Reduced order finite difference scheme based on POD for fractional stochastic advection-diffusion equation

Zoleikha Soori[1] and A. Aminataei[2]

[1]KN Toosi University of Technology
[2]K N Toosi University of Technology Faculty of General Science

February 22, 2024

Abstract

This paper introduce a numerical solution of time fractional stochastic advection-diffusion equa- tion (FSA-DE) wherein time fractional derivative is described in Caputo sence of order α $(0 < \alpha < 1)$. First, a L1 approximation is employed to estimate the Caputo derivative. Then, the spatial derivative is discretized by a second-order finite difference scheme. Moreover, we combine the implicit finite difference (IFD) scheme with the proper orthogonal decomposition (POD) method to reduce the used cpu time. In other words, we obtain POD based reduced-order IFD scheme. As a result, the new scheme can be viewed as the modification of the exiting job (Mirzaee et al., 2020 [23]). The numerical results provide to verify the feasibility and efficiency of the new method.

Reduced order finite difference scheme based on POD for fractional stochastic advection-diffusion equation

Z. Soori* and **A. Aminataei†**

Faculty of Mathematics, K. N. Toosi University of Technology, P. O. Box: 1676-53381, Tehran, Iran

Abstract

This paper introduce a numerical solution of time fractional stochastic advection-diffusion equation (FSA-DE) wherein time fractional derivative is described in Caputo sence of order α $(0 < \alpha < 1)$. First, a L1 approximation is employed to estimate the Caputo derivative. Then, the spatial derivative is discretized by a second-order finite difference scheme. Moreover, we combine the implicit finite difference (IFD) scheme with the proper orthogonal decomposition (POD) method to reduce the used cpu time. In other words, we obtain POD based reduced-order IFD scheme. As a result, the new scheme can be viewed as the modification of the exiting job (Mirzaee et al., 2020 [23]). The numerical results provide to verify the feasibility and efficiency of the new method.

Keywords: Fractional stochastic advection-diffusion equation, Implicit finite difference scheme, Reduced implicit finite difference scheme, Proper orthogonal decomposition method.

2020 Mathematics Subject Classification: 37L55, 65M06, 35R11

1 Introduction

Based on [24], the question arises why one should learn more about stochastic calculus? Stochastic partial differential equations (SPDEs) play an important role in a wide range of active research in mathematics, chemistry, fluid mechanics, microelectronics, theoretical physic and finance.

Fractional differential equation has attracted increasing attention due to their application in several field including mathematics, mechanics, physics, chemistry, engineering, control theory and finance.

In contrast, the huge number of studies in deterministic fractional differential equations, there have been only a few papers in connection with fractional stochastic differential equations (FSDEs) especially Caputo fractional time derivative. Previous studies have been limited to the existence and uniqueness of mild solution. Interesting papers for existence and uniqueness are found in [3, 25, 28].

In recent years, there has been a big development in numerical solution of SPDEs. For example, authors of [30] employed a compact finite difference method for solving stochastic space fractional advection-diffusion equation of Itô. They used Fourier analysis to prove stability and convergence of presented scheme similar to [27]. In [33], a Galerkin finite element method is considered for time fractional stochastic diffusion equation with multiplicative noise. A numerical scheme for the nonlinear time fractional stochastic reaction-diffusion equation is carried out in [12] wherein mixed finite element and BDF2-θ is considered to discretize in spatial and temporal directions, respectively. Kamrani [8] investigated the numerical solution of FSDEs using Galerkin method based on Jacobi polynomials. The main aim of [34] is to develop a fourth-order central difference scheme and the semi-implicit Crank-Nicolson scheme for obtaining a new fully discrete scheme of space fractional wave equation by additive and multiplicative

*Email: zsoori@mail.kntu.ac.ir (Corresponding Author)

†Email: `ataei@kntu.ac.ir`

noise. In [26], a nonlocal backward problem is proposed for fractional stochastic diffusion equations wherein the eigenfunction expansion of the solution is reduced to an integral equation. Authors of [5] considered an approximate controllability for fractional stochastic wave equation of Riemann-Liouville type.

The POD technique has a long history. The POD method is derived from eigenvector analysis method which was initially proposed by K. Pearson in 1901. The POD [6] is an effective technique to decrease the degrees of freedom. The essential features of the POD-based reduced are in the following two aspects, i.e., preserve the accuracy of numerical solutions and reduce the computational time. It has been used to develop some POD-based reduced-order numerical computational methods for the time dependent PDEs. Known studies are reduced-order finite differences, the reduced-order finite element methods and reduced-order finite volume element methods which are found in the book [13]. The POD method are needed more for scheme that are on fine grids. For example, there is more computation cost in the Richardson extrapolation algorithms on fine grids. Hence, we able to use POD technique instead of parallel computing to overcome with this problem.

A reduced-order finite element method based on POD scheme for fractional Tricomi-type equation is considered in [11]. Authors of [1, 2, 32] presented some efficient compact finite difference schemes based on POD for PDEs such as two-dimensional distributed-order Riesz space-fractional diffusion equation, multi-dimensional parabolic equation and Korteweg-de Vries equation. Some reduced finite difference schemes based on POD scheme for parabolic and hyperbolic equations are utilized in [16, 29]. Fu et al. [4] investigated a reduced-order for time fractional diffusion equation based on POD technique and disceret empirical interpolation method. Luo et al. [18] developed a reduced-order extrapolation and finite difference scheme by POD for two dimensional time-space tempered diffusion-wave equation. Authors of [9, 10] utilized a Galerkin POD for parabolic problems and general equation in fluid dynamics. For more information about POD method, one can refer to [14, 15, 17, 19, 20, 21, 22].

