^6\|u\|^6),$$

which indicates that there exists $\tau_u>0$ such that $t_{\tilde{u}_r}\ge\tau_u$.

Thirdly, we talk about the monotonicity of $\int_{\mathbb{R}^3}\phi_{v_r}\bar{v}_r^2dx$ respect to r. Notice that

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{v_r}\bar{v}_r^2dx&=\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{v_r^2(x)\bar{v}_r^2(y)}{|x-y|}dxdy\\&=\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x',x_3+R+r)u^2(y',-y_3+R+r)}{|x-y|}dxdy\\&=\frac{1}{4\pi}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u^2(x)u^2(y)}{|(x',x_3-R-r)-(y',-y_3+R+r)|}dxdy\\&=\frac{1}{4\pi}\int_{B_R(0)}\int_{B_R(0)}\frac{u^2(x)u^2(y)}{\sqrt{|x'-y'|^2+|x_3+y_3-2(R+r)|^2}}dxdy\end{aligned}$$

and $|x'-y'|^2+|x_3+y_3-2(R+r)|^2$ is strictly increasing about $r>0$ for any $x,y\in B_R(0)$. We see that $\int_{\mathbb{R}^3}\phi_{v_r}\bar{v}_r^2dx$ is strictly decreasing about $r>0$. Hence, the application of Lemma 2.1 *(ii)* signifies that $t_{\tilde{u}_r}$ is strictly decreasing with respect to r, which guarantees the existence of $\hat{t}\ge\tau_u$ such that $t_{\tilde{u}_r}\to\hat{t}$ as $r\to+\infty$. Meanwhile, taking into account $\mathrm{dist}(\mathrm{supp}(v_r),\mathrm{supp}(\bar{v}_r))\ge 2r$ and (3.2), we have

$$\begin{aligned}\int_{\mathbb{R}^3}\phi_{v_r}\bar{v}_r^2dx&=\frac{1}{4\pi}\int_{\mathrm{supp}(v_r)}\int_{\mathrm{supp}(\bar{v}_r)}\frac{v_r^2(x)\bar{v}_r^2(y)}{|x-y|}dxdy\\&\le\frac{1}{8\pi r}\left(\int_{\mathbb{R}^3}|u|^2dx\right)^2\\&\le\frac{2(c_0+\varepsilon)^2}{\pi r}\to 0\text{ as }r\to+\infty.\end{aligned}\tag{3.4}$$

Therefore, based on (3.3) and (3.4), we are led to

$$\hat{t}^2\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx+\hat{t}^4\int_{\mathbb{R}^3}\phi_u u^2dx-\hat{t}^{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx-\hat{t}^6\int_{\mathbb{R}^3}|u|^6dx=0.$$

Since $u\in\mathcal{N}$, it immediately gives that $\hat{t}=1$. Thus, $t_{\tilde{u}_r}\to 1$ as $r\to+\infty$. As a result, due to $t_{\tilde{u}_r}\tilde{u}_r\in\mathcal{N}$, we deduce that

$$\begin{aligned}I(t_{\tilde{u}_r}\tilde{u}_r)&=\frac14 t_{\tilde{u}_r}^2\int_{\mathbb{R}^3}(|\nabla\tilde{u}_r|^2+|\tilde{u}_r|^2)dx+\left(\frac14-\frac{1}{p+1}\right)t_{\tilde{u}_r}^{p+1}\int_{\mathbb{R}^3}|\tilde{u}_r|^{p+1}dx+\frac{1}{12}t_{\tilde{u}_r}^6\int_{\mathbb{R}^3}|\tilde{u}_r|^6dx\\&=\frac12 t_{\tilde{u}_r}^2\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx+\left(\frac12-\frac{2}{p+1}\right)t_{\tilde{u}_r}^{p+1}\int_{\mathbb{R}^3}|u|^{p+1}dx+\frac16 t_{\tilde{u}_r}^6\int_{\mathbb{R}^3}|u|^6dx\\&\to\frac12\int_{\mathbb{R}^3}(|\nabla u|^2+|u|^2)dx+\left(\frac12-\frac{2}{p+1}\right)\int_{\mathbb{R}^3}|u|^{p+1}dx+\frac16\int_{\mathbb{R}^3}|u|^6dx\end{aligned}$$

as $r \to +\infty$. This together with (3.2) implies that for any $\epsilon > 0$ there exists $r_\epsilon > 0$ large enough such that

$$I(t_{\tilde{u}_r}\tilde{u}_r) \leq 2c_{\mathcal{N}} + 3\epsilon, \quad \forall r > r_\epsilon. \tag{3.5}$$

Since $\tilde{u}_r \in H^1_{odd}(\mathbb{R}^3)$, equivalently, $t_{\tilde{u}_r}\tilde{u}_r \in \mathcal{N}_{odd}$. Therefore, taking the limit in (3.5), we deduce that $c_{odd} \leq 2c_{\mathcal{N}}$.

Taking into account **Step 1** and **Step 2**, we readily reach to the following contradiction

$$2c_{\mathcal{N}} < c_{\mathcal{M}} \leq c_{odd} \leq 2c_{\mathcal{N}},$$

by observing that $\mathcal{N}_{odd} \subset \mathcal{M}$ implies $c_{odd} \geq c_{\mathcal{M}}$. Naturally, we complete the proof of Theorem 1.1. □

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