In this paper, we study the time FSA-DE of order α $(0<\alpha<1)$ as follows:

$$ {}_0^{\mathrm{C}}\mathcal{D}_t^{\alpha}u(x,t)=(\beta+\gamma\frac{dB(t)}{dt})\frac{\partial^2 u(x,t)}{\partial x^2}+\sigma\frac{\partial u(x,t)}{\partial x}+f(x,t),\quad (x,t)\in[a,b]\times[0,T], \tag{1} $$

with the initial condition:

$$ u(x,0)=\psi(x),\quad x\in[a,b], $$

and the boundary conditions:

$$ \begin{aligned} u(a,t)&=\varphi_1(t),\quad t\in[0,T],\\ u(b,t)&=\varphi_2(t),\quad t\in[0,T], \end{aligned} $$

where ${}_0^{\mathrm{C}}\mathcal{D}_t^{\alpha}$ is the $\alpha-th$ Caputo fractional derivative defined by

$$ {}_0^{\mathrm{C}}\mathcal{D}_t^{\alpha}u(x,t)=\frac{1}{\Gamma(1-\alpha)}\int_0^t\frac{\partial^2 u(x,t)}{\partial s^2}(t-s)^{-\alpha}\mathrm{ds}, $$

here, β,γ,σ are real constants, $\psi(x),\varphi_1(t),\varphi_2(t)$ are the stochastic process defined on the propability space $(\Omega,\mathcal{F},\mathcal{P})$, $f(x,t)$ is a known function and $u(x,t)$ is an unknown stochastic process which should be estimated. The term $B(t)$ denots one-dimensional standard Brownian motion process which satisfy in the following properties :
1) $B(0)=0$.
2) For all $0\le s<t<T$, $B(t)-B(s)$ is random variable with expectation zero and variance $t-s$. Therefore, $B(t)-B(s)\sim\sqrt{t-s}\mathcal{N}(0,1)$ denotes normal distribution with expactation zero and variance 1.
3) For $0\le s<t<u<v<T$, the increments $B(t)-B(s)$ and $B(v)-B(u)$ are independent.

We point that the Brownian motion is a function very commonly used in stochastic calculus. It is a continuous process but it is not a differentiable function.

In this paper, first, we employ the classical L1 formula to approximate the Caputo fractional derivative of order α $(0<\alpha<1)$ and the second-order IFD scheme for discretization of spatial derivatives.

Then, combination of POD technique and IFD scheme is considered for FSA-DE wherein POD-IFD is constructed. It can not be only reduced into a scheme with lower dimension number, but also gurantee high accuracy. The error analysis is discussed as well. What distinguishes the current paper from previous works is its numerical solution aspect. To our knowledge, the POD-IFD scheme has never employed to solve time FSA-DE.

the outline of this paper is as follows: In Section 2, the IFD scheme is employed to approximate spatial derivatives and the classical L1 formula to discretize time Caputo fractional derivative. In Section 3, we introduce the POD method and then we combine the POD technique with the IFD scheme. In Section 4, The analysis of errors for the IFD and RIFD schemes are discussed. Two numerical examples have been included in Section 5 to verify the accuracy and efficiency of our proposed method. Finally, concluding remarks are given in Section 6.

2 Numerical scheme

In this section, we construct difference approximation to the Eq. (1). For this purpose, we define $x_i = a + ih$, $i = 0, 1, \ldots, M$, $t_n = n\tau$, $n = 0, 1, \ldots, N$, wherein $h = \frac{b-a}{M}$ and $\tau = \frac{T}{N}$ are the spatial and temporal step sizes, respectively, and M, N are some given positive integers. For any grid function $u = \{u_i^n | 1 \le i \le M, 0 \le n \le N\}$, denote

$$\hat{\delta}_x u_i^n = \frac{u_{i+1}^n - u_{i-1}^n}{2h}, \qquad \delta_x^2 u_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2}, \qquad \frac{dB}{dt} = \frac{B_n - B_{n-1}}{\tau}. \tag{2}$$

Here, we employ the L1 formula [35] to approximate the Caputo fractional derivative as follows:

$$\begin{aligned}
{}_0^{\mathrm{C}}\mathcal{D}_t^{\alpha} u(x, t_n) &= \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} \frac{\partial u(x,s)}{\partial s} (t_n - s)^{-\alpha} \mathrm{ds} \\
&\approx \frac{1}{\Gamma(1-\alpha)} \sum_{k=0}^{n-1} \frac{u(x, t_{k+1}) - u(x, t_k)}{\tau} \int_{t_k}^{t_{k+1}} (t_n - s)^{-\alpha} \mathrm{ds} \\
&= \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \sum_{k=0}^{n-1} a_k \big[u(x, t_{n-k}) - u(x, t_{n-k-1})\big] + O(\tau^{1-\alpha}),
\end{aligned} \tag{3}$$

Lemma 1 *The coefficients* $a_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}, \quad k = 0, 1, \ldots, n-1$, *satisfy*

$(1) 1 = a_0 > a_1 > a_2 > \ldots > a_k \ldots \longrightarrow 0,$

$(2) (1-\alpha)(k+1)^{-\alpha} < a_k < (1-\alpha)(k)^{-\alpha}.$

Substituting Eqs. (2) and (3) into Eq. (1), the IFD is obtained as follows:

$$\begin{aligned}
\frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \big[u_i^n - \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) u_i^k - a_{n-1} u_i^0\big] &= (\beta + \gamma \frac{B_n - B_{n-1}}{\tau}) \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} \\
&\quad + \sigma \frac{u_{i+1}^n - u_{i-1}^n}{2h} + f_i^n, \\
&\quad 1 \le i \le M-1, \quad 1 \le n \le N.
\end{aligned} \tag{4}$$

After simplification, the above equation can be rewritten in the following form :

$$\begin{aligned}
&\Big[-\frac{\mu}{h^2} (\beta + \gamma (\frac{B_n - B_{n-1}}{\tau})) - \frac{\mu\sigma}{2h} \Big] u_{i+1}^n + \Big[1 + \frac{2\mu}{h^2} (\beta + \gamma (\frac{B_n - B_{n-1}}{\tau})) \Big] u_i^n \\
&+ (-\frac{\mu}{h^2} (\beta + \gamma (-\frac{B_n - B_{n-1}}{\tau})) + \frac{\mu\sigma}{2h}) u_{i-1}^n = \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) u_i^k - a_{n-1} u_i^0 + \mu f_i^n, \\
&i = 1, 2, \ldots, M-1,
\end{aligned} \tag{5}$$

where $\mu = \tau^{\alpha}\Gamma(2-\alpha)$. In order to facilitate computations, the difference scheme (5) can be represented to the following matrix-vector multiplication:

$$\begin{cases} K_n u^1 = \mu F^1 + G^1, \\ K_n u^n = c_1 u^{n-1} + c_2 u^{n-2} + \ldots + c_{n-1} u^{n-1} + a_{n-1} u^0 + \mu F^n + G^n, \qquad n > 1, \end{cases} \tag{6}$$

where the tridiagonal matrices in (6) are given by

$$K_n = \text{tri}\Big[-\frac{\mu}{h^2}\big(\beta + \gamma(\frac{B_n - B_{n-1}}{\tau})\big) - \frac{\mu\sigma}{2h}, 1 + \frac{2\mu}{h^2}\big(\beta + \gamma(\frac{B_n - B_{n-1}}{\tau})\big), -\frac{\mu}{h^2}\big(\beta + \gamma(\frac{B_n - B_{n-1}}{\tau})\big) + \frac{\mu\sigma}{2h}\Big],$$

$$F^n = \Big[f_1^n, f_2^n, \ldots, f_{M-1}^n \Big]^T,$$

$$G^n = \Big[-\frac{\mu}{h^2}\big(\beta + \gamma(\frac{B_n - B_{n-1}}{\tau})\big) + \frac{\mu\sigma}{2h}, 0, \ldots, 0, -\frac{\mu}{h^2}\big(\beta + \gamma(\frac{B_n - B_{n-1}}{\tau})\big) - \frac{\mu\sigma}{2h}\Big]^T,$$

and $c_n = a_{n-1} - a_n (n = 1, 2, \ldots, N)$. The approximate solutions $\{u_i^n\} \quad (i = 1, 2, \ldots, M-1)$ are obtained from solving IFD scheme (6).

3 The RIFD scheme based on POD method

in this section, we employ the POD technique for creating the RIFD scheme. There exist different interpretations for the POD method. Three of the most methods are Karhunen-Loeve decomposition (KLD), the principal component analysis (PCA), and the singular value decomposition (SVD). In this paper, we use the direct form of the POD based on SVD [16].

3.1 Formulate the POD basis

Step 1. Form snapshots

For this aim, we choose first $L \ll N$ sequence of solutions $\{u_i^n\}_{n=1}^{L} \quad (i = 1, 2, \ldots, M-1)$ from the N sequence of approximate solutions $\{u_i^n\}_{n=1}^{N} \quad (i = 1, 2, \ldots, M-1)$ of IFD (6).

$$S = \begin{pmatrix} u_1^1 & u_1^2 & \ldots & u_1^L \\ u_2^1 & u_2^2 & \ldots & u_2^L \\ \vdots & \vdots & \ddots & \vdots \\ u_M^1 & u_M^2 & \ldots & u_M^L \end{pmatrix}_{M-1 \times L}, \tag{7}$$

Step 2. apply the SVD form on Matrix S

$$S = U \begin{pmatrix} D_r & 0 \\ 0 & 0 \end{pmatrix} V^T,$$

where $D_r = \text{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$. The singular values σ_i can be arranged as $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$ and $r = \text{rank(S)}$. $U = U_{M-1\times M-1}$ and $V = V_{L\times L}$ are orthogonal matrices. The matrice $U = (\Phi_1, \Phi_2, \ldots, \Phi_{M-1})$ and $V = (\psi_1, \psi_2, \ldots, \psi_{M-1})$ contain the orthogonal eigenvalues to the SS^T and $S^T S$, respectively and $\lambda_i = \sigma_i^2 (i = 1, 2, \ldots, r)$. We define a projection P_M by

$$P_M(S^L) = \sum_{j=1}^{m} (\Phi_j, S^l)\phi_j, \qquad (l = 1, 2, \ldots, L), \tag{8}$$

where $S^l = \big[u_1^l, u_2^l, \ldots, u_{M-1}^l\big]$, besides, $m < r$ and (Φ_j, S^l) is inner product of vectors ϕ_j and S^l. The following inequality for orthogonal projection are result:

$$\big|s^l - P_m(s^l)\big|_2 \leq \sigma_{m+1} = \sqrt{\lambda_{m+1}}. \tag{9}$$

The $\{\Phi_i\}_{i=1}^m$ is a set of optimal basis and $\Phi = (\Phi_1, \Phi_2, \dots, \Phi_m)$ is s matrix created by the orthogonal eigenvectors such that $\Phi^T\Phi = \mathrm{I}$. Now, we construct a RIFD scheme, if u_n of (6) is substituted by

$$P_m(u^n) = u^{*n} = \Phi w^n = \Phi_{(M-1)\times m}(w^n)_{m\times 1}, \qquad n = 0, 1, \dots, N. \tag{10}$$

Considering $\Phi^T\Phi = \mathrm{I}$, we obtain RIFD scheme as follows:

$$\begin{cases} Hw^1 = \mu H\Phi^T F^1 + \Phi^T G^1, \\ Hw^n = c_1 w^{n-1} + c_2 w^{n-2} + \dots + c_{n-1}w^{n-1} + a_{n-1}w^0 + \mu\Phi^T F^n + \Phi^T G^n, \qquad n > 1, \end{cases} \tag{11}$$

where $H = \Phi^T K \Phi$. Having computed w^n from (11), we obtain POD optimal solution $u^{*n} = \Phi w^n$. The RIFD only contains $m \times N$ equations, while IFD contains $(M-1)\times N$ equations (usually $m \ll M-1$). In fact, the number of degrees of freedom in RIFD scheme (11) reduces in comparision with IFD (11). Henece, we use RIFD method.

4 Error estimation

This section is devoted in analysing the errors of the IFD and RIFD solutions. First, we state the following remark and that which is basic in the whole theory.

Remark

Matrix K_n in IFD scheme (6) is not a symmetric tridiagonal matrix. By Numerical computations, assume that each product of off-diagonal entires is strictly positive $b_i c_i$.
A transform matrix D define as follows:

$$D = \mathrm{diag}(\delta_1, \dots, \delta_n),$$

and

$$\delta_i = \begin{cases} 1, & i = 1, \\ \sqrt{\frac{c_{i-1}\dots c_1}{b_{i-1}\dots b_1}}, & i = 2, \dots n-1. \end{cases}$$

Based on [36], a symmetric tridiagonal matrix J can be obtained as follows:

$$J = D^{-1}TD = \begin{pmatrix} a_1 & \mathrm{sgn} b_1\sqrt{b_1c_1} & & & \\ \mathrm{sgn} b_1\sqrt{b_1c_1} & a_2 & \mathrm{sgn} b_2\sqrt{b_2c_2} & & \\ & \ddots & \ddots & \ddots & \\ & & & & \mathrm{sgn} b_{n-1}\sqrt{b_{n-1}c_{n-1}} \\ & & & \mathrm{sgn} b_{n-1}\sqrt{b_{n-1}c_{n-1}} & a_n \end{pmatrix}, \tag{12}$$

For simplicity, we assume that th entires of K_n define as follows:

$$T = \begin{pmatrix} a_1 & b_1 & & & \\ c_1 & a_2 & b_2 & & \\ & \ddots & \ddots & \ddots & \\ & & & & b_{n-1} \\ & & & c_{n-1} & a_n \end{pmatrix}.$$

Now, matrices of J and T have the same eigenvalues. Here, matrix $K_n = T$. Now, we can obtain the eigenvalues of martix K_n by Thomas algorithm [31]:

$$\lambda_i(K_n) = 1 + \frac{2\mu}{h^2}\left(\beta + \gamma(\frac{B_n - Bn - 1}{\tau})\right) + 2\sqrt{\frac{\mu}{h^2}(\frac{B_n - Bn - 1}{\tau}) - \frac{\mu^2\sigma^2}{4h^2}}\cos(\frac{i\pi}{M}). \tag{13}$$

Theorem 2 *Let u^n be the solution of (6) and U^n the exact solution (6), then*

$$\|U^n - u^n\| \leq C_n \sum_{k=1}^{n} \theta_{n,k}(h^2 + \tau^{1-\alpha}).$$

Proof. From (6), we get:

$$K_n U^n = \sum_{k=1}^{n} (a_{n-k-1} - a_{n-k}) U^k + a_{n-1} U^0 + \mu F^n + G^n + T^n, \quad 1 \leq n \leq N, \tag{14}$$

where T^n be the local truncation error. Let $e^n = U^n - u^n$ and $e^0 = 0$. Subtracting (6) from (14), we obtain:

$$K_n e^n = \sum_{k=1}^{n} (a_{n-k-1} - a_{n-k}) e^k + T^n, \quad 1 \leq n \leq N,$$

utilizing the inner product with e^n, we get:

$$(K_n e^n, e^n) = \sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k})(e^k, e^n) + (T, e^n), \quad 1 \leq n \leq N. \tag{15}$$

For any symmetric matrix R, we have the following properties of Rayleigh-Ritz ratio from [7] as follows:

$$\lambda_{\min}(R) \leq \frac{(Rv, v)}{(v, v)} \leq \lambda_{\max}(R), \tag{16}$$

which v is a vector in R^{M-1} and $v \neq 0$. Hence, we have:

$$\lambda_{\min}(K_n)\|e^n\|_2^2 = \lambda_{\min}(K_n)(e^n, e^n) \leq (K_n e^n, e^n).$$

From the above equation, Eq. (15) becomes:

$$\|e^n\|_2 \leq \frac{1}{\lambda_{\min}(K_n)} \left(\sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \|e^k\|_2 + \|T^n\|_2 \right), \quad 1 \leq n \leq N.$$

From (12), we have $\frac{1}{\lambda_{\min}(K_n)} \leq 1 + 2\frac{\mu}{h^2}\left(\beta + \gamma\left(\frac{B_n - B_{n-1}}{\tau}\right)\right)$. Therefore, we obtain:

$$\|e^n\|_2 \leq C_n \left(\sum_{k=1}^{n-1} (a_{n-k-1} - a_{n-k}) \|e^k\|_2 + \|T^n\|_2 \right), \quad 1 \leq n \leq N,$$

where $C_n = 1 + 2\frac{\mu}{h^2}\left(\beta + \gamma\left(\frac{B_n - B_{n-1}}{\tau}\right)\right)$. By mathematical induction, we can obtain:

$$\|e^n\|_2 \leq C_n \sum_{k=1}^{n-1} \theta_{n,k} \|T^k\|_2, \quad 1 \leq n \leq N,$$

where

$$\theta_{n,j} = \sum_{k=1}^{n-j} C_{k+j-1}(a_{n-(k+j-1)-1} - a_{n-(j+j-1)})\theta_{k+j-1,j}.$$

From Eqs. (2) and (3) , we obtain:

$$\|e^n\| \leq C_n \sum_{k=1}^{n} \theta_{n,k}(h^2 + \tau^{1-\alpha}).$$

Theorem 3 *Let u^n and u^{*n} be the solution vectors of (6) and (11), respectively. If we consider the first $L \ll N$ sequence of solutions $\{u_i^n\}_{n=1}^L, i = 1, 2, \dots, M-1$, from the N sequence solutions $\{u_i^n\}_{n=1}^N, i = 1, 2, \dots, M-1$, as snapshots, then*

$$\|u^{*n} - u^n\|_2 \le \sigma_{m+1}, \quad n = 1, 2, \dots, L,$$

and

$$\|u^{*n} - u^n\|_2 \le CL\sigma_{m+1}, \quad n = L+1, \dots, N.$$

Proof. Let $e^{*n} = u^{*n} - u^n$. From (9), we get:

$$\|e^{*n}\|_2 = \|u^{*n} - u^n\|_2 \le \sigma_{m+1}, \quad n = 1, 2, \dots, L, \tag{17}$$

once $n = L+1, \dots, N$, replacing u^n in (6) by u^{*n}, we obtain:

$$K_n u^{*n} = \sum_{k=1}^{n-1}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})u^{*n} + a_{n-1}u^{*0} + \mu F^n + G^n, \tag{18}$$

By subtracting (18) from (6) and utilizing the inner product with e^{*n}, we can obtain:

$$(K_n e^{*n}, e^{*n}) = \sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})(K_n e^{*k}, e^{*n}), \quad n = L+1, \dots, N.$$

From (12), the above equation can be rewritten as:

$$\|e^{*n}\|_2 \le \frac{1}{\lambda_{\min}(K_n)}\sum_{k=1}^{n-1}(a_{n-k-1} - a_{n-k})\|e^{*k}\|_2.$$

From Lemmma 1, it follows that

$$a_{n-k-1} - a_{n-k} \le a_{n-k-1} < a_0 \le 1.$$

Now, From the above inequality and mathematical induction, we obtain:

$$\begin{aligned}\|e^{*n}\|_2 \le C_n \sum_{k=1}^{n-1} \|e^{*k}\|_2 =\le C_n \sum_{k=1}^{L} \|e^{*k}\|_2 + \sum_{k=L+1}^{n-1} \|e^{*k}\| \\ \le C_n(L\sigma_{m+1} + \sum_{k=L+1}^{n-1} \|e^{*k}\|_2) \le C_n(CL\sigma_{m+1}).\end{aligned}$$

Therefore, the theorem is proved.

Theorem 4 *Under the conditions of Theorem 3, let u^{*n} be the solution vector of the RIFD scheme (11) and u^n be the solution vector of the IFD scheme(6), then we have:*

$$\|u^{n*} - u^n\| \le \sigma_{m+1} + C_n(h^2 + \tau^{1-\alpha}), \quad n = 1, 2, \dots, L,$$

and

$$\|u^{n*} - u^n\| \le C_n(CL\sigma_{m+1}) + C_n(h^2 + \tau^{1-\alpha}), \quad n = L+1, 2, \dots, N,$$

where m is the number of POD bases.

5 Numerical experiments

In this section, we present two experiments to verify the feasibility and efficiency of the RIFD scheme based on the POD method.

Experiment 1:

Consider the time fractional stochastic advection-diffusion type equation as follows:

$$ {}_0^C\mathcal{D}_t^{\alpha} u(x,t) = \left(\frac{1}{\pi^2} + \frac{dB(t)}{dt} \right) \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad (x,t) \in [0,1] \times [0,1], \tag{19} $$

with the initial condition:

$$ u(x,0) = 0, $$

and the boundary conditions:

$$ u(0,t) = 0, $$
$$ u(1,t) = 0, $$

where $f(x,t) = \frac{2\tau^{2-\alpha}\sin(\pi x)}{\Gamma(3-\alpha)} + \left(\frac{1}{\pi^2} + \frac{\mathrm{dB}}{\mathrm{dt}}\right)\pi^2 t^2 \sin(\pi x) - \pi t^2 \cos(\pi x)$. The exact solution is $u(x,t) = t^2 \sin(\pi x)$.

Table 1: Comparison of exact and approximate solutions with $h = \tau = \frac{1}{200}$ of Experiment 1.

x	Exact	Approximate			
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
0.1	0.30901699	0.30904178	0.30904202	0.30904760	0.30903757
0.2	0.58778525	0.58783269	0.58783312	0.58784369	0.58782973
0.3	0.80901699	0.80908265	0.80908321	0.80909770	0.80908988
0.4	0.95105652	0.95113401	0.95113470	0.95115169	0.95116014
0.5	1.00000000	1.00008110	1.00008256	1.00010037	1.00013076
0.6	0.95105694	0.95113489	0.95113536	0.95115226	0.95120253
0.7	0.80901694	0.80908400	0.80908435	0.80909869	0.80915854
0.8	0.58778525	0.58783420	0.58783441	0.58784480	0.58789851
0.9	0.30901699	0.30904287	0.30904297	0.30904842	0.30908019

Eq. (19) is solved with the help of IFD scheme (6) with the $M, N = 100$ wherein exact and approximate solutions for different values of $\alpha = 0.2, 0.4, 0.6$ and 0.8 are tested. Table 1 confirms that the approximate solutions are colse to exact solutions. Figure 1 verify the above mentioned solutions for values of $\alpha = 0.2, 0.4, 0.6$ and 0.8.

Table 2: The maximum absolute errors and CPU time in scheme (6) of Experiment 1.

N	M	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
25	50	3.45×10^{-4}	4.05×10^{-4}	7.29×10^{-4}
		$0.28s$	$0.24s$	$0.23s$
50	100	7.94×10^{-5}	6.04×10^{-5}	7.10×10^{-5}
		$6.08s$	$6.38s$	$6.48s$
100	200	2.03×10^{-5}	1.06×10^{-4}	7.10×10^{-5}
		$39.01s$	$36.30s$	$41.66s$

Table 3: The maximum absolute errors and CPU time in scheme (11) of Experiment 1.

(N, L, m)	M	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
$(25, 5, 4)$	50	3.45×10^{-4}	4.05×10^{-4}	7.29×10^{-4}
		$0.09s$	$0.14s$	$0.1s$
$(50, 5, 4)$	100	7.94×10^{-5}	6.04×10^{-5}	7.10×10^{-5}
		$1.10s$	$1.12s$	$1.06s$
$(100, 5, 4)$	200	2.03×10^{-5}	1.06×10^{-4}	7.10×10^{-5}
		$2.71s$	$3.02s$	$2.93s$

What extracts from Tables 2 and 3 is that in RIFD scheme the computational time is less than IFD scheme while the accuracy is preserved. The above tables confirm preference of the RIFD than IFD. Figure 2 shows this fact for $m = 4$ and $L = 7$.

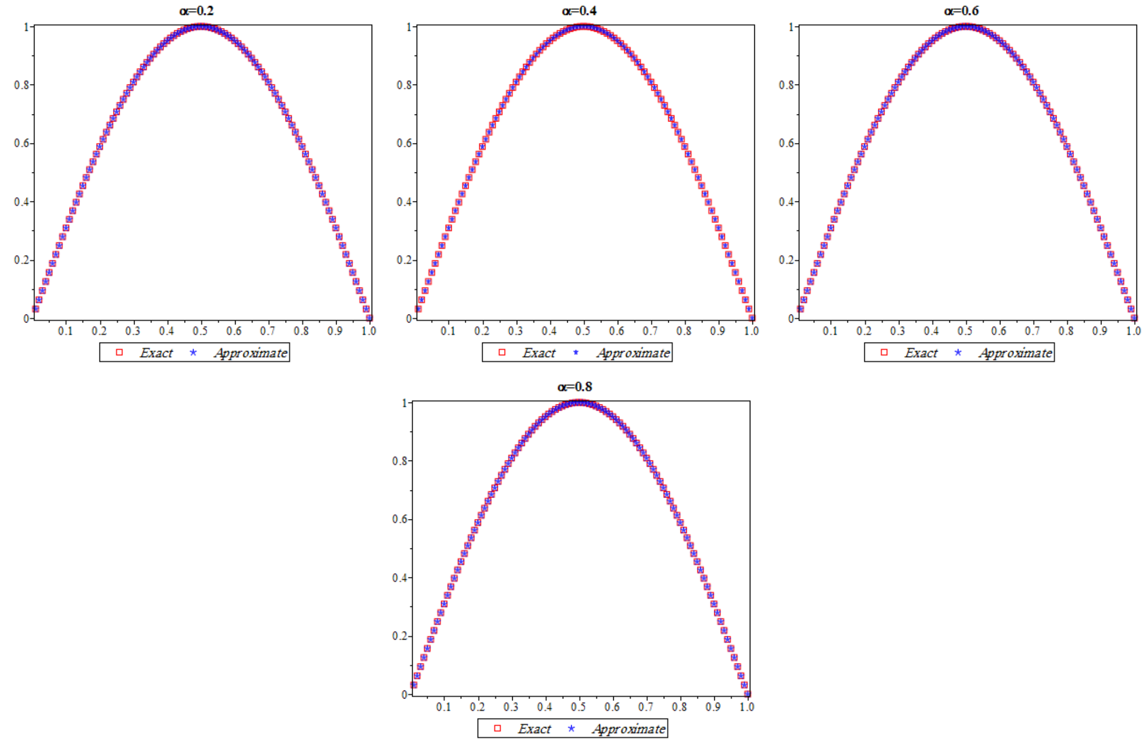

Figure 1:
Plots of the exact solution and numerical solution at $T = 1$ with $h = \tau = \frac{1}{200}$ of Experiment 1.

Experiment 2:

Consider the time fractional stochastic advection-diffusion type equation as follows:

$$ {}_{0}^{\mathrm{C}}\mathcal{D}_t^{\alpha} u(x,t) = \left(1 + \frac{dB(t)}{dt}\right) \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} + f(x,t), \quad (x,t) \in [0,1] \times [0,1], \tag{20} $$

with the initial condition :

$$ u(x,0) = x^3 \sin^2(x), \quad x \in [0,1], $$

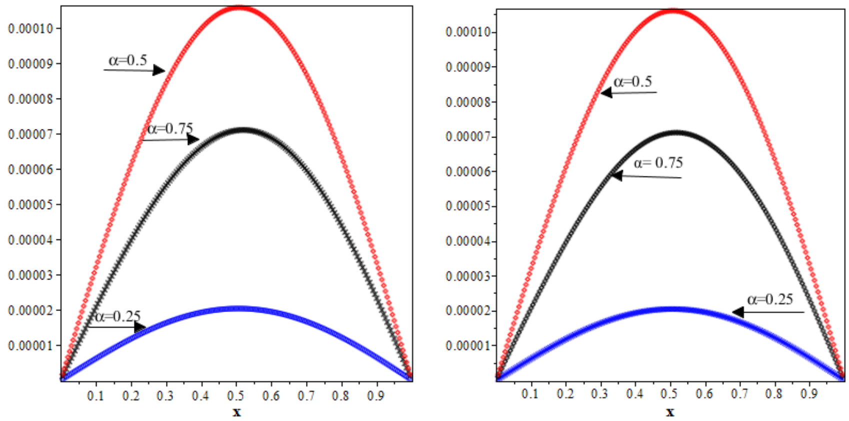

Figure 2:
The error curves RIFD (right) and IFD (left) schemes at $T = 1$ with $h = \tau = \frac{1}{200}$ of Experiment1.

and the boundary conditions:

$$\begin{aligned} &u(0,t) = 0, \\ &u(1,t) = (t+1)^3 \sin^2(1), \quad t \in [0,1]. \end{aligned}$$

The exact solution is $u(x,t) = (t+x)^3 \sin^2(x)$.

Table 4: Comparison of exact and approximate solutions with $h = \tau = \frac{1}{200}$ of Experiment 2.

x	Exact	Approximate			
		$\alpha = 0.2$	$\alpha = 0.4$	$\alpha = 0.6$	$\alpha = 0.8$
0.1	0.01326569	0.01326496	0.01327203	0.01327957	0.01254461
0.2	0.06819635	0.06819635	0.06821064	0.08822284	0.06688063
0.3	0.19186883	0.19185156	0.19187281	0.19188511	0.19039249
0.4	0.41611839	0.41608838	0.41611604	0.41612591	0.41512638
0.5	0.77573986	0.77569711	0.77572984	0.77574066	0.77543421
0.6	1.30589132	1.30583877	1.30587406	1.30589462	1.30592716
0.7	2.03897571	2.03891953	2.03895351	2.03899038	2.03930321
0.8	3.00114581	3.00109540	3.00112301	3.00117005	3.00218402
0.9	4.20868958	4.20865723	4.20867307	4.20870845	4.20996679

In Table 4, exact and approximate solutions for different values of $\alpha = 0.2, 0.4, 0.6$ and 0.8 for $M, N = 100$ are tested. Table 4 confirms that the approximate solutions are colse to exact solutions that this fact is shown in Figure 3.

Table 5: The maximum absolute errors and CPU time in scheme (6) of Experiment 2.

N	M	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
50	50	2.38×10^{-4}	3.58×10^{-4}	8.29×10^{-4}
		$4.82s$	$5.00s$	$5.14s$
100	100	1.70×10^{-5}	3.79×10^{-5}	3.69×10^{-5}
		$29.95s$	$27.18s$	$28.45s$

Table 6: The maximum absolute errors and CPU time in scheme (11) of Experiment 2.

(N, L, m)	M	$\alpha = 0.25$	$\alpha = 0.5$	$\alpha = 0.75$
$(50, 10, 6)$	50	2.38×10^{-4}	3.58×10^{-4}	8.29×10^{-4}
		$2.15s$	$2.10s$	$1.81s$
$(100, 11, 7)$	100	1.70×10^{-5}	3.79×10^{-5}	3.69×10^{-5}
		$5.53s$	$5.64s$	$5.42s$

From Tables 5 and 6, we conclude that RIFD scheme is better than in sence that time taken is less compared with IFD scheme.

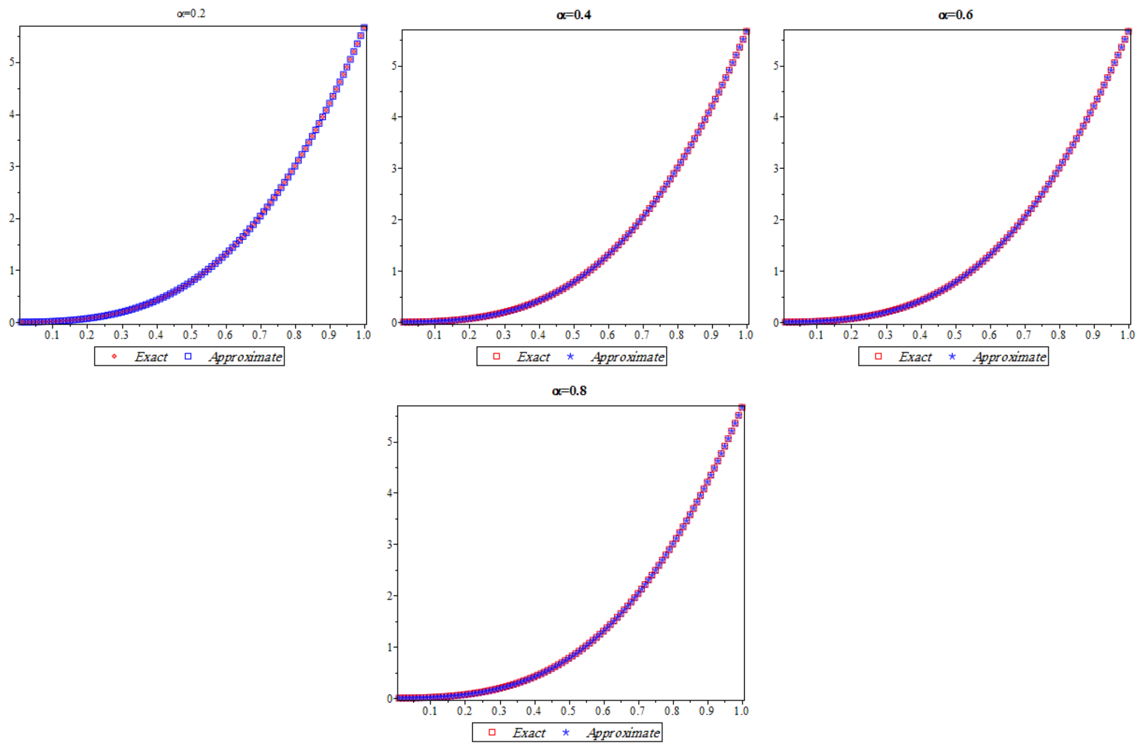

Figure 3:
Plots of the exact solution and approximate solution at $T = 1$ with $h = \tau = \frac{1}{200}$ of Experiment 2.

Figure 4 shows the error curves for the IFD scheme with $h = t = \frac{1}{200}$ and with $L = 11, m = 7$ for RIFD scheme which are observing alike.

6 Conclusions

In this paper, we have benefitted from the POD technique to derive reduced IFD scheme for FSA-DE in order to make the proposed scheme better and useful than previous studies. The main features of the paper is to introduce a new scheme for FSA-DE in order to preserve accuracy and alleviate cpu time. We have tested the correctness of our scheme with two numerical experiments. Tables and figures confirm the efficiency of the presented scheme.

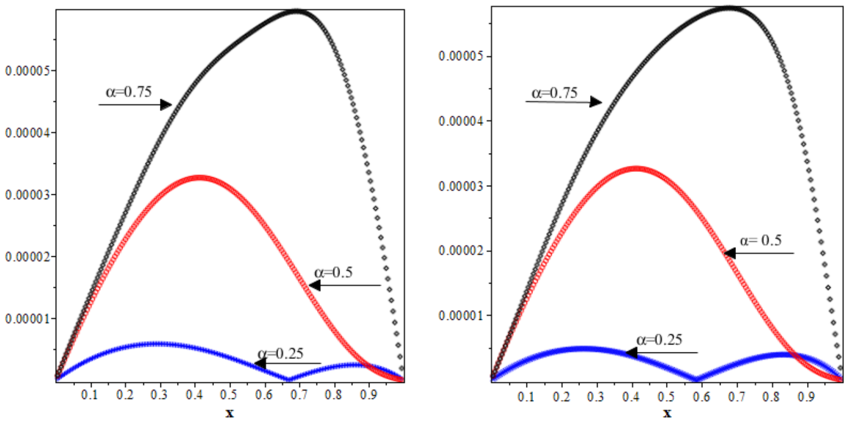

Figure 4:
The error curves RIFD (right) and IFD (left) schemes at $T = 1$ with $h = \tau = \frac{1}{200}$ of Experiment 2.

References

[1] M. Abbaszadeh, M. Dehghan, A POD-based reduced-order Crank-Nicolson/fourth-order alternating direction implicit (ADI) finite difference scheme for solving the two-dimensional distributed-order Riesz space-fractional diffusion equation, Appl. Numer. Math., 158 (2020) 271-291.

[2] B. Xu, X. Zhang, An efficient high-order compact finite difference scheme based on proper orthogonal decomposition for the multi-dimensional parabolic equation, Adv. Diff. Equ., 341 (2019) 1-22.

[3] A.T. Ciprian, Analysis of Variations for Self-Similar Processes: A Stochastic Calculus Approach, Springer, 2013.

[4] H. Fu, H. Wang, Z. Wang, POD/DEIM reduced-order modeling of time-fractional partial differential equations with applications in parameter identication, J. Sci. Comput., doi 10.1007/s10915-017-0433-8.

[5] J.W. He, L. Peng, Approximate controllability for a class of fractional stochastic wave equations, Comput. Math. Appl., 78 (2019) 1463-1476.

[6] P. Holmes , J. Lumley, G. Berkooz, C.W. Rowely, Turbulence, Coherent Structures, Dynamical Systems and Symmetry, 2nd Ed., Cambridge University Press, 2012.

[7] R. A. Horn and C. R. Johnson, Matrix Analysis, Cambridge University, Cambridge, 1985.

[8] M. Kamrani, Numerical solution of stochastic fractional differential equations, Numer. Algorithms., 68 (2015) 81-93.

[9] K. Kunisch , S. Volkwein , Galerkin proper orthogonal decomposition methods for parabolic problems, Numer. Math., 90 (2001) 117-148 .

[10] K. Kunisch , S. Volkwein , Galerkin proper orthogonal decomposition methods for a general equation in fluid dynamics, SIAM J. Numer. Anal., 40 (2002) 492-515.

[11] J. Liu, H. Ll, Y. Liu, Z. Fang, Reduced-order finite element method based on POD for fractional Tricomi-type equation, Appl. Math. Mech., doi 10.1007/s10483-016-2078-8.

[12] X. Liu, X. Yang, Mixed finite element method for the nonlinear time-fractional stochastic fourth-order reaction-diffusion equation, Comput. Math. Appl., 84 (2021) 39-55.

[13] Z. Luo, G. Chen, Proper Orthogonal Decomposition Methods for Partial Differential Equations, Mathematics in Science and Engineering, 2019.

[14] Z.D. Luo , J. Chen , I.M. Navon , X. Yang , Mixed finite element formulation and error estimates based on proper orthogonal decomposition for the non-stationary Navier-Stokes equations, SIAM J. Numer. Anal., 47 (2008) 1-19.

[15] Z.D. Luo , J. Chen , P. Sun , X. Yang , Finite element formulation based on proper orthogonal decomposition for parabolic equations, Sci. China Ser. A. Math., 52 (2009) 587-596 .

[16] Z. Luo, S. Jin, J. Chen, A reduced-order extrapolation central scheme based on POD for two-dimensional fourth-order hyperbolic equations, Appl. Math. Comput., 289 (2016) 396-408.

[17] Z.D. Luo , Q.L. Ou , J.R. Wu , Z.H. Xie , A reduced FE formulation based on POD for two-dimensional hyperbolic equation, Acta Math. Sci., 32 (2012) 1997-2009.

[18] Z. D. Luo, H. Wang, A highly efficient reduced-order extrapolated finite difference algorithm for time-space tempered fractional diffusion-wave equation, Appl. Math. Lett., 102 (2020) 1-8.

[19] Z.D. Luo , J. Zhu , R.W. Wang , I.M. Navon , Proper orthogonal decomposition approach and error estimation of mixed finite element methods for the tropical pacic ocean reduced gravity model, Comput. Meth. Appl. Mech. Eng., 196 (2007) 4184-4195.

[20] Z.D. Luo , Y.J. Zhou , X. Yang , A reduced finite element formulation based on proper orthogonal decomposition for burgers equation, Appl. Numer. Math., 59 (2009) 1933-1946.

[21] Z.D. Luo , H. Li , Y.J. Zhou , Z.H. Xie , A reduced finite element formulation and error estimates based on POD method for two-dimensional solute transport problems, J. Math. Anal. Appl., 385 (2012) 371-383.

[22] Z.D. Luo , Z.H. Xie , Y.Q. Shang , J. Chen , A reduced finite volume element formulation based on POD for parabolic equations, J. Comput. Appl. Math., 235 (2011) 2098-2111 .

[23] F. Mirzaee, K. Sayevand, S. Rezaei and N. Samadyar, Finite difference and spline approximation for solving fractional stochastic advection-diffusion equation, Iran. J. Sci. Technol. Trans. Sci., doi10.1007/s40995-020-01036-6.

[24] B. Oksendal, Stochastic Differential Equations, Springer, New York, 2000.

[25] J. C. Pedjeu, G. S. Ladde, Stochastic fractional differential equations: modeling, method and analysis, Chaos, Solitons Fractals, 45 (2012) 279-293.

[26] L. Peng, Y. Huang, On nonlocal backward problems for fractional stochastic diffusion equations, Comput. Math. Appl., 78 (2019) 1450-1462.

[27] C. Roth, Difference methods for stochastic partial differential equations. Z. Angew. Math. Mech., 82(2002) 821-830.

[28] R. Sakthivel, P. Revathi, Y. Ren, Existence of solutions for nonlinear fractional stochastic differential equations. Nonlinear Anal. TMA., 81 (2013) 70-86.

[29] P. Sun, Z. Luo, Y. Zhou, Some reduced finite difference shcemes based on a proper orthogonal decomposition technique for parabolic equations, Appl. Numer. Math., 60 (2010) 154-164.

[30] N.H. Sweilam, D.M. El-Sakout, M.M. Muttardi, Compact finite difference method to numerically solving a stochastic fractional advection-diffusion equation, Adv. Differ. Equ., 189 (2020) 1-20.

[31] J. W. Thomas, Numerical Partial Differential Equations: Finite Difference Methods, Texts Appl. Math., 22, Springer, New York, 1995.

[32] X. Zhang, P. Zhang, A reduced high-order compact finite difference scheme based on proper orthogonal decomposition technique for KdV equation, Appl. Math. Comput., 339 (2018) 535-545.

[33] G. Zou, Galerkin finite element method for time-fractional stochastic diffusion equations, Comp. Appl. Math., https://doi.org/10.1007/s40314-018-0609-3.

[34] Y. Zhou, J. Xie, Z. Zhang, Highly efficient difference methods for stochastic space fractional wave equation driven by additive and multiplicative noise, Appl. Math. Lett., 116 (2021) 1-8.

[35] P. Zhuang, F. Liu, Implicit difference approximation for the time fractional diffusion equation, J. Appl. Math. Comput., 22 (2006) 87-99.

[36] www.math.hkbu.edu.hk math lecture